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Research article

Multivalued contraction maps on fuzzy b-metric spaces and an application

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Abstract: In this article, the concept of a Hausdorff fuzzy b-metric space is introduced. The new notion is used to establish some fixed point results for multivalued mappings in G-complete fuzzy b-metric spaces satisfying a suitable requirement of contractiveness. An illustrative example is formulated to support the results. Eventually, an application for the existence of a solution for an integral inclusion is established which involves showing the materiality of the obtained results. These results are more general and some theorems proved by of Shehzad et al. are their special cases.

Keywords: fuzzy b-metric space; Hausdorff fuzzy b-metric space; fixed point; multivalued mapping
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1. Introduction

Fixed point theory has a vital role in mathematics and applied sciences. Also, this theory has lot of applications in differential equations and integral equations to guarantee the existence and uniqueness of the solutions [1, 2]. The Banach contraction principle [3] has an imperative role in fixed point theory. Since after the appearance of this principle, it has become very popular and there has been a lot of activity in this area. On the other hand, to establish Banach contraction principle in a more general structure, the notion of a metric space was generalized by Bakhtin [4] in 1989 by introducing the idea of a b-metric space. Afterwards, the same idea was further investigated by Czerwik [5] to establish different results in b-metric spaces. The study of b-metric spaces holds a prominent place in fixed point theory. Banach contraction principle is generalized in many ways by changing the main platform of the metric space [6–9].

Zadeh [10] introduced the notion of a fuzzy set theory to deal with the uncertain states in daily life. Motivated by the concept, Kramosil and Michálek [11] defined the idea of fuzzy metric spaces. Grabiec [12] gave contractive mappings on a fuzzy metric space and extended fixed point theorems of Banach and Edelstein in such spaces. Successively, George and Veeramani [13] slightly altered the concept of a fuzzy metric space introduced by Kramosil and Michálek [11] and then attained a Hausdorff topology and a first countable topology on it. Numerous fixed point theorems have been constructed in fuzzy metric spaces. For instance, see [14–20].

Nădăban [21] studied the notion of a fuzzy b-metric space and proved some results. Rakić et al. [22] (see also [23]) proved some new fixed point results in b-fuzzy metric spaces. The notion of a Hausdorff fuzzy metric on compact sets is introduced in [24] and recently studied by Shahzad et al. [25] to establish fixed point theorems for multivalued mappings in complete fuzzy metric spaces. In this paper, we use the idea of a fuzzy b-metric space and establish some fixed point results for multivalued mappings in Hausdorff fuzzy b-metric spaces. Some fixed point theorems are also derived from these results. Finally, we investigate the applicability of the obtained results to integral equations.

Throughout the article, fuzzy metric space and fuzzy b-metric space are denoted by FMS and FBMS, respectively.

2. Preliminaries

Bakhtin [4] defined the notion of a b-metric space as follows:

Definition 2.1. [4] Let Ω be a non-empty set. For any real number $b \geq 1$, a function $d_b \colon \Omega \times \Omega \longrightarrow \mathbb{R}$ is called a b-metric if it satisfies the following properties for all $\zeta_1, \zeta_2, \zeta_3 \in \Omega$:

$BM1$: $d_b(\zeta_1, \zeta_2) \geq 0$;

$BM2$: $d_b(\zeta_1, \zeta_2) = 0$ if and only if $\zeta_1 = \zeta_2$;

$BM3$: $d_b(\zeta_1, \zeta_2) = d_b(\zeta_2, \zeta_1)$ for all $\zeta_1, \zeta_2 \in \Omega$;

$BM4$: $d_b(\zeta_1, \zeta_3) \leq b\Big[d_b(\zeta_1, \zeta_2) + d_b(\zeta_2, \zeta_3)\Big]$.

The pair (Ω, d_b) is called a b-metric space.

Definition 2.2. [26] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a **continuous t-norm** if it satisfies the following conditions:

(1) $*$ is associative and commutative;

(2) $*$ is continuous;

(3) $\zeta * 1 = \zeta$ for all $\zeta \in [0, 1]$;

(4) $\zeta_1 * \zeta_2 \leq \zeta_3 * \zeta_4$ wherever $\zeta_1 \leq \zeta_3$ and $\zeta_2 \leq \zeta_4$, for all $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [0, 1]$.

Example 2.1. Define a mapping $* : [0,1] \times [0,1] \to [0,1]$ by

$$\zeta_1 * \zeta_2 = \zeta_1 \zeta_2 \quad \text{for } \zeta_1, \zeta_2 \in [0,1].$$

It is then obvious that $*$ is a continuous t-norm, known as the product norm.

