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Research article

Moderate deviation principle for m-dependent random variables under the sub-linear expectation

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Abstract: Let $\{X_n, n \geq 1\}$ be a sequence of m-dependent strictly stationary random variables in a sub-linear expectation $(\Omega, \mathcal{H}, \mathbb{E})$. In this article, we give the definition of m-dependent sequence of random variables under sub-linear expectation spaces taking values in $\mathbb{R}$. Then we establish moderate deviation principle for this kind of sequence which is strictly stationary. The results in this paper generalize the result that in the case of independent identically distributed samples. It provides a basis to discuss the moderate deviation principle for other types of dependent sequences.

Keywords: moderate deviation principle; m-dependent random variables; sub-linear expectation; stationary sequences
Mathematics Subject Classification: 60F10

1. Introduction

Moderate deviation principle (MDP) and large deviation principle (LDP) are applied to many aspects such as physics, finance and communication because of their typical significance. Especially in risk measurement and model uncertainty in statistics and finance. LDP and MDP can measure the asymptotic properties of the probability of rare event, particularly LDP and MDP for independent identically distributed (IID) random variables. Although LDP and MDP for the sequence of IID random variables are similar in form, there are essential differences between LDP and MDP. Generally speaking, LDP describes the ergodic phenomena that depend on the law of large numbers, while the MDP describes the asymptotic behavior of probabilities that between the law of large numbers and the central limit theorem. Specifically, the rate function of LDP depends on the distribution of random variables, while the rate function of MDP only depends on the limit density form of the central limit theorem of random variables, so MDP is more universal compared with LDP.

Petrov [23] constructed the law of the iterated logarithm and probabilities of moderate deviations of sums of dependent random variables. The main aim of Feng et al. [8] was to study self-normalized moderate deviations for transient random walk in random scenery under a much weaker condition than a finite moment-generating function of the scenery variables. Chen and Zhang [1] mainly discussed moderate deviations for the total population arising from a nearly unstable sub-critical Galton-Watson process with immigration. Xue [25] presented MDP for the paths of a certain class of density-dependent Markov chains. Miao et al. [18] obtained the conclusion that MDP for m-dependent random variables.

Peng [20] came up with a new notion of sub-linear expectation, referred to as G-expectation and the related G-normal distribution. Through the research of scholars, sub-linear expectation can be regarded as an extension of the classical linear expectation. There are two important nonlinear expectations, one is the capacity which was introduced by Choquet [4] and the other is sub-linear expectation spaces which was initiated by Peng [22]. Peng [22] also gave some basic concepts simultaneously. Furthermore, Denis et al. [5] presented the function spaces and capacity related to a sub-linear expectation: Application to G-Brownian motion paths. Liu and Zhang [17] established central limit theorem and invariance principle for linear processes generated by IID random variables under sub-linear expectation. Many related results have been investigated in the sub-linear expectation space. These results can be found in the work of Guo and Zhang [11], Lin and Feng [14], Liu and Zhang [16], and references therein. Recently, some researchers have already got LDP and MDP under the sub-linear expectation framework. Tan and Zong [24] obtained the LDP for random variables in sub-linear expectation spaces taking values in $\mathbb{R}^d$, which do not have to satisfy the independent identical distributed condition. Chen and Feng [2] put forward LDP for negatively dependent random variables under sub-linear expectation, moreover they gained the upper bound of MDP. Associated with Peng's CLT, Gao and Xu [9] obtained LDP and MDP for independent random variables under sub-linear expectations. Zhou and Logachov [26] investigated MDP for a sequence of weak independent but not identically distributed random variables under sub-linear expectations. It can be regarded as an extension of the Theorem 3.1 and Corollary 3.2 in Gao and Xu [9] with $d = 1$ and $Y_i = 0$.

The main aim of this paper is to prove MDP for m-dependent random variables under the sub-linear expectation. This theorem can be considered as a generalization of Gao and Xu's conclusion (in [9]). This paper is organized as follows: In Section 2, we introduce the framework of sub-linear expectation spaces and some basic settings. In Section 3, we give the main result and proofs of this paper.

2. Basic settings

First we review some definitions related symbols and properties on sub-linear expectation space. More detailed information are mentioned in Peng [20–22]. Let Ω be a complete separable metric space equipped with the the Borel σ-algebra $\mathcal{B}(\Omega)$. We use the framework and notations of Peng [19–22].

Let $(\Omega, \mathcal{F})$ be a given measurable space. Let $\mathcal{H}$ be a linear space of real functions defined on Ω such that if $(X_1, X_2, ..., X_n) \in \mathcal{H}$, then $\varphi(X_1, X_2, ..., X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz continuous functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leqslant C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ. In this paper, the space $\mathcal{H}$ will be used as the space of random variables. Often a random variable $X = (X_1, \ X_2, ..., X_n)$ is called a n-dimensional random vector, denoted by $X \in \mathcal{H}^n$.

Definition 2.1. A sub-linear expectation on $\mathcal{H}$ is a functional $\mathbb{E}:\mathcal{H}\to\bar{\mathbb{R}}$ satisfying the following properties: For any $X, Y\in\mathcal{H}$, we have

(1) Monotonicity: $X\geqslant Y$ implies $\mathbb{E}[X]\geqslant\mathbb{E}[Y]$.

(2) Constant preserving: $\mathbb{E}[c]=c,\ \forall c\in\mathbb{R}$.

(3) Sub-additivity: $\mathbb{E}[X+Y]\leqslant\mathbb{E}[X]+\mathbb{E}[Y]$.

