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Research article

Existence of stable standing waves for the nonlinear Schrödinger equation with attractive inverse-power potentials

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Abstract: In this paper, we consider the following nonlinear Schrödinger equation with attractive inverse-power potentials

$$i\partial_t\psi + \Delta\psi + \gamma|x|^{-\sigma}\psi + |\psi|^\alpha\psi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $N \geq 3$, $0 < \gamma < \infty$, $0 < \sigma < 2$ and $\frac{4}{N} < \alpha < \frac{4}{N-2}$. By using the concentration compactness principle and considering a local minimization problem, we prove that there exists a $\gamma_0 > 0$ sufficiently small such that $0 < \gamma < \gamma_0$ and for any $a \in (0, a_0)$, there exist stable standing waves for the problem in the L^2-supercritical case. Our results are complement to the result of Li-Zhao in [23].

Keywords: nonlinear Schrödinger equation; inverse-power potentials; stable standing waves
Mathematics Subject Classification: 35Q55

1. Introduction and main results

In this paper, we consider the following Cauchy problem for the nonlinear Schrödinger equation with an attractive inverse-power potential

$$\begin{cases} i\partial_t\psi + \Delta\psi + \gamma|x|^{-\sigma}\psi + |\psi|^\alpha\psi = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\ \psi(0, x) = \psi_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 3$, $\psi : [0, T^*) \times \mathbb{R}^N \to \mathbb{C}$ is an unknown complex valued function with $0 < T^* \leq \infty$, $\psi_0 \in H^1(\mathbb{R}^N)$, $\gamma \in (0, +\infty)$, $\sigma \in (0, 2)$ and $\frac{4}{N} < \alpha < \frac{4}{N-2}$.

In the case $\sigma = 1$, i.e., the operator $\Delta + \frac{\gamma}{|x|}$ with Coulomb potential, (1.1) describes the situation where the wave function of an electron, satisfying the Schrödinger evolution equation, is influenced by m nuclei, see [19] for a broader introduction. In the case $0 < \sigma < 2$, i.e., the operator $\Delta + \frac{\gamma}{|x|^\sigma}$ with slowly decaying potentials, we refer the readers to [15] and references therein. It also attracted a great deal of attention from mathematicians in recent years, see, e.g. [4, 13, 14, 23].

Recently, this type of equations has been studied widely in [1, 4–11, 13, 17, 20, 22–26]. In particular, Eq (1.1) enjoys a class of special solutions, which are called standing waves, namely solutions of the form $\psi(t,x) = e^{i\omega t}u(x)$, where $\omega \in \mathbb{R}$ is a frequency and $u \in H^1(\mathbb{R}^N)$ satisfies the elliptic equation

$$-\Delta u + \omega u - \gamma|x|^{-\sigma}u - |u|^{\alpha}u = 0. \tag{1.2}$$

The Eq (1.2) is variational and its action functional is defined by

$$S_{\omega}(u) := E_{\gamma}(u) + \frac{\omega}{2}\|u\|_{L^2}^2, \tag{1.3}$$

where the corresponding energy functional $E_{\gamma}(u)$ is defined by

$$E_{\gamma}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{\gamma}{2}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{\sigma}}dx - \frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}. \tag{1.4}$$

For the evolutional type Eq (1.1), one of the most interesting problems is to consider the stability of standing waves, which is defined as follows:

Definition 1.1. *Assume u is a solution of* (1.2). *The standing wave $e^{i\omega t}u(x)$ is called orbitally stable in $H^1(\mathbb{R}^N)$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\psi_0 \in H^1(\mathbb{R}^N)$ satisfies*

$$\|\psi_0 - u\|_{H^1} \le \delta,$$

then the solution $\psi(t)$ to (1.1) *with $\psi|_{t=0} = \psi_0$ satisfies*

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R},y\in\mathbb{R}^N}\|\psi(t,\cdot) - e^{i\theta}u(\cdot - y)\|_{H^1} \le \epsilon.$$

There are two main methods to study the stability of standing waves. One is the stability or instability criterion in [16] proposed by Grillakis, Shatah and Strauss. It says that the standing wave $e^{i\omega t}u_{\omega}$ is stable if $\frac{\partial}{\partial\omega}\|u_{\omega}\|_{L^2}^2 > 0$ and unstable if not, see [12] for more details. However, it is hard to estimate the sign of $\frac{\partial}{\partial\omega}\|u_{\omega}\|_{L^2}^2$ for nonlinear Schrödinger equations without scaling invariance, e.g. (1.1). The other is the constrained minimization approach introduced by Cazenave and Lions in [3]. In this paper, we thus take into account the orbital stability of the set of minimizers by using the method from Cazenave and Lions in [3].

Now we recall the following definition of the orbital stability of the set $\mathcal{M}$.

Definition 1.2. *The set $\mathcal{M} \subset H^1(\mathbb{R}^N)$ is called orbitally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for any initial data $\psi_0 \in H^1(\mathbb{R}^N)$ satisfying*

$$\inf_{u\in\mathcal{M}}\|\psi_0 - u\|_{H^1} < \delta,$$

the corresponding solution u to (1.1) *satisfies*

$$\inf_{u\in\mathcal{M}}\|\psi(t) - u\|_{H^1} < \epsilon,\ \forall\ t > 0.$$

Based on the above definition, in order to study the stability, we require that the solution of (1.1) exists globally, at least for initial data ψ_0 enough close to $\mathcal{M}$. In the L^2-subcritical case, all solutions for (1.1) exist globally. Therefore, the stability of standing waves has been studied widely in this case, see, e.g. [3, 4, 14, 23]. However, in the L^2-supercritical case, we know that the solution of (1.1) with small initial data exists globally, but for some large initial data, the solution of (1.1) may blow up in finite time by the local well-posedness theory of NLS, see [2] for further inference.