George and Veermani [13] defined a fuzzy metric space as follows:

Definition 2.3. [13] Consider a nonempty set Ω, then $(\Omega, F, *)$ is a fuzzy metric space if $*$ is a continuous t-norm and F is a fuzzy set on $\Omega \times \Omega \times [0, +\infty)$ satisfying the following for all $\zeta_1, \zeta_2, \zeta_3 \in \Omega$ and $\alpha, \beta > 0$:

$[F1]$: $F(\zeta_1, \zeta_2, \alpha) > 0$;

$[F2]$: $F(\zeta_1, \zeta_2, \alpha) = 1$ if and only if $\zeta_1 = \zeta_2$;

$[F3]$: $F(\zeta_1, \zeta_2, \alpha) = F(\zeta_2, \zeta_1, \alpha)$;

$[F4]$: $F(\zeta_1, \zeta_3, \alpha + \beta) \geq F(\zeta_1, \zeta_2, \alpha) * F(\zeta_2, \zeta_3, \beta)$;

$[F5]$: $F(\zeta_1, \zeta_2, .)\colon [0, +\infty) \to [0,1]$ is continuous.

In [27], the idea of a fuzzy b-metric space is given as:

Definition 2.4. [27] Let $\Omega \neq \phi$ be a set, $b \geq 1$ be a real number and $*$ be a continuous t-norm. A fuzzy set F_b on $\Omega \times \Omega \times [0, +\infty)$ is called a fuzzy b-metric on Ω if for all $\zeta_1, \zeta_2, \zeta_3 \in \Omega$, the following conditions hold:

$[Fb1]$: $F_b(\zeta_1, \zeta_2, \alpha) > 0$;

$[Fb2]$: $F_b(\zeta_1, \zeta_2, \alpha) = 1$, for all $\alpha > 0$ if and only if $\zeta_1 = \zeta_2$;

$[Fb3]$: $F_b(\zeta_1, \zeta_2, \alpha) = F_b(\zeta_2, \zeta_1, \alpha)$;

$[Fb4]$: $F_b(\zeta_1, \zeta_3, b(\alpha + \beta)) \geq F_b(\zeta_1, \zeta_2, \alpha) * F_b(\zeta_2, \zeta_3, \beta)$ for all $\alpha, \beta \geq 0$;

$[Fb5]$: $F_b(\zeta_1, \zeta_2, .)\colon (0, +\infty) \to [0,1]$ is left continuous.

Example 2.2. Let (Ω, d_b) be a b-metric space. Define a mapping $F_b\colon \Omega \times \Omega \times [0, +\infty) \to [0,1]$ by

$$F_b(\zeta_1, \zeta_2, \alpha) = \begin{cases} \dfrac{\alpha}{\alpha + d_b(\zeta_1, \zeta_2)} & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Then $(\Omega, F_b, \wedge)$ is a fuzzy b-metric space, where

$$\zeta_1 \wedge \zeta_2 = \min\{\zeta_1, \zeta_2\} \quad \text{for all } \zeta_1, \zeta_2 \in [0,1].$$

$\wedge$ is a t-norm, known as the minimum t-norm.

Following Grabiec [12], we extend the idea of a G-Cauchy sequence and the notion of completeness in the FBMS as follows:

Definition 2.5. Let $(\Omega, F_b, *)$ be a FBMS.

1) A sequence $\{\zeta_n\}$ in Ω is said to be a G-Cauchy sequence if $\lim_{n\to+\infty} F_b(\zeta_n, \zeta_{n+q}, \alpha) = 1$ for $\alpha > 0$ and $q > 0$.
2) A FBMS in which every G-Cauchy sequence is convergent is called a G-complete FBMS.

Similarly, for a FBMS $(\Omega, F_b, *)$, a sequence $\{\zeta_n\}$ in Ω is said to be convergent if there exits $\zeta \in \Omega$ such that for all $\alpha > 0$,

$$\lim_{n\to+\infty} F_b(\zeta_n, \zeta, \alpha) = 1.$$

Definition 2.6. [25] Let $\mho \neq \phi$ be a subset of a FBMS $(\Omega, F, *)$ and $\alpha > 0$, then the fuzzy distance $\mathcal{F}$ of an element $\varrho_1 \in \Omega$ and the subset $\mho \subset \Omega$ is

$$\mathcal{F}(\varrho_1, \mho, \alpha) = \sup\{F(\varrho_1, \varrho_2, \alpha) \colon \varrho_2 \in \mho\}.$$

Note that $\mathcal{F}(\varrho_1, \mho, \alpha) = \mathcal{F}(\mho, \varrho_1, \alpha)$.

