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Research article

Numerical differentiation for two-dimensional scattered data on arbitrary domain base on Hermite extension with an implicit iteration process

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Abstract: In this paper, we develop a method for numerical differentiation of two-dimensional scattered input data on arbitrary domain. A Hermite extension approach is used to realize the approximation and a modified implicit iteration method is proposed to stabilize the approximation process. For functions with various smooth conditions, the numerical solution process of the method is uniform. The error estimates are obtained and numerical results show that the new method is effective. The advantage of the method is that it can solve the problem in any domain.

Keywords: numerical differentiation; ill posed problem; regularization; implicit iteration; Hermite extension
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1. Introduction

In the application of many mathematical physics problems, we need to estimate derivatives of an unknown function from given noisy data. It turns out to be an ill-posed problem, which means, the small errors in the measurement data can induce huge errors in its computed derivatives. Many methods and techniques have been proposed regarding this topic [1–17]. According to type of regularization techniques, these methods can be classified into finite difference methods, mollification methods, differentiation by integration method and Tikhonov methods. Most of these methods are for one-dimensional case, but only a few are for high-dimensional cases [18–20]. These methods of dealing with two-dimensional problems are basically aimed at the regular region, and there are few methods that can deal with the numerical differentiation problem on the irregular domain. For example, if the data are given at scatter points of the domain in Figure 1, numerical implementation of most existing methods is difficult.

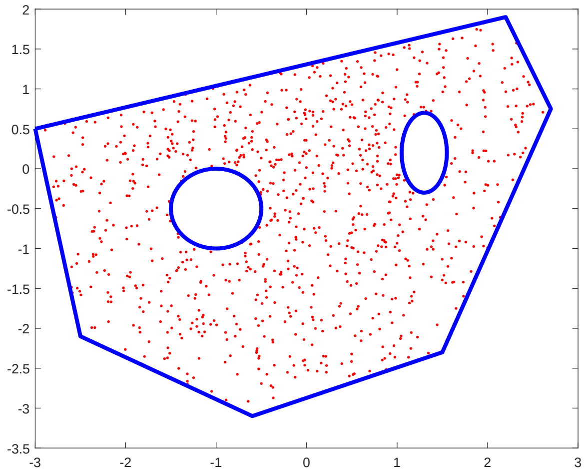


Figure 1. Irregular region.

By the extension theorem [21], any function on sub-domain of $\mathbb{R}^2$ can be extended to $\mathbb{R}^2$ without losing its smoothness. There are many work of extension problem has been developed since Whitney's seminal work [22–25]. And it is well known that functions with high smoothness can be approximated very precisely by their Hermite expansion. So in this paper, we consider a Hermite extension method to deal with the numerical differentiation problem on the irregular domain. That is to say, we take a function defined on any domain as part of a function on $\mathbb{R}^2$. For the one-dimensional case, the numerical differentiation method along the line of this method has been given in [26]. The numerical process of extension is usually unstable, so regularization technology is needed. As an alternative regularization method, we use a modified implicit iteration method in this paper. Compared with Tikhonov method which is used in [17], the implicit iteration method can select larger regularization parameters in each iteration, which makes the calculation process more stable. In [27], Jin has given the detailed theory and numerical implementation of implicit iteration method in Hilbert scales. But the application of the method to our problem still needs some improvement. The existing implicit iterative method can only deal with the problem of finite smoothness, and the numerical implementation of the algorithm is inconsistent for different smoothness. In this paper, we will present a modified form of implicit iteration method. It can deal with numerical differentiation of functions with any smoothness and the solution process is uniform.

This paper is organized as follows: In the next section, we introduce some preliminary materials. In Section 3, we describe the modified implicit iteration method for numerical differentiation problem and give some auxiliary results. The convergence estimate of the approximation solution can be founded in Section 4. In Section 5, some numerical examples are given. Conclusion is then given in the final Section 6.

2. Preliminaries

In this section, we introduce some notations and preliminaries that will be used throughout the paper.

2.1. Hermite functions in $\mathbb{R}^2$

Let $\mathbf{x} = (x_1, x_2)$, $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$. Let

$$\hat{f}(\boldsymbol{\omega}) = \mathscr{F}[f(\mathbf{x})] = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{x})e^{-\mathbf{i}\boldsymbol{\omega}\cdot\mathbf{x}}d\mathbf{x}$$

be the 2-dimensional Fourier transform of the function $f(\mathbf{x}) \in L^2(\mathbb{R}^2)$. The corresponding inverse Fourier transform of the function $\hat{f}(\boldsymbol{\omega})$ is

$$f(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega})e^{\mathbf{i}\boldsymbol{\omega}\cdot\mathbf{x}}d\boldsymbol{\omega}.$$

And $\|\cdot\|_p$ denotes the norm of the Sobolev space $H^p(\mathbb{R}^2)$ with $p \geq 0$ defined by

$$\|f\|_p = \left(\int_{\mathbb{R}^2} (1 + |\omega|^2)^p |\hat{f}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right)^{\frac{1}{2}}. \tag{2.1}$$

Particularly, for $p = 0$ we can recover the $L^2(\mathbb{R}^2)$ norm, i.e.,

$$\|f\| = \left(\int_{\mathbb{R}^2} |\hat{f}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right)^{\frac{1}{2}}.$$

Let $\boldsymbol{\ell} = (\ell_1, \ell_2)$, $\ell_i (i = 1, 2)$ being non-negative integers. Set $|\boldsymbol{\ell}|_1 = \ell_1 + \ell_2$ and $\boldsymbol{\ell} \cdot \mathbf{x} = \ell_1 x_1 + \ell_2 x_2$. The normalized 2-dimensional Hermite function is defined by

$$H_{\boldsymbol{\ell}}(\mathbf{x}) = H_{\ell_1}(x_1)H_{\ell_2}(x_2), \tag{2.2}$$

where

$$\begin{aligned} &H_0(x_i) \equiv \pi^{-1/4}\exp(-(1/2)x_i^2),\\ &H_1(x_i) \equiv \pi^{-1/4}\sqrt{2}x_i\exp(-(1/2)x_i^2),\\ &H_{\ell_i+1}(x_i) = \sqrt{\tfrac{2}{\ell_i+1}}x_i H_{\ell_i}(x_i) - \sqrt{\tfrac{\ell_i}{\ell_i+1}}H_{\ell_i-1}(x_i), \ell_i \geq 1. \end{aligned} \tag{2.3}$$

We know that the Fourier transform of the Hermite functions can be given as [28]:

$$\hat{H}_{\boldsymbol{\ell}}(\boldsymbol{\omega}) = (-\mathbf{i})^{|\boldsymbol{\ell}|_1} H_{\boldsymbol{\ell}}(\boldsymbol{\omega}). \tag{2.4}$$

The set of Hermite functions satisfy the orthogonality relations

$$\int_{\mathbb{R}^2} H_{\boldsymbol{\ell}}(\mathbf{x})H_{\mathbf{m}}(\mathbf{x})d\mathbf{x} = \delta_{\boldsymbol{\ell},\mathbf{m}}. \tag{2.5}$$

The Hermite expansion of a function $f \in L^2(\mathbb{R}^2)$ is as

$$f(\mathbf{x}) = \sum_{|\boldsymbol{\ell}|_1=0}^{\infty} \mathsf{f}_{\boldsymbol{\ell}} H_{\boldsymbol{\ell}}(\mathbf{x}), \tag{2.6}$$

with the Fourier-Hermite coefficients

$$\mathsf{f}_{\boldsymbol{\ell}} = \int_{\mathbb{R}^2} f(\mathbf{x})H_{\boldsymbol{\ell}}(\mathbf{x})d\mathbf{x}, \quad |\boldsymbol{\ell}|_1 = 0, 1, \ldots. \tag{2.7}$$

2.2. Problem description

Suppose that Λ is a subdomain of $\mathbb{R}^2$. For any two-tuples $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, |\alpha|_1 = \alpha_1 + \alpha_2$. The notation ∂_j stands for $\frac{\partial}{\partial x_j}$ and $D^\alpha f = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f$. For any positive integer q, we define $D^q f =: \{D^\alpha f : |\alpha|_1 = q\}$ and

$$|D^q f| = \left(\sum_{|\alpha|_1=q} |D^\alpha f|^2\right)^{1/2}. \tag{2.8}$$

The norm $\|\cdot\|_{s,\Lambda}$ in Sobolev space $H^s(\Lambda)$ is defined as

$$\|f\|_{s,\Lambda} := \left(\int_\Lambda |f|^2 + |D^s f|^2 d\mathbf{x}\right)^{1/2}, \tag{2.9}$$

where $s = 0$ and $\|\cdot\|_{0,\Lambda}$ denotes the $L^2(\Lambda)$ norm.