For Eq (1.1), in the mass subcritical and critical cases, i.e., $0 < \alpha \leq \frac{4}{N}$, Dinh in [4] and Li-Zhao in [23] studied the stability of set of minimizers by using the concentration compactness principle. In the mass supercritical case, i.e., $\frac{4}{N} < \alpha < \frac{4}{N-2}$, Fukaya-Ohta in [14] proved the strong instability of standing wave $e^{i\omega t}u$ under the assumption $\partial^2_\lambda E_\gamma(u_\lambda)|_{\lambda=1} \leq 0$ with $u_\lambda(x) := \lambda^{\frac{N}{2}} u(\lambda x)$. Therefore, whether there are stable standing waves is an interesting problem in the mass supercritical case. In this paper, we will solve this problem by considering the following minimization problem

$$m_\gamma(a) := \inf_{u \in S(a)} E_\gamma(u), \tag{1.5}$$

where

$$S(a) = \{u \in H^1(\mathbb{R}^N),\ \|u\|^2_{L^2} = a\}. \tag{1.6}$$

In the L^2-supercritical case, the energy $E_\gamma(u)$ is unbounded from below on $S(a)$. Actually, when $\frac{4}{N} < \alpha < \frac{4}{N-2}$, taking $u \in S(a)$ and setting $u_\lambda(x) := \lambda^{\frac{N}{2}} u(\lambda x)$, then $\|u\|^2_{L^2} = \|u_\lambda\|^2_{L^2} = a$, and we will find that

$$E_\gamma(u_\lambda) = \frac{\lambda^2}{2}\|\nabla u\|^2_{L^2} - \frac{\lambda^\sigma \gamma}{2}\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^\sigma} dx - \frac{\lambda^{\frac{N\alpha}{2}}}{\alpha+2}\|u\|^{\alpha+2}_{L^{\alpha+2}} \to -\infty,$$

as $\lambda \to \infty$, where $2 < \frac{N\alpha}{2} < \frac{2N}{N-2}$. Thus, we can not discuss the global minimization problem (1.5) to study the existence and stability of standing waves for (1.1). However, inspired by the thought in [18], we consider the further constrained minimization problem:

$$m_\gamma(a, r_0) := \inf_{u \in V(a)} E_\gamma(u). \tag{1.7}$$

The sets $V(a)$ and $\partial V(a)$ are given by

$$V(a) := \{u \in S(a) : \|\nabla u\|^2_{L^2} < r_0\}, \quad \partial V(a) := \{u \in S(a) : \|\nabla u\|^2_{L^2} = r_0\},$$

for an appropriate $r_0 > 0$, depending only on a_0 but not on $a \in (0, a_0)$. However, compared with the case (i.e., $\gamma = 0$) considered in [18], the energy functional of (1.2) is not invariant under the scaling transform due to the inhomogeneous nonlinearity $\gamma|x|^{-\sigma}u$. Furthermore, we cannot prove the preconceived limiting problem of the energy functional under translation sequences. In fact, we can solve the minimization problem (1.7) by proving the boundedness of the translation sequences. We now denote all the energy minimizers of (1.7) by

$$\mathcal{M}(a) := \{u \in V(a) : E_\gamma(u) = m_\gamma(a, r_0)\}.$$

Our main results are as follows:

Theorem 1.3. *Let $N \geq 3$, $0 < \sigma < 2$ and $\frac{4}{N} < \alpha < \frac{4}{N-2}$. Then there exists a $\gamma_0 > 0$ sufficiently small such that $0 < \gamma < \gamma_0$ and for any $a \in (0, a_0)$, the following properties hold:*

(i) $\emptyset \neq \mathcal{M}(a) \subset V(a)$.

(ii) $\mathcal{M}(a)$ is orbitally stable.

This paper is organized as follows: In Section 2, we firstly give some variational problems and then prove that the solution $\psi(t)$ of (1.1) with the initial data ψ_0 exists globally. In Section 3, we will show that $\mathcal{M}(a)$ is orbitally stable.

2. The variational problem

In this section, we first establish the following classical inequalities: If $N \geq 2$ and $\alpha \in [2, \frac{2N}{N-2})$, then the following Gagliardo-Nirenberg inequality holds that

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} \leq C(\alpha)\|u\|_{L^2}^{\alpha+2-\frac{N\alpha}{2}}\|\nabla u\|_{L^2}^{\frac{N\alpha}{2}}, \quad \forall\ u \in H^1(\mathbb{R}^N), \tag{2.1}$$

where $C(\alpha)$ is the sharp constant. Let $1 \leq p < \infty$, if $\sigma < N$ is such that $0 \leq \sigma \leq p$, then $\frac{|u(\cdot)|^p}{|\cdot|^\sigma} \in L^1(\mathbb{R}^N)$ for any $u \in H^1(\mathbb{R}^N)$. Moreover,

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^\sigma} dx \leq C_1\|u\|_{L^p}^{p-\sigma}\|\nabla u\|_{L^p}^{\sigma},\ \forall\ u \in H^1(\mathbb{R}^N), \tag{2.2}$$

where $C_1 = \left(\frac{p}{N-\sigma}\right)^\sigma$.