(4) Positive homogeneity: $\mathbb{E}[\lambda X]=\lambda\mathbb{E}[X],\ \forall\lambda>0$.

Here $\bar{\mathbb{R}}=[-\infty,+\infty]$. The triple $(\Omega,\mathcal{H},\mathbb{E})$ is called a sub-linear expectation space.

We give some useful properties of sub-linear expectation $\mathbb{E}$, the proof can be found in Peng [22].

Remark 2.1. *Properties (3) and (4) are called sub-linearity, they imply*

(5) Convexity: $\mathbb{E}[\alpha X+(1-\alpha)Y]\leqslant\alpha\mathbb{E}[X]+(1-\alpha)\mathbb{E}[Y]$, *for* $\alpha\in[0,1]$.

If a nonlinear expectation $\mathbb{E}$ *satisfies the convexity property, we call it a convex expectation.*

Properties (2) and (3) imply

(6) Cash translatability: $\mathbb{E}[X+c]=\mathbb{E}[X]+c,\ \forall c\in\mathbb{R}$.

(7) From the definition, it is easily shown that $\mathbb{E}[X-Y]\geqslant\mathbb{E}[X]-\mathbb{E}[Y],\ \forall X,Y\in\mathcal{H}$ *with* $\mathbb{E}[Y]$ *being finite.*

Definition 2.2. (Identical distribution) (Peng [22]) Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1,\mathcal{H}_1,\mathbb{E}_1)$ and $(\Omega_2,\mathcal{H}_2,\mathbb{E}_2)$. They are called identically distributed, denoted by $\mathbf{X}_1\overset{d}{=}\mathbf{X}_2$, if

$$\mathbb{E}_1[\varphi(\mathbf{X}_1)]=\mathbb{E}_2[\varphi(\mathbf{X}_2)],\ \forall\varphi\in C_{l,Lip}(\mathbb{R}^n),$$

whenever the sub-expectations are finite. A sequence of random variables $\{X_n, n\geqslant 1\}$ is said to be identically distributed if $X_i\overset{d}{=}X_1$ for each $i\geqslant 1$.

Definition 2.3. (Independence) (Chen and Feng [2]) Let $(X_1,\ldots,X_{n+1})$ be real random variables on $(\Omega,\mathcal{H})$. X_{n+1} is said to be independent of $(X_1,\ldots,X_n)$ under $\mathbb{E}[\cdot]$, if for every non negative measurable function $\varphi_i(\cdot)$ on $\mathbb{R}$ with $\mathbb{E}\left[\varphi_i\left(X_i\right)\right]<\infty, i=1,\ldots,n+1$, we have

$$\mathbb{E}\left[\prod_{i=1}^{n+1}\varphi_i\left(X_i\right)\right]=\mathbb{E}\left[\prod_{i=1}^{n}\varphi_i\left(X_i\right)\right]\mathbb{E}\left[\varphi_{n+1}\left(X_{n+1}\right)\right].$$

$\{X_n\}_{n=1}^{\infty}$ is said to be a sequence of independent random variables, if X_{n+1} is independent of $(X_1,\ldots,X_n)$ for all $n\in\mathbb{N}$.

Definition 2.4. (IID random variables) A sequence of random variables $\{X_n, n\geqslant 1\}$ is said to be independent and identically distributed, if $X_i\overset{d}{=}X_1$ and X_{i+1} is independent to $(X_1,...,X_i)$ for each $i\geqslant 1$.

Definition 2.5. (m-dependent random variables) (Li [13]) The sequence$\{X_i\}_{i=1}^{\infty}$ is called m-dependent if there exits an integer m such that for every n and every $j\geqslant m+1$, $(X_{n+m+1},\cdots,X_{n+j})$ is independent from $(X_1,...,X_n)$. In particular, if $m=0$, $\{X_i\}_{i=1}^{\infty}$ is called independent sequence.

Next we introduce a definition that generalizes the concept of strictly stationary sequence to the case of sequence in a sub-linear expectation space.

Definition 2.6. (Liu and Zhang [15]) $\{\xi_n,\ n\in\mathbb{N}\}$ is said to be a sequence of strictly stationary random variables on the $(\Omega,\mathcal{H},\mathbb{E})$, if for any function $\phi_n\in C_{l,Lip}(\mathbb{R}^n):\mathbb{R}^n\to\mathbb{R}$, then

$$\mathbb{E}[\phi_n(\xi_1,\xi_2,...,\xi_n)]=\mathbb{E}[\phi_n(\xi_{1+k},\xi_{2+k},...,\xi_{n+k})],\quad\forall n\geq 1,\ k\in\mathbb{N}.$$

In this paper, we always assume $\mathbb{E}$ is regular, i.e., for all $\{X_n, n \geqslant 1\} \subset \mathcal{H}$,

$$X_n(\omega) \downarrow 0 \text{ for all } \omega \in \Omega \Longrightarrow \lim_{n\to\infty} \mathbb{E}(X_n) = 0.$$

Now we recall the definition of the upper expectation and corresponding capacity. For more details, refer to chapter 6 of Peng [22]. Let $\mathcal{M}$ be the set of all probability measures on Ω and denote $\mathcal{P}$ as relatively compact subset of $\mathcal{M}$.

- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions.
- $B_b(\Omega)$: all bounded functions in $L^0(\Omega)$.
- $C_b(\Omega)$: all continuous functions in $B_b(\Omega)$.