Suppose that $g(x) \in H^p(\mathbb{R})$, $p \geq 2$ and we only know its approximate function g^δ on Λ such that

$$\|g^\delta - g\|_{0,\Lambda} \leq \delta, \tag{2.10}$$

where $\delta > 0$ is a given constant called the error level. Our problem is to calculate approximate derivatives of g on Λ from the noisy data g^δ, or, equivalently, to construct a function $f^\delta(x)$ from $g^\delta(x)$ which is close to $g(x)$ in the sense that

$$\lim_{\delta\to 0} \|f^\delta - g\|_{r,\Lambda} = 0, \quad r \geq 1. \tag{2.11}$$

3. Implicit iteration method for numerical differentiation

For any vector $\vec{\mathsf{f}} = \{\mathsf{f}_\ell\}_{|\ell|_1=0}^\infty \in l^2$, if we let

$$\mathcal{H}\vec{\mathsf{f}} := \sum_{|\ell|=0}^{\infty} \mathsf{f}_\ell H_\ell(\mathbf{x}), \tag{3.1}$$

then the process of constructing an approximation function f^δ from data g^δ can be transformed to solving the following equations

$$\mathcal{H}\vec{\mathsf{f}} = g^\delta. \tag{3.2}$$

In this paper, we present an modified implicit iteration method to solve the above equations. For this purpose, we introduce the following operator:

$$\mathcal{R}\vec{\mathsf{f}} := \mathcal{H}^{-1}\mathscr{F}^{-1}\left[e^{|\omega|}\widehat{\mathcal{H}\vec{\mathsf{f}}}(\omega)\right]. \tag{3.3}$$

It is obvious that $\mathcal{R}$ is unbounded self-adjoint strictly positive definite operator. Then we choose

$$f_n^\delta = \mathcal{H}\vec{\mathsf{f}}_n^\delta, \tag{3.4}$$

as the approximation of g, where $\vec{\mathsf{f}}_n^\delta$ is determined by the following implicit iteration process

$$\begin{aligned} &\vec{\mathsf{f}}_0^\delta = 0, \\ &\vec{\mathsf{f}}_k^\delta = \vec{\mathsf{f}}_{k-1}^\delta - (\mathcal{H}^*\mathcal{H} + \beta_k\mathcal{R}^2)^{-1}\mathcal{H}^*(\mathcal{H}\vec{\mathsf{f}}_{k-1}^\delta - g^\delta), \quad k = 1, 2, \ldots, n, \end{aligned} \tag{3.5}$$

where $\beta_k > 0$ are properly chosen real numbers. For reference [27], the positive number

$$\sigma_n := \sum_{k=1}^{n} \frac{1}{\beta_k} \tag{3.6}$$

plays the role of the regularization parameter and we will chosen it as the solution of the nonlinear equation

$$d(\sigma_n) := \|\mathcal{H}\vec{f}_n^\delta - g^\delta\|_{0,\Lambda} = C\delta, \tag{3.7}$$

with a constant $C \geq 1$. If we let $\mathcal{T} = \mathcal{H}\mathcal{R}^{-1}$, then $\vec{f}_n^\delta$ possesses the representation [27]

$$\vec{f}_n^\delta = \mathcal{R}^{-1} s_n(\mathcal{T}^*\mathcal{T})\mathcal{T}^* g^\delta \quad \text{with} \quad s_n(\lambda) = \frac{1}{\lambda}\left(1 - \prod_{k=1}^{n} \frac{\beta_k}{\lambda + \beta_k}\right). \tag{3.8}$$

Remark 3.1. *If we use the operator* $\mathcal{B} = \left(\sum_{|\alpha|_1=q} D^\alpha\right)$ *with some a constant* q *instead of* $\mathcal{R}$*, then we return to the framework in [27] and the convergence results can be obtained accordingly. When* p *and* q *satisfy a certain relation, the result is order optimal. It should be noticed that for large* q*, the numerical process of the method is difficult. We will point that the method is always order optimal when we use the operator* $\mathcal{R}$ *and the numerical process is uniform for any* p*.*

The following lemma holds for $s_n(\lambda)$.

Lemma 3.1. *[27] The function* $s_n : (0, c] \to (0, \infty)$ *with* $c = \|\mathcal{T}\|^2$ *and the corresponding residual function* $r_n(\lambda) := 1 - \lambda s_n(\lambda)$ *obey the properties*

$$\begin{aligned} s_n(\lambda) &\leq \sigma_n, & \lambda s_n(\lambda) &\leq 1, \\ \lambda r_n(\lambda) &\leq \sigma_n^{-1}, & r_n(\lambda) &\leq 1. \end{aligned} \tag{3.9}$$

From above lemma, we can deduce the following results.

Lemma 3.2.

$$\sqrt{\lambda} s_n(\lambda) \leq \sqrt{\sigma_n}, \quad \sqrt{\lambda} r_n(\lambda) \leq \sqrt{\sigma_n^{-1}}. \tag{3.10}$$

Proof. For $\lambda \leq \sigma^{-1}$,

$$\sqrt{\lambda} s_n(\lambda) \leq \sqrt{\lambda}\sigma_n \leq \sqrt{\sigma_n} \tag{3.11}$$

and

$$\sqrt{\lambda} r_n(\lambda) \leq \sqrt{\lambda} \leq \sqrt{\sigma_n^{-1}}. \tag{3.12}$$

Moreover, for $\lambda \geq \sigma^{-1}$,

$$\sqrt{\lambda} s_n(\lambda) = \frac{\sqrt{\lambda}}{\lambda}\lambda s_n(\lambda) \leq \frac{\sqrt{\lambda}}{\lambda} \leq \sqrt{\sigma_n} \tag{3.13}$$

and

$$\sqrt{\lambda} r_n(\lambda) \leq \frac{\sqrt{\lambda}}{\lambda}\lambda r_n(\lambda) \leq \frac{\sigma_n^{-1}}{\sqrt{\lambda}} \leq \sqrt{\sigma_n^{-1}}. \tag{3.14}$$

□

4. Error estimate

Owing to $g \in H^p(\mathbb{R})$, we suppose that

$$\|g\|_p \le E, \tag{4.1}$$

where E is a constant. Set the vector $\vec{\mathsf{g}}$ contains all Fourier-Hermite coefficients of g, i.e.,

$$g(\mathbf{x}) = \left(\mathcal{H}\vec{\mathsf{g}}\right)(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^2. \tag{4.2}$$

Let

$$\vec{\mathsf{g}}_N = \mathcal{P}_N\vec{\mathsf{g}} \quad \text{and} \quad g_N = \mathcal{H}\vec{\mathsf{g}}_N. \tag{4.3}$$

We define the vector $\vec{\mathsf{f}}_n$ as

$$\vec{\mathsf{f}}_n = \mathcal{R}^{-1}s_n(\mathcal{T}^*\mathcal{T})\mathcal{T}^*g_N, \tag{4.4}$$

then we have

$$\mathcal{H}(\vec{\mathsf{f}}_n^\delta - \vec{\mathsf{f}}_n) = \mathcal{T}s_n(\mathcal{T}^*\mathcal{T})\mathcal{T}^*(g^\delta - g_N), \tag{4.5}$$

$$\mathcal{H}(\vec{\mathsf{g}} - \vec{\mathsf{f}}_n) = \mathcal{T}r_n(\mathcal{T}^*\mathcal{T})\mathcal{R}\vec{\mathsf{g}}_N, \tag{4.6}$$

$$g^\delta - \mathcal{H}\vec{\mathsf{f}}_n^\delta = r_n(\mathcal{T}\mathcal{T}^*)g^\delta, \tag{4.7}$$

$$\mathcal{R}(\vec{\mathsf{f}}_n^\delta - \vec{\mathsf{f}}_n) = s_n(\mathcal{T}^*\mathcal{T})\mathcal{T}^*(g^\delta - g_N), \tag{4.8}$$

$$\mathcal{R}(\vec{\mathsf{g}} - \vec{\mathsf{f}}_n) = r_n(\mathcal{T}^*\mathcal{T})\mathcal{R}\vec{\mathsf{g}}_N. \tag{4.9}$$

In our further analysis, we shall make use of the following lemmas.

Lemma 4.1. *If the condition (4.1) holds, then*

$$\|g - g_N\| \le N^{-p}E \quad \textit{and} \quad \|\mathcal{R}\vec{\mathsf{g}}_N\|_{l^2} \le C_N E, \tag{4.10}$$

where

$$C_N = \max\left(1, \frac{e^N}{N^p}\right). \tag{4.11}$$

Proof. From (2.1) and (3.3), we can obtain

$$\|g - g_N\|^2 = \int_{|\omega|>N} |\hat{g}(\omega)|^2\, d\omega \le N^{-2p}\int_{|\omega|>N}(1 + |\omega|^2)^p\, |\hat{g}|^2\, d\omega \le N^{-2p}\|g\|_p^2 \tag{4.12}$$

and

$$\begin{aligned} \|\mathcal{R}\vec{\mathsf{g}}_N\|_{l^2} &= \int_{|\omega|\le N} e^{2|\omega|}\, |\hat{g}(\omega)|^2\, d\omega \\ &= \int_{|\omega|\le N} \frac{e^{2|\omega|}}{(1 + |\omega|^2)^p}(1 + |\omega|^2)^p\, |\hat{g}(\omega)|^2\, d\omega \\ &\le \max\left(1, \frac{e^{2N}}{N^{2p}}\right)\|g\|_p^2. \end{aligned} \tag{4.13}$$

□

Lemma 4.2. *If the condition (4.1) holds, we have*

$$\|\mathcal{H}(\vec{\mathrm{f}}_n^\delta - \vec{\mathrm{g}}_N)\|_{0,\Lambda} \leq (C+1)\delta + N^{-p}E, \tag{4.14}$$

$$\|\mathcal{R}(\vec{\mathrm{f}}_n^\delta - \vec{\mathrm{g}}_N)\|_{l^2} \leq \sqrt{\sigma_n}(\delta + N^{-p}E) + C_N E \tag{4.15}$$

and

$$\|\mathcal{H}\vec{\mathrm{f}}_n^\delta - g^\delta\|_{0,\Lambda} \leq \delta + N^{-p}E + \sqrt{\sigma_n^{-1}}C_N E. \tag{4.16}$$