Secondly, by (2.1) and (2.2), we have

$$\begin{aligned} E_\gamma(u) \geq &\|\nabla u\|_{L^2}^2 \left(\frac{1}{2} - \frac{\gamma C_1}{2}\|u\|_{L^2}^{2-\sigma}\|\nabla u\|_{L^2}^{\sigma-2} - \frac{C(\alpha)}{\alpha+2}\|u\|_{L^2}^{\alpha+2-\frac{N\alpha}{2}}\|\nabla u\|_{L^2}^{\frac{N\alpha}{2}-2}\right) \\ = &\|\nabla u\|_{L^2}^2 f(\|u\|_{L^2}^2, \|\nabla u\|_{L^2}^2). \end{aligned} \tag{2.3}$$

Next, setting

$$\eta_0 = 2-\sigma, \quad \eta_1 = \sigma - 2, \quad \eta_2 = \alpha + 2 - \frac{N\alpha}{2}, \quad \eta_3 = \frac{N\alpha}{2} - 2,$$

we now consider the function $f(a, r)$ defined on $(0, \infty) \times (0, \infty)$ by

$$f(a, r) = \frac{1}{2} - \frac{\gamma C_1}{2} a^{\frac{\eta_0}{2}} r^{\frac{\eta_1}{2}} - \frac{C(\alpha)}{\alpha+2} a^{\frac{\eta_2}{2}} r^{\frac{\eta_3}{2}}, \tag{2.4}$$

and, for each $a \in (0, \infty)$, its restriction $g_a(r)$ defined on $(0, \infty)$ by $r \mapsto g_a(r) = f(a, r)$. For further reference, note that for any $N \geq 3$, $\eta_0 \in (0, 2)$, $\eta_1 \in (-2, 0)$, $\eta_2 \in (0, \frac{4}{N})$ and $\eta_3 \in (0, \frac{4}{N-2})$.

Lemma 2.1. *[2, Theorem 9.2.6] Let $N \geq 3$, $0 < \sigma < 2$ and $\frac{4}{N} < \alpha < \frac{4}{N-2}$. Then for any initial data $\psi_0 \in H^1(\mathbb{R}^N)$, there exists $T = T(\|\psi_0\|_{H^1})$ such that* (1.1) *admits a unique solution with $\psi(0) = \psi_0$. Let $[0, T^*)$ be the maximal time interval on which the solution ψ is well-defined, if $T^* < \infty$, then $\|\psi(t)\|_{H^1} \to \infty$ as $t \uparrow T^*$. Moreover, for all $0 < t < T^*$, the solution $\psi(t)$ satisfies the following conservations of mass and energy*

$$\|\psi(t)\|_{L^2}^2 = \|\psi_0\|_{L^2}^2, \ \ E_\gamma(\psi(t)) = E_\gamma(\psi_0),$$

where the energy $E_\gamma(u)$ is defined by (1.4).

Lemma 2.2. *For every $a > 0$, the function $g_a(r) = f(a, r)$ has a unique global maximum and the maximum value satisfies*

$$\begin{cases} \max\limits_{r>0} g_a(r) > 0, & if \;\; a < a_0, \\ \max\limits_{r>0} g_a(r) = 0, & if \;\; a = a_0, \\ \max\limits_{r>0} g_a(r) < 0, & if \;\; a > a_0, \end{cases}$$

where

$$a_0 := \left[\frac{1}{2K}\right]^{\frac{2(\eta_3-\eta_1)}{\eta_0\eta_3-\eta_1\eta_2}} > 0, \tag{2.5}$$

with

$$K := \frac{\gamma C_1}{2}\left[-\frac{C_1\eta_1(\alpha+2)}{2C(\alpha)\eta_3}\right]^{\frac{\eta_1}{\eta_3-\eta_1}} + \frac{C(\alpha)}{\alpha+2}\left[-\frac{C_1\eta_1(\alpha+2)}{2C(\alpha)\eta_3}\right]^{\frac{\eta_3}{\eta_3-\eta_1}}. \tag{2.6}$$

Proof. According to the definition of $g_a(r)$, we have

$$g_a'(r) = -\frac{\gamma C_1\eta_1}{4}a^{\frac{\eta_0}{2}}r^{\frac{\eta_1}{2}-1} - \frac{C(\alpha)\eta_3}{2(\alpha+2)}a^{\frac{\eta_2}{2}}r^{\frac{\eta_3}{2}-1}.$$

Hence, the equation $g_a'(r) = 0$ has a unique solution given by

$$r_a = \left[-\frac{\gamma C_1\eta_1(\alpha+2)}{2C(\alpha)\eta_3}\right]^{\frac{2}{\eta_3-\eta_1}} a^{\frac{\eta_0-\eta_2}{\eta_3-\eta_1}}. \tag{2.7}$$

Noticing that $g_a(r) \to -\infty$ as $r \to 0$ and $g_a(r) \to -\infty$ as $r \to \infty$, we obtain that r_a is the unique global maximum point of $g_a(r)$. Actually, the maximum value is

$$\begin{aligned} \max_{r>0} g_a(r) &= \frac{1}{2} - \frac{\gamma C_1}{2}a^{\frac{\eta_0}{2}}r_a^{\frac{\eta_1}{2}} - \frac{C(\alpha)}{\alpha+2}a^{\frac{\eta_2}{2}}r_a^{\frac{\eta_3}{2}} \\ &= \frac{1}{2} - \frac{\gamma C_1}{2}a^{\frac{\eta_0}{2}}\left[-\frac{C_1\eta_1(\alpha+2)}{2C(\alpha)\eta_3}\right]^{\frac{\eta_1}{\eta_3-\eta_1}} a^{\frac{\eta_1(\eta_0-\eta_2)}{2(\eta_3-\eta_1)}} \\ &\quad - \frac{C(\alpha)}{\alpha+2}a^{\frac{\eta_2}{2}}\left[-\frac{C_1\eta_1(\alpha+2)}{2C(\alpha)\eta_3}\right]^{\frac{\eta_3}{\eta_3-\eta_1}} a^{\frac{\eta_3(\eta_0-\eta_2)}{2(\eta_3-\eta_1)}} \\ &= \frac{1}{2} - Ka^{\frac{\eta_0\eta_3-\eta_1\eta_2}{2(\eta_3-\eta_1)}}. \end{aligned}$$

According to the definition of a_0, we have $\max\limits_{r>0} g_{a_0}(r) = 0$. Hence, the lemma follows. □

Now let $a_0 > 0$ be given by (2.5) and $r_0 := r_{a_0} > 0$ being determined by (2.7). Note that by the proof of Lemma 2.2, we have $f(a_0, r_0) = 0$ and $f(a, r_0) \geq 0$ for all $a \in (0, a_0)$. We denote

$$B(r_0) := \{u \in H^1(\mathbb{R}^N) : \|\nabla u\|_{L^2}^2 < r_0\} \quad and \quad V(a) := S(a) \cap B(r_0).$$

