Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 5971–5990.
DOI: 10.3934/math.2022333
Received: 26 October 2021
Revised: 12 December 2021
Accepted: 10 January 2022
Published: 14 January 2022

Research article

Weighted and endpoint estimates for commutators of bilinear pseudo-differential operators

Yanqi Yang*, **Shuangping Tao and Guanghui Lu**

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: yangyq@nwnu.edu.cn; Tel: +8618709498755.

Abstract: In this paper, by applying the accurate estimates of the Hörmander class, the authors consider the commutators of bilinear pseudo-differential operators and the operation of multiplication by a Lipschitz function. By establishing the pointwise estimates of the corresponding sharp maximal function, the boundedness of the commutators is obtained respectively on the products of weighted Lebesgue spaces and variable exponent Lebesgue spaces with $\sigma \in \mathcal{BBS}^{1}_{1,1}$. Moreover, the endpoint estimate of the commutators is also established on $L^\infty \times L^\infty$.

Keywords: bilinear pseudo-differential operator; commutator; Lipschitz function; the product of variable exponent Lebesgue space
Mathematics Subject Classification: 42B20, 42B25, 47G30

1. Introduction and main results

Let T be a linear operator. Given a function a, the commutator $[T, a]$ is defined by

$$[T, a](f) := T(af) - aT(f).$$

There is an increasing interest to the study of T being a pseudo-differential operator because of its theory plays an important role in many aspects of harmonic analysis and it has had quite a success in linear setting. As one of the most meaningful branches, the study of bilinear pseudo-differential operators was motivated not only as generalizations of the theory of linear ones but also its natural appearance and important applications. This topic is continuously attracting many researchers.

Let a be a Lipschitz function and $1 < p < \infty$. The estimates of the form

$$\|[T, a](f)\|_{L^p} \lesssim \|a\|_{\mathrm{Lip}^1} \|f\|_{L^p}, \quad \text{for all} \quad f \in L^p(\mathbb{R}^n) \tag{1.1}$$

have been studied extensively. In particular, Calderón proved that (1.1) holds when T is a pseudo-differential operator whose kernel is homogeneous of degree of $-n-1$ in [7]. Coifman and Meyer showed (1.1) when $T = T_\sigma$ and σ is a symbol in the Hörmander class $S^1_{1,0}$ go back to [10, 11], this result was later extended by Auscher and Taylor in [4] to $\sigma \in \mathcal{B}S^1_{1,1}$, where the class $\mathcal{B}S^1_{1,1}$, which contains $S^1_{1,0}$ modulo symbols associated to smoothing operators, consists of symbols whose Fourier transforms in the first n-dimensional variable are appropriately compactly supported. The method in the proofs of [10, 11] was mainly showed that, for each Lipschitz continuous functions a on $\mathbb{R}^n$, $[T, a]$ is a Calderón-Zygmund singular integral whose kernel constants are controlled by $\|a\|_{\mathrm{Lip}^1}$. For another thing, Auscher and Taylor proved (1.1) in two different ways: one method is based on the paraproducts while the other is based on the Calderón-Zygmund singular integral operator approach that relies on the $T(1)$ theorem. Fore a more systematic study of these (and even more general) spaces, we refer the readers to see [38, 39].

Given a bilinear operator T and a function a, the following two kinds commutators are respectively defined by

$$[T, a]_1(f, g) = T(af, g) - aT(f, g)$$

and

$$[T, a]_2(f, g) = T(f, ag) - aT(f, g).$$

In 2014, Bényi and Oh proved that (1.1) is also valid to this bilinear setting in [6]. More precisely, given a bilinear pseudo-differential operator T_σ with σ in the bilinear Hörmander class $BS^1_{1,0}$ and a Lipschitz function a on $\mathbb{R}^n$, it was proved in [6] that $[T, a]_1$ and $[T, a]_2$ are bilinear Calderón-Zygmund operators. The main aim of this paper is to study (1.1) of $[T_\sigma, a]_j (j = 1, 2)$ on the products of weighted Lebesgue spaces and variable exponent Lebesgue spaces with $\sigma \in \mathcal{B}BS^1_{1,1}$. Before stating our main results, we need to recall some definitions and notations. We say that a function a defined on $\mathbb{R}^n$ is Lipschitz continuous if

$$\|a\|_{\mathrm{Lip}^1} := \sup_{x,y\in\mathbb{R}^n} \frac{|a(x) - a(y)|}{|x - y|} < \infty.$$

Let $\delta \geq 0$, $\rho > 0$ and $m \in \mathbb{R}$. An infinitely differentiable function $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to the bilinear Hörmander class $BS^m_{\rho,\delta}$ if for all multi-indices $\alpha, \beta, \gamma \in \mathbb{N}^n_0$ there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$|\partial^\alpha_x \partial^\beta_\xi \partial^\gamma_\eta \sigma(x, \xi, \eta)| \leq C(1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}.$$

Given a $\sigma(x, \xi, \eta) \in BS^m_{\rho,\delta}$, the bilinear pseudo-differential operator associated to σ is defined by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x, \xi, \eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)} \mathrm{d}\xi \mathrm{d}\eta, \text{ for all } x \in \mathbb{R}^n, f, g \in \mathcal{S}(\mathbb{R}^n).$$

In 1980, Meyer [34] firstly introduced the linear $BS^m_{1,1}$, and corresponding boundedness of $[T_\sigma, a]_j (j = 1, 2)$ is obtained by Bényi-Oh in [6] , that is, given $m \in \mathbb{R}$ and $r > 0$, an infinitely differentiable function $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to $\mathcal{B}_r BS^m_{1.1}$ if

$$\sigma \in BS^m_{1,1}, supp(\hat{\sigma}^1) \subset \{(\tau, \xi, \eta) \in \mathbb{R}^{3n} : |\tau| \leq r(|\xi| + |\eta|)\},$$

where $\hat{\sigma}^1$ denotes the Fourier transform of σ with respect to its first variable in $\mathbb{R}^n$, that is, $\hat{\sigma}^1(\tau, \xi, \eta) = \widehat{\sigma(\cdot, \xi, \eta)}(\tau)$, for all $\tau, \xi, \eta \in \mathbb{R}^n$. The class $\mathcal{B}BS^m_{1,1}$ is defined as

$$\mathcal{B}BS^m_{1,1} = \bigcup_{r\in(0,\frac{1}{7})} \mathcal{B}_r BS^m_{1,1}.$$

Recently, many authors are interested in bilinear operators, which is a natural generalization of linear case. With the further research, Árpád Bényi and Virginia Naibo proved that boundedness for the commutators of bilinear pseudo-differential operators and Lipschitz functions with $\sigma \in \mathcal{BBS}^{1}_{1,1}$ on the Lebesgue spaces in [5]. In 2018, Tao and Li proved that the boundedness of the commutators of bilinear pseudo-differential operators was also true on the classical and generalized Morrey spaces in [40]. Motivated by the results mentioned above, a natural and interesting problem is to consider whether or not (1.1) is true on the weighted Lebesgue spaces and variable exponent Lebesgue spaces with $\sigma \in \mathcal{BBS}^{1}_{1,1}$. The purpose of this paper is to give an surely answer. And also, the endpoint estimate is obtained on $L^\infty \times L^\infty$. Our proofs are based on the pointwise estimates of the sharp maximal function proved in the next section.

Many results involving bilinear pseudo-differential operators theory have been obtained in parallel with the linear ones but some new interesting phenomena have also been observed. One aspect developed rapidly is the one related to the compactness of the bilinear pseudo-differential operators, especially, the properties of compactness for the commutators of bilinear pseudo-differential operators and Lipschitz functions. As the commutators $[T_\sigma, a]_j$ $(j = 1, 2)$ are bilinear Calderón-Zygmund operators if $\sigma \in \mathcal{BBS}^{1}_{1,1}$, similar to the proof of [15] (Theorem A and Theorem 2.12), we can obtain easily that $[T_\sigma, a]_j$ and $[[T_\sigma, a]_j, b]_i$ $(i, j = 1, 2)$ are compact operators on the Lebesgue spaces and the Morrey spaces. For the sake of convenience, there are no further details below.

Suppose that $\sigma \in \mathcal{BBS}^{1}_{1,1}$. Let K and K_j denote the kernel of T_σ and $[T_\sigma, a]_j$ $(j = 1, 2)$, respectively. We have

$$K(x, y, z) = \int\int e^{i\xi\cdot(x-y)} e^{i\eta\cdot(x-z)} \sigma(x, \xi, \eta) d\xi d\eta,$$

$$K_1(x, y, z) = (a(y) - a(x))K(x, y, z), \quad K_2(x, y, z) = (a(z) - a(x))K(x, y, z).$$

Then the following consequences are true.

Theorem A. [6] If $x \neq y$ or $x \neq z$, then we have

(1) $| \partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x, y, z) | \lesssim (|x - y| + |x - z|)^{-2n-1-|\alpha|-|\beta|-|\gamma|}$;

(2) $|K_j(x, y, z)| \lesssim \|a\|_{\mathrm{Lip}^1} (|x - y| + |x - z| + |y - z|)^{-2n}$.

The statement of our main theorems will be presented in follows.

Theroem 1.1. Let $q' > 1$, $\sigma \in \mathcal{BBS}^{1}_{1,1}$ and a be a Lipschitz function on $\mathbb{R}^n$. Suppose for fixed $1 \leq r_1, r_2 \leq q'$ with $1/r = 1/r_1 + 1/r_2$, $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{r_1} \times L^{r_2}$ into $L^{r,\infty}$ with norm controlled by $\|a\|_{\mathrm{Lip}^1}$. If $0 < \delta < 1/2$, then

$$M^\sharp_\delta([T_\sigma, a]_j(f, g))(x) \leq C\|a\|_{\mathrm{Lip}^1} M_{q'}(f)(x) M_{q'}(g)(x), \quad j = 1, 2$$

for all f, g of bounded measurable functions with compact support.