Proof. From (2.10), (3.7), (4.10) and the triangle inequality

$$\|\mathcal{H}(\vec{\mathrm{f}}_n^\delta - \vec{\mathrm{g}}_N)\|_{0,\Lambda} \leq \|\mathcal{H}\vec{\mathrm{f}}_n^\delta - g^\delta\|_{0,\Lambda} + \|g^\delta - g\|_{0,\Lambda} + \|g - g_N\|_{0,\Lambda} \leq (C+1)\delta + N^{-p}E. \tag{4.17}$$

And by using the triangle inequality, (2.10), (3.10) and (4.8)–(4.10)

$$\begin{aligned} \|\mathcal{R}(\vec{\mathrm{f}}_n^\delta - \vec{\mathrm{g}}_N)\|_{l^2} &\leq \|\mathcal{R}(\vec{\mathrm{f}}_n^\delta - \vec{\mathrm{f}}_n)\|_{l^2} + \|\mathcal{R}(\vec{\mathrm{f}}_n - \vec{\mathrm{g}}_N)\|_{l^2} \\ &= \|s_n(\mathcal{T}^*\mathcal{T})\mathcal{T}^*(g^\delta - g_N)\|_{l^2} + \|r_n(\mathcal{T}^*\mathcal{T})\mathcal{R}\vec{\mathrm{g}}_N\|_{l^2} \\ &\leq \sqrt{\sigma_n}\|(g^\delta - g_N)\|_{0,\Lambda} + \|\mathcal{R}\vec{\mathrm{g}}_N\|_{l^2} \\ &\leq \sqrt{\sigma_n}(\delta + N^{-p}E) + C_N E. \end{aligned} \tag{4.18}$$

Moreover, in terms of the triangle inequality, (2.10), (3.9), (3.10) and (4.7), we have

$$\begin{aligned} \|\mathcal{H}\vec{\mathrm{f}}_n^\delta - g^\delta\|_{0,\Lambda} &= \|r_n(\mathcal{T}\mathcal{T}^*)g^\delta\|_{0,\Lambda} \\ &\leq \|r_n(\mathcal{T}\mathcal{T}^*)(g^\delta - g)\|_{0,\Lambda} + \|r_n(\mathcal{T}\mathcal{T}^*)(g - g_N)\|_{0,\Lambda} + \|r_n(\mathcal{T}\mathcal{T}^*)g_N\|_{0,\Lambda} \\ &\leq \delta + \|g - g_N\|_{0,\Lambda} + \|r_n(\mathcal{T}\mathcal{T}^*)\mathcal{T}\| \cdot \|\mathcal{R}\vec{\mathrm{g}}_N\| \\ &\leq \delta + N^{-p}E + \sqrt{\sigma_n^{-1}}C_N E. \end{aligned} \tag{4.19}$$

□

Lemma 4.3. *[21] Let Ω be a domain in $\mathbb{R}^2$ satisfying the cone condition. There exists a constant K depending on ϵ_0 and j, s, such that for any $0 < \epsilon \leq \epsilon_0$ and $0 \leq j \leq s$*

$$\|f\|_{j,\Omega} \leq K(\epsilon\|f\|_{s,\Omega} + \epsilon^{-j/(s-j)}\|f\|_{0,\Omega}). \tag{4.20}$$

Lemma 4.4. *Suppose that the vector sequence $\vec{\mathrm{h}}_n^\delta = \left\{\mathrm{h}_{\boldsymbol{\ell}}^\delta\right\}_{|\boldsymbol{\ell}|_1=0}^\infty$ satisfies*

$$\|\mathcal{H}\vec{\mathrm{h}}^\delta\|_{0,\Lambda} \leq k_1\delta, \quad \|\mathcal{R}\vec{\mathrm{h}}^\delta\|_{l^2} \leq k_2 e^{k_3\delta^{-\frac{1}{p}}}\delta, \qquad \delta \to 0, \tag{4.21}$$

then for any $\Omega \subseteq \Lambda$ satisfying the cone condition, there exists a constant M

$$\|\mathcal{H}\vec{\mathrm{h}}^\delta\|_{p,\Omega} \leq M. \tag{4.22}$$

Proof. It is easy to deduce that there exist a constant δ_0 such that

$$e^{k_3\delta^{-\frac{1}{p}}} > \frac{k_3^p}{\delta}, \quad \forall\delta < \delta_0. \tag{4.23}$$

And for simplicity, we prove the theorem with $\delta < \delta_0$. Let

$$N_0 = k_3\delta^{-\frac{1}{p}}, \tag{4.24}$$

and we have

$$\begin{aligned}\left\|\mathcal{H}\vec{\mathsf{h}}^{\delta}\right\|_{p,\Omega} &\le \left\|\mathcal{H}\left(\mathcal{P}_{N_0}\vec{\mathsf{h}}^{\delta}\right)\right\|_{p,\Omega} + \left\|\mathcal{H}\left[(\mathcal{I}-\mathcal{P}_{N_0})\vec{\mathsf{h}}^{\delta}\right]\right\|_{p,\Omega} \\ &= I_1 + I_2.\end{aligned} \tag{4.25}$$

By Parseval's formula, we can see that the second term I_2 satisfies

$$\begin{aligned}\left\|\mathcal{H}\left[(\mathcal{I}-\mathcal{P}_{N_0})\vec{\mathsf{h}}^{\delta}\right]\right\|_{p,\Omega}^2 &\le \left\|\mathcal{H}\left[(\mathcal{I}-\mathcal{P}_{N_0})\vec{\mathsf{h}}^{\delta}\right]\right\|^2 \\ &= \int_{|\omega|>N_0}(1+|\omega|^2)^p|\widehat{\mathcal{H}\vec{\mathsf{h}}^{\delta}}(\omega)|^2 d\omega \\ &= \int_{|\omega|>N_0}\frac{(1+|\omega|^2)^p}{e^{2|\omega|}}|e^{2|\omega|}\widehat{\mathcal{H}\vec{\mathsf{h}}^{\delta}}(\omega)|^2 d\omega \\ &\le \frac{(N_0+1)^{2p}}{e^{2N_0}}\int_{|\omega|>N_0}|e^{2|\omega|}\widehat{\mathcal{H}\vec{\mathsf{h}}^{\delta}}(\omega)|^2 d\omega \\ &\le \frac{N_0^{2p}}{e^{2(N_0-1)}}\|\mathcal{R}\vec{\mathsf{h}}^{\delta}\|_{l^2}^2 \\ &\le e^{-2}k_3^{2p}\frac{1}{\delta^2}\cdot k_2^2\delta^2 = e^{-2}k_3^{2p}k_2^2.\end{aligned} \tag{4.26}$$

Hence

$$I_2 \le 2e^{-1}k_3^p k_2. \tag{4.27}$$

So all we need is to prove there exist a constant M_1 such that

$$I_1 < M_1, \qquad \delta \to 0. \tag{4.28}$$

Note that

$$\|\mathcal{H}\left(\mathcal{P}_{N_0}\vec{\mathsf{h}}^{\delta}\right)\|_{0,\Omega} \le \|\mathcal{H}\vec{\mathsf{h}}^{\delta}\|_{0,\Omega} + \|\mathcal{H}\left[(\mathcal{I}-\mathcal{P}_{N_0})\vec{\mathsf{h}}^{\delta}\right]\|_{0,\Omega} \tag{4.29}$$

and

$$\begin{aligned}\left\|\mathcal{H}(\mathcal{I}-\mathcal{P}_{N_0})\vec{\mathsf{h}}^{\delta}\right\|_{0,\Omega} &\le \left\|\mathcal{H}\left[(\mathcal{I}-\mathcal{P}_{N_0})\vec{\mathsf{h}}^{\delta}\right]\right\|^2 \\ &= \int_{|\omega|>N_0}|\widehat{\mathcal{H}\vec{\mathsf{h}}^{\delta}}(\omega)|^2 d\omega \\ &= \int_{|\omega|>N_0}\frac{1}{e^{2|\omega|}}|e^{|\omega|}\widehat{\mathcal{H}\vec{\mathsf{h}}^{\delta}}(\omega)|^2 d\omega \\ &\le \frac{1}{e^{2N_0}}\|\mathcal{R}\vec{\mathsf{h}}^{\delta}\|_{l^2}^2 \le k_2^2\delta^2.\end{aligned} \tag{4.30}$$

Therefore

$$\|\mathcal{H}\left(\mathcal{P}_{N_0}\vec{\mathsf{h}}^{\delta}\right)\|_{0,\Omega} \le (k_1+2k_2)\delta. \tag{4.31}$$

Now we prove (4.28) by using reduction to absurdity, if (4.28) does not hold, then for any $q > p$ there exist a sequence δ_i such that

$$\|\mathcal{H}\left(\mathcal{P}_{N_0}\vec{\mathsf{h}}^{\delta}\right)\|_q \ge 2k_2k_3^p\left(\frac{\delta_i}{k_3^p}\right)^{\frac{p-q}{p}}, \quad \delta_i \to 0. \tag{4.32}$$