Lemma 2.1. *[28] If $\Lambda \in CB(\Omega)$, then $\zeta_1 \in \Lambda$ if and only if $\mathcal{F}(\Lambda, \zeta_1, \alpha) = 1$ for all $\alpha > 0$, where $CB(\Omega)$ is the collection of closed bounded subsets of Ω.*

Definition 2.7. [25] Let $(\Omega, F, *)$ be a FMS. Define a function $\Theta_{\mathcal{F}}$ on $\hat{C}_0(\Omega) \times \hat{C}_0(\Omega) \times (0, +\infty)$ by

$$\Theta_{\mathcal{F}}(\Lambda, \mho, \alpha) = \min\{\inf_{\varrho_1 \in \Lambda} \mathcal{F}(\varrho_1, \mho, \alpha), \inf_{\varrho_2 \in \mho} \mathcal{F}(\Lambda, \varrho_2, \alpha)\}$$

for all $\Lambda, \mho \in \hat{C}_0(\Omega)$ and $\alpha > 0$, where $\hat{C}_0(\Omega)$ is the collection of all nonemty compact subsets of Ω.

Lemma 2.2. *[29] Let $(\Omega, F, *)$ be a complete FMS and $F(\zeta_1, \zeta_2, k\alpha) \geq F(\zeta_1, \zeta_2, \alpha)$ for all $\zeta_1, \zeta_2 \in \Omega$, $k \in (0, 1)$ and $\alpha > 0$ then $\zeta_1 = \zeta_2$.*

Lemma 2.3. *[25] Let $(\Omega, F, *)$ be a complete FMS such that $(\hat{C}_0, \Theta_{\mathcal{F}}, *)$ is a Hausdorff FMS on $\hat{C}_0$. Then for all $\Lambda, \mho \in \hat{C}_0$, for each $\zeta \in \Lambda$ and for $\alpha > 0$, there exists $\varrho_\zeta \in \mho$ so that $\mathcal{F}(\zeta, \mho, \alpha) = F(\zeta, \varrho_\zeta, \alpha)$ then*

$$\Theta_{\mathcal{F}}(\Lambda, \mho, \alpha) \leq F(\zeta, \varrho_\zeta, \alpha).$$

The notion of a Hausdorff FMS in Definition 2.6 of [25] can be extended naturally for a Hausdorff FBMS on $\hat{C}_0$ as follows:

Definition 2.8. Let $(\Omega, F_b, *)$ be a FBMS. Define a function $\Theta_{\mathcal{F}_b}$ on $\hat{C}_0(\Omega) \times \hat{C}_0(\Omega) \times (0, +\infty)$ by

$$\Theta_{\mathcal{F}_b}(\Lambda, \mho, \alpha) = \min\{\inf_{\zeta \in \Lambda} \mathcal{F}_b(\zeta, \mho, \alpha), \inf_{\varrho \in \mho} \mathcal{F}_b(\Lambda, \varrho, \alpha)\}$$

for all $\Lambda, \mho \in \hat{C}_0(\Omega)$ and $\alpha > 0$.

3. Main results

This section deals with the idea of Hausdorff FBMS and certain new fixed point results in a fuzzy FBMS. Note that, one can easily extend Lemma 2.1 to 2.3 in the setting of fuzzy b-metric spaces.

Lemma 3.1. *If $\Lambda \in C\mho(\Omega)$, then $\zeta \in \Lambda$ if and only if $\mathcal{F}_b(\Lambda, \zeta, \alpha) = 1$ for all $\alpha > 0$.*

Proof. Since

$$\mathcal{F}_b(\Lambda,\zeta,\alpha) = \sup\{F_b(\zeta,\varrho,\alpha) \colon \varrho \in \Lambda\} = 1,$$

there exists a sequence $\{\varrho_n\} \subset \Lambda$ such that $F_b(\zeta,\varrho_n,\alpha) > 1 - \frac{1}{n}$. Letting $n \to +\infty$, we get $\varrho_n \to \zeta$. From $\Lambda \in C\mho(\Omega)$, it follows that $\zeta \in \Lambda$. Conversely, if $\zeta \in \Lambda$, we have

$$\mathcal{F}_b(\Lambda,\zeta,\alpha) = \sup\{F_\theta(\zeta,\varrho,\alpha) \colon \varrho \in \Lambda\} > F_b(\zeta,\zeta,\alpha) = 1.$$

□

Again, due to [17], the following fact follows from [$Fb5$].

Lemma 3.2. *Let* $(\Omega, F_b, *)$ *be a G-complete FBMS. If for two elements* $\zeta,\varrho \in \Omega$ *and for a number* $k < 1$

$$F_b(\zeta,\varrho,k\alpha) \geq F_b(\zeta,\varrho,\alpha),$$

then $\zeta = \varrho$.