Theorem 1.2. Let $q' > 1$, $\sigma \in \mathcal{BBS}^{1}_{1,1}$ and a be a Lipschitz function on $\mathbb{R}^n$. Suppose for fixed $1 \leq r_1, r_2 \leq q'$ with $1/r = 1/r_1 + 1/r_2$, $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{r_1} \times L^{r_2}$ into $L^{r,\infty}$ with norm controlled by $\|a\|_{\mathrm{Lip}^1}$. If $b \in \mathrm{BMO}$, $0 < \delta < 1/2$, $\delta < \varepsilon < \infty$, $q' < s < \infty$, then

$$M^\sharp_\delta([[T_\sigma, a]_j, b]_i)(x) \leq C\|b\|_{\mathrm{BMO}} \Big((M_\varepsilon([T_\sigma, a]_j(f, g))(x) + \|a\|_{\mathrm{Lip}^1} (M_s(f)(x))(M_s(g)(x)) \Big),$$

where $i, j = 1, 2$ and above inequality is valid for all f, g of bounded measurable functions with compact support.

Theorem 1.3. Let $q' > 1$, $\sigma \in \mathcal{BBS}^{1}_{1,1}$ and a be a Lipschitz function on $\mathbb{R}^n$. Suppose for fixed $1 \le r_1, r_2 \le q'$ with $1/r = 1/r_1 + 1/r_2$, $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{r_1} \times L^{r_2}$ into $L^{r,\infty}$ with norm controlled by $\|a\|_{\mathrm{Lip}^1}$. If $(\omega_1, \omega_2) \in (A_{p_1/q'}, A_{p_2/q'})$ and $\omega = \omega_1^{\frac{p}{p_1}} \omega_2^{\frac{p}{p_2}}$, then for $q' < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{p_1}(\omega) \times L^{p_2}(\omega)$ into $L^p(\omega)$.

Theorem 1.4. Let $q' > 1$, $\sigma \in \mathcal{BBS}^{1}_{1,1}$ and a be a Lipschitz function on $\mathbb{R}^n$. Suppose for fixed $1 \le r_1, r_2 \le q'$ with $1/r = 1/r_1 + 1/r_2$, $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{r_1} \times L^{r_2}$ into $L^{r,\infty}$ with norm controlled by $\|a\|_{\mathrm{Lip}^1}$. If $b \in \mathrm{BMO}$, $(\omega_1, \omega_2) \in (A_{p_1/q'}, A_{p_2/q'})$ and $\omega = \omega_1^{\frac{p}{p_1}} \omega_2^{\frac{p}{p_2}}$, then for $q' < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, $[[T_\sigma, a]_j, b]_i (i, j = 1, 2)$ is bounded from $L^{p_1}(\omega) \times L^{p_2}(\omega)$ into $L^p(\omega)$.

Theorem 1.5. Let $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, and q_0^j be given as in Lemma 4.4 for $p_j(\cdot)$, j=1,2. Suppose that $\sigma \in \mathcal{BBS}^{1}_{1,1}$, a is a Lipschitz function on $\mathbb{R}^n$ and $1 < q' \le \min\{q_0^1, q_0^2\}$. If for fixed $1 \le r_1, r_2 \le q'$ with $1/r = 1/r_1 + 1/r_2$, $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{r_1} \times L^{r_2}$ into $L^{r,\infty}$ with norm controlled by $\|a\|_{\mathrm{Lip}^1}$, then $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ into $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 1.6. Let $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, and q_0^j be given as in Lemma 4.4 for $p_j(\cdot)$, j=1,2. Suppose that $\sigma \in \mathcal{BBS}^{1}_{1,1}$, a is a Lipschitz function on $\mathbb{R}^n$ and $1 < q' \le \min\{q_0^1, q_0^2\}$. If for fixed $1 \le r_1, r_2 \le q'$ with $1/r = 1/r_1 + 1/r_2$, $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{r_1} \times L^{r_2}$ into $L^{r,\infty}$ with norm controlled by $\|a\|_{\mathrm{Lip}^1}$, and $b \in \mathrm{BMO}$, then $[[T_\sigma, a]_j, b]_i (i, j = 1, 2)$ is bounded from $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ into $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 1.7. Let $\sigma \in \mathcal{BBS}^{1}_{1,1}$ and a be a Lipschitz function. Suppose for fixed $1 \le r_1, r_2 \le q'$ with $1/r = 1/r_1 + 1/r_2$, $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{r_1} \times L^{r_2}$ into $L^{r,\infty}$ with norm controlled by $\|a\|_{\mathrm{Lip}^1}$. Then $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^\infty \times L^\infty$ into BMO.

We use the following notation: For $1 \le p \le \infty$, p' is the conjugate index of p, that is, $1/p + 1/p' = 1$. $B(x, R)$ denotes the ball centered at x with radius $R > 0$ and $f_B = \frac{1}{|B(x,R)|} \int_{B(x,R)} f(y) \mathrm{d}y$. The paper is organized as follows. The pointwise estimates of the sharp maximal functions are presented in Section 2. The weighted boundedness is given in Section 3. The proofs of the boundedness on the product of variable exponent Lebesgue spaces are showed in Section 4. The endpoint estimate is proved in Section 5.

2. Pointwise estimates for the sharp maximal functions

In this section, we shall prove Theorems 1.1 and 1.2. In order to do this, let's recall some definitions.

Given a function $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$, the sharp maximal function is defined by

$$M^\sharp(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B| \mathrm{d}y \approx \sup_{x \in B} \inf_{a \in \mathbb{C}} \frac{1}{|B|} \int_B |f(y) - a| \mathrm{d}x,$$

where the supremum is taken over all balls B containing x. Let $0 < \delta < \infty$. We denote by $M_\delta^\sharp$ the operator

$$M_\delta^\sharp(f) = [M^\sharp(|f|^\delta)]^{1/\delta}.$$

Similarly, we use M_δ to denote the operator $M^\delta(f) = [M(|f|^\delta)]^{1/\delta}$, where M is the Hardy-Littlewood maximal function defined by

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|} \int_B f(y)\mathrm{d}y.$$

The operator $M^{\sharp}_{\delta}$ was appeared implicitly in a paper by John [20] and was introduced by Strömberg [37]. The sharp maximal function $M^{\sharp}$ and $M^{\sharp}_{\delta}$ not only have close relation to BMO, but also are important tools to obtain pointwise inequalities regarding many operators in harmonic analysis (see [3, 12, 21, 25, 26, 36]).

To prove the Theorems 1.1 and 1.2, we need the following Kolmogorov's inequality and the inequality regarding the BMO functions.

Lemma 2.1. [19, 28] Let $0 < p < q < \infty$. Then there is a constant $C = C_{p,q} > 0$, such that

$$|Q|^{-1/p}\|f\|_{L^p(Q)} \le C|Q|^{-1/q}\|f\|_{L^{q,\infty}(Q)}$$

for all measurable functions f.

Lemma 2.2. [27] Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose $1 \le p < \infty$, $r_1 > 0$, $r_2 > 0$ and $x \in \mathbb{R}^n$. Then

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_{B(x,r_2)}|^p \mathrm{d}y\right)^{1/p} \le C\left(1 + \left|\ln \frac{r_1}{r_2}\right|\right)\|f\|_{\mathrm{BMO}},$$

where C is a positive constant independent of f, x, r_1 and r_2.

Lemma 2.3. [5] If $\sigma \in \mathcal{B}BS^{1}_{1,1}$ and a is a Lipschitz function on $\mathbb{R}^n$, then the commutators $[T_\sigma, a]_j$, $j = 1, 2$ are bilinear Calderón-Zygmund operators. In particular, $[T_\sigma, a]_j$, $j = 1, 2$ are bounded from $L^{p_1} \times L^{p_2}$ into L^p for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 < p_1,\ p_2 < \infty$ and verify appropriate end-point boundedness properties. Moreover, the corresponding norms of the operators are controlled by $\|a\|_{\mathrm{Lip}^1}$.

Proof of Theorem 1.1. Let f, g be bounded measurable functions with compact support. Then for any ball $B = B(x_0, r_B)$ containing x, we decompose f and g as follows:

$$f = f\chi_{16B} + f\chi_{(16B)^c} := f^1 + f^2, \quad g = g\chi_{16B} + g\chi_{(16B)^c} := g^1 + g^2.$$

Choose a $z_0 \in 3B\backslash 2B$. Then

$$\begin{aligned}
&\left(\frac{1}{|B|}\left||[T_\sigma, a]_j(f, g)(z)|^\delta - |[T_\sigma, a]_j(f^2, g^2)(z_0)|^\delta\right|\mathrm{d}z\right)^{1/\delta}\\
\le\ & C\left(\frac{1}{|B|}|[T_\sigma, a]_j(f, g)(z) - [T_\sigma, a]_j(f^2, g^2)(z_0)|^\delta \mathrm{d}z\right)^{1/\delta}\\
\le\ & C\left(\frac{1}{|B|}|[T_\sigma, a]_j(f^1, g^1)(z)|^\delta \mathrm{d}z\right)^{1/\delta} + C\left(\frac{1}{|B|}\int_B |[T_\sigma, a]_j(f^2, g^1)(z)|^\delta \mathrm{d}z\right)^{1/\delta}\\
& + C\left(\frac{1}{|B|}|[T_\sigma, a]_j(f^1, g^2)(z)|^\delta \mathrm{d}z\right)^{1/\delta}\\
& + C\left(\frac{1}{|B|}\int_B |[T_\sigma, a]_j(f^2, g^2)(z) - [T_\sigma, a]_j(f^2, g^2)(z_0)|^\delta \mathrm{d}z\right)^{1/\delta}\\
:=\ & \sum_{s=1}^{4} I_s.
\end{aligned}$$