If not, $\exists\bar{q}$ for any $\delta \to 0$

$$\|\mathcal{H}\left(\mathcal{P}_{N_0}\vec{\mathsf{h}}^{\delta}\right)\|_{\bar{q}} < 2k_2k_3^p\left(\frac{\delta}{k_3^p}\right)^{\frac{p-\bar{q}}{p}}, \tag{4.33}$$

then (4.28) can be derived by 4.29 and Lemma 4.3 with $\epsilon = \left(\frac{\delta_i}{k_3^p}\right)^{\frac{\bar{q}-p}{p}}$, $s = \bar{q}$ and $j = p$. Then

$$\begin{aligned}
\int_{|\omega|<N_0}\left(\sum_{k=0}^{N_0}\frac{|\omega|^k}{k!}\right)^2\left|\widehat{\mathcal{H}\vec{\mathsf{h}}}^{\delta_i}(\omega)\right|^2 d\omega &= \int_{|\omega|<N_0}\frac{\left(\sum_{k=0}^{N_0}\frac{|\omega|^k}{k!}\right)^2}{(1+|\omega|^2)^{N_0}}(1+|\omega|^2)^{N_0}\left|\widehat{\mathcal{H}\vec{\mathsf{h}}}^{\delta_i}(\omega)\right|^2 d\omega \\
&\geq \int_{|\omega|<N_0}\frac{\left(\sum_{k=0}^{N_0}\frac{|\omega|^k}{k!}\right)^2}{(1+|\omega|)^{2N_0}}(1+|\omega|^2)^{N_0}\left|\widehat{\mathcal{H}\vec{\mathsf{h}}}^{\delta_i}(\omega)\right|^2 d\omega \\
&\geq \int_{|\omega|<N_0}\frac{\left(\sum_{k=0}^{N_0-1}\frac{|\omega|^k}{k!}\right)^2}{|\omega|^{2N_0}}(1+|\omega|^2)^{N_0}\left|\widehat{\mathcal{H}\vec{\mathsf{h}}}^{\delta_i}(\omega)\right|^2 d\omega \\
&\geq \int_{|\omega|<N_0}\frac{\left(\sum_{k=0}^{N_0-1}\frac{N_0{}^k}{k!}\right)^2}{N_0^{2N_0}}(1+|\omega|^2)^{N_0}\left|\widehat{\mathcal{H}\vec{\mathsf{h}}}^{\delta_i}(\omega)\right|^2 d\omega \\
&\geq \frac{\left(\sum_{k=0}^{N_0-1}\frac{N_0{}^k}{k!}\right)^2}{N_0^{2N_0}}\left\|\mathcal{H}\left(\mathcal{P}_{N_0}\vec{\mathsf{h}}^{\delta_i}\right)\right\|_{N_0}^2 \\
&\geq k_2^2\left(\sum_{k=0}^{N_0}\frac{\left(k_3\delta_i^{-\frac{1}{p}}\right)^k}{k!}\right)^2\delta_i^2.
\end{aligned} \tag{4.34}$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}^2} e^{2|\omega|}\left|\widehat{\mathcal{H}\vec{\mathsf{h}}}^{\delta_i}(\omega)\right|^2 d\omega &= \lim_{\delta_i\to 0}\int_{|\omega|<N_0(\delta_i)}\left(\sum_{k=0}^{N_0(\delta_i)}\frac{|\omega|^k}{k!}\right)^2\left|\widehat{\mathcal{H}\vec{\mathsf{h}}}^{\delta_i}(\omega)\right|^2 d\omega \\
&\geq k_2^2\left(\sum_{k=0}^{N_0}\frac{\left(k_3\delta_i^{-\frac{1}{p}}\right)^k}{k!}\right)^2\delta_i^2 \\
&= k_2^2\lim_{\delta_i\to 0} e^{2k_3\delta_i^{-\frac{1}{p}}}\delta_i^2.
\end{aligned} \tag{4.35}$$

So there exists a $\bar{\delta}$ such that

$$\|\mathcal{R}\vec{\mathsf{h}}^{\bar{\delta}}\|_{l^2}^2 = \int_{\mathbb{R}^2} e^{2|\omega|}\left|\widehat{\mathcal{H}\mathsf{h}^{\bar{\delta}}}(\omega)\right|^2 d\omega > \int_{\mathbb{R}^2} e^{2|\omega|}\left|\widehat{\mathcal{H}\mathsf{h}^{\bar{\delta}}}(\omega)\right|^2 d\omega \geq k_2^2 e^{2k_3\hat{\delta}^{-\frac{1}{p}}}\bar{\delta}^2, \tag{4.36}$$

which contradicts the assumptions of the Lemma. □

Theorem 4.1. *Suppose that the conditions (2.10) and (4.1) hold, f_n^δ is defined by (3.4) and (3.7) then for any $\Omega \subseteq \Lambda$ satisfying the cone condition and $0 < j \leq p$,*

$$\|f_n^\delta - g\|_{j,\Omega} = O\left(\delta^{\frac{p-j}{p}}\right). \tag{4.37}$$

Proof. Let

$$N_0 = \left(\frac{2E}{(C-1)\delta}\right)^{\frac{1}{p}}. \tag{4.38}$$

Then from Lemma 4.2, we have

$$\|\mathcal{H}(\vec{\mathsf{f}}_n^\delta - \vec{\mathsf{g}}_{N_0})\|_{0,\Lambda} \le \frac{3C+1}{2}\delta, \tag{4.39}$$

$$\|\mathcal{R}(\vec{\mathsf{f}}_n^\delta - \vec{\mathsf{g}}_{N_0})\|_{l^2} \le \frac{C+1}{2} e^{(\frac{2E}{C-1})^{\frac{1}{p}}\delta^{-\frac{1}{p}}}\delta. \tag{4.40}$$

Thus, by using Lemma 4.4, there exists a constant M

$$\|\mathcal{H}(\vec{\mathsf{f}}_n^\delta - \vec{\mathsf{g}}_{N_0})\|_{0,\Omega} \le M. \tag{4.41}$$

Then

$$\begin{aligned} \|\mathcal{H}\vec{\mathsf{f}}_n^\delta - g\|_{p,\Omega} &\le \|\mathcal{H}(\vec{\mathsf{f}}_n^\delta - \vec{\mathsf{g}}_{N_0})\|_{0,\Omega} + \|g - g_N\|_{p,\Omega} \\ &\le \|\mathcal{H}(\vec{\mathsf{f}}_n^\delta - \vec{\mathsf{g}}_{N_0})\|_{0,\Omega} + \|g\|_p \\ &\le M + E. \end{aligned} \tag{4.42}$$

Moreover, by using (2.10), (3.7) and the triangle inequality

$$\|\mathcal{H}\vec{\mathsf{f}}_n^\delta - g\|_{0,\Omega} \le \|\mathcal{H}\vec{\mathsf{f}}_n^\delta - g^\delta\|_{0,\Omega} + \|g^\delta - g\|_{0,\Omega} \le \|\mathcal{H}\vec{\mathsf{f}}_n^\delta - g^\delta\|_{0,\Lambda} + \|g^\delta - g\|_{0,\Lambda} \le (C+1)\delta. \tag{4.43}$$

The assertion of theorem follows from (4.42), (4.43) and Lemma 4.3.

□

5. Numerical realization

The data are usually given at scatter points in practical applications. Let $\mathbf{x}_i \in \Lambda (i = 1, 2, \dots, m)$ be the given points and

$$\mathsf{g}^\delta = (g^\delta(\mathbf{x}_1), g^\delta(\mathbf{x}_2), \dots, g^\delta(\mathbf{x}_m))^T$$

be the noisy data vector. Let $\boldsymbol{\sigma}^{(j)} = (\sigma^{(j_1)}, \sigma^{(j_2)}), (0 \le j_1, j_2 \le n)$ being the Hermite-Gauss type interpolation points and $\boldsymbol{\rho}^{(j)}$ are the corresponding Hermite-Gauss weights. For $f, g \in L^2(\mathbb{R}^2)$, we define the discrete inner product

$$\langle f, g\rangle_n := \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} \boldsymbol{\rho}^{(j)} f(\boldsymbol{\sigma}^{(j)})\overline{g(\boldsymbol{\sigma}^{(j)})},$$

and

$$\breve{H}_{\boldsymbol{\ell}}(\boldsymbol{\omega}) := (-1)^{|\boldsymbol{\ell}|_1} e^{|\boldsymbol{\omega}|} H_{\boldsymbol{\ell}}(\boldsymbol{\omega}).$$

Let

$$\mathscr{H}_n = \operatorname{span}\{H_{(0,0)}(\mathbf{x}), H_{(1,0)}(\mathbf{x}), H_{(1,1)}(\mathbf{x}), \dots, H_{(n,n-1)}(\mathbf{x}), H_{(n,n)}(\mathbf{x})\},$$

then we give the matrices $\mathsf{A}_{(n+1)^2\times(n+1)^2}$, $\mathsf{R}_{(n+1)^2\times(n+1)^2}$, $\mathsf{H}_{m\times(n+1)^2}$ as

$$\begin{aligned} &\mathsf{A}_{|\boldsymbol{\ell}|_1+1,|\boldsymbol{k}|_1+1} = \textstyle\sum_{i=1}^m H_{\boldsymbol{\ell}}(\mathbf{x}_i) H_{\boldsymbol{k}}(\mathbf{x}_i), \qquad \mathsf{R}_{|\boldsymbol{\ell}|_1+1,|\boldsymbol{k}|_1+1} = \langle \breve{H}_{\boldsymbol{\ell}}, \breve{H}_{\boldsymbol{k}}\rangle_n, \\ &\mathsf{H}_{i,|\boldsymbol{\ell}|_1+1} = H_{\boldsymbol{\ell}}(\mathbf{x}_i), \quad i = 1, 2, \dots, m; |\boldsymbol{\ell}|_1, |\boldsymbol{k}|_1 = 0, 1, \dots n. \end{aligned}$$