We define the upper probability $\mathbb{V}$

$$\mathbb{V}(A) = \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Under this framework, $\mathbb{V}(\cdot)$ is a Choquet capacity (cf. Denis and Martini [6], Dellacherie [7]) which satisfies the following properties.

Proposition 2.1. $\mathbb{V}(\cdot)$ *is a Choquet capacity, i.e.,*

(1) $0 \leqslant \mathbb{V}(A) \leqslant 1, \ \forall A \in \Omega.$

(2) If $A \subset B$, *then* $\mathbb{V}(A) \leqslant \mathbb{V}(B)$.

(3) If $(A_n)_{n=1}^{\infty}$ *is a sequence in* $\mathcal{B}(\Omega)$, *then* $\mathbb{V}\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mathbb{V}(A_n)$.

(4) If $(A_n)_{n=1}^{\infty}$ *is an increasing sequence in* $\mathcal{B}(\Omega)$: $A_n \uparrow A = \bigcup_{n=1}^{\infty} A_n$, *then* $\mathbb{V}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mathbb{V}(A_n)$.

We will consider the upper expectation $\mathbb{E}$ generated by $\mathcal{P}$, for any $X \in L^0(\Omega)$ such that $E_P[X]$ exists for each $P \in \mathcal{P}$,

$$\mathbb{E}[X] := \sup_{P\in\mathcal{P}} E_P[X], \quad \mathbb{V}(A) = \mathbb{E}[I_A], \quad A \in \mathcal{B}(\Omega),$$

and $\mathbb{L}^1 = \{X \in B(\Omega);\ \mathbb{E}[|X|] < \infty\}$. The upper expectation $\mathbb{E}[\cdot]$ is a sub-linear expectation which is introduced in Definition 2.1 on $B_b(\Omega)$ as well as on $C_b(\Omega)$.

In the end of this section, we recall two important lemmas which will be used in the following proof.

Lemma 2.1. *(Chen et al. [3]) Let* X, Y *be real* $\mathcal{F}$*-measurable random variables on upper expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$.

(1) Hölder's inequality: For $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$\mathbb{E}[|XY|] \leqslant \left(\mathbb{E}\left[|X|^p\right]\right)^{\frac{1}{p}} \cdot \left(\mathbb{E}\left[|Y|^q\right]\right)^{\frac{1}{q}}.$$

(2) Chebyshev's inequality: Let $f(x) > 0$ *be a non-decreasing nonnegative function on* $\mathbb{R}$, *then for any* x,

$$\mathbb{V}(X \geqslant x) \leqslant \frac{\mathbb{E}[f(X)]}{f(x)}.$$

3. Main results and proofs

Some notations and definitions that will be used in this section are presented before we give the main results.

Definition 3.1. A function $I : \mathbb{R} \to [0, \infty)$ is called a rate function, if for all $l \geqslant 0$, the set $\{x : I(x) \leqslant l\}$ is a compact subset of $\mathbb{R}$.

We will consider a sequence of m-independent random variables $\{X_i\}_{i=1}^{\infty}$ in a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Assume that b_n is a sequence of positive real numbers satisfying $\lim\limits_{n\to\infty} \frac{b_n}{\sqrt{n}} = \infty$, $\lim\limits_{n\to\infty} \frac{b_n}{n} = 0$. Denote $S_n = \sum\limits_{i=1}^{n} X_i$, $\overline{S}_n = \frac{S_n}{b_n}$, for $n \in \mathbb{N}$.

We adopt the framework and notations in Section 2 and recall the definition of LDP under sub-linear expectations with an upper probability $\mathbb{V}$.

Definition 3.2. (Gao and Xu [10]) Let Ω be a topology space and $\mathbb{R}$ be a σ-algebra on Ω. Let $(V_n, n \geqslant 1)$ be a family of measurable maps from Ω into $\mathbb{R}$ and $a(n)$, $n \geqslant 1$ be a positive function satisfying $a(n) \to \infty$ as $n \to \infty$.

$\{\mathbb{V}(V_n \in \cdot), n \geqslant 1\}$ is said to satisfy LDP with speed $a(n)$ and with rate function $I(x)$ if for any measurable closed subset $F \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{a(n)} \log \mathbb{V}\left(V_n \in F\right) \leqslant -\inf_{x\in F} I(x), \tag{3.1}$$

and for any measurable open set $O \subset \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{a(n)} \log \mathbb{V}\left(V_n \in O\right) \geqslant -\inf_{x\in O} I(x), \tag{3.2}$$

(3.1) and (3.2) are referred respectively to as upper bound of large deviations (ULD) and lower bound of large deviations (LLD).

If (3.1) and (3.2) are satisfied with $a(n) = \frac{b_n^2}{n}$ and $V_n = \frac{S_n}{b_n}$, we say that V_n satisfies the LDP with rate function $I(\cdot)$ and speed $\frac{n}{b_n^2}$, also say that V_n satisfies MDP.

Next we give the main result of this paper.