For any $0 < \delta < r < \infty$, it follows from Lemma 2.1 that

$$\begin{aligned} I_1 &\leq C|B|^{-1/\delta}\|[T_\sigma, a]_j(f^1, g^1)\|_{L^\delta(B)} \\ &\leq C|B|^{-1/r}\|[T_\sigma, a]_j(f^1, g^1)\|_{L^{r,\infty}(B)} \\ &\leq C\|a\|_{\mathrm{Lip}^1}\left(\frac{1}{|16B|}\int_{16B}|f(y_1)|^{r_1}\mathrm{d}y_1\right)^{\frac{1}{r_1}}\left(\frac{1}{|16B|}\int_{16B}|g(y_2)|^{r_2}\mathrm{d}y_2\right)^{\frac{1}{r_2}} \\ &\leq C\|a\|_{\mathrm{Lip}^1}M_{r_1}(f)(x)M_{r_2}(g)(x) \\ &\leq C\|a\|_{\mathrm{Lip}^1}M_{q'}(f)(x)M_{q'}(g)(x). \end{aligned}$$

If $z \in B,\ y_1 \in (16B)^c,\ y_2 \in 16B$, noticing that $|z-y_1|+|z-y_2|+|y_1-y_2| \sim |z-y_1|+|z-y_2| \geq |z-y_1|$, then we have by Theorem A,

$$\begin{aligned} I_2 &\leq C\left(\frac{1}{|B|}\int_B\left(\int_{(16B)^c}\int_{16B}|K(z,y_1,y_2)|f(y_1)||g(y_2)\mathrm{d}y_2\mathrm{d}y_1\right)^\delta \mathrm{d}z\right)^{1/\delta} \\ &\leq C\left(\frac{1}{|B|}\int_B\left(\int_{(16B)^c}\left(\int_{16B}|g(y_2)\mathrm{d}y_2\right)\|a\|_{\mathrm{Lip}^1}\frac{f(y_1)}{|z-y_1|^{2n}}\mathrm{d}y_1\right)^\delta \mathrm{d}z\right)^{1/\delta} \\ &\leq C\left(\int_{16B}|g(y_2)\mathrm{d}y_2\right)\|a\|_{\mathrm{Lip}^1}\sum_{k=4}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{f(y_1)}{|x_0-y_1|^{2n}}\mathrm{d}y_1 \\ &\leq C\|a\|_{\mathrm{Lip}^1}\left(\frac{1}{|16B|}\int_{16B}|g(y_2)\mathrm{d}y_2\right)\sum_{k=4}^{\infty}2^{-kn}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(y_1)|\mathrm{d}y_1 \\ &\leq C\|a\|_{\mathrm{Lip}^1}M(f)(x)M(g)(x)\sum_{k=4}^{\infty}2^{-kn} \\ &\leq C\|a\|_{\mathrm{Lip}^1}M_{q'}(f)(x)M_{q'}(g)(x). \end{aligned}$$

By the similar way, we can get that

$$I_3 \leq C\|a\|_{\mathrm{Lip}^1}M_{q'}(f)(x)M_{q'}(g)(x).$$

As $z \in B$ and $y_1,\ y_2 \in (16B)^c$, then $|y_1-z_0| \geq 2|z-z_0|$, $|y_2-z_0| \geq 2|z-z_0|$ and $r_B \leq |z-z_0| \leq 4r_B$. It follows from Hölder's inequality that

$$\begin{aligned} I_4 &\leq C\left(\frac{1}{|B|}\int_B\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|K(z,y_1,y_2)-K(z_0,y_1,y_2)||f^2(y_1)||g^2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\right)^\delta \mathrm{d}z\right)^{1/\delta} \\ &\leq C(\frac{1}{|B|}\int_B(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\int_{2^{k_2}|z-z_0|\leq|y_2-z_0|\leq 2^{k_2+1}|z-z_0|}\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|z-z_0|} \\ &\quad \times|K(z,y_1,y_2)-K(z_0,y_1,y_2)||f(y_1)||g(y_2)|\mathrm{d}y_1\mathrm{d}y_2)^\delta \mathrm{d}z)^{1/\delta} \\ &\leq C(\frac{1}{|B|}\int_B(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\int_{2^{k_2}|z-z_0|\leq|y_2-z_0|\leq 2^{k_2+1}|z-z_0|}|g(y_2)| \\ &\quad \times\left(\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)||f(y_1)|^q\mathrm{d}y_1\right)^{\frac{1}{q}} \end{aligned}$$

$$\times(\int_{2^{k_1+4}B}|f(y_1)|^{q'}\mathrm{d}y_1)^{\frac{1}{q'}}\mathrm{d}y_2)^{\delta}\mathrm{d}z)^{\frac{1}{\delta}}$$

$$\leq C(\frac{1}{|B|}\int_B(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}(\int_{2^{k_1+4}B}|f(y_1)|^{q'}\mathrm{d}y_1)^{\frac{1}{q'}}(\int_{2^{k_2+4}B}|g(y_2)|^{q'}\mathrm{d}y_2)^{\frac{1}{q'}}$$

$$\times(\int_{2^{k_2}|z-z_0|\leq|y_2-z_0|\leq 2^{k_2+1}|z-z_0|}\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|z-z_0|}|K(z,y_1,y_2)$$

$$-K(z_0,y_1,y_2)|^q\mathrm{d}y_1\mathrm{d}y_2)^{\frac{1}{q}})^{\delta}\mathrm{d}z)^{\frac{1}{\delta}}$$

$$\leq C\|a\|_{\mathrm{Lip}^1}(\frac{1}{|B|}\int_B(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}(\frac{1}{|2^{k_1+4}B|}\int_{2^{k_1+4}B}|f(y_1)|^{q'}\mathrm{d}y_1)^{\frac{1}{q'}}$$

$$\times(\frac{1}{|2^{k_1+4}B|}\int_{2^{k_2+4}B}|g(y_2)|^{q'}\mathrm{d}y_2)^{\frac{1}{q'}}$$

$$\times|2^{k_1+4}B|^{1/q'}|2^{k_2+4}B|^{1/q'}|z-z_0|^{-\frac{2n}{q'}}C_{k_1}2^{-\frac{k_1 n}{q'}}C_{k_2}2^{-\frac{k_2 n}{q'}})^{\delta}\mathrm{d}z)^{1/\delta}$$

$$\leq C\|a\|_{\mathrm{Lip}^1}M_{q'}(f)(x)M_{q'}(g)(x)\left(\sum_{k_1=1}^{\infty}C_{k_1}\right)\left(\sum_{k_2=1}^{\infty}C_{k_2}\right)$$

$$\leq C\|a\|_{\mathrm{Lip}^1}M_{q'}(f)(x)M_{q'}(g)(x), \tag{2.1}$$

where we use the fact of a weaker size condition of standard m-linear Calderón-Zygmund kernel than its classical size condition given in [31], that is: For any $k_1,\cdots,k_m\in\mathbb{N}_+$, there are positive constant C_{k_i}, $i=1,\cdots,m$, such that

$$(\int_{2^{k_m}|y_0-y_0'|\leq|y_m-y_0|\leq 2^{k_m+1}|z_0-z_0'|}\cdots\int_{2^{k_1}|y_0-y_0'|\leq|y_1-y_0|\leq 2^{k_1+1}|z_0-z_0'|}$$

$$|K(y_0,y_1\cdots y_m)-K(y_0',y_1\cdots y_m)|^q\mathrm{d}y_1\cdots\mathrm{d}y_m)^{\frac{1}{q}}$$

$$\leq C|y_0-y_0'|^{-\frac{mn}{q'}}\prod_{i=1}^{m}C_{k_i}2^{-\frac{n}{q'}k_i}, \tag{2.2}$$

where $\sum\limits_{k_i=1}^{\infty}C_{k_i}<\infty,\ \ i=1,2,\ 1<q<\infty$. Together with the commutators $[T_\sigma,a]_j$, $j=1,2$ are bilinear Calderón-Zygmund operators and Theorem A, then we obtain the fact that

$$(\int_{2^{k_2}|y_0-y_0'|\leq|y_2-y_0|\leq 2^{k_2+1}|z_0-z_0'|}\int_{2^{k_1}|y_0-y_0'|\leq|y_1-y_0|\leq 2^{k_1+1}|z_0-z_0'|}$$

$$|K(y_0,y_1,y_2)-K(y_0',y_1,y_2)|^q\mathrm{d}y_1\mathrm{d}y_2)^{\frac{1}{q}}$$

$$\leq C\|a\|_{\mathrm{Lip}^1}|y_0-y_0'|^{-\frac{2n}{q'}}\prod_{i=1}^{2}C_{k_i}2^{-\frac{n}{q'}k_i}. \tag{2.3}$$

Thus, we have

$$M_\delta^\sharp([T_\sigma,a]_j(f,g))(x)\approx\sup_{x\in B}\inf_{a\in\mathbb{C}}\left(\frac{1}{|B|}\int_B|[T_\sigma,a]_j(f,g)(z)|^\delta-a|\mathrm{d}z\right)^{1/\delta}$$

$$\leq \sup_{x\in B}\left(\frac{1}{|B|}\int_B \big||[T_\sigma,a]_j(f,g)(z)|^\delta - |[T_\sigma,a]_j(f^2,g^2)(z_0)|^\delta\big| dz\right)^{1/\delta}$$
$$\leq C\|a\|_{\mathrm{Lip}^1} M_{q'}(f)(x)M_{q'}(g)(x).$$

Thus we finish the proof of Theorem 1.1.