With these preparations, the discrete form of the implicit iteration method can be given as

$$\begin{aligned} &\mathsf{f}_0^\delta = 0, \\ &\mathsf{f}_k = \mathsf{f}_{k-1} - (\mathsf{H}^T\mathsf{H} + \beta_k \mathsf{R})^{-1}\mathsf{H}^T(\mathsf{H}\mathsf{f}_{k-1}^\delta - \mathsf{g}^\delta), \quad k = 1, 2, \dots, n. \end{aligned} \tag{5.1}$$

Suppose that

$$\left(\sum_{i=1}^{m}(g^{\delta}(\mathbf{x}_i)-g(\mathbf{x}_i)^2\right)^{1/2}\leq\boldsymbol{\delta}. \tag{5.2}$$

Similar to what is done in [27], we take $\beta_1 = 1, \beta_k = q^{k-1}\beta_1$ with some $q < 1$ and choose n as the first integer for which

$$\|\mathsf{Hf}_n^{\delta}-\mathsf{g}^{\delta}\|\leq C_1\boldsymbol{\delta}<\|\mathsf{Hf}_k^{\delta}-\mathsf{g}^{\delta}\|,\quad 0\leq k<n, \tag{5.3}$$

and then adjust the parameter β_n such that

$$C_2\delta\leq\|\mathsf{Hf}_n^{\delta}-\mathsf{g}^{\delta}\|\leq C_1\boldsymbol{\delta}, \tag{5.4}$$

where C_1, C_2 are two constants that obey $1 \leq C_1 \leq C_2$.

Remark 5.1. *It should be noted that the Hermite-Gauss points are only used to calculate the matrix* R*, regardless of the location of the noisy data.*

6. Numerical tests

In this section, we give some numerical tests to verify the effect of the new method. All tests are realized on Windows 10 system with Memory 16GB, CPU Intel(R) Core(TM)i7-8550U by using Matlab 2017b. Let $\mathsf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)^T$ and the perturbed data are generated by

$$g^{\delta}(\mathsf{x}) = g(\mathsf{x}) + \mathrm{randn}(\mathrm{size}(\mathsf{x}))\cdot\epsilon, \tag{6.1}$$

where ϵ is the error level and randn(size(·)) is Matlab functions. In all cases we choose the parameter $n = 64$, $q = 1/2$ and $C = 1.01$. We have tested these parameters with other values, and the results are similar. In order to adapt to the characteristic of Hermite function approximation, the scaling factor [29] is used in the numerical processing.

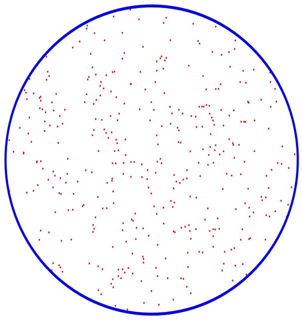
Example 6.1. *[18] Let* $\Lambda = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\}$ *is a disk and scatter nodes are given as Figure 2a. We choose the exact function as* $g(\mathbf{x}) = (x_1^2 + x_2^2 - 2)^3$ *and set* $\epsilon = 0.01$*. The numerical results are exhibited in Figures 2c–3h.*

Example 6.2. *[18] Let* $\Lambda = \{(x_1, x_2) \mid 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq \pi\}$ *is a rectangle and scatter nodes are given as Figure 3i. We choose the exact function as* $g(\mathbf{x}) = (x_1^2 + x_2^2 - 2)^3$ *and set* $\epsilon = 0.01$*. We have shown the numerical results in Figures 3k–4h.*

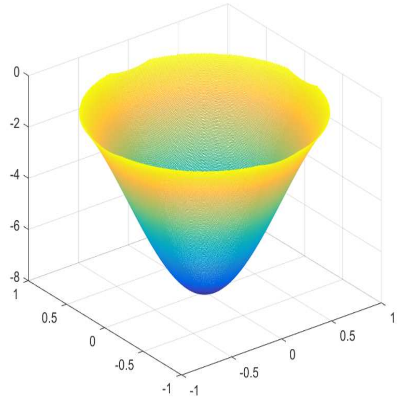
Example 6.3. *[19] Let* $\Lambda = \{(x_1, x_2) \mid -1 \leq x_1 \leq 3,\ -1 \leq x_2 \leq 3\}$*, the data are given at the equidistant nodes whose sampling step was* 0.1×0.1*. The exact function is chosen as* $g(\mathbf{x}) = \sin\left(\frac{1}{2}x_1^2 + \frac{1}{4}x_2^2 + 3\right)\cos(2x_1 + 1 - \exp(x_2))$*. We have shown the numerical results in Figures 4i–5h with* $\epsilon = 0.01$*.*

Example 6.4. *Now we let* Λ *is a irregular domain and scatter nodes are given as Figure 5i. We choose the exact function as* $g(\mathbf{x}) = \cos(x_1 \cdot x_2)$ *and set* $\epsilon = 0.01$*. The numerical approximations and corresponding errors are shown in Figures 5k–6h.*

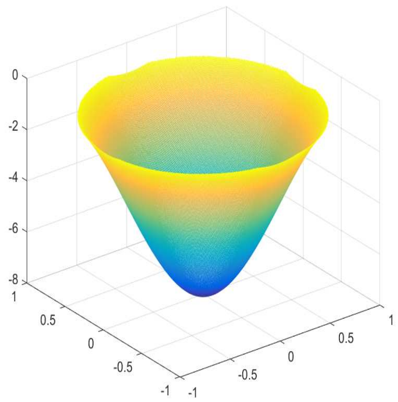
All the above numerical results show that the proposed method is effective.



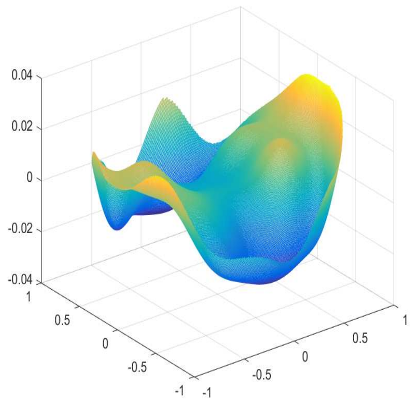
a Λ and nodes



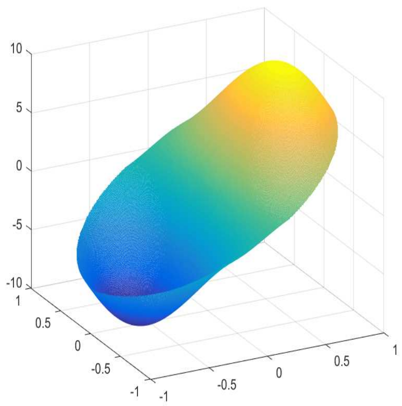
b the exact function g



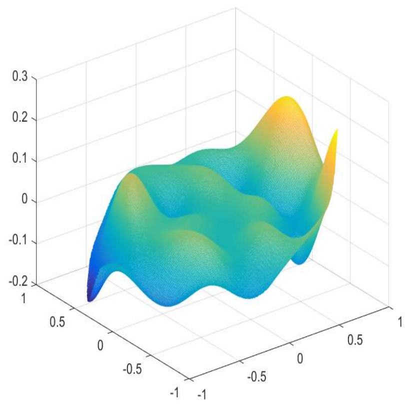
c the constructed function of g



d the constructed error function of g

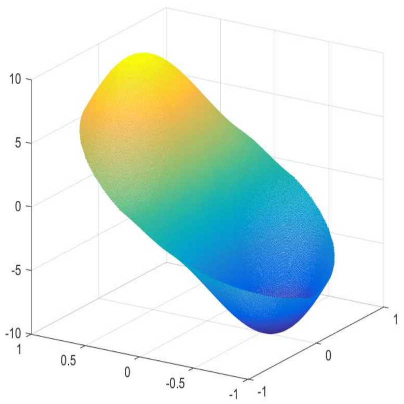


e the constructed function of g_x

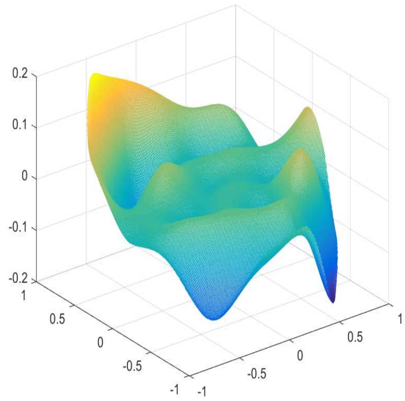


f the constructed error function of g_x

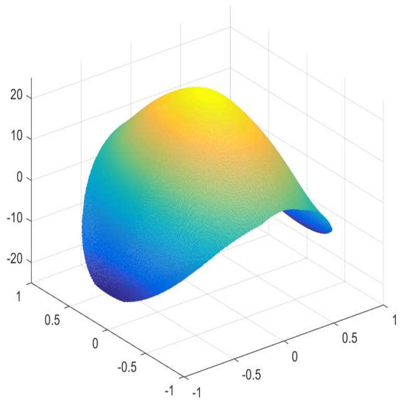
Figure 2. Example 1.