Theorem 3.1. *Let $\{X_i\}_{i=1}^{\infty}$ be a strictly stationary m-dependent random variable sequence under sub-linear expectation, $\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$, $\lim\limits_{n\to\infty} \frac{1}{n}\mathbb{E}[\sum\limits_{i=1}^{n} X_i]^2 = \sigma^2 < \infty$, there exists $\delta > 0$, such that $\mathbb{E}\exp\{\delta|X_1|\} < \infty$. $\{\mathbb{V}(\overline{S}_n \in \cdot), n \longrightarrow \infty\}$ is said to satisfy the MDP with a rate function $I(x)$.*

(1) For every closed set $F \subset \mathbb{R}$ such that

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\overline{S}_n \in F\right) \leqslant -\inf_{x\in F} I(x); \tag{3.3}$$

(2) for every open set $G \subset \mathbb{R}$ such that

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left(\overline{S}_n \in G\right) \geqslant -\inf_{x\in G} I(x), \tag{3.4}$$

where $I(x) := \sup\limits_{y\in\mathbb{R}} \left\{xy - \frac{y^2\sigma^2}{2}\right\} = \frac{x^2}{2\sigma^2}$.

Before starting our proof, we need to mention the following lemmas which are useful to prove the above results.

Lemma 3.1. *(Liu and Zhang [15]) Suppose that* $\{\theta_i, 1 \leqslant i \leqslant n\}$ *is a sequence of* $\mathbb{R}$*-valued random variables on* $(\Omega, \mathcal{H}, \mathbb{E})$ *and* $\zeta_i \in [0,1]$, $1 \leqslant i \leqslant n$, *such that* $\sum_{i=1}^{n} \zeta_i = 1$. *Then*

$$\log \mathbb{E} \exp\left\{\sum_{i=1}^{n} \zeta_i \theta_i\right\} \leqslant \sum_{i=1}^{n} \zeta_i \log \mathbb{E} \exp\{\theta_i\}. \tag{3.5}$$

Lemma 3.2. *Under the assumption of Theorem 3.1, for any* $x \in \mathbb{R}$, *we have*

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp\left(x \frac{b_n}{n} S_n\right) = \frac{\sigma^2 x^2}{2}. \tag{3.6}$$

Proof. Fix $K \geqslant m+1, n \geqslant K+m$, define $l = \left[\frac{n}{K+m}\right]$. Note

$$\varepsilon_t = \sum_{i=1}^{K} X_{t(K+m)+i}, \quad \eta_t = \sum_{i=1}^{m} X_{t(K+m)+K+i}, \quad t = 0, 1, \cdots, l-1.$$

We have defined that $\{X_i, i \geqslant 1\}$ is zero-mean m-dependent random variables, therefore $\{\varepsilon_t, t \geqslant 0\}$ and $\{\eta_t, t \geqslant 0\}$ are both independent identically distributed random variables, in addition $\mathbb{E}\varepsilon_1 = 0$, $\mathbb{E}\eta_1 = 0$. Thus, based on the definition of S_n, we have

$$S_n = \sum_{t=0}^{l-1} \varepsilon_t + \sum_{t=0}^{l-1} \eta_t + \sum_{i=l(K+m)+1}^{n} X_i := I_1 + I_2 + I_3, \tag{3.7}$$

when $p_1 > 1, p_2 > 1$, choose $q_1 > 1, q_2 > 1$, such that $\frac{1}{p_1} + \frac{1}{q_1} = 1, \frac{1}{p_2} + \frac{1}{q_2} = 1$. By the means of (3.7) and (1) in Lemma 2.1 (Hölder inequality), for any $x \in \mathbb{R}$, that

$$\begin{aligned}
\frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x \frac{b_n}{n} S_n\right\} &\leqslant \frac{1}{q_1} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x q_1 \frac{b_n}{n} I_3\right\} + \frac{1}{p_1} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x p_1 \frac{b_n}{n} (I_1 + I_2)\right\} \\
&\leqslant \frac{1}{q_1} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x q_1 \frac{b_n}{n} I_3\right\} + \frac{1}{p_1 p_2} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x p_1 p_2 \frac{b_n}{n} I_1\right\} \\
&\quad + \frac{1}{p_1 q_2} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x p_1 q_2 \frac{b_n}{n} I_2\right\} \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{3.8}$$

Above all, we deal with J_1. By using the definition of I_3, Lemma 3.1 and the strictly stationary of $\{X_i\}_{i=1}^{\infty}$, it's obvious that

$$\begin{aligned}
\limsup_{n\to\infty} J_1 &\leqslant \lim_{n\to\infty} \frac{1}{q_1} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{|x| q_1 \frac{b_n}{n} \sum_{i=l(K+m)+1}^{(l+1)(K+m)} |X_i|\right\} \\
&= \limsup_{n\to\infty} \frac{1}{q_1} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{|x| q_1 \frac{b_n}{n} (K+m) \sum_{i=1}^{K+m} \frac{|X_i|}{K+m}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \limsup_{n\to\infty} \frac{1}{q_1}\cdot\frac{n}{b_n^2}\sum_{i=1}^{K+m}\frac{1}{K+m}\log\mathbb{E}\exp\left\{|x|q_1\frac{b_n}{n}(K+m)|X_i|\right\}\\
&=\limsup_{n\to\infty} \frac{1}{q_1}\cdot\frac{n}{b_n^2}\log\mathbb{E}\exp\left\{|x|q_1\frac{b_n}{n}(K+m)|X_1|\right\}\\
&\leqslant \limsup_{n\to\infty} \frac{1}{q_1}\cdot\frac{n}{b_n^2}\log\mathbb{E}\exp\{\delta|X_1|\}\\
&=0,
\end{aligned}\tag{3.9}
$$

when n is sufficient big, then $|x|q_1\frac{b_n}{n}(K+m)\leqslant\delta$.