Proof of Theorem 1.2. Without loss of generality, we consider the case $i = 1$, the proof of the case $i = 2$ is similar. Let f_1, f_2 be bounded measurable functions with compact support. As in the proof of Theorem 1.1, we write f and g as

$$f = f\chi_{16B} + f\chi_{(16B)^c} := f^1 + f^2, \ \ g = g\chi_{16B} + g\chi_{(16B)^c} := g^1 + g^2.$$

Then

$$\begin{aligned}
[[T_\sigma,a]_j,b]_1(f,g)(z) &= (b(z)-b_{16B})[T_\sigma,a]_j(f,g)(z) - [T_\sigma,a]_j((b-b_{16B})f,g)(z)\\
&= (b(z)-b_{16B})[T_\sigma,a]_j(f,g)(z) - [T_\sigma,a]_j((b-b_{16B})f^1,g^1)(z)\\
&\quad -[T_\sigma,a]_j((b-b_{16B})f^1,g^2)(z) - [T_\sigma,a]_j((b-b_{16B})f^2,g^1)(z)\\
&\quad -[T_\sigma,a]_j((b-b_{16B})f^2,g^2)(z),
\end{aligned}$$

where $b_{16B} = \frac{1}{|16B|}\int_{16B} b(z)dz$. Therefore, for any fixed $z_0 \in 3B\backslash 2B$, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \big|[[T_\sigma,a]_j,b]_1(f,g)(z) + [T_\sigma,a]_j((b-b_{16B})f^2,g^2)(z_0)\big|^\delta dz\right)^{\frac{1}{\delta}}\\
&\leq C\left(\frac{1}{|B|}\int_B |(b(z)-b_{16B})[T_\sigma,a]_j(f,g)(z)|^\delta dz\right)^{\frac{1}{\delta}}\\
&+C\left(\frac{1}{|B|}\int_B |[T_\sigma,a]_j((b-b_{16B})f^1,g^1))(z)|^\delta dz\right)^{\frac{1}{\delta}}\\
&+C\left(\frac{1}{|B|}\int_B |[T_\sigma,a]_j((b-b_{16B})f^1,g^2)(z)|^\delta dz\right)^{\frac{1}{\delta}}\\
&+\ C\left(\frac{1}{|B|}\int_B |[T_\sigma,a]_j((b-b_{16B})f^2,g^1)(z)|^\delta dz\right)^{\frac{1}{\delta}}\\
&+\ C\left(\frac{1}{|B|}\int_B |[T_\sigma,a]_j((b-b_{16B})f^2,g^2)(z) - [T_\sigma,a]_j((b-b_{16B})f^2,g^2)(z_0)|^\delta dz\right)^{\frac{1}{\delta}}\\
&:= \sum_{t=1}^{5} II_t.
\end{aligned}$$

Since $0 < \delta < 1/2$ and $\delta < \varepsilon < \infty$, there exists an l such that $1 < l < \min\{\frac{\varepsilon}{\delta}, \frac{1}{1-\delta}\}$. Then $\delta l < \varepsilon$ and $\delta l' > 1$. By Hölder's inequality, we have

$$II_1 \leq C\left(\frac{1}{|B|}\int_B |(b(z)-b_{16B}|^{\delta l'} dz\right)^{\frac{1}{\delta l'}}\left(\frac{1}{|B|}\int_B |[T_\sigma,a]_j(f,g)(z)|^{\delta l} dz\right)^{\frac{1}{\delta l}}$$

$$
\begin{aligned}
&\leq C\|b\|_{\mathrm{BMO}}\left(\frac{1}{|B|}\int_B |[T_\sigma,a]_j(f,g)(z)|^\varepsilon \mathrm{d}z\right)^{\frac{1}{\varepsilon}}\\
&\leq C\|b\|_{\mathrm{BMO}}M_\varepsilon([T_\sigma,a]_j(f,g))(x).
\end{aligned}
$$

Since $q' < s < \infty$, denoting $t = s/q'$, then $1 < t < \infty$. Noticing that $0 < \delta < r < \infty$, it follows from Lemmas 2.1 and 2.3 that

$$
\begin{aligned}
&II_2 \leq C|B|^{-1/\delta}\|[T_\sigma,a]_j((b-b_{16B})f^1,g^1)\|_{L^\delta(B)}\\
&\leq C|B|^{-1/r}\|[T_\sigma,a]_j((b-b_{16B})f^1,g^1)\|_{L^{r,\infty}(B)}\\
&\leq C\|a\|_{\mathrm{Lip}^1}(\frac{1}{|16B|}\int_{16B}|b(y_1)-b_{16B}|^{r_1}|f(y_1)|^{r_1}dy_1)^{\frac{1}{r_1}}(\frac{1}{|16B|}\int_{16B}|g(y_2)|^{r_2}dy_2)^{\frac{1}{r_2}}\\
&\leq C\|a\|_{\mathrm{Lip}^1}(\frac{1}{|16B|}\int_{16B}|b(y_1)-b_{16B}|^{r_1t'}dy_1)^{\frac{1}{r_1t'}}(\frac{1}{|16B|}\int_{16B}|f(y_1)|^{r_1t}dy_2)^{\frac{1}{r_1t}}\\
&\times(\frac{1}{|16B|}\int_{16B}|g(y_2)|^{r_2}dy_2)^{\frac{1}{r_2}}\\
&\leq C\|a\|_{\mathrm{Lip}^1}\|b\|_{\mathrm{BMO}}(\frac{1}{|16B|}\int_{16B}|f(y_1)|^{s}dy_2)^{\frac{1}{s}}\times(\frac{1}{|16B|}\int_{16B}|g(y_2)|^{s}dy_2)^{\frac{1}{s}}\\
&\leq C\|a\|_{\mathrm{Lip}^1}\|b\|_{\mathrm{BMO}}M_s(f)(x)M_s(g)(x).
\end{aligned}
$$

By Theorem A, as $z \in B$, $y_1 \in (16B)$, $y_2 \in 16B^c$, noticing that $|z-y_1|+|z-y_2|+|y_1-y_2| \sim |z-y_1|+|z-y_2| \geq |z-y_2|$, then we have

$$
\begin{aligned}
&II_3 \leq C(\frac{1}{|B|}\int_B(\int_{(16B)^c}\int_{(16B)}|K(z,y_1,y_2)||b(y_1-b_{16B})||f(y_1)||g(y_2)|\mathrm{d}y_1\mathrm{d}y_2)^\delta \mathrm{d}z)^{1/\delta}\\
&\leq C\|a\|_{\mathrm{Lip}^1}(\frac{1}{|B|}\int_B(\int_{(16B)^c}(\int_{(16B)}|b(y_1-b_{16B})||f(y_1)|\mathrm{d}y_1)\frac{f(y_2)}{|z-y_2|^{2n}}\mathrm{d}y_2)^\delta\mathrm{d}z)^{1/\delta}\\
&\leq C\|a\|_{\mathrm{Lip}^1}\left(\int_{(16B)}|b(y_1-b_{16B})||f(y_1)|\mathrm{d}y_1\right)\sum_{k=4}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{f(y_2)}{|z_0-y_2|^{2n}}\mathrm{d}y_2\\
&\leq C\|a\|_{\mathrm{Lip}^1}\left(\int_{(16B)}|b(y_1-b_{16B})|^q\mathrm{d}y_1\right)^{1/q}\left(\int_{(16B)}|f(y_1)|^{q'}\mathrm{d}y_1\right)^{1/q'}\\
&\times\sum_{k=4}^{\infty}2^{-kn}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|g(y_2)|dy_2\\
&\leq C\|a\|_{\mathrm{Lip}^1}\|b\|_{\mathrm{BMO}}M_{q'}(f)(x)M(g)(x)\sum_{k=4}^{\infty}2^{-kn}\\
&\leq C\|a\|_{\mathrm{Lip}^1}\|b\|_{\mathrm{BMO}}M_s(f)(x)M_s(g)(x).
\end{aligned}
$$

Similar to estimate II_3, by Lemma 2.2, we can get that

$$
\begin{aligned}
II_4 &\leq C\|a\|_{\mathrm{Lip}^1}\left(\int_{(16B)}|g(y_2)|\mathrm{d}y_2\right)\sum_{k=4}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|b(y_1b_{16B})||f(y_1)|}{|x_0-y_q|^{2n}}\mathrm{d}y_1\\
&\leq C\|a\|_{\mathrm{Lip}^1}M(g)(x)\sum_{k=4}^{\infty}2^{-kn}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(y_1-b_{16B})||f(y_1)|\mathrm{d}y_1
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|a\|_{\mathrm{Lip}^1} M(g)(x)\sum_{k=4}^{\infty}2^{-kn}\left(\frac{1}{|2^{k+1}B|}|b(y_1-b_{16B})|^q\mathrm{d}y_1\right)^{1/q}\\
&\quad\times\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(y_1)|^{q'}dy_1\right)^{\frac{1}{q'}}\\
&\le C\|a\|_{\mathrm{Lip}^1}\|b\|_{\mathrm{BMO}}M_{q'}(f)(x)M(g)(x)\sum_{k=4}^{\infty}2^{-kn}\\
&\le C\|a\|_{\mathrm{Lip}^1}\|b\|_{\mathrm{BMO}}M_s(f)(x)M_s(g)(x).
\end{aligned}
$$