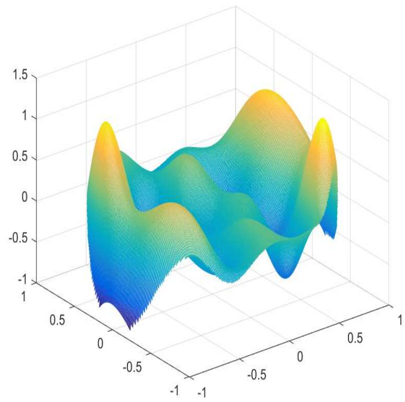
a the constructed function of g_y



b the constructed error function of g_y

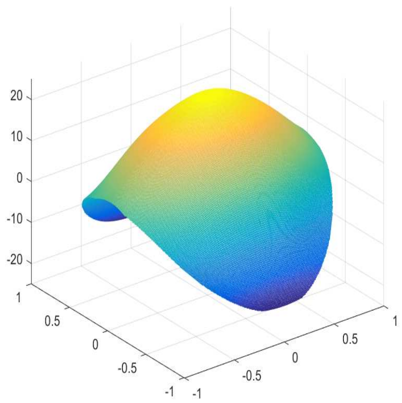


c the constructed function of g_{xx}

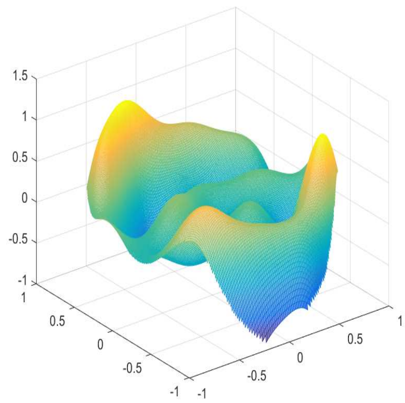


d the constructed error function of g_{xx}

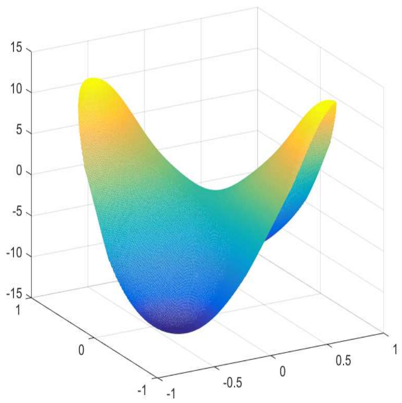
Figure 2. Continued.



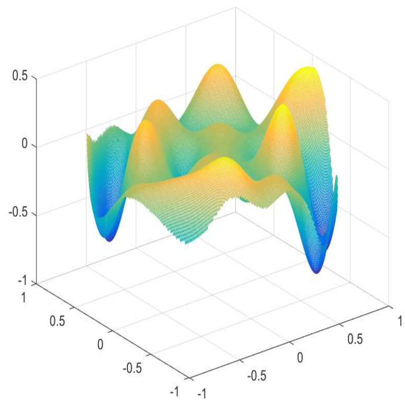
e the constructed function of g_{yy}



f the constructed error function of g_{yy}

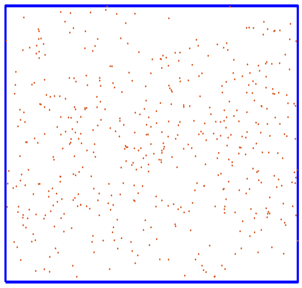


g the constructed function of g_{xy}

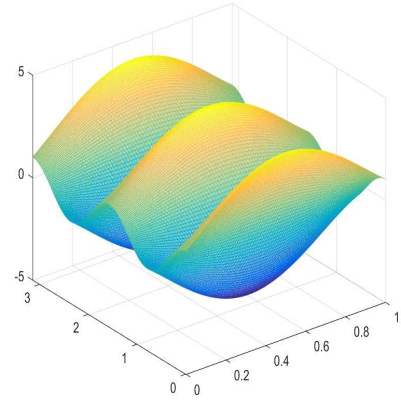


h the constructed error function of g_{xy}

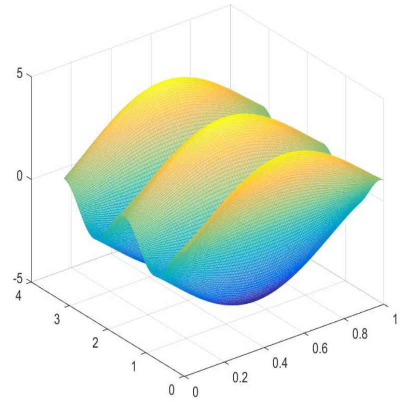
Figure 2. Continued.



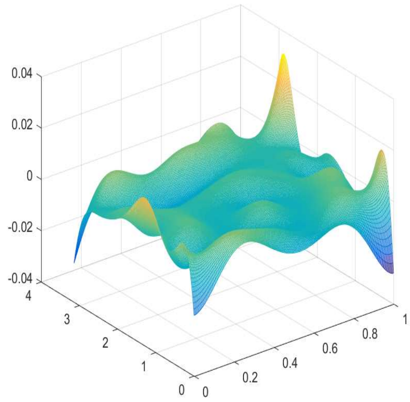
i Λ and nodes



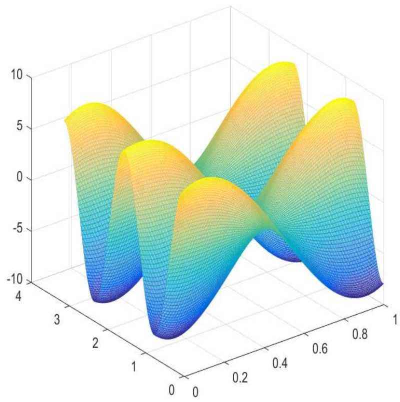
j the exact function g



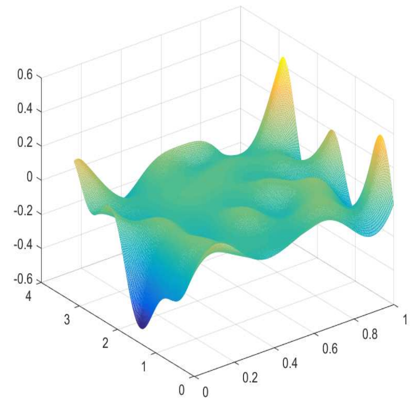
k the constructed function of g



l the constructed error function of g

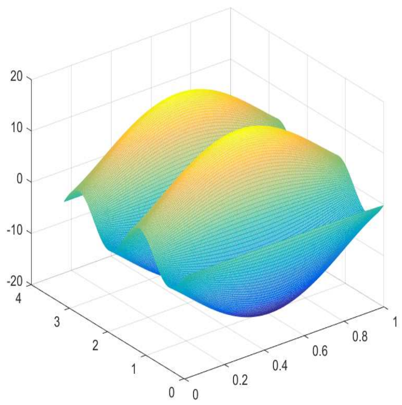


m the constructed function of g_x

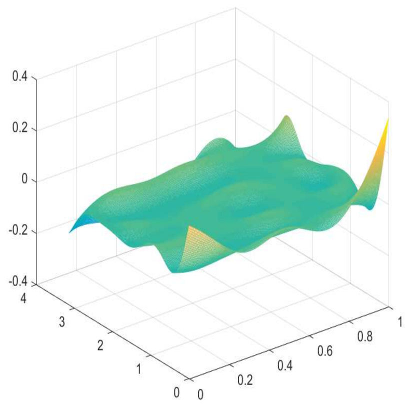


n the constructed error function of g_x

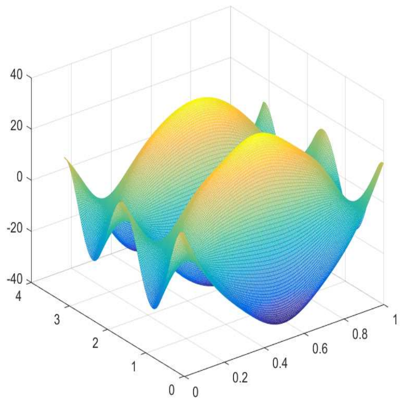
Figure 3. Example 2.