We now consider the following local minimization problem:

$$m_\gamma(a, r_0) := \inf_{u \in V(a)} E_\gamma(u), \quad \forall\, a \in (0, a_0). \tag{2.8}$$

And if $(a_1, r_1) \in (0, \infty) \times (0, \infty)$ be such that $f(a_1, r_1) \geq 0$, then for any $a_2 \in (0, a_1]$, by Lemma 2.2 and direct calculations, we have

$$f(a_2, r_2) \geq 0 \quad \text{if} \quad r_2 \in (\frac{a_2}{a_1}r_1, r_1). \tag{2.9}$$

Theorem 2.3. *Let $N \geq 3$, $0 < \sigma < 2$, $\frac{4}{N} < \alpha < \frac{4}{N-2}$ and $a \in (0, a_0)$, there exists a $\gamma_0 > 0$ sufficiently small such that $0 < \gamma < \gamma_0$. Then, there exists $u \in H^1(\mathbb{R}^N)$ such that $E_\gamma(u) = m_\gamma(a, r_0)$.*

Lemma 2.4. *For any $a \in (0, a_0)$, the following property holds,*

$$\inf_{u \in V(a)} E_\gamma(u) < 0 \leq \inf_{u \in \partial V(a)} E_\gamma(u). \tag{2.10}$$

Proof. For any $u \in \partial V(a)$, we have $\|\nabla u\|_{L^2}^2 = r_0$. Thus, by (2.3) and Lemma 2.2, we have

$$E_\gamma(u) \geq \|\nabla u\|_{L^2}^2 f(a, \|\nabla u\|_{L^2}^2) = r_0 f(a, r_0) \geq r_0 f(a_0, r_0) = 0. \tag{2.11}$$

Now let $u \in S(a)$ be arbitrary but fixed. We denote $u_\lambda(x) := \lambda^{\frac{N}{2}} u(\lambda x)$ for $\lambda \in (0, \infty)$. It is obvious that $u_\lambda \in S(a)$ for any $\lambda \in (0, \infty)$. We set the map on $(0, \infty)$ by

$$E_\gamma(u_\lambda) = \frac{\lambda^2}{2}\|\nabla u\|_{L^2}^2 - \frac{\gamma\lambda^\sigma}{2}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^\sigma} dx - \frac{\lambda^{\frac{N\alpha}{2}}}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}. \tag{2.12}$$

Noticing that $0 < \sigma < 2$, $2 < \frac{N\alpha}{2} < \frac{2N}{N-2}$, thus $E_\gamma(u_\lambda) < 0$ and $\|\nabla u_\lambda\|_{L^2}^2 = \lambda^2\|\nabla u\|_{L^2}^2 < r_0$ for sufficiently small $\lambda > 0$. The proof follows. □

Lemma 2.5. *Let $\frac{4}{N} < \alpha < \frac{4}{N-2}$, $r_0 > 0$ be determined as Lemma 2.2 and $0 < a < a_0$ be as in Lemma 2.4. Then, there exists $\delta > 0$ such that for any initial data $\psi_0 \in H^1(\mathbb{R}^N)$ and $\inf_{u \in \mathcal{M}(a)} \|\psi_0 - u\|_{H^1} < \delta$, the solution $\psi(t)$ of (1.1) with the initial data ψ_0 exists globally.*

Proof. Firstly, we denote the right hand of (2.10) by A. According to the continuity of energy functional $E_\gamma(u)$ with respect to $u \in \mathcal{M}(a)$, we deduce $E_\gamma(u) = m_\gamma(a, r_0) < A$ and $\|\nabla u\|_{L^2}^2 < r_0$. Moreover, there exists $\delta > 0$ such that for any $\psi_0 \in H^1(\mathbb{R}^N)$ and $\|\psi_0 - u\|_{H^1} < \delta$, we have

$$E_\gamma(\psi_0) < A \text{ and } \|\nabla\psi_0\|_{L^2}^2 < r_0. \tag{2.13}$$

Secondly, let us prove this result by contradiction. If not, there exists $\psi_0 \in H^1(\mathbb{R}^N)$ such that $\|\psi_0 - u\|_{H^1} < \delta$ and the solution $\psi(t)$ with an initial value of ψ_0 blows up in finite time. By continuity, there exists $T_1 > 0$ such that $\|\nabla\psi(T_1)\|_{L^2}^2 > r_0$. We now assume the initial value $\tilde{\psi}_0 = \frac{\sqrt{a}}{\|\psi_0\|_{L^2}}\psi_0$. When $\delta > 0$ is sufficiently small, we have

$$E_\gamma(\tilde{\psi}_0) < A \quad \text{and} \quad \tilde{\psi}_0 \in S(a).$$

When $\sqrt{a} < \|\psi_0\|_{L^2}$, $\|\nabla\tilde{\psi}_0\|_{L^2}^2 < \|\nabla\psi_0\|_{L^2}^2 < r_0$. When $\sqrt{a} > \|\psi_0\|_{L^2}$, considering $0 < a < a_0$, we have $\|\nabla\tilde{\psi}_0\|_{L^2}^2 < r_0$. This implies that $\tilde{\psi}_0 \in V(a)$. Since the solution of (1.1) is continuously dependent on the initial data and $\|\nabla\psi(T_1)\|_{L^2}^2 > r_0$, there exists $T_2 > 0$ such that $\|\nabla\tilde{\psi}(T_2)\|_{L^2}^2 > r_0$, where $\tilde{\psi}(t)$ is the solution of (1.1) that satisfies $\tilde{\psi}(t)|_{t=0} = \tilde{\psi}_0$. We consequently deduce from the continuity that there exists $T_3 > 0$ such that $\|\nabla\tilde{\psi}(T_3)\|_{L^2}^2 = r_0$. This indicates that $\tilde{\psi}(T_3) \in \partial V(a)$. Then we infer from Lemma 2.4 that $A > E_\gamma(\tilde{\psi}_0) > E_\gamma(\tilde{\psi}(T_3)) \geq \inf_{u \in \partial V(a)} E_\gamma(u)$, which is a contradiction. The proof follows. □

Lemma 2.6. *It holds that*

(i) $a \in (0, a_0)$, $a \mapsto m_\gamma(a, r_0)$ is a continuous mapping.