As $z\in B$ and $y_1,y_2\in(16B)^c$, then $|y_1-z_0|\ge 2|z-z_0|$, $|y_2-z_0|\ge 2|z-z_0|$ and $r_B\le|z-z_0|\le 4r_B$. Noticing that $\frac{1}{q}+\frac{1}{tq'}+\frac{1}{t'q'}=1$. It follows from Hölder's inequality, Theorem A and the fact (2.3) that

$$
\begin{aligned}
II_5 &\le C\Big(\frac{1}{|B|}\int_B\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|K(z,y_1,y_2)-K(z_0,y_1,y_2)||b(y_1)-b_{16B}|\\
&\quad\times|f^2(y_1)||g^2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\Big)^{\delta}\Big)^{1/\delta}\\
&\le C\Big(\frac{1}{|B|}\int_B\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\int_{2^{k_2}|z-z_0|\le|y_2-z_0|\le 2^{k_2+1}|z-z_0|}\int_{2^{k_1}|z-z_0|\le|y_1-z_0|\le 2^{k_1+1}|z-z_0|}\\
&\quad\times|K(z,y_1,y_2)-K(z_0,y_1,y_2)||b(y_1)-b_{16B}||f(y_1)||g(y_2)|\mathrm{d}y_1\mathrm{d}y_2\Big)^{\delta}\mathrm{d}z\Big)^{1/\delta}\\
&\le C\Big(\frac{1}{|B|}\int_B\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\int_{2^{k_2}|z-z_0|\le|y_2-z_0|\le 2^{k_2+1}|z-z_0|}|g(y_2)|\\
&\quad\times\left(\int_{2^{k_1}|z-z_0|\le|y_1-z_0|\le 2^{k_1+1}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|^q\mathrm{d}y_1\right)^{\frac{1}{q}}\\
&\quad\times\Big(\int_{2^{k_1+4}B}|b(y_1)-b_{16B}|^{t'q'}\mathrm{d}y_1\Big)^{\frac{1}{t'q'}}\Big(\int_{2^{k_1+4}B}|f(y_1)|^{tq'}\mathrm{d}y_1\Big)^{\frac{1}{tq'}}\mathrm{d}y_2\Big)^{\delta}\mathrm{d}z\Big)^{\frac{1}{\delta}}\\
&\le C\Big(\frac{1}{|B|}\int_B\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\int_{2^{k_1+4}B}|b(y_1)-b_{16B}|^{t'q'}\mathrm{d}y_1\Big)^{\frac{1}{t'q'}}\\
&\quad\times\Big(\int_{2^{k_2+4}B}|f(y_1)|^{tq'}\mathrm{d}y_1\Big)^{\frac{1}{tq'}}\Big(\int_{2^{k_2+4}B}|g(y_2)|^{q'}\mathrm{d}y_2\Big)^{\frac{1}{q'}}\\
&\quad\times\Big(\int_{2^{k_2}|z-z_0|\le|y_2-z_0|\le 2^{k_2+1}|z-z_0|}\int_{2^{k_1}|z-z_0|\le|y_1-z_0|\le 2^{k_1+1}|z-z_0|}|K(z,y_1,y_2)\\
&\quad-K(z_0,y_1,y_2)|^q\mathrm{d}y_1\mathrm{d}y_2\Big)^{\frac{1}{q}}\Big)^{\delta}\mathrm{d}z\Big)^{\frac{1}{\delta}}\\
&\le C\|a\|_{\mathrm{Lip}^1}\Big(\frac{1}{|B|}\int_B\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\frac{1}{|2^{k_1+4}B|}\int_{2^{k_1+4}B}|b(y_1)-b_{16B}|^{t'q'}\mathrm{d}y_1\Big)^{\frac{1}{t'q'}}\\
&\quad\times\Big(\frac{1}{|2^{k_1+4}B|}\int_{2^{k_1+4}B}|f(y_1)|^{tq'}\mathrm{d}y_1\Big)^{\frac{1}{tq'}}\times\Big(\frac{1}{|2^{k_1+4}B|}\int_{2^{k_2+4}B}|g(y_2)|^{q'}\mathrm{d}y_2\Big)^{\frac{1}{q'}}\\
&\quad\times|2^{k_1+4}B|^{1/q'}|2^{k_2+4}B|^{1/q'}|z-z_0|^{-\frac{2n}{q'}}C_{k_1}2^{-\frac{k_1n}{q'}}C_{k_2}2^{-\frac{k_2n}{q'}}\Big)^{\delta}\mathrm{d}z\Big)^{1/\delta}\\
&\le C\|a\|_{\mathrm{Lip}^1}\|b\|_{\mathrm{BMO}}M_s(f)(x)M_{q'}(g)(x)\left(\sum_{k_1=1}^{\infty}C_{k_1}\right)\left(\sum_{k_2=1}^{\infty}C_{k_2}\right)
\end{aligned}
$$

$$\leq \quad C\|a\|_{\mathrm{Lip}^1}\|b\|_{\mathrm{BMO}}M_s(f)(x)M_s(g)(x).$$

Combining the estimate of II_j, $j = 1, 2, 3, 4, 5$, we get

$$\begin{aligned}&\Big(\frac{1}{|B|}\int_B |[[T_\sigma, a]_j, b]_1(f,g)(z) + [T_\sigma, a]_j((b - b_{16B})f^2, g^2)(z_0)|^\delta \mathrm{d}z\Big)^{\frac{1}{\delta}}\\ \leq\ & C\|b\|_{\mathrm{BMO}}(M_\varepsilon([T_\sigma, a]_j(f,g))(x) + \|a\|_{\mathrm{Lip}^1}M_s(f)(x)M_s(g)(x)).\end{aligned}$$

Similarly, for the case $i = 2$, we can obtain that

$$\begin{aligned}&\Big(\frac{1}{|B|}\int_B |[[T_\sigma, a]_j, b]_2(f,g)(z) + [T_\sigma, a]_j((b - b_{16B})f^2, g^2)(z_0)|^\delta \mathrm{d}z\Big)^{\frac{1}{\delta}}\\ \leq\ & C\|b\|_{\mathrm{BMO}}(M_\varepsilon([T_\sigma, a]_j(f,g))(x) + \|a\|_{\mathrm{Lip}^1}M_s(f)(x)M_s(g)(x)).\end{aligned}$$

Thus,

$$\begin{aligned}M^\sharp_\delta\ & ([[T_\sigma, a]_j, b]_i(f,g))(x) \approx \sup_{x\in B}\inf_{a\in\mathbb{C}}\left(\frac{1}{|B|}\int_B |[[T_\sigma, a]_j, b]_1(f,g)(z)|^\delta - a|\mathrm{d}z\right)^{1/\delta}\\ \leq\ & C\|b\|_{\mathrm{BMO}}(M_\varepsilon([T_\sigma, a]_j(f,g))(x) + \|a\|_{\mathrm{Lip}^1}M_s(f)(x)M_s(g)(x)).\end{aligned}$$

This finishes the proof of Theorem 1.2.

3. Boundedness on product of weighted Lebesgue spaces

The theory of weighted estimates has played very important roles in modern harmonic analysis with lots of extensive applications in the others fields of mathematics, which has been extensively studied (see [35, 29, 30, 33], for instance). In this section, for the commutators of bilinear pseudo-differential operators and Lipschitz functions, we will establish its boundedness of product of weighted Lebesgue spaces owning to the pointwise estimate of its sharp maximal function, that is, Theorem 1.1. The boundedness of the corresponding bilinear commutators with BMO function on the product of weighted Lebesgue spaces is also obtained by using Theorem 1.1 and Theorem 1.2.

Let us recall the definition of the class of Muckenhoupt weights A_p before proving Theorems 1.3 and 1.4. Let $1 < p < \infty$ and ω be a non-negative measurable function. We say $\omega \in A_p$ if for every cube Q in $\mathbb{R}^n$, there exists a positive C independent of Q such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\mathrm{d}x\right)^{p-1} \leq C.$$

Denote by $A_\infty = \bigcup_{p\geq 1} A_p$. It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$, and $\omega \in A_p$ for some q, $1 < q < p$.

To prove Theorems 1.3 and 1.4, we need the following inequality regarding maximal functions which is a version of the classical ones due to Fefferman and Stein in (see [17]), and a property of A_p.

Lemma 3.1. [17] Let $0 < p, \delta < \infty$, and $\omega \in A_\infty$. Then there exists a positive constant C depending on the A_∞ constant of ω such that

$$\int_{\mathbb{R}^n} [M_\delta(f)(x)]^p\omega(x)\mathrm{d}x \leq C\int_{\mathbb{R}^n} [M^\sharp_\delta(f)(x)]^p\omega(x)\mathrm{d}x,$$

for every function f such that the left-hand side is finite.

Lemma 3.2 [18] For $(\omega_1, \cdots, \omega_m) \in (A_{p_1}, \cdots, A_{p_m})$ with $1 \le p_1, \cdots, p_m < \infty$, and for $0 < \theta_1, \cdots, \theta_m < 1$ such that $\theta_1 + \cdots + \theta_m = 1$, we have $\omega_1^{\theta_1} \cdots \omega_m^{\theta_m} \in A_{\max\{p_1,\cdots,p_m\}}$.

Proof of Theorem 1.3. It follows from Lemma 3.2 that $\omega \in A_{\max\{p_1/q',p_2/q'\}} \subset A_\infty$. Take a δ such that $0 < \delta < 1/2$. Then by Lemma 3.1 and Theorem 1.1, we get

$$\begin{aligned}
\|[T_\sigma, a]_j(f,g)\|_{L^p(\omega)} &\le \|M_\delta([T_\sigma, a]_j(f,g))\|_{L^p(\omega)} \\
&\le C\|M^\sharp_\delta([T_\sigma, a]_j(f,g))\|_{L^p(\omega)} \\
&\le C\|a\|_{\mathrm{Lip}^1}\|M_{q'}(f)M_{q'}(g)\|_{L^p(\omega)} \\
&\le C\|a\|_{\mathrm{Lip}^1}\|M_{q'}(f)\|_{L^{p_1}(\omega_1)}\|M_{q'}(g)\|_{L^{p_2}(\omega_2)} \\
&= C\|a\|_{\mathrm{Lip}^1}\|M(|f|^{q'})\|^{1/q'}_{L^{p_1/q'}(\omega_1)}M(|g|^{q'})\|^{1/q'}_{L^{p_2/q'}(\omega_2)} \\
&\le C\|a\|_{\mathrm{Lip}^1}\||f|^{q'}\|^{1/q'}_{L^{p_1/q'}(\omega_1)}\||g|^{q'}\|^{1/q'}_{L^{p_2/q'}(\omega_2)} \\
&= C\|a\|_{\mathrm{Lip}^1}\|f\|_{L^{p_1}(\omega_1)}\|g\|_{L^{p_2}(\omega_2)}.
\end{aligned}$$

We complete the proof of the Theorem 1.3.