Next is to prove J_2. According to Lemma 1 in Zhou and Logachov [26], we know that

$$
\lim_{n\to\infty}\frac{n^2}{b_n^2}\log\mathbb{E}\exp\left\{y\frac{b_n}{n}\varepsilon_1\right\}=\frac{\sigma_1^2y^2}{2},\quad \forall y\in R,\tag{3.10}
$$

where $\sigma_1^2=\mathbb{E}\varepsilon_1^2$. On the one hand, by the definition of l, that $\lim\limits_{n\to\infty}\frac{l}{n}=\frac{1}{K+m}$. On the other hand, $\{\varepsilon_i\}$ is independent identically distributed and $\{X_i\}$ is strictly stationary, that

$$
\begin{aligned}
\lim_{n\to\infty}J_2&=\frac{1}{p_1p_2}\lim_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\exp\left\{xp_1p_2\frac{b_n}{n}\sum_{t=0}^{l-1}\varepsilon_t\right\}\\
&=\frac{1}{p_1p_2}\lim_{n\to\infty}l\frac{n}{b_n^2}\log\mathbb{E}\exp\left\{xp_1p_2\frac{b_n}{n}\varepsilon_0\right\}\\
&=\frac{1}{p_1p_2}\cdot\frac{1}{K+m}\lim_{n\to\infty}\frac{n^2}{b_n^2}\log\mathbb{E}\exp\left\{xp_1p_2\frac{b_n}{n}\varepsilon_0\right\}\\
&=p_1p_2\cdot\frac{1}{K+m}\cdot\frac{\sigma_1^2x^2}{2}\\
&=p_1p_2\cdot\frac{1}{K+m}\cdot\frac{x^2}{2}\mathbb{E}\left[\sum_{i=1}^{K}X_i\right]^2\\
&=p_1p_2\cdot\frac{K}{K+m}\cdot\frac{x^2}{2}\cdot\frac{1}{K}\mathbb{E}\left[\sum_{i=1}^{K}X_i\right]^2.
\end{aligned}
$$

Firstly we order $K\to\infty$, then let $p_1\to1, p_2\to1$, that

$$
\begin{aligned}
\lim_{p_1\to1,p_2\to1}\lim_{K\to\infty}\lim_{n\to\infty}J_2&=\lim_{p_1\to1,p_2\to1}\lim_{K\to\infty}\frac{p_1p_2x^2}{2}\cdot\frac{K}{K+m}\cdot\frac{1}{K}\mathbb{E}\left[\sum_{i=1}^{K}X_i\right]^2\\
&=\lim_{p_1\to1,p_2\to1}\frac{p_1p_2x^2}{2}\cdot\sigma^2\\
&=\frac{\sigma^2x^2}{2}.
\end{aligned}\tag{3.11}
$$

Now we prove J_3. Because $\{\eta_i\}$ is also independent identically distributed, so we use the same method with J_2, such that

$$
\lim_{n\to\infty}J_3=\frac{1}{p_1q_2}\lim_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{E}\exp\left\{xp_1q_2\frac{b_n}{n}\sum_{t=0}^{l-1}\eta_t\right\}
$$

$$
\begin{aligned}
&= \frac{1}{p_1 q_2} \lim_{n\to\infty} l \frac{n}{b_n^2} \log \mathbb{E} \exp \left\{ x p_1 q_2 \frac{b_n}{n} \eta_0 \right\} \\
&= \frac{1}{p_1 q_2} \cdot \frac{1}{K+m} \lim_{n\to\infty} \frac{n^2}{b_n^2} \log \mathbb{E} \exp \left\{ x p_1 q_2 \frac{b_n}{n} \eta_0 \right\} \\
&= \frac{p_1 q_2}{K+m} \cdot \frac{x^2 \mathbb{E} \eta_1^2}{2} \\
&= \frac{p_1 q_2}{K+m} \cdot \frac{x^2}{2} \mathbb{E} \left[\sum_{i=1}^{m} X_i \right]^2 .
\end{aligned}
$$

Let $K \to \infty$, so we can obtain

$$
\lim_{K\to\infty} \lim_{n\to\infty} J_3 = 0. \tag{3.12}
$$

Combining with (3.9), (3.11) and (3.12), we can get

$$
\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp \left\{ x \frac{b_n}{n} S_n \right\} \leqslant \frac{\sigma^2 x^2}{2}. \tag{3.13}
$$

According to (1) in Lemma 2.1 (Hölder inequality), that

$$
\begin{aligned}
\log \mathbb{E} \exp \left\{ x \frac{b_n}{p_1 p_2 n} I_1 \right\} &= \log \mathbb{E} \exp \left\{ x \frac{b_n}{p_1 p_2 n} (I_1 + I_2 - I_2) \right\} \\
&\leqslant \frac{1}{p_1} \log \mathbb{E} \exp \left\{ \frac{b_n}{p_2 n} (I_1 + I_2) \right\} + \frac{1}{q_1} \log \mathbb{E} \exp \left\{ -x \frac{q_1 b_n}{p_1 p_2 n} I_2 \right\} \\
&= \frac{1}{p_1} \log \mathbb{E} \exp \left\{ x \frac{b_n}{p_2 n} (I_1 + I_2 + I_3 - I_3) \right\} \\
&\quad + \frac{1}{q_1} \log \mathbb{E} \exp \left\{ -x \frac{q_1 b_n}{p_1 p_2 n} I_2 \right\} \\
&\leqslant \frac{1}{p_1 p_2} \log \mathbb{E} \exp \left\{ x \frac{b_n}{n} (I_1 + I_2 + I_3) \right\} \\
&\quad + \frac{1}{p_1 q_2} \log \mathbb{E} \exp \left\{ -x \frac{q_2 b_n}{p_2 n} I_3 \right\} + \frac{1}{q_1} \log \mathbb{E} \exp \left\{ -x \frac{q_1 b_n}{p_1 p_2 n} I_2 \right\}.
\end{aligned} \tag{3.14}
$$