Lemma 3.3. *Let* $(\Omega, F_b, *)$ *be a G-complete FBMS, such that* $(\hat{C}_0, \Theta_{\mathcal{F}_b}, *)$ *is a Hausdorff FBMS on* $\hat{C}_0$*. Then for all* $\Lambda, \mho \in \hat{C}_0$*, for each* $\zeta \in \Lambda$ *and for* $\alpha > 0$ *there exists* $\varrho_\zeta \in \mho$*, satisfying* $\mathcal{F}_b(\zeta,\mho,\alpha) = F_b(\zeta,\varrho_\zeta,\alpha)$ *also*

$$\Theta_{\mathcal{F}_b}(\Lambda,\mho,\alpha) \leq F_b(\zeta,\varrho_\zeta,\alpha).$$

Proof. If

$$\Theta_{\mathcal{F}_b}(\Lambda,\mho,\alpha) = \inf_{\zeta\in\Lambda} \mathcal{F}_b(\zeta,\mho,\alpha),$$

then

$$\Theta_{\mathcal{F}_b}(\Lambda,\mho,\alpha) \leq \mathcal{F}_b(\zeta,\mho,\alpha).$$

Since for each $\zeta \in \Lambda$, there exists $\varrho_\zeta \in \mho$ satisfying

$$\mathcal{F}_b(\zeta,\mho,\alpha) = F_b(\alpha,\varrho_\zeta,\alpha).$$

Hence,

$$\Theta_{\mathcal{F}_b}(\Lambda,\mho,\alpha) \leq F_b(\zeta,\varrho_\zeta,\alpha).$$

Now, if

$$\begin{aligned}\Theta_{\mathcal{F}_b}(\Lambda,\mho,\alpha) &= \inf_{\varrho\in\mho} \mathcal{F}_b(\Lambda,\varrho,\alpha)\\ &\leq \inf_{\zeta\in\Lambda} \mathcal{F}_b(\zeta,\mho,\alpha)\\ &\leq \mathcal{F}_b(\zeta,\mho,\alpha) = F_b(\zeta,\varrho_\zeta,\alpha),\end{aligned}$$

this implies

$$\Theta_{\mathcal{F}_b}(\Lambda,\mho,\alpha) \leq F_b(\zeta,\varrho_\zeta,\alpha)$$

for some $\varrho_\zeta \in \mho$. Hence, in both cases, the result is proved. □

Theorem 3.1. *Let $(\Omega, F_b, *)$ be a G-complete FBMS with $b \geqslant 1$ and $\Theta_{\mathcal{F}_b}$ be a Hausdorff FBMS. Let $S : \Omega \to \hat{C}_0(\Omega)$ be a multivalued mapping satisfying*

$$\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha) \geq F_b(\zeta, \varrho, \alpha) \tag{3.1}$$

for all $\zeta, \varrho \in \Omega$, where $bk < 1$, then S has a fixed point.

Proof. For $a_0 \in \Omega$, we choose a sequence $\{\zeta_n\}$ in Ω as follows: Let $a_1 \in \Omega$ such that $a_1 \in Sa_0$. By using Lemma 3.3, we can choose $a_2 \in Sa_1$ such that

$$F_b(a_1, a_2, \alpha) \geqslant \Theta_{\mathcal{F}_b}(Sa_0, Sa_1, \alpha) \text{ for all } \alpha > 0.$$

By induction, we have $a_{n+1} \in Sa_n$ satisfying

$$F_b(a_n, a_{n+1}, \alpha) \geqslant \Theta_{\mathcal{F}_b}(Sa_{n-1}, Sa_n, \alpha) \text{ for all } n \in \mathbb{N}.$$

Now, by (3.1) together with Lemma 3.3, we have

$$\begin{aligned} F_b(a_n, a_{n+1}, \alpha) &\geq \Theta_{\mathcal{F}_b}(Sa_{n-1}, Sa_n, \alpha) \geq F_b\left(a_{n-1}, a_n, \frac{\alpha}{k}\right) \\ &\geq \Theta_{\mathcal{F}_b}\left(Sa_{n-2}, Sa_{n-1}, \frac{\alpha}{k}\right) \geq F_b\left(a_{n-2}, a_{n-1}, \frac{\alpha}{k^2}\right) \\ &\vdots \\ &\geq \Theta_{\mathcal{F}_b}\left(Sa_0, Sa_1, \frac{\alpha}{k^{n-1}}\right) \geq F_b\left(a_0, a_1, \frac{\alpha}{k^n}\right). \end{aligned} \tag{3.2}$$

For any $q \in \mathbb{N}$, writing $\alpha = \frac{\alpha}{qb} + \frac{\alpha(q-1)}{qb}$ and using $[Fb4]$ to get

$$F_b(a_n, a_{n+q}, \alpha) \geq F_b\left(a_n, a_{n+1}, \frac{\alpha}{qb}\right) * F_b\left(a_{n+1}, a_{n+q}, \frac{(q-1)\alpha}{qb}\right).$$

Again, writing $\frac{(q-1)\alpha}{qb} = \frac{\alpha}{qb} + \frac{\alpha(q-2)}{qb}$ together with $[Fb4]$, we have

$$F_b(a_n, a_{n+q}, \alpha) \geq F_b\left(a_n, a_{n+1}, \frac{\alpha}{qb}\right) * F_b\left(a_{n+1}, a_{n+2}, \frac{\alpha}{qb^2}\right) * F_b\left(a_{n+2}, a_{n+q}, \frac{(q-2)\alpha}{qb^2}\right).$$