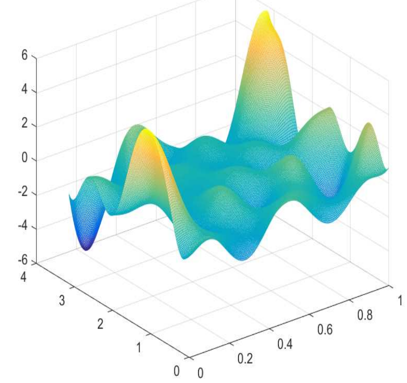
a the constructed function of g_y



b the constructed error function of g_y

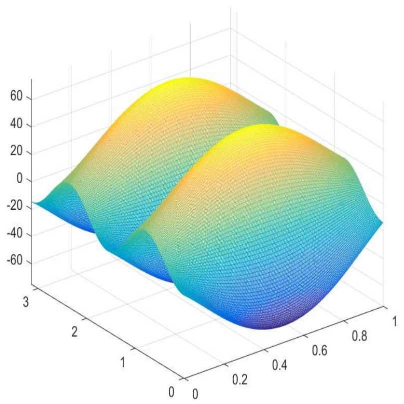


c the constructed function of g_{xx}

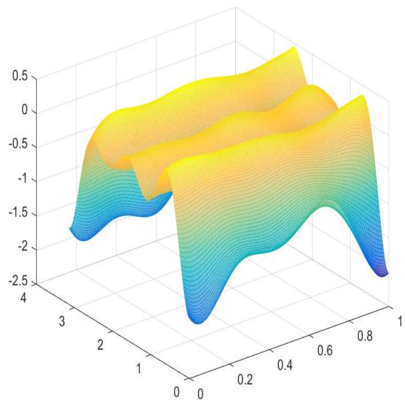


d the constructed error function of g_{xx}

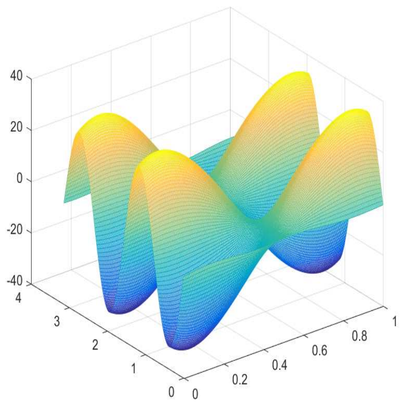
Figure 3. Continued.



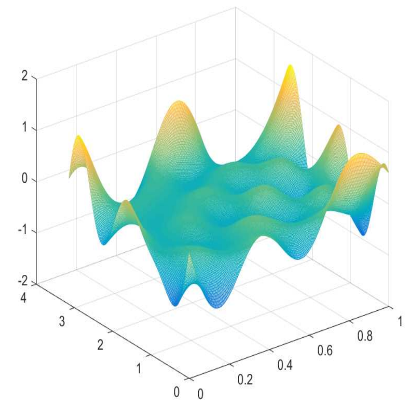
e the constructed function of g_{yy}



f the constructed error function of g_{yy}

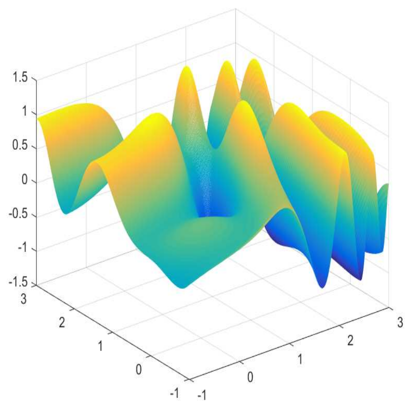


g the constructed function of g_{xy}

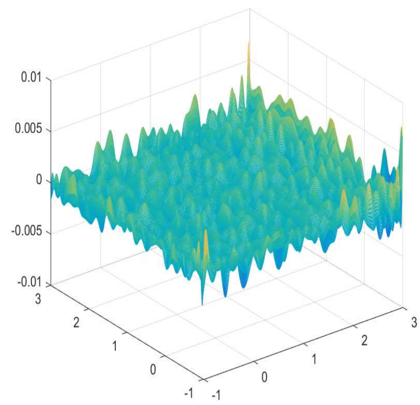


h the constructed error function of g_{xy}

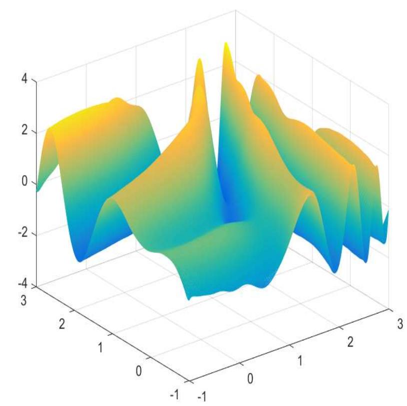
Figure 3. Continued.



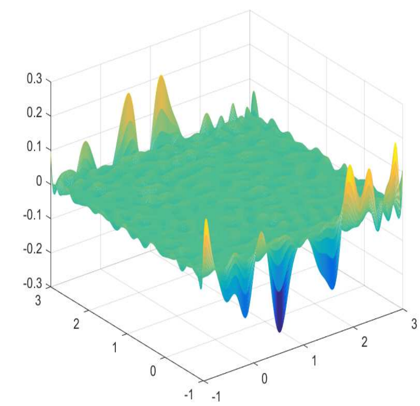
i the constructed function of g



j the constructed error function of g

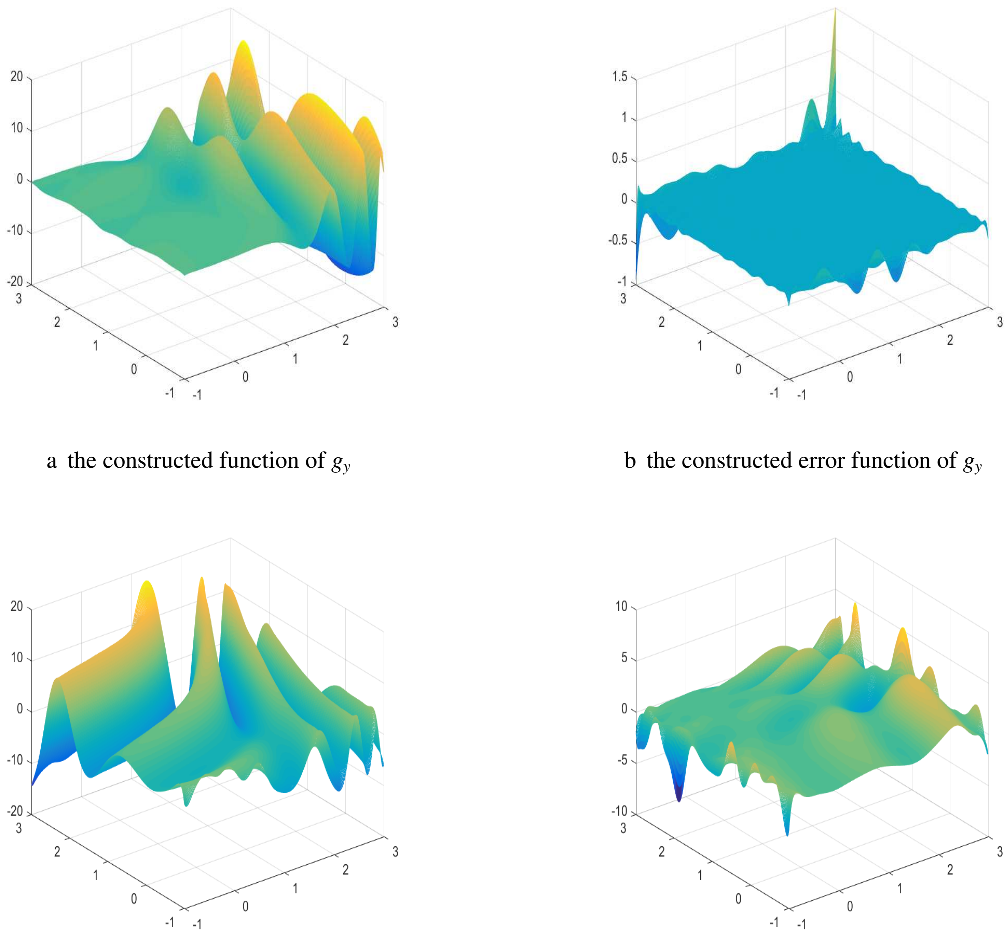


k the constructed function of g_x



l the constructed error function of g_x

Figure 4. Example 3.



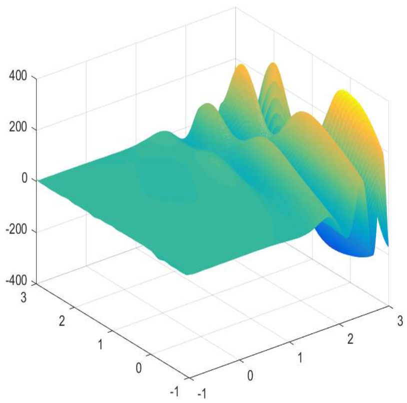
a the constructed function of g_y

b the constructed error function of g_y

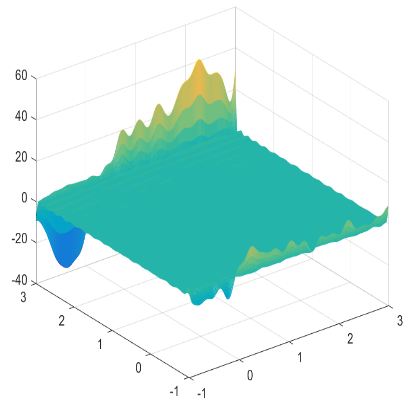
c the constructed function of g_{xx}

d the constructed error function of g_{xx}

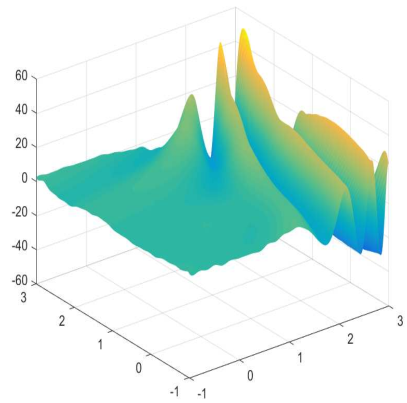
Figure 4. Continued.