Similar to (3.9), that

$$
\limsup_{n\to\infty} \frac{1}{p_1 q_2} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp \left\{ -x \frac{q_2 b_n}{p_2 n} I_3 \right\} \leqslant 0. \tag{3.15}
$$

Similar to (3.11), that

$$
\lim_{p_1\to1, p_2\to1} \lim_{K\to\infty} \lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp \left\{ x \frac{b_n}{p_1 p_2 n} I_1 \right\} = \frac{\sigma^2 x^2}{2}. \tag{3.16}
$$

Similar to (3.12), that

$$
\lim_{K\to\infty} \lim_{n\to\infty} \frac{1}{q_1} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp \left\{ -x \frac{q_1 b_n}{p_1 p_2 n} I_2 \right\} = 0. \tag{3.17}
$$

Combining with (3.14)–(3.17), we conclude that

$$\begin{aligned}\frac{\sigma^2 x^2}{2} &= \lim_{p_1\to 1, p_2\to 1} \lim_{K\to\infty} \lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x \frac{b_n}{p_1 p_2 n} I_1\right\} \\ &\leqslant \lim_{p_1\to 1, p_2\to 1} \liminf_{n\to\infty} \frac{1}{p_1 p_2} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x \frac{b_n}{n} S_n\right\} \\ &\quad + \limsup_{n\to\infty} \frac{1}{p_1 q_2} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{-x \frac{q_2 b_n}{p_2 n} I_3\right\} \\ &\quad + \lim_{K\to\infty} \lim_{n\to\infty} \frac{1}{q_1} \cdot \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{-x \frac{q_1 b_n}{p_1 p_2 n} I_2\right\} \\ &= \liminf_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{x \frac{b_n}{n} S_n\right\}, \end{aligned} \tag{3.18}$$

and together with (3.13) and (3.18), then follows (3.6). Thus we finish the proof. □

Proof of Theorem 3.1

Proof. We divide the proof into two steps in this section.
Step 1. Let F be any given closed set. It's obvious we can get the result when $F = \emptyset$. So we suppose that $F \neq \emptyset$. Note

$$x_- := \sup\{x \in F : x < 0\} \leqslant 0, \quad x_+ := \inf\{x \in F : x \geqslant 0\} \geqslant 0,$$

thus $F \subseteq (-\infty, x_-] \cup [x_+, +\infty)$. Let $x_- = -\infty$, if $F \cap (-\infty, 0] = \emptyset$ and $x_+ = +\infty$, if $F \cap [0, +\infty) = \emptyset$. Then, if $F \neq \emptyset$, there exist a finite x_- or x_+. Hence it is sufficient to see that

$$\begin{aligned}\log \mathbb{V}\left(\overline{S}_n \in F\right) &= \log \mathbb{E} I_{\{\overline{S}_n \in F\}} \\ &\leqslant \log\left(\mathbb{E} I_{\{\overline{S}_n \in (-\infty, x_-]\}} + \mathbb{E} I_{\{\overline{S}_n \in [x_+, +\infty)\}}\right) \\ &\leqslant \log\left(2 \max\left(\mathbb{E} I_{\{\overline{S}_n \in (-\infty, x_-]\}}, \mathbb{E} I_{\{\overline{S}_n \in [x_+, +\infty)\}}\right)\right). \end{aligned} \tag{3.19}$$

Now we estimate $\mathbb{E} I_{\{\overline{S}_n \in (-\infty, x_-]\}}$ primarily. Let $f(x) = e^x$, for all $\lambda > 0$, by (2) in Lemma 2.1 (Chebyshev's inequality) we can get

$$\mathbb{E} I_{\{\overline{S}_n \in (-\infty, x_-]\}} = \mathbb{E} I_{\left\{-\lambda \frac{b_n}{n} S_n \geqslant -\lambda \frac{b_n^2}{n} x_-\right\}} \leqslant \frac{\mathbb{E} e^{-\lambda \frac{b_n}{n} S_n}}{e^{-\lambda \frac{b_n^2}{n} x_-}},$$

from Lemma 3.2, that,

$$\begin{aligned}\frac{n}{b_n^2} \log \mathbb{V}\left\{\overline{S}_n \in (-\infty, x_-]\right\} &\leqslant \frac{n}{b_n^2} \log \frac{\mathbb{E} e^{-\lambda \frac{b_n}{n} S_n}}{e^{-\lambda \frac{b_n^2}{n} x_-}} \\ &= \frac{n}{b_n^2} \log \mathbb{E} e^{-\lambda \frac{b_n}{n} S_n} + \lambda x_-, \end{aligned}$$

therefore

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left\{\overline{S}_n \in (-\infty, x_-]\right\} \leqslant \limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{-\lambda \frac{b_n}{n} S_n\right\} + \lambda x_-$$

$$= \frac{\sigma^2\lambda^2}{2} + \lambda x_-. \tag{3.20}$$

Here we choose $\lambda = -\frac{x_-}{\sigma^2}$ in (3.20) thus

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left\{\overline{S}_n \in (-\infty, x_-]\right\} \leqslant -\frac{x_-^2}{2\sigma^2}. \tag{3.21}$$