Proof of Theorem 1.4. It follows from Lemma 3.2 that $\omega \in A_\infty$. Take δ and ε such that $0 < \delta < \varepsilon < 1/2$. Then by Lemma 3.1 and Theorem 1.1, let $\vec{f} = (f_1, f_2)$, we get

$$\begin{aligned}
\|M_\varepsilon([T_\sigma, a]_j(\vec{f}))\|_{L^p(\omega)} &\le C\|M^\sharp_\varepsilon([T_\sigma, a]_j(\vec{f}))\|_{L^p(\omega)} \\
&\le C\|a\|_{\mathrm{Lip}^1}\|\prod_{t=1}^{2} M_{q'}(f_t)\|_{L^p(\omega)}.
\end{aligned}$$

Since $\omega_t \in A_{p_t/q'}$, $t = 1, 2$, there exists an l_t such that $1 < l_t < p_t/q'$ and $\omega_t \in A_{l_t}$. It follows from $q' < p_t/l_t$ that there is an s_t such that $q' < s_t < p_t/l_t < p_t$. Let $s = \min\{s_1, s_2\}$. Then $s > q'$ and $s < p_t$.

Since $l_t < p_t/s_t \le p_t/s$, then $\omega_i \in A_{l_t} \subset A_{p_t/s}$, $t = 1, 2$. It follows from Lemma 3.1 and Theorem 1.2 that

$$\begin{aligned}
\|[[T_\sigma, a]_j, b]_i(\vec{f})\|_{L^p(\omega)} &\le \|M_\delta([[T_\sigma, a]_j, b]_i(\vec{f}))\|_{L^p(\omega)} \le C\|M^\sharp_\delta([[T_\sigma, a]_j, b]_i(\vec{f}))\|_{L^p(\omega)} \\
&\le C\|b\|_{\mathrm{BMO}}\left(\|M_\varepsilon([T_\sigma, a]_j(\vec{f}))\|_{L^p(\omega)} + \|a\|_{\mathrm{Lip}^1}\|\prod_{t=1}^{2} M_s(f_t)\|_{L^p(\omega)}\right) \\
&\le C\|b\|_{\mathrm{BMO}}\left(\|a\|_{\mathrm{Lip}^1}\|\prod_{t=1}^{2} M_{q'}(f_t)\|_{L^p(\omega)} + \|a\|_{\mathrm{Lip}^1}\|\prod_{t=1}^{2} M_s(f_t)\|_{L^p(\omega)}\right) \\
&\le C\|b\|_{\mathrm{BMO}}\|a\|_{\mathrm{Lip}^1}\left(\|\prod_{t=1}^{2} M_s(f_t)\|_{L^p(\omega)}\right) \\
&= C\|b\|_{\mathrm{BMO}}\|a\|_{\mathrm{Lip}^1}\prod_{t=1}^{2}\|M_s(f_t)\|_{L^{p_t}(\omega_t)} \\
&\le C\|b\|_{\mathrm{BMO}}\|a\|_{\mathrm{Lip}^1}\prod_{t=1}^{2}\|M(|f_t|^s)\|^{1/s}_{L^{p_t/s}(\omega_t)}
\end{aligned}$$

$$= C\|b\|_{\mathrm{BMO}}\|a\|_{\mathrm{Lip}^1}\prod_{t=1}^{2}\|f_t\|_{L^{p_t}(\omega_t)}.$$

We complete the proof of the Theorem 1.4.

4. Boundedness on product of variable exponent Lebesgue spaces

The spaces with variable exponent have been widely studied in recent ten years. The results show that they are not only the generalized forms of the classical function spaces with invariable exponent, but also there are some new breakthroughs in the research techniques. These new real variable methods help people further understand the function spaces. Due to the fundamental paper [24] by Kovóčik and Rákosník, Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ becomes one of the important class function spaces. The theory of the variable exponent function spaces have been applied in fluid dynamics, elastlcity dynamics, calculus of variations and differential equations with non-standard growth conditions (for example, see [1, 2, 16]). In [8], authors proved the extrapolation theorem which leads the boundedness of some classical operators including the commutators on $L^{p(\cdot)}(\mathbb{R}^n)$. Karlovich and Lerner also obtained the bundedness of the singular integral commutators in [23]. The boundedness of some typical operators is being studied with keen interest on spaces with variable exponent (see [9, 22, 41–43]).

In this section, we will establish the boundedness of $[T_\sigma, a]_j$ and $[[T_\sigma, a]_j, b]_i (i, j = 1, 2)$ on the product of variable exponent Lebesgue spaces, that is, we shall prove Theorems 1.5 and 1.6.

Denote $\mathcal{P}(\mathbb{R}^n)$ to be the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ with

$$p_- =: \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 1 \text{ and } p_+ =: \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty,$$

and $\mathcal{B}(\mathbb{R}^n)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that the Hardy-littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 4.1. [23] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f \text{ measurable} : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \lambda > 0 \right\}.$$

As $p(\cdot) = p$ is a constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ coincides with the usual Lebesgue space. It is pointed out in [23] that $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{ \eta > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x \leq 1 \right\}.$$

Lemma 4.2. [13] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if M_{q_0} is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for some $1 < q_0 < \infty$, where $M_{q_0}(f) = [M(|f|^{q_0})]^{1/q_0}$.

Lemma 4.3. [32] Let $p(\cdot), p_1(\cdot), \cdots, p_m(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ so that $1/p(x) = 1/p_1(x) + \cdots + 1/p_m(x)$. Then for any $f_j \in L^{p_j}(\mathbb{R}^n)$, $j = 1, 2, \cdots, m$, there has

$$\|\prod_{j=1}^{m} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2^{m-1}\prod_{j=1}^{m}\|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

Lemma 4.4. [14] Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose for some fixed $0 < p_0 < \infty$, every $(f, g) \in \mathcal{F}$ and every $\omega \in A_1$,

$$\int_{\mathbb{R}^n} |f(x)|^{p_0}\omega(x)\mathrm{d}x \leq C_0 \int_{\mathbb{R}^n} |g(x)|^{p_0}\omega(x)\mathrm{d}x.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $p_0 \leq p_-$. If $(\frac{p(\cdot)}{p_0})' \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all $(f, g) \in \mathcal{F}$, $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 4.5. [14] If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then C_0^∞ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 4.6. [13] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following conditions are equivalent.

(1) $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$;

(2) $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$;

(3) $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < p_0 < p_-$;

(4) $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < p_0 < p_-$.

Proof of Theorem 1.5. Here we note $\vec{f} = (f_1, f_2)$, where f_1 and f_2 are bounded measurable functions with compact support. Since $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then by Lemma 4.6, there exists a p_0 such that $1 < p_0 < p_-$ and $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$. Take a δ such that $0 < \delta < 1/2$. For any $\omega \in A_1$, it follows from Lemma 3.1 and Theorem 1.1 that

$$\begin{aligned}
\int_{\mathbb{R}^n} |[T_\sigma, a]_j(\vec{f})|^{p_0}\omega(x)\mathrm{d}x &\leq C\int_{\mathbb{R}^n} [M_\delta([T_\sigma, a]_j(\vec{f}))(x)]^{p_0}\omega(x)\mathrm{d}x \\
&\leq C\int_{\mathbb{R}^n} [M_\delta^\sharp([T_\sigma, a]_j(\vec{f}))(x)]^{p_0}\omega(x)\mathrm{d}x \\
&\leq C\|a\|_{\mathrm{Lip}^1}\int_{\mathbb{R}^n}\left[\prod_{t=1}^{2} M_{q'}(f_t)(x)\right]^{p_0}\omega(x)\mathrm{d}x \\
&\leq C\|a\|_{\mathrm{Lip}^1}\int_{\mathbb{R}^n}\left[\prod_{t=1}^{2} M_{q_0^j}(f_t)(x)\right]^{p_0}\omega(x)\mathrm{d}x.
\end{aligned}$$

Applying Lemma 4.4 to the pair $([T_\sigma, a]_j(\vec{f}), \prod_{t=1}^{2} M_{q_0^t}(f_t))$, we can get

$$\|[T_\sigma, a]_j(\vec{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|a\|_{\mathrm{Lip}^1}\left\|\prod_{t=1}^{2} M_{q_0^j}(f_t))\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Then by Lemmas 4.2 and 4.3, we have

$$\|[T_\sigma, a]_j(\vec{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|a\|_{\mathrm{Lip}^1}\left\|\prod_{t=1}^{2} M_{q_0^j}(f_t))\right\|_{L^{p_t(\cdot)}(\mathbb{R}^n)} \leq C\|a\|_{\mathrm{Lip}^1}\prod_{t=1}^{2}\|f_t\|_{L^{p_t(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof of the Theorem 1.5.