Continuing in the same way and using $[Fb4]$ repeatedly for $(q-2)$ more steps, we obtain

$$F_b(a_n, a_{n+q}, \alpha) \geq F_b\left(a_n, a_{n+1}, \frac{\alpha}{qb}\right) * F_b\left(a_{n+1}, a_{n+2}, \frac{\alpha}{qb^2}\right) * \ldots * F_b\left(a_{n+q-1}, a_{n+q}, \frac{\alpha}{qb^q}\right).$$

Using (3.2) and $[Fb5]$, we get

$$\begin{aligned} F_b(a_n, a_{n+q}, \alpha) \geq{}& F_b\left(a_0, a_1, \frac{\alpha}{qbk^n}\right) * F_b\left(a_0, a_1, \frac{\alpha}{q(b)^2k^{n+1}}\right) * F_b\left(a_0, a_1, \frac{\alpha}{q(b)^3k^{n+2}}\right) * \ldots * \\ & F_b\left(a_0, a_1, \frac{\alpha}{q(b)^q k^{n+q-1}}\right). \end{aligned}$$

Consequently,

$$F_b(a_n, a_{n+q}, \alpha) \geq F_b\left(a_0, a_1, \frac{\alpha}{q(bk)k^{n-1}}\right) * F_b\left(a_0, a_1, \frac{\alpha}{q(bk)^2k^{n-1}}\right) * F_b\left(a_0, a_1, \frac{\alpha}{q(bk)^3k^{n-1}}\right) * \ldots * F_b\left(a_0, a_1, \frac{\alpha}{q(bk)^qk^{n-1}}\right).$$

Since for all $n, q \in \mathbb{N}$, we have $bk < 1$, taking limit as $n \to +\infty$, we get

$$\lim_{n\to+\infty} F_b(a_n, a_{n+q}, \alpha) = 1 * 1 * \ldots * 1 = 1.$$

Hence, $\{a_n\}$ is a G-Cauchy sequence. Then, the G-completeness of Ω implies that there exists $z \in \Omega$ such that

$$\begin{aligned}
F_b(z, Sz, \alpha) &\geq F_b\left(z, a_{n+1}, \frac{\alpha}{2b}\right) * F_b\left(a_{n+1}, Sz, \frac{\alpha}{2b}\right) \\
&\geq F_b\left(z, a_{n+1}, \frac{\alpha}{2b}\right) * \Theta_{\mathcal{F}_b}\left(Sa_n, Sz, \frac{\alpha}{2b}\right) \\
&\geq F_b\left(z, a_{n+1}, \frac{\alpha}{2b}\right) * F_b\left(a_n, z, \frac{\alpha}{2bk}\right) \\
&\longrightarrow 1 \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

By Lemma 3.1, we have $z \in Sz$. Hence, z is a fixed point for S. □

Example 3.1. Let $\Omega = [0, 1]$ and $F_b(\zeta, \varrho, \alpha) = \dfrac{\alpha}{\alpha + (\zeta - \varrho)^2}$.
It is easy to verify that $(\Omega, F_b, *)$ is a G-complete FBMS with $b \geq 1$.
For $k \in (0, 1)$, define a mapping $S : \Omega \to \hat{C}_0(\Omega)$ by

$$S(\zeta) = \begin{cases} \{0\} & \text{if } \zeta = 0 \\ \{0, \frac{\sqrt{k}\zeta}{2}\} & \text{otherwise.} \end{cases}$$

In the case $\zeta = \varrho$, we have

$$\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha) = 1 = F_b(\zeta, \varrho, \alpha).$$

For $\zeta \neq \varrho$, we have the following cases:

If $\zeta = 0$ and $\varrho \in (0, 1]$, we have

$$\begin{aligned}
&\Theta_{\mathcal{F}_b}(S(0), S(\varrho), k\alpha) \\
&\quad = \min\{\inf_{a\in S(0)} \mathcal{F}_b(a, S(\varrho), k\alpha), \inf_{b\in S(\varrho)} \mathcal{F}_b(S(0), b, k\alpha)\} \\
&\quad = \min\left\{\inf_{a\in S(0)} \mathcal{F}_b\left(a, \left\{0, \frac{\sqrt{k}\varrho}{2}\right\}, k\alpha\right), \inf_{b\in S(\varrho)} \mathcal{F}_b\left(\{0\}, b, k\alpha\right)\right\} \\
&\quad = \min\left\{\inf\left\{\mathcal{F}_b\left(0, \{0, \frac{\sqrt{k}\varrho}{2}\}, k\alpha\right)\right\}, \inf\left\{\mathcal{F}_b\left(\{0\}, 0, k\alpha\right), \mathcal{F}_b\left(\{0\}, \frac{\sqrt{k}\varrho}{2}, k\alpha\right)\right\}\right\}
\end{aligned}$$