Similarly, it is obvious that

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{V}\left\{\overline{S}_n \in [x_+, +\infty)\right\} \leqslant -\frac{x_+^2}{2\sigma^2}. \tag{3.22}$$

Hence, from (3.19), (3.21) and (3.22) it is sufficient to show that

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \ln\left(\mathbb{V}\left(\overline{S}_n \in F\right)\right) \leqslant -\min\left(\frac{x_-^2}{2\sigma^2}, \frac{x_+^2}{2\sigma^2}\right) \leqslant -\inf_{x\in F} I(x).$$

Finally, $\inf_{x\in F} I(x) \leqslant \min\left(\frac{x_-^2}{2\sigma^2}, \frac{x_+^2}{2\sigma^2}\right)$ is remain to prove. In fact, under the condition we have given that $I(x) := \sup_{y\in\mathbb{R}}\left\{xy - \frac{y^2\sigma^2}{2}\right\} = \frac{x^2}{2\sigma^2}$, it yields

$$\inf_{x\in F} I(x) \leqslant \min\left(I\left(x_-\right), I\left(x_+\right)\right) = \min\left(\frac{x_-^2}{2\sigma^2}, \frac{x_+^2}{2\sigma^2}\right).$$

Therefore, we have finish the proof of the inequality (3.3) in Theorem 3.1.

Step 2. Next we have to prove that for any open set G inequality (3.4) holds with $I(x)$ defined in Theorem 3.1. Let G be any open set. It is obvious we can gain the result when $G = \emptyset$. In order to get the following assumption we choose $G \neq \emptyset$. It's apparent that for any $h \geqslant 0$, the set

$$T_h := \{x : I(x) \leqslant h\},$$

is compact. According to $G \neq \emptyset$, then $G \cap T_{h_G} \neq \emptyset$ as $h_G > 0$.

Due to G is an open set, for any $\mu > 0$ there exists $x \in G \cap T_{h_G}$, we have

$$\inf_{y\in G} I(y) \geqslant I(x) - \mu. \tag{3.23}$$

For any $\xi > 0$ denote

$$x^{(\xi)} := (x - \xi, x + \xi).$$

For any $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, and denote $O(\lambda, n) := \log \mathbb{E} e^{\lambda \frac{b_n}{n} S_n}$. Hence we can obtain

$$I_{\left\{\overline{S}_n \in x^{(\xi)}\right\}} e^{\lambda \frac{b_n}{n} S_n} = I_{\left\{\overline{S}_n \in x^{(\xi)}\right\}} e^{\lambda \frac{b_n^2}{n} \overline{S}_n} \geqslant I_{\left\{\overline{S}_n \in x^{(\xi)}\right\}} e^{\frac{b_n^2}{n}(\lambda x - |\lambda|\xi)}. \tag{3.24}$$

By (3.24) and (7) in Remark 2.1, for any $\lambda \in \mathbb{R}$ and sufficiently small $\xi > 0$, it yields

$$\log \mathbb{V}\left(\overline{S}_n \in G\right) \geqslant \log \mathbb{V}\left(\overline{S}_n \in x^{(\xi)}\right)$$

$$
\begin{aligned}
&= \log \mathbb{E} I_{\{\overline{S}_n \in x^{(\xi)}\}} \\
&\geqslant \log \mathbb{E}\left(I_{\{\overline{S}_n \in x^{(\xi)}\}} e^{\lambda \frac{b_n}{n} S_n - O(\lambda,n)} e^{-\lambda \frac{b_n^2}{n} x + O(\lambda,n)} e^{-|\lambda| \frac{b_n^2}{n} \xi} \right) \\
&= -\lambda \frac{b_n^2}{n} x + O(\lambda,n) - |\lambda| \frac{b_n^2}{n} \xi + \log \mathbb{E}\left\{ \left(1 - I_{\{\overline{S}_n \notin x^{(\xi)}\}}\right) e^{\lambda \frac{b_n}{n} S_n - O(\lambda,n)} \right\} \\
&\geqslant -\lambda \frac{b_n^2}{n} x + O(\lambda,n) - |\lambda| \frac{b_n^2}{n} \xi + \log\left(1 - \mathbb{E} I_{\{\overline{S}_n \notin x^{(\xi)}\}} e^{\lambda \frac{b_n}{n} S_n - O(\lambda,n)}\right). \qquad (3.25)
\end{aligned}
$$

For all $\gamma > 0$, we obtain that

$$
\begin{aligned}
\mathbb{E} I_{\{\overline{S}_n \notin x^{(\xi)}\}} e^{\lambda \frac{b_n}{n} S_n - O(\lambda,n)} &\leqslant \mathbb{E} I_{\{\overline{S}_n \geqslant x+\xi\}} e^{\lambda \frac{b_n}{n} S_n - O(\lambda,n)} + \mathbb{E} I_{\{\overline{S}_n \leqslant x-\xi\}} e^{\lambda \frac{b_n}{n} S_n - O(\lambda,n)} \\
&\leqslant \frac{\mathbb{E} e^{\frac{\gamma b_n}{n\sigma^2} S_n} e^{\lambda \frac{b_n}{n} S_n}}{e^{\frac{\gamma b_n^2 (x+\xi)}{\sigma^2 n}} e^{O(\lambda,n)}} + \frac{\mathbb{E} e^{-\frac{\gamma b_n}{n\sigma^2} S_n} e^{\lambda \frac{b_n}{n} S_n}}{e^{\frac{\gamma b_n^2 (\xi - x)}{\sigma^2 n}} e^{O(\lambda,n)}} \\
&= \frac{\mathbb{E} e^{\left(\frac{\gamma}{\sigma^2} + \lambda\right) \frac{b_n}{n} S_n}}{e^{\frac{\gamma b_n^2 (x+\xi)}{\sigma^2 n}} e^{O(\lambda,n)}} + \frac{\mathbb{E} e^{\left(\lambda - \frac{\gamma}{\sigma^2}\right) \frac{b_n}{n} S_n}}{e^{\frac{\gamma b_n^2 (\xi - x)}{\sigma^2 n}} e^{O(\lambda,n)}} \\
&= H_1 + H_2.
\end{aligned}
$$