Proof of Theorem 1.6. Denote $q_0 = \min\{q_0^1, q_0^2\}$, then $q' < q_0 < \infty$. Let $\vec{f} = (f_1, f_2)$, where f_1 and f_2 are bounded measurable functions with compact support. Since $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then by Lemma 4.6, there exists a p_0 such that $1 < p_0 < p_-$ and $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$. Take δ and ε such that $0 < \delta < \varepsilon < 1/2$. For any $\omega \in A_1$, it follows from Lemma 3.1 , Theorem 1.1 and Theorem 1.2 that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |[[T_\sigma, a]_j, b]_i(\vec{f})|^{p_0}\omega(x)\mathrm{d}x \le C\int_{\mathbb{R}^n} [M_\delta([[T_\sigma, a]_j, b]_i(\vec{f}))(x)]^{p_0}\omega(x)\mathrm{d}x\\
\le\ & C\int_{\mathbb{R}^n} [M_\delta^\sharp([[T_\sigma, a]_j, b]_i(\vec{f}))(x)]^{p_0}\omega(x)\mathrm{d}x\\
\le\ & C\|b\|_{\mathrm{BMO}}^{p_0}\int_{\mathbb{R}^n}\left(M_\varepsilon([T_\sigma, a]_j(\vec{f}))(x)\mathrm{d}x + \|a\|_{\mathrm{Lip}^1}\prod_{t=1}^{2} M_{q_0}(f_t)(x)\right)^{p_0}\omega(x)\mathrm{d}x\\
\le\ & C\|b\|_{\mathrm{BMO}}^{p_0}\left(\int_{\mathbb{R}^n}[M_\varepsilon^\sharp([T_\sigma, a]_j(\vec{f}))]^{p_0}\omega(x)\mathrm{d}x + \|a\|_{\mathrm{Lip}^1}^{p_0}\int_{\mathbb{R}^n}\left[\prod_{t=1}^{2} M_{q'}(f_t)(x)\right]^{p_0}\omega(x)\mathrm{d}x\right)\\
\le\ & C\|b\|_{\mathrm{BMO}}^{p_0}\|a\|_{\mathrm{Lip}^1}^{p_0}\left(\int_{\mathbb{R}^n}\left[\prod_{t=1}^{2} M_{q'}(f_t)(x)\right]^{p_0}\omega(x)\mathrm{d}x + \int_{\mathbb{R}^n}\left[\prod_{t=1}^{2} M_{q_0}(f_t)(x)\right]^{p_0}\omega(x)\mathrm{d}x\right)\\
\le\ & C\|b\|_{\mathrm{BMO}}^{p_0}\|a\|_{\mathrm{Lip}^1}^{p_0}\int_{\mathbb{R}^n}\left[\prod_{t=1}^{2} M_{q_0^j}(f_t)(x)\right]^{p_0}\omega(x)\mathrm{d}x.
\end{aligned}$$

Applying Lemma 4.4 to the pair $([[T_\sigma, a]_j, b]_i(\vec{f}), \prod_{t=1}^{2} M_{q_0^t}(f_t))$, we can get

$$\|[[T_\sigma, a]_j, b]_i(\vec{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|a\|_{\mathrm{Lip}^1}\left\|\prod_{t=1}^{2} M_{q_0^j}(f_t))\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Then by Lemmas 4.2 and 4.3, we have

$$\begin{aligned}
\|[[T_\sigma, a]_j, b]_i(\vec{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C\|a\|_{\mathrm{Lip}^1}\left\|\prod_{t=1}^{2} M_{q_0^t}(f_t))\right\|_{L^{p_t(\cdot)}(\mathbb{R}^n)}\\
&\le C\|a\|_{\mathrm{Lip}^1}\prod_{t=1}^{2}\|f_t\|_{L^{p_t(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

We complete the proof of Theorem 1.6.

5. Endpoint estimate

In this section, we will show the endpoint estimate for the $[T_\sigma, a]_j$ with $j = 1, 2$, that is, we will give the proof of Theorem 1.7.

Proof of Theorem 1.7. Take p_1, p_2 such that $\max\{q', 2\} < p_1, p_2 < \infty$. Let $1/p = 1/p_1 + 1/p_2$. Then $1 < p < \infty$. It follows from Lemma 2.3 that $[T_\sigma, a]_j (j = 1, 2)$ is bounded from $L^{p_1} \times L^{p_2}$ into L^p.

Let $f, g \in L^\infty$. Then for any ball $B = B(x_0, r_B)$ with $r_B > 0$, we decompose f and g as follows

$$f = f\chi_{2B} + f\chi_{(2B)^c} := f^1 + f^2, \quad g = g\chi_{2B} + g\chi_{(2B)^c} := g^1 + g^2.$$

Then

$$\begin{aligned}
& \frac{1}{|B|}\int_B |[T_\sigma, a]_j(f,g)(z)| - [T_\sigma, a]_j(f^2, g^2)(x_0)|\mathrm{d}z \\
\leq & \frac{1}{|B|}\int_B |[T_\sigma, a]_j(f^1, g^1)(z)| + \frac{1}{|B|}\int_B |[T_\sigma, a]_j(f^2, g^1)(z)|\mathrm{d}z \\
& + \frac{1}{|B|}\int_B |[T_\sigma, a]_j(f^1, g^2)(z)|\mathrm{d}z \\
& + \frac{1}{|B|}\int_B |[T_\sigma, a]_j(f^2, g^2)(z) - [T_\sigma, a]_j(f^2, g^2)(x_0)|\mathrm{d}z \\
:= & \sum_{s=1}^4 J_s.
\end{aligned}$$

It follows from the Hölder's inequality and Lemma 2.3 that

$$\begin{aligned}
J_1 &\leq \left(\frac{1}{|B|}\int_B |[T_\sigma, a]_j(f^1, g^1)(z)|^p\right)^{1/p} \\
&\leq C|B|^{-1/p}\|a\|_{\mathrm{Lip}^1}\|f^1\|_{L^{p_1}}\|g^1\|_{L^{p_2}} \\
&\leq \|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty.
\end{aligned}$$

By the size conditions in Theorem A of the kernel, we have

$$\begin{aligned}
J_2 &\leq \frac{1}{|B|}\int_B \left(\int_{(2B)^c}\int_{2B} |K(z, y_1, y_2)|f(y_1)||g(y_2)|\mathrm{d}y_2\mathrm{d}y_1\right)\mathrm{d}z \\
&\leq C\|a\|_{\mathrm{Lip}^1}\frac{1}{|B|}\int_B \left(\int_{(2B)^c}\left(\int_{2B}|g(y_2)|\mathrm{d}y_2\right)\frac{|f(y_1)|}{|z-y_1|^{2n}}\mathrm{d}y_1\right)\mathrm{d}z \\
&\leq C\|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty\left(\int_{2B}\mathrm{d}y_2\right)\left(\int_{(2B)^c}\frac{1}{|x_0-y_1|^{2n}}\mathrm{d}y_1\right) \\
&\leq C\|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty.
\end{aligned}$$

Similarly, we can obtain that

$$J_3 \leq C\|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty.$$

Noting that as $z \in B$, and $y_1, y_2 \in (2B)^c$, then $|y_1 - x_0| \geq 2|z - x_0|$ and $|y_2 - x_0| \geq 2|z - x_0|$. It follows from the Hölder's inequality and (2.3) that

$$\begin{aligned}
J_4 &\leq \frac{1}{|B|}\int_B \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(z, y_1, y_2)|f(y_1) - K(x_0, y_1, y_2)|f^2(y_1)||g^2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\right)\mathrm{d}z \\
&\leq C\sum_{k_1=1}^\infty\sum_{k_2=1}^\infty \frac{1}{|B|}\int_B \int_{2^{k_2}|z-z_0|\leq|y_2-z_0|\leq 2^{k_2}|z-z_0|}\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|z-z_0|}
\end{aligned}$$

$$
\begin{aligned}
&|K(z,y_1,y_2)-K(x_0,y_1,y_2)||f(y_1)||g(y_2)|\mathrm{d}y_1\mathrm{d}y_2|\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z\\
\le\;& C\|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\frac{1}{|B|}\int_B\int_{2^{k_2}|z-z_0|\le|y_2-z_0|\le 2^{k_2+1}|z-z_0|}(2^{k_1}|z-x_0|)^{\frac{n}{q'}}\\
&\times\left(\int_{2^{k_1}|z-z_0|\le|y_1-z_0|\le 2^{k_1+1}|z-z_0|}|K(z,y_1,y_2)-K(x_0,y_1,y_2)|^q\mathrm{d}y_1\right)^{1/q}\mathrm{d}y_2\mathrm{d}z\\
\le\;& C\|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\frac{1}{|B|}(2^{k_1}|z-x_0|)^{\frac{n}{q'}}(2^{k_2}|z-x_0|)^{\frac{n}{q'}}\\
&\times(\int_{2^{k_2}|z-z_0|\le|y_2-z_0|\le 2^{k_2+1}|z-z_0|}\int_{2^{k_1}|z-z_0|\le|y_1-z_0|\le 2^{k_1+1}|z-z_0|}\\
&\times|K(z,y_1,y_2)-K(x_0,y_1,y_2)|^q\mathrm{d}y_1\mathrm{d}y_2)^{1/q}\mathrm{d}z\\
\le\;& C\|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}2^{\frac{k_1n}{q'}}2^{\frac{k_2n}{q'}}(C_{k_1}2^{-\frac{k_1n}{q'}})(C_{k_2}2^{-\frac{k_2n}{q'}})\\
\le\;& C\|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|[T_\sigma,a]_j(f,g)\|_{\mathrm{BMO}} &= \sup_B\frac{1}{|B|}\int_B|[T_\sigma,a]_j(f,g)(z)-([T_\sigma,a]_j(f,g))_B|\mathrm{d}z\\
&\le \sup_B\frac{1}{|B|}\int_B|[T_\sigma,a]_j(f,g)(z)-[T_\sigma,a]_j(f^2,g^2)(x_0)|\mathrm{d}z\\
&\le C\|a\|_{\mathrm{Lip}^1}\|f\|_\infty\|g\|_\infty.
\end{aligned}
$$

Which completes the proof of the Theorem 1.7.

6. Conclusions

In this paper, we consider the commutators of bilinear pseudo-differential operators and the operation of multiplication by a Lipschitz function. By establishing the pointwise estimates of the corresponding sharp maximal function, the boundedness of the commutators is obtained respectively on the products of weighted Lebesgue spaces and variable exponent Lebesgue spaces with $\sigma\in\mathcal{BBS}^{1}_{1,1}$. Moreover, the endpoint estimate of the commutators is also established on $L^\infty\times L^\infty$.