$$= \min\left\{\inf\left\{\sup\left\{\mathcal{F}_b(0,0,k\alpha), \mathcal{F}_b(0, \frac{\sqrt{k}\varrho}{2}, k\alpha)\right\}\right\}, \inf\left\{\mathcal{F}_b(0,0,k\alpha), \mathcal{F}_b(0, \frac{\sqrt{k}\varrho}{2}, k\alpha)\right\}\right\}$$
$$= \min\left\{\inf\left\{\sup\left\{1, \frac{\alpha}{\alpha + \frac{\varrho^2}{4}}\right\}\right\}, \inf\left\{1, \frac{\alpha}{\alpha + \frac{\varrho^2}{4}}\right\}\right\}$$
$$= \min\left\{\inf\{1\}, \frac{\alpha}{\alpha + \frac{\varrho^2}{4}}\right\}$$
$$= \min\{1, \frac{\alpha}{\alpha + \frac{\varrho^2}{4}}\} = \frac{\alpha}{\alpha + \frac{\varrho^2}{4}}.$$

It follows that $\Theta_{\mathcal{F}_b}(S(0), S(\varrho), k\alpha) > F_b(0, \varrho, \alpha) = \dfrac{\alpha}{\alpha + \varrho^2}$.
If ζ and $\varrho \in (0, 1]$, an easy calculation with either possibility of supremum and infimum, yield that

$$\Theta_{\mathcal{F}_b}(S(\zeta), S(\varrho), k\alpha) = \min\left\{\sup\left\{\frac{\alpha}{\alpha + \frac{\zeta^2}{4}}, \frac{\alpha}{\alpha + \frac{(\zeta-\varrho)^2}{4}}\right\}, \sup\left\{\frac{\alpha}{\alpha + \frac{\varrho^2}{4}}, \frac{\alpha}{\alpha + \frac{(\zeta-\varrho)^2}{4}}\right\}\right\}$$
$$\geq \frac{\alpha}{\alpha + \frac{(\zeta-\varrho)^2}{4}} > \frac{\alpha}{\alpha + (\zeta - \varrho)^2} = F_b(\zeta, \varrho, \alpha).$$

Thus, for all cases, we have

$$\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha) \geq F_b(\zeta, \varrho, \alpha).$$

Hence, all the conditions of Theorem 3.1 are satisfied and 0 is a fixed point of S.

Theorem 3.2. *Let $(\Omega, F_b, *)$ be a G-complete FBMS with $b \geqslant 1$ and $\Theta_{\mathcal{F}_b}$ be a Hausdorff FBMS. Let $S : \Omega \to \hat{C}_0(\Omega)$ be a multivalued mapping which satisfies*

$$\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha) \geq \min\left\{\frac{\mathcal{F}_b(\varrho, S\varrho, \alpha)\left[1 + \mathcal{F}_b(\zeta, S\varrho, \alpha)\right]}{1 + F_b(\zeta, \varrho, \alpha)}, F_b(\zeta, \varrho, \alpha)\right\} \tag{3.3}$$

for all $\zeta, \varrho \in \Omega$, where $bk < 1$, then S has a fixed point.

Proof. In the same way as in Theorem 3.1 for $a_0 \in \Omega$, we choose a sequence $\{a_n\}$ in Ω as follows: Let $a_1 \in \Omega$ such that $a_1 \in S a_0$. By Lemma 2.3, we can choose $a_2 \in S a_1$ such that

$$F_b(a_1, a_2, \alpha) \geqslant \Theta_{\mathcal{F}_b}(S a_0, S a_1, \alpha) \quad \text{for all } \alpha > 0.$$

By induction, we have $a_{n+1} \in S a_n$ satisfying

$$F_b(a_n, a_{n+1}, \alpha) \geqslant \Theta_{\mathcal{F}_b}(S a_{n-1}, S a_n, \alpha) \quad \text{for all } n \in \mathbb{N}.$$

Now, by (3.3) together with Lemma 3.3, we have

$$F_b(a_n, a_{n+1}, \alpha) \geq \Theta_{\mathcal{F}_b}(S a_{n-1}, S a_n, \alpha)$$
$$\geq \min\left\{\frac{\mathcal{F}_b\left(a_n, S a_n, \frac{\alpha}{k}\right)\left[1 + \mathcal{F}_b\left(a_{n-1}, S a_{n-1}, \frac{\alpha}{k}\right)\right]}{1 + F_b\left(a_{n-1}, a_n, \frac{\alpha}{k}\right)}, F_b\left(a_{n-1}, a_n, \frac{\alpha}{k}\right)\right\}$$

$$\geq \min\left\{\frac{F_b\left(a_n, a_{n+1}, \frac{\alpha}{k}\right)\left[1 + F_b\left(a_{n-1}, a_n, \frac{\alpha}{k}\right)\right]}{1 + F_b\left(a_{n-1}, a_n, \frac{\alpha}{k}\right)}, F_b\left(a_{n-1}, a_n, \frac{\alpha}{k}\right)\right\}$$
$$\geq \min\left\{F_b\left(a_n, a_{n+1}, \frac{\alpha}{k}\right), F_b\left(a_{n-1}, a_n, \frac{\alpha}{k}\right)\right\}. \tag{3.4}$$