(ii) Let $a \in (0, a_0)$. For all $\mu \in (0, a)$, we have

$$m_\gamma(a, r_0) < m_\gamma(\mu, r_0) + m_\gamma(a - \mu, r_0).$$

Proof. (i) Let $a \in (0, a_0)$ be arbitrary and $\{a_n\}_{n\geq 1} \subset (0, a_0)$ be such that $a_n \to a$. According to the definition of $m_\gamma(a_n, r_0)$, we know that there exists $u_n \in V(a)$ such that

$$E_\gamma(u_n) \leq m_\gamma(a_n, r_0) + \varepsilon \text{ and } E_\gamma(u_n) < 0 \text{ for any } \varepsilon > 0 \text{ sufficiently small.} \tag{2.14}$$

We set $v_n := \sqrt{\frac{a}{a_n}} u_n$ and hence $v_n \in S(a)$. We find that $v_n \in V(a)$. Indeed, if $a_n \geq a$, then

$$\|\nabla v_n\|_{L^2}^2 = \frac{a}{a_n}\|\nabla u_n\|_{L^2}^2 \leq \|\nabla u_n\|_{L^2}^2 < r_0.$$

If $a_n < a$, by Lemma 2.2, (2.4) and (2.9) we have $f(a_n, r) \geq f(a_n, r_0) \geq f(a_0, r_0) = 0$ for any $r \in [\frac{a_n}{a} r_0, r_0]$. Indeed, since $f(a, r)$ is a non-increasing function, then we have

$$f(a_n, r_0) \geq f(a_0, r_0) = 0. \tag{2.15}$$

And then by direct calculations we have

$$f(a_n, r) \geq f(a_n, r_0). \tag{2.16}$$

Hence, we deduce from (2.3) and (2.14) that $f(a_n, \|\nabla u_n\|_{L^2}^2) < 0$, thus $\|\nabla u_n\|_{L^2}^2 < \frac{a_n}{a} r_0$ and

$$\|\nabla v_n\|_{L^2}^2 = \frac{a}{a_n}\|\nabla u_n\|_{L^2}^2 < \frac{a}{a_n}\frac{a_n}{a} r_0 = r_0.$$

Since $v_n \in V(a)$ we can write

$$m_\gamma(a, r_0) \leq E_\gamma(v_n) = E_\gamma(u_n) + [E_\gamma(v_n) - E_\gamma(u_n)],$$

where

$$E_\gamma(v_n) - E_\gamma(u_n) = \frac{1}{2}(\frac{a}{a_n} - 1)\|\nabla u_n\|_{L^2}^2 - \frac{\gamma}{2}(\frac{a}{a_n} - 1)\int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^\sigma} dx - \frac{1}{\alpha+2}\left[(\frac{a}{a_n})^{\frac{\alpha+2}{2}} - 1\right]\|u_n\|_{L^{\alpha+2}}^{\alpha+2}.$$

Since $\|\nabla u_n\|_{L^2}^2 < r_0$, also $\|u_n\|_{L^{\alpha+2}}^{\alpha+2}$ and $\int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^\sigma} dx$ are uniformly bounded. Thus, we have

$$m_\gamma(a, r_0) \leq E_\gamma(v_n) = E_\gamma(u_n) + o_n(1) \quad \text{as } \mathrm{n} \to \infty. \tag{2.17}$$

Combining (2.14) and (2.17), we get

$$m_\gamma(a, r_0) \leq m_\gamma(a_n, r_0) + \varepsilon + o_n(1).$$

Now, let $u \in V(a)$ be such that

$$E_\gamma(u) \leq m_\gamma(a, r_0) + \varepsilon \quad \text{and} \quad E_\gamma(u) < 0.$$

Set $u_n := \sqrt{\frac{a_n}{a}} u$ and hence $u_n \in S(a_n)$. Clearly, $\|\nabla u\|_{L^2}^2 < r_0$ and $a_n \to a$ imply $\|\nabla u_n\|_{L^2}^2 < r_0$ for n large enough, so that $u_n \in V(a)$. Also, $E_\gamma(u_n) \to E_\gamma(u)$. We thus have

$$m_\gamma(a_n, r_0) \leq E_\gamma(u_n) = E_\gamma(u) + [E_\gamma(u_n) - E_\gamma(u)] \leq m_\gamma(a, r_0) + \varepsilon + o_n(1). \tag{2.18}$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $m_\gamma(a_n, r_0) \to m_\gamma(a, r_0)$. The point (i) follows.
(ii) Note that, fixed $\mu \in (0, a)$, it is sufficient to prove that the following holds

$$m_\gamma(\theta\mu, r_0) < \theta m_\gamma(\mu, r_0), \quad \forall\, \theta \in \left(1, \frac{a}{\mu}\right]. \tag{2.19}$$

Indeed, if (2.19) holds, then we have

$$m_\gamma(a, r_0) < \frac{a-\mu}{a} \cdot \frac{a}{a-\mu} m_\gamma(a-\mu, r_0) + \frac{\mu}{a} \cdot \frac{a}{\mu} m_\gamma(\mu, r_0) = m_\gamma(a-\mu, r_0) + m_\gamma(\mu, r_0).$$