Acknowledgments

This work is supported by the Doctoral Scientific Research Foundation of Northwest Normal University (202003101203), Young Teachers′ Scientific Research Ability Promotion Project of Northwest Normal University (NWNU-LKQN2021-03) and National Natural Science Foundation of China (11561062).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. E. Acerbi, G. Mingione, Regularity results for stationary electrorheological fluids, *Arch. Ration. Mech. Anal.,* **164** (2002), 213–259. http://dx.doi.org/10.1007/s00205-002-0208-7
2. E. Acerbi, G. Mingione, Gradient estimates for a class of parabolic systems, *Duke Math. J.,* **136** (2007), 285–320. http://dx.doi.org/10.1215/S0012-7094-07-13623-8
3. J. Alvarez, C. Pérez, Estimates with A_∞ weights for various singular integral operators, *Boll. Unione Mat. Ital.,* **8** (1994), 123–133.
4. P. Auscher, M. Taylor, Paradifferential operators and commutator estimates, *Commun. Part. Diff. Eq.,* **20** (1995), 1743–1775. http://dx.doi.org/10.1080/03605309508821150
5. Á. Bényi, V. Naibo, Commutators of bilinear pseudodifferential operators and Lipschitz functions, *J. Fourier Anal. Appl.,* **24** (2018), 759–779. http://dx.doi.org/10.1007/s00041-016-9519-1
6. Á. Bényi, T. Oh, Smoothing of commutators for a Hörmander class of bilinear pseudodifferential operators, *J. Fourier Anal. Appl.,* **20** (2014), 282–300. http://dx.doi.org/10.1007/s00041-013-9312-3
7. A. Calderón, Commutators of singular integral operators, *P. Natl. Acad. Sci. USA,* **53** (1965), 1092–1099. http://dx.doi.org/10.1073/pnas.53.5.1092
8. D. Cruz-Uribe, A. Fiorenza, J. M. Martell, C. Pérez, The boundedness of classical operators on variable L^p spaces, *Ann. Acad. Sci. Fenn.-M.,* **31** (2006), 239–264.
9. D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer, Weighted norm inequalities for the maximal operators on variable Lebesgue spaces, *J. Math. Anal. Appl.,* **394** (2012), 744–760. http://dx.doi.org/10.1016/j.jmaa.2012.04.044
10. R. Coifman, Y. Meyer, Au delà des opérateurs pseudo-différentiels, *Astérisque,* **57** (1978).
11. R. Coifman, Y. Meyer, Commutateurs d'intégrales singulières et opérateurs multilinéaires, *Ann. I. Fourier,* **28** (1978), 177–202. http://dx.doi.org/10.5802/aif.708
12. D. Cruz-Uribe, J. M. Martell, C. Pérez, Sharp two-weight inequalities for singular integrals, with applications to the Hilbert transform and the Sarason conjecture, *Adv. Math.,* **216** (2007), 647–676. http://dx.doi.org/10.1016/j.aim.2007.05.022
13. L. Diening, Maximal function on Musielak-Orlicz spaces and generlaized Lebesgue spaces, *Bull. Sci. Math.,* **129** (2005), 657–700. http://dx.doi.org/10.1016/j.bulsci.2003.10.003
14. L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Math., Springer-Verlag, Berlin, 2011. http://dx.doi.org/10.1007/978-3-642-18363-8
15. Y. Ding, T. Mei, Boundedness and compactness for the commutators of bilinear operators on Morrey spaces, *Potential Anal.,* **42** (2015), 717–748. http://dx.doi.org/10.1007/s11118-014-9455-0

16. L. Diening, M. Ružička, Calderón-Zygmund operators on generalized Lebesgue spaces L^p and problem related to fluid dynamics, *J. Reine. Angew. Math.,* **563** (2003), 197–220. http://dx.doi.org/10.1515/crll.2003.081

17. C. Fefferman, E. M. Stein, H^p spaces of several variables, *Acta Math.,* **129** (1972), 137–193. http://dx.doi.org/10.1007/bf02392215

18. L. Grafakos, J. M. Martell, Extrapolation of weighted norm inequalities for multivariable operators and application, *J. Geom. Anal.,* **14** (2004), 19–46. http://dx.doi.org/ 10.1007/BF02921864

19. J. García-Cuerva, J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math. Studies, 116, North-Holland Publishing Co., Amsterdam, 1985.

20. F. John, Quasi-isometric mappings, Seminari 1962/63 di Analisi Algebra, Geometria e Topologia, *Ist. Naz. Alta Mat. Ediz.,* **2** (1965), 462–473. http://dx.doi.org/ 10.1002/cpa.3160220209

21. B. Jawerth, A. Torchinsky, Local sharp maximal functions, *J. Approx. Theory,* **43** (1985), 231–270. http://dx.doi.org/10.1016/0021-9045(85)90102-9

22. Ö. Kulak, The inclusion theorems for variable exponent Lorentz spaces, *Turkish J. Math.,* **40** (2016), 605–619. http://dx.doi.org/10.3906/mat-1502-23

23. A. Y. Karlovich, A. K. Lerner, Commutators of singular integrals on generalized L^p spaces with variable exponent, *Publ. Mat.,* **49** (2005), 111–125. http://dx.doi.org/10.5565/PUBLMAT_49105_05

24. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.,* **41** (1991), 592–618.

25. A. K. Lerner, Weighted norm inequalities for the local sharp maximal function, *J. Fourier Anal. Appl.,* **10** (2004), 465–474. http://dx.doi.org/10.1007/s00041-004-0987-3

26. A. K. Lerner, A pointwise estimate for the local sharp maximal function with applications to singular integrals, *Bull. London Math. Soc.,* **42** (2010), 843–856. http://dx.doi.org/10.1112/blms/bdq042

27. Y. Lin, S. Lu, Strongly singular Calderón-Zygmund operators and their commutators, *Jordan J. Math. Stat.,* **1** (2008), 31–49.

28. A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, *Adv. Math.,* **220** (2009), 1222–1264. http://dx.doi.org/10.1016/j.aim.2008.10.014

29. K. Li, W. Sun, Weighted estimates for multilinear pseudodifferential operators, *Acta Math. Sin., Engl. Ser.,* **30** (2014), 1281–1288. http://dx.doi.org/10.1007/s10114-014-3027-5

30. Y. Lin, M. Xu, Endpoint estimates for Marcinkiewicz integrals on weighted weak Hardy spaces, *Acta Math. Sin., Engl. Ser.,* **31** (2015), 430–444. http://dx.doi.org/10.1007/s10114-015-4277-6

31. Y. Lin, Y. Xiao, Multilinear singular integral operators with generalized kernels and their multilinear commutators, *Acta Math. Sin.,* **33** (2017), 1443–1462. http://dx.doi.org/10.1007/s10114-017-7051-0

32. G. Lu, P. Zhang, Multilinear Calderón-Zygmund operator with kernels of Dinis type and applications, *Nonlinear Anal.-Theor.,* **107** (2014), 92–117. http://dx.doi.org/10.1016/j.na.2014.05.005

33. Y. Lin, G. Zhang, Weighted estimates for commutators of strongly singular Calderón-Zygmund operators, *Acta Math. Sin.,* **32** (2016), 1297–1311. http://dx.doi.org/10.1007/s10114-016-6154-3

34. Y. Meyer, Régularité des solutions des équations aux dérivées partielles nonlinéaires, *Springer LNM,* **842** (1980), 293–302. http://dx.doi.org/10.1007/BFb0089941

35. D. Maldonado, D. Naibo, Weighted norm inequalities for paraproducts and bilinear pseudodifferentialoperators with mild regularity, *J. Fourier Anal. Appl.,* **15** (2009), 218–261. http://dx.doi.org/10.1007/s00041-008-9029-x

36. C. Pérez, R. H. Torres, Sharp maximal function estimates for multilinear singular integrals, *Contemp. Math.,* **320** (2003), 323–331. http://dx.doi.org/10.1090/conm/320/05615

37. J. O. Strömberg, Bounded mean oscillation with Orlicz norms and duality of Hardy spaces, *Indiana Univ. Math. J.,* **28** (1979), 511–544. http://dx.doi.org/10.1512/iumj.1979.28.28037

38. M. E. Taylor, Commutator estimates, *P. Am. Math. Soc.,* **131** (2003), 1501–1507. http://dx.doi.org/10.1090/S0002-9939-02-06723-0

39. M. E. Taylor, Commutator estimates for Hölder continuous and bmo-Sobolev multipliers, *P. Am. Math. Soc.,* **143** (2015), 5265–5274. http://dx.doi.org/10.1090/proc/12825

40. S. Tao, Q. Li, Commutators of bilinear pseudodifferential operators with Hörmander symbol on generalized Morrey spaces, *J. Jilin Univ.,* **56** (2018), 469–474. http://dx.doi.org/10.13413/j.cnki.jdxblxb.2018.03.01

41. L. Wang, S. Tao, Boundedness of Littlewood-Paley operators and their commutators on Herz-Morrey spaces with variable exponent, *J. Inequal. Appl.* **227** (2014), 1–17. http://dx.doi.org/10.1186/1029-242X-2014-227

42. L. Wang, S. Tao, Parameterized littlewood-paley operators and their commutators on Lebesgue spaces with variable exponent, *Anal. Theory Appl.,* **31** (2015), 13–24. http://dx.doi.org/10.4208/ata.2015.v31.n1.2

43. L. Wang, S. Tao, θ-type Calderón-Zygmund operators and commutators in variable exponents Herz space, *Open Math.,* **16** (2018), 1607–1620. http://dx.doi.org/10.1515/math-2018-0133

AIMS Press