If

$$\min\left\{F_b\left(a_n, a_{n+1}, \frac{\alpha}{k}\right), F_b\left(a_{n-1}, a_n, \frac{\alpha}{k}\right)\right\} = F_b\left(a_n, a_{n+1}, \frac{\alpha}{k}\right),$$

then (3.4) implies

$$F_b(a_n, a_{n+1}, \alpha) \geq F_b(a_n, a_{n+1}, \frac{\alpha}{k}).$$

Then nothing to prove by Lemma 3.2. If

$$\min\left\{F_b(a_n, a_{n+1}, \frac{\alpha}{k}), F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\} = F_b(a_{n-1}, a_n, \frac{\alpha}{k}),$$

then from (3.4) we have

$$F_b(a_n, a_{n+1}, \alpha) \geq F_b(a_{n-1}, a_n, \frac{\alpha}{k}) \geqslant \ldots \geqslant F_b(a_0, a_1, \frac{\alpha}{k^n}).$$

By adopting the same procedure as in Theorem 3.1 after inequality (3.2), we can complete the proof. □

Remark 3.1. By taking $b = 1$ in Theorem 3.2, Theorem 2.1 of [25] can be obtained.

Theorem 3.3. *Let $(\Omega, F_b, *)$ be a G-complete FBMS (with $b \geqslant 1$) and $\Theta_{\mathcal{F}_b}$ be a Hausdorff FBMS. Let $S : \Omega \to \hat{C}_0(\Omega)$ be a multivalued map which satisfies*

$$\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha) \geq \min\left\{\frac{\mathcal{F}_b(\varrho, S\varrho, \alpha)\left[1 + \mathcal{F}_b(\zeta, S\zeta, \alpha) + \mathcal{F}_b(\varrho, S\zeta, \alpha)\right]}{2 + F_b(\zeta, \varrho, \alpha)}, F_b(\zeta, \varrho, \alpha)\right\} \tag{3.5}$$

for all $\zeta, \varrho \in \Omega$, where $bk < 1$, then S has a fixed point.

Proof. Starting same way as in Theorem 3.1, we have

$$F_b(a_1, a_2, \alpha) \geqslant \Theta_{\mathcal{F}_b}(Sa_0, Sa_1, \alpha) \quad \text{for all } \alpha > 0.$$

By induction, we have $a_{n+1} \in Sa_n$ satisfying

$$F_b(a_n, a_{n+1}, \alpha) \geqslant \Theta_{\mathcal{F}_b}(Sa_{n-1}, Sa_n, \alpha) \quad \text{for all } n \in \mathbb{N}.$$

Now, by (3.5) together with Lemma 3.3, we have

$$F_b(a_n, a_{n+1}, \alpha) \geq \Theta_{\mathcal{F}_b}(Sa_{n-1}, Sa_n, \alpha)$$
$$\geq \min\left\{\frac{\mathcal{F}_b(a_n, Sa_n, \frac{\alpha}{k})\left[1 + \mathcal{F}_b(a_{n-1}, Sa_{n-1}, \frac{\alpha}{k}) + \mathcal{F}_b(a_n, Sa_{n-1}, \frac{\alpha}{k})\right]}{2 + F_b(a_{n-1}, a_n, \frac{\alpha}{k})}, F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\}$$
$$\geq \min\left\{\frac{F_b(a_n, a_{n+1}, \frac{\alpha}{k})\left[1 + F_b(a_{n-1}, a_n, \frac{\alpha}{k}) + F_b(a_n, a_n, \frac{\alpha}{k})\right]}{2 + F_b(a_{n-1}, a_n, \frac{\alpha}{k})}, F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\}$$

$$\geq \min\left\{\frac{F_b(a_n, a_{n+1}, \frac{\alpha}{k})\left[1 + F_b(a_{n-1}, a_n, \frac{\alpha}{k}) + 1\right]}{2 + F_b(a_{n-1}, a_n, \frac{\alpha}{k})}, F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\}$$

$$\geq \min\left\{\frac{F_b(a_n, a_{n+1}, \frac{\alpha}{k})\left[2 + F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right]}{2 + F_b(a_{n-1}, a_n, \frac{\alpha}{k})}, F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\}$$

$$\geq \min\left\{F_b(a_n, a_{n+1}, \frac{\alpha}{k}), F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\}. \tag{3.6}$$