Next, we prove that (2.19) holds. According to the definition of $m_\gamma(\mu, r_0)$, we have that there exists $u \in V(\mu)$ such that

$$E_\gamma(u) \le m_\gamma(\mu, r_0) + \varepsilon \quad \text{and} \quad E_\gamma(u) < 0 \quad \text{for any } \varepsilon > 0 \text{ sufficiently small.} \tag{2.20}$$

By (2.9), $f(a, r) \ge 0$ for any $r \in [\frac{\mu}{a} r_0, r_0]$. Hence, we can deduce from (2.3) and (2.20) that

$$\|\nabla u\|_{L^2}^2 < \frac{\mu}{a} r_0. \tag{2.21}$$

Now we set $v(x) := u(\theta^{-\frac{1}{N}} x)$. On the one hand, we note that $\|v\|_{L^2}^2 = \theta \|u\|_{L^2}^2 = \theta\mu$ and also, because of (2.21), $\|\nabla v\|_{L^2}^2 = \theta \|\nabla u\|_{L^2}^2 < r_0$. Thus $v \in V(\theta\mu)$. On the other hand, we obtain from (2.2) that

$$\gamma \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^\sigma} dx \le \gamma C_1 \mu^{\frac{2-\sigma}{2}} \|\nabla u\|_{L^2}^\sigma < \gamma_0 C_1 a^{\frac{2-\sigma}{2}} \|\nabla u\|_{L^2}^\sigma,$$

and it follows easily that

$$\|\nabla u\|_{L^2}^2 - \gamma \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^\sigma} dx > \|\nabla u\|_{L^2}^2 - \gamma_0 C_1 a^{\frac{2-\sigma}{2}} \|\nabla u\|_{L^2}^\sigma > 0$$

for $0 < \gamma < \gamma_0$ and γ_0 small sufficiently. We can obtain that

$$\begin{aligned}
m_\gamma(\theta\mu, r_0) &\le \liminf_{n\to\infty} E_\gamma(v) \\
&= \liminf_{n\to\infty} \left(\theta E_\gamma(u) + \|\nabla u\|_{L^2}^2 \Big(\frac{\theta^{1-\frac{\sigma}{N}}}{2} - \frac{\theta}{2}\Big) - \gamma \Big(\frac{\theta^{1-\frac{\sigma}{N}}}{2} - \frac{\theta}{2}\Big) \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^\sigma} dx \right) \\
&= \theta m_\gamma(\mu, r_0) + \Big(\frac{\theta^{1-\frac{\sigma}{N}}}{2} - \frac{\theta}{2}\Big) \liminf_{n\to\infty} \left(\|\nabla u\|_{L^2}^2 - \gamma \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^\sigma} dx \right) \\
&< \theta m_\gamma(\mu, r_0).
\end{aligned}$$

□

Lemma 2.7. *Let $\{v_n\}_{n\ge1} \subset B(r_0)$ be such that $\int_{\mathbb{R}^N} \frac{|v_n|^2}{|x|^\sigma} dx \to 0$. Then, there exist a $\beta_0 > 0$ such that*

$$E_\gamma(v_n) \ge \beta_0 \|\nabla v_n\|_{L^2}^2 + o_n(1).$$

Proof. Indeed, using the Sobolev inequality, we obtain that

$$\begin{aligned} E_\gamma(v_n) =& \frac{1}{2}\|\nabla v_n\|_{L^2}^2 - \frac{1}{\alpha+2}\|v_n\|_{L^{\alpha+2}}^{\alpha+2} + o_n(1) \\ \geq& \|\nabla v_n\|_{L^2}^2 \left(\frac{1}{2} - \frac{C(\alpha)}{\alpha+2} a^{\frac{\eta_2}{2}} r_a^{\frac{\eta_3}{2}} \right) + o_n(1). \end{aligned}$$

Now, since $f(a_0, r_0) = 0$, we have

$$\beta_0 := \frac{1}{2} - \frac{C(\alpha)}{\alpha+2} a_0^{\frac{\eta_2}{2}} r_0^{\frac{\eta_3}{2}} = \frac{\gamma C_1}{2} a_0^{\frac{\eta_0}{2}} r_0^{\frac{\eta_1}{2}} > 0.$$

□

Lemma 2.8. *For any $a \in (0, a_0)$, let $\{u_n\}_{n\geq 1} \subset B(r_0)$ be such that $\|u_n\|_{L^2}^2 \to a$ and $E_\gamma(u_n) \to m_\gamma(a, r_0)$. Then, there exists a $\beta_1 > 0$ and a sequence $\{y_n\}_{n\geq 1} \subset \mathbb{R}^N$ such that*

$$\int_{B(y_n,R)} |u_n|^2 dx \geq \beta_1 > 0, \quad \text{for some R} > 0. \tag{2.22}$$

Proof. We assume by contradiction that (2.22) does not hold. Since $\{u_n\}_{n\geq 1} \subset B(r_0)$ and $\|u_n\|_{L^2}^2 \to a$, the sequence $\{u_n\}_{n\geq 1}$ is bounded in $H^1(\mathbb{R}^N)$. From Lemma I.1 in [21] and since $\frac{4}{N} + 2 < \alpha + 2 < \frac{2N}{N-2}$, we deduce that $\|u_n\|_{L^{\alpha+2}} \to 0$ as $n \to \infty$. At this point, Lemma 2.7 implies $E_\gamma(u_n) \geq o_n(1)$. This contradicts the fact that $m_\gamma(a, r_0) < 0$ and the lemma follows. □

Theorem 2.9. *For any $a \in (0, a_0)$, if $\{u_n\}_{n\geq 1} \subset B(r_0)$ is such that $\|u_n\|_{L^2}^2 \to a$ and $E_\gamma(u_n) \to m_\gamma(a, r_0)$ then, up to translation, $u_n \to u$ in $H^1(\mathbb{R}^N)$.*

Proof. Let $\{u_n\}_{n\geq 1} \subset B(r_0)$ be a minimizing sequence of the energy functional $E_\gamma(u)$, that is,

$$\|u_n\|_{L^2}^2 = a \ \text{ and } \ \lim_{n\to\infty} E_\gamma(u_n) = m_\gamma(a, r_0).$$