The following point is to proof H_1 and H_2, respectively. From Lemma 3.2 we know that

$$
\lim_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{E} \exp\left\{ \lambda \frac{b_n}{n} S_n \right\} = \frac{\lambda^2 \sigma^2}{2},
$$

there exist N, such that $n > N$, for any $\epsilon \in \mathbb{R}$, we have

$$
\mathbb{E} e^{\lambda \frac{b_n}{n} S_n} \leqslant e^{\frac{b_n^2}{n} \frac{\lambda^2 \sigma^2}{2} (1+\epsilon)}.
$$

For n large enough, thus

$$
\begin{aligned}
H_1 &= \frac{\mathbb{E} e^{\left(\frac{\gamma}{\sigma^2} + \lambda\right) \frac{b_n}{n} S_n}}{e^{\frac{b_n^2}{n} \frac{\gamma x + \gamma \xi}{\sigma^2}} e^{O(\lambda,n)}} \\
&\leqslant \frac{e^{\frac{b_n^2}{n} \left(\frac{\left(\gamma/\sigma^2 + \lambda\right)^2}{2} \sigma^2 (1+\epsilon) \right)}}{e^{\frac{b_n^2}{n} \left(\frac{\gamma x + \gamma \xi}{\sigma^2} + \frac{\lambda^2}{2} \sigma^2 (1+\epsilon) \right)}} \\
&= \exp\left\{ \frac{b_n^2}{n} \left[\left(\frac{\gamma^2}{2\sigma^2} + \gamma \lambda \right)(1+\epsilon) - \frac{\gamma x + \gamma \xi}{\sigma^2} \right] \right\},
\end{aligned}
$$

by the choice of $\gamma = \xi$, let $\epsilon \to 0$ thus

$$
H_1 \leqslant \exp\left\{ \frac{b_n^2}{n} \left[-\frac{\xi^2}{2\sigma^2} + \xi \left(\lambda - \frac{x}{\sigma^2} \right) \right] \right\}.
$$

Similarly

$$
H_2 \leqslant \exp\left\{ \frac{b_n^2}{n} \left[-\frac{\xi^2}{2\sigma^2} + \xi \left(\frac{x}{\sigma^2} - \lambda \right) \right] \right\}.
$$

It can be concluded that

$$\mathbb{E}I_{\{\overline{S}_n\notin x^{(\xi)}\}}e^{\lambda\frac{b_n}{n}S_n-O(\lambda,n)} \leqslant 2\exp\left\{\frac{b_n^2}{n}\left[-\frac{\xi^2}{2\sigma^2}+\xi\left|\frac{x}{\sigma^2}-\lambda\right|\right]\right\},$$

we choose $\lambda=\frac{x}{\sigma^2}$, thus

$$\begin{aligned}\frac{n}{b_n^2}\log(1-\mathbb{E}I_{\{\overline{S}_n\notin x^{(\xi)}\}}e^{\lambda\frac{b_n}{n}S_n-O(\lambda,n)}) &\geqslant \frac{n}{b_n^2}\log\{1-(H_1+H_2)\}\\ &\geqslant \frac{n}{b_n^2}\log\left\{1-2\exp\left\{\frac{b_n^2}{n}\left[-\frac{\xi^2}{2\sigma^2}+\xi\left|\frac{x}{\sigma^2}-\lambda\right|\right]\right\}\right\}\\ &\to 0.\end{aligned} \tag{3.26}$$

Combining (3.25), (3.26) and Lemma 3.2 , we get

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\overline{S}_n\in G\right)\geqslant -\frac{x^2}{2\sigma^2}-\frac{|x|}{\sigma^2}\xi,$$

let $\xi\to 0$, since $x\in T_{h_G}$ and the set T_{h_G} does not depend on ξ, so we obtain

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\overline{S}_n\in G\right)\geqslant -\frac{x^2}{2\sigma^2}. \tag{3.27}$$

By (3.27), for any $\mu>0$

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\log\mathbb{V}\left(\overline{S}_n\in G\right)\geqslant -\frac{x^2}{2\sigma^2}=-I(x)\geqslant -\inf_{x\in G}I(x)-\mu. \tag{3.28}$$

Consequently, when $\mu\to 0$ the result is deduced from (3.28). Combining Step 1 and Step 2, the proof of Theorem 3.1 is obtained. □

4. Conclusions

In this paper, the author establish MDP for m-dependent random variables under the sub-linear expectation. The results extend Gao and Xu's conclusion (in [9]) from independent random variables to m-dependent random variables under the sub-linear expectations. The research about MDP for dependent sequences is a new trend in probability and statistics, one can refer to [9,18,25,26] and references therein for details.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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