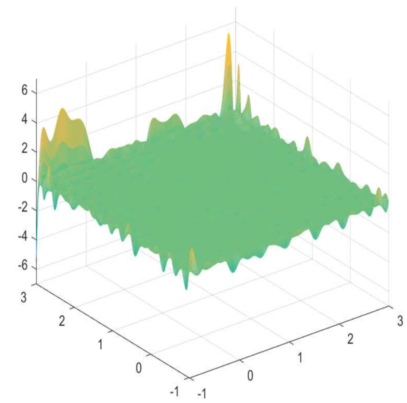
e the constructed function of g_{yy}



f the constructed error function of g_{yy}

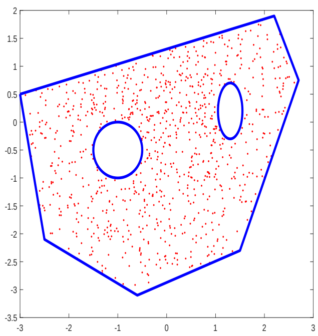


g the constructed function of g_{xy}

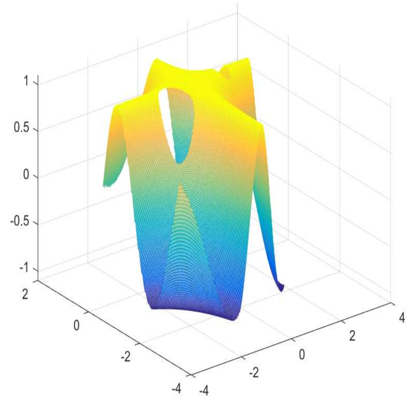


h the constructed error function of g_{xy}

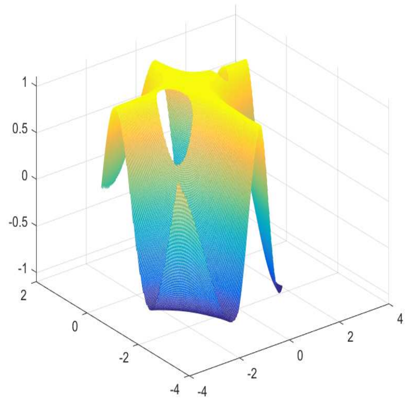
Figure 4. Continued.



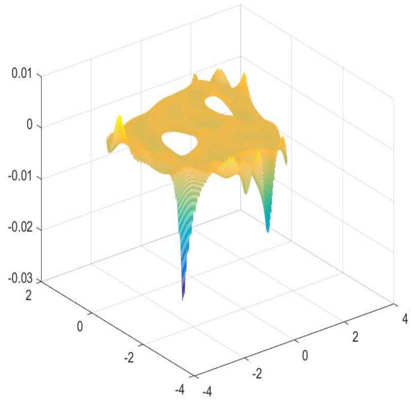
i Λ and nodes



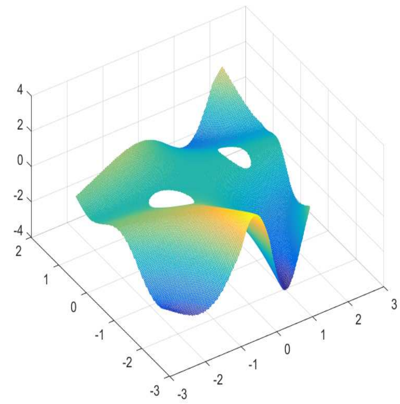
j the exact function g



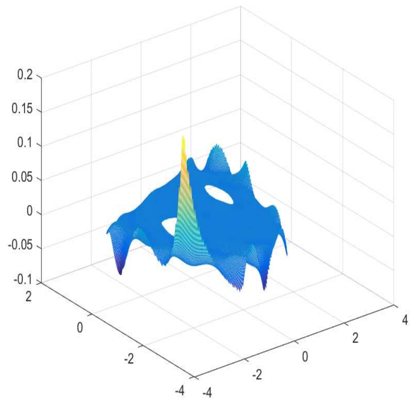
k the constructed function of g



l the constructed error function of g

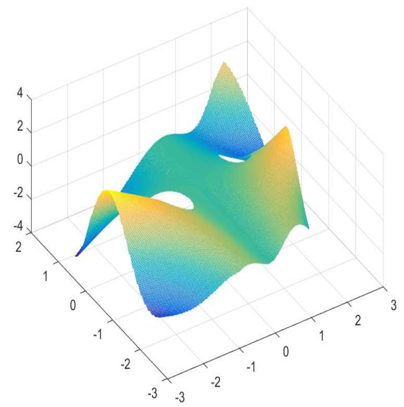


m the constructed function of g_x

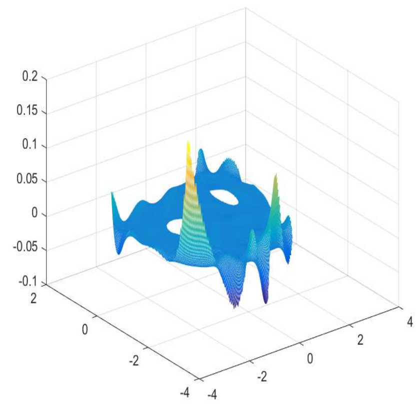


n the constructed error function of g_x

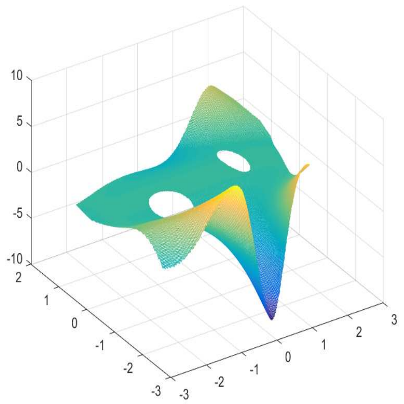
Figure 5. Example 4.



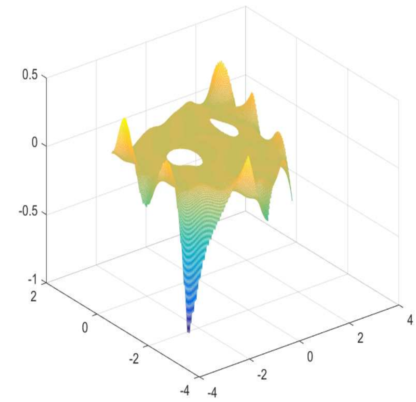
a the constructed function of g_y



b the constructed error function of g_y

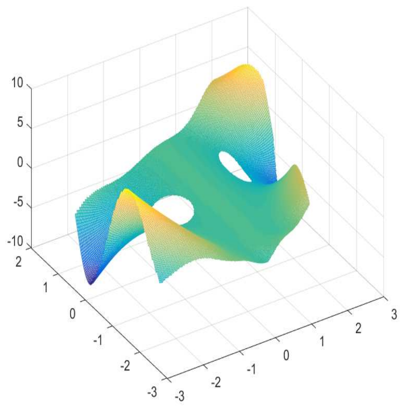


c the constructed function of g_{xx}

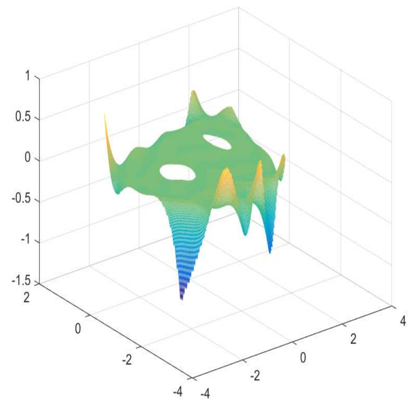


d the constructed error function of g_{xx}

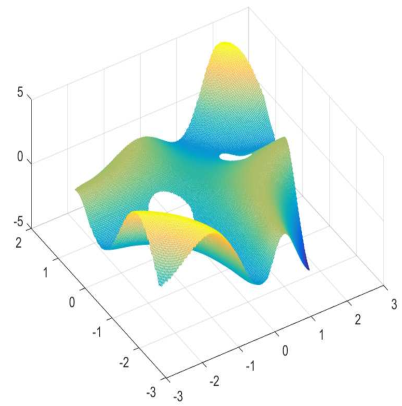
Figure 5. Continued.



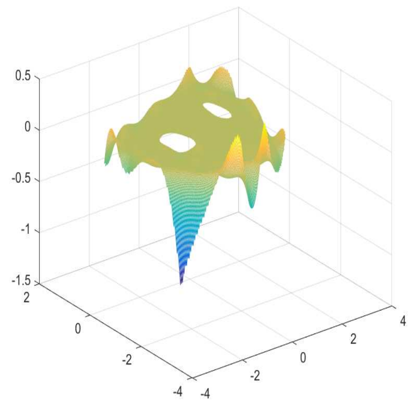
e the constructed function of g_{yy}



f the constructed error function of g_{yy}



g the constructed function of g_{xy}



h the constructed error function of g_{xy}

Figure 5. Continued.

7. Conclusions

In this paper, we present a Hermite extension method with an implicit iteration process for numerical differentiation of two-dimensional functions. Because the method can directly deal with the data given on any domain, it is more convenient than other methods in practical application. The theoretical results show that the convergence rates of the method is self-adaptive.

Conflict of interest

The authors declared that they have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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