By similar proof as [23], we known that $\{u_n\}_{n\geq 1}$ is bounded in $H^1(\mathbb{R}^N)$. By Lemma 2.6, Lemma 2.8 and Rellich compactness theorem, there exist sequences $\{u_{n_k}\}_{k\geq 1} \subset H^1(\mathbb{R}^N)$ and $\{y_{n_k}\}_{k\geq 1} \subset \mathbb{R}^N$ such that for any $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that for all $k \geq 1$,

$$\int_{B(y_{n_k}, R(\epsilon))} |u_{n_k}|^2 dx \geq \beta_2 - \epsilon > 0. \tag{2.23}$$

Denote $\tilde{u}_{n_k}(\cdot) = u_{n_k}(\cdot + y_{n_k})$, then there exists $\tilde{u}$ such that $\tilde{u}_{n_k} \rightharpoonup \tilde{u}$ weakly in $H^1(\mathbb{R}^N)$, $\tilde{u}_{n_k} \to \tilde{u}$ strongly in $L^r_{loc}(\mathbb{R}^N)$ with $r \in [2, \frac{2N}{N-2})$, combining with (2.23) we have

$$\int_{B(0,R(\epsilon))} |\tilde{u}|^2 dx \geq \beta_2 - \epsilon > 0.$$

Thus $\int_{\mathbb{R}^N} |\tilde{u}|^2 dx = \beta_2$. Indeed, we assume by contradiction that $\int_{\mathbb{R}^N} |\tilde{u}|^2 dx = \bar{\beta}_2 < \beta_2$. According to Brézis-Lieb Lemma and $\tilde{u}_{n_k} = (\tilde{u}_{n_k} - \tilde{u}) + \tilde{u}$, we have

$$\|\tilde{u}_{n_k}\|_{H^1}^2 = \|\tilde{u}_{n_k} - \tilde{u}\|_{H^1}^2 + \|\tilde{u}\|_{H^1}^2 + o_n(1),$$

$$\|\tilde{u}_{n_k}\|_{L^q}^q = \|\tilde{u}_{n_k} - \tilde{u}\|_{L^q}^q + \|\tilde{u}\|_{L^q}^q + o_n(1), \quad 1 \le q < \infty,$$

$$\|\tilde{u}_{n_k}\|_{L^{p+2}}^{p+2} = \|\tilde{u}_{n_k} - \tilde{u}\|_{L^{p+2}}^{p+2} + \|\tilde{u}\|_{L^{p+2}}^{p+2} + o_n(1),$$

$$\int_{\mathbb{R}^N} \frac{|\tilde{u}_{n_k}|^2}{|x|^\sigma} dx = \int_{\mathbb{R}^N} \frac{|\tilde{u}_{n_k} - \tilde{u}|^2}{|x|^\sigma} dx + \int_{\mathbb{R}^N} \frac{|\tilde{u}|^2}{|x|^\sigma} dx + o_n(1).$$

Then,

$$E(\tilde{u}_{n_k}) = E(\tilde{u}_{n_k} - \tilde{u}) + E(\tilde{u}) + o_n(1).$$

Hence,

$$m_\gamma(\beta_2, r_0) \ge m_\gamma(\beta_2 - \bar{\beta}_2, r_0) + m_\gamma(\bar{\beta}_2, r_0),$$

which contradicts Lemma 2.6. Thus $\int_{\mathbb{R}^N} |\tilde{u}|^2 dx = \beta_2$, i.e., $\tilde{u}_{n_k} \to \tilde{u}$ strongly in $L^2(\mathbb{R}^N)$. By the Gagliardo-Nirenberg inequality, $\tilde{u}_{n_k} \to \tilde{u}$ strongly in $L^s(\mathbb{R}^N)$, where $s \in [2, \frac{2N}{N-2})$. Next, our aim is to show that $\{y_{n_k}\}_{k\ge1} \subset \mathbb{R}^N$ is bounded. If it was not the case, we deduce that

$$\int_{\mathbb{R}^N} \frac{|u_{n_k}|^2}{|x|^\sigma} dx = \int_{\mathbb{R}^N} \frac{|\tilde{u}_{n_k}|^2}{|x + y_{n_k}|^\sigma} dx \to 0 \quad \text{as } \; k \to \infty.$$

Hence $\lim\limits_{n\to\infty} E_\gamma(u_{n_k}) = \lim\limits_{n\to\infty} E_\gamma^\infty(u_{n_k})$, where $E_\gamma^\infty(u_{n_k})$ is the corresponding energy functional of the sequence $\{u_{n_k}\}_{k\ge1} \subset H^1(\mathbb{R}^N)$ via translation transformation. On the other hand, we know that $\lim\limits_{n\to\infty} E_\gamma^\infty(u_n)$ is attained by a nontrivial function $u_{r_0} \in B(r_0)$. Hence,

$$\begin{aligned}
\lim_{k\to\infty} E_\gamma^\infty(u_{n_k}) &= \lim_{n\to\infty} E_\gamma^\infty(\tilde{u}_{n_k}) \\
&= \lim_{k\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \tilde{u}_{n_k}|^2 dx - \lim_{k\to\infty} \frac{1}{\alpha+2} \int_{\mathbb{R}^N} |\tilde{u}_{n_k}|^{\alpha+2} dx \\
&\ge \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx - \frac{1}{\alpha+2} \int_{\mathbb{R}^N} |\tilde{u}|^{\alpha+2} dx.
\end{aligned}$$

According to the definition of $E_\gamma^\infty(u_{n_k})$, we see that $\tilde{u}$ is a minimizer of $\lim\limits_{k\to\infty} E_\gamma^\infty(u_{n_k})$. Consequently,

$$\lim_{k\to\infty} E_\gamma(u_{n_k}) < \lim_{k\to\infty} E_\gamma^\infty(u_{n_k}) - \int_{\mathbb{R}^N} \frac{|u_{r_0}|^2}{|x|^\sigma} dx < \lim_{k\to\infty} E_\gamma^\infty(u_{n_k}),$$

which contradicts $\lim\limits_{k\to\infty} E_\gamma(u_{n_k}) \ge \lim\limits_{k\to\infty} E_\gamma^\infty(u_{n_k})$. Thus, $\{y_{n_k}\}_{k\ge1} \subset \mathbb{R}^N$ is bounded. We can deduce, up to subsequences, $\lim\limits_{k\to\infty} y_{n_k} = y_0$ for some $y_0 \in \mathbb{R}^N$. Consequently,

$$\begin{aligned}
\|u_{n_k}(x) - \tilde{u}(x - y_0)\|_{L^s} \le& \|u_{n_k}(x) - \tilde{u}(x - y_{n_k})\|_{L^s} + \|\tilde{u}(x - y_{n_k}) - \tilde{u}(x - y_0)\|_{L^s} \\
=& \|u_{n_k}(x + y_{n_k}) - \tilde{u}(x)\|_{L^s} + \|\tilde{u}(x + y_0 - y_{n_k}) - \tilde{u}(x)\|_{L^s} \\
\to& \, 0
\end{aligned} \tag{2.24}$$

as $k \to \infty$ for any $s \in [2, \frac{2N}{N-2})$, that is, u_{n_k} converges strongly to $\tilde{u}(x-y_0)$ in $L^s(\mathbb{R}^N)$ for $s \in [2, \frac{2N}{N-2})$. We denote $u(x) = \tilde{u}(x-y_0)$. Hence,

$$\lim_{k\to\infty} E_\gamma(u_{n_k}) \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\gamma}{2}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^\sigma} dx - \frac{1}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2} dx.$$

We see that $E_\gamma(u) = m_\gamma(a, r_0)$ and hence $u_{n_k} \to u$ in $H^1(\mathbb{R}^N)$.

□

Proof of Theorem 2.3. The existence of γ_0 and a minimizer for $E_\gamma(u)$ on $V(a)$ were proved in Lemma 2.6, Lemma 2.7 and Theorem 2.9.

3. Orbital stability

Proof of Theorem 1.3. (i) The property that $\mathcal{M}(a) \subset V(a)$ is non-empty follows from Theorem 2.3.

(ii) We will show that $\mathcal{M}(a)$ is orbitally stable. Firstly, we note that the solution ψ of (1.1) exists globally by Lemma 2.5. Now we suppose by contradiction that there exist $\epsilon_0 > 0$, a sequence of initial data $\{\psi_{0,n}\}_{n\geq 1} \subset H^1(\mathbb{R}^N)$ and a sequence $\{t_n\}_{n\geq 1} \subset \mathbb{R}$ such that for all $n \geq 1$,

$$\inf_{u\in\mathcal{M}(a)} \|\psi_{0,n} - u\|_{H^1} < \frac{1}{n}, \quad \inf_{u\in\mathcal{M}(a)} \|\psi_n(t_n) - u\|_{H^1} \geq \epsilon_0, \tag{3.1}$$

where $\psi_n(t)$ is the solution to (1.1) with initial data $\psi_{0,n}$. Next, we claim that there exists $v \in \mathcal{M}(a)$ such that

$$\lim_{n\to\infty} \|\psi_{0,n} - v\|_{H^1} = 0.$$

Indeed, by (3.1), we see that for each $n \geq 1$, there exists $v_n \in V(a)$ such that

$$\|\psi_{0,n} - v_n\|_{H^1} < \frac{2}{n}. \tag{3.2}$$

Since $\{v_n\}_{n\geq 1} \subset \mathcal{M}(a)$ is a minimizing sequence of (1.7) and by the same argument as in Lemma 2.3, there exists $v \in \mathcal{M}(a)$ such that

$$\lim_{n\to\infty} \|v_n - v\|_{H^1} = 0. \tag{3.3}$$

By (3.2) and (3.3), the claim follows. And then, due to $v \in V(a)$, we have $\tilde{\psi}_n = \frac{\sqrt{a}\psi_n(t_n)}{\|\psi_n(t_n)\|_{L^2}} \in V(a)$ and

$$\lim_{n\to\infty} E_\gamma(\tilde{\psi}_n) = \lim_{n\to\infty} E_\gamma(\psi_n(t_n)) = \lim_{n\to\infty} E_\gamma(\psi_{0,n}) = E_\gamma(v) = m_\gamma(a, r_0),$$

which implies that $\{\tilde{\psi}_n\}_{n\geq 1}$ is a minimizing sequence for (1.7). Thanks to the compactness of all minimizing sequence of (1.7), there is a $\tilde{u} \in V(a)$ satisfies

$$\tilde{\psi}_n \to \tilde{u} \ \ in \ \ H^1(\mathbb{R}^N).$$

Moreover, by the definition of $\tilde{\psi}_n$, it follows easily that

$$\tilde{\psi}_n - \psi_n(t_n) \to 0 \ \ in \ \ H^1(\mathbb{R}^N).$$

Consequently, we have

$$\psi_n(t_n) \to \tilde{u} \ \ in \ \ H^1(\mathbb{R}^N),$$

which contradicts (3.1). The proof is now complete.

4. Conclusions

In this work, we study the stability of the set of energy minimizers in the mass supercritical case. In the mass supercritical case, the energy functional is unbounded from below on $S(a)$. Thus, we consider the further constrained minimization problem to study the existence and stability of standing waves for (1.1). And, the energy functional is not invariant under the scaling transform due to inverse-power potentials $\gamma|x|^{-\sigma}u$. It is intrinsically difficult for us to prove the compactness of minimizing sequences. By a rather delicate analysis, we can overcome this difficulty by proving the boundedness of any translation sequence.

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Conflict of interest

The author declares no conflicts of interest.

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