If

$$\min\left\{F_b(a_n, a_{n+1}, \frac{\alpha}{k}), F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\} = F_b(a_n, a_{n+1}, \frac{\alpha}{k}),$$

then (3.6) implies

$$F_b(a_n, a_{n+1}, \alpha) \geq F_b(a_n, a_{n+1}, \frac{\alpha}{k}).$$

Then nothing to prove by Lemma 3.2.
If

$$\min\left\{F_b(a_n, a_{n+1}, \frac{\alpha}{k}), F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\} = F_b(a_{n-1}, a_n, \frac{\alpha}{k}),$$

then from (3.6), we have

$$F_b(a_n, a_{n+1}, \alpha) \geq F_b(a_{n-1}, a_n, \frac{\alpha}{k}) \geqslant \ldots \geqslant F_b(a_0, a_1, \frac{\alpha}{k^n}).$$

By adopting the same procedure as in Theorem 3.1 after inequality (3.2), we can complete the proof. □

Next, a corollary of Theorem 3.3 is given.

Corollary 3.1. Let $(\Omega, F, *)$ be a G-complete FMS and $\Theta_{\mathcal{F}}$ be a Hausdorff FMS. Let $S : \Omega \to \hat{C}_0(\Omega)$ be a multivalued mapping satisfying

$$\Theta_{\mathcal{F}}(S\zeta, S\varrho, k\alpha) \geq \min\left\{\frac{\mathcal{F}(\varrho, S\varrho, \alpha)\left[1 + \mathcal{F}(\zeta, S\zeta, \alpha) + \mathcal{F}(\varrho, S\zeta, \alpha)\right]}{2 + F(\zeta, \varrho, \alpha)}, F(\zeta, \varrho, \alpha)\right\}$$

for all $\zeta, \varrho \in \Omega$, where $0 < k < 1$, then S has a fixed point.

Proof. Taking $b = 1$ in Theorem 3.3, one can complete the proof. □

Theorem 3.4. *Let $(\Omega, F_b, *)$ be a G-complete FBMS with $b \geqslant 1$ and $\Theta_{\mathcal{F}_b}$ be a Hausdorff FBMS. Let $S : \Omega \to \hat{C}_0(\Omega)$ be a multivalued map which satisfies*

$$\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha) \geq \min\left\{\frac{\mathcal{F}_b(\zeta, S\zeta, \alpha)\left[1 + \mathcal{F}_b(\varrho, S\varrho, \alpha)\right]}{1 + \mathcal{F}_b(S\zeta, S\varrho, \alpha)}, \frac{\mathcal{F}_b(\zeta, S\varrho, \alpha)\left[1 + \mathcal{F}_b(\zeta, S\zeta, \alpha)\right]}{1 + F_b(\zeta, \varrho, \alpha)}, \right.$$
$$\left. \frac{\mathcal{F}_b(\zeta, S\zeta, \alpha)\left[2 + \mathcal{F}_b(\zeta, S\varrho, \alpha)\right]}{1 + F_b(\zeta, S\varrho, \alpha) + \mathcal{F}_b(\varrho, S\zeta, \alpha)}, F_b(\zeta, \varrho, \alpha)\right\} \tag{3.7}$$

for all $\zeta, \varrho \in \Omega$, where $bk < 1$, then S has a fixed point.

Proof. For $a_0 \in \Omega$, we choose a sequence $\{x_n\}$ in Ω as follows: Let $a_1 \in \Omega$ such that $a_1 \in Sa_0$. By using Lemma 2.3, we can choose $a_2 \in Sa_1$ such that

$$F_b(a_1, a_2, \alpha) \geqslant \Theta_{\mathcal{F}_b}(Sa_0, Sa_1, \alpha) \quad \text{for all } \alpha > 0.$$

By induction, we have $a_{n+1} \in Sa_n$ satisfying

$$F_b(a_n, a_{n+1}, \alpha) \geqslant \Theta_{\mathcal{F}_b}(Sa_{n-1}, Sa_n, \alpha) \quad \text{for all } n \in \mathbb{N}.$$

Now, by (3.7) together with 3.3, we have

$$\begin{aligned}
F_b(a_n, a_{n+1}, \alpha) &\geq \Theta_{\mathcal{F}_b}(Sa_{n-1}, Sa_n, \alpha)\\
&\geq \min\Bigg\{\frac{\mathcal{F}_b(a_{n-1}, Sa_{n-1}, \frac{\alpha}{k})\left[1 + \mathcal{F}_b(a_n, S_{a_n}, \frac{\alpha}{k})\right]}{1 + \mathcal{F}_b(Sa_{n-1}, Sa_n, \frac{\alpha}{k})}, \frac{\mathcal{F}_b(a_n, Sa_n, \frac{\alpha}{k})\left[1 + \mathcal{F}_b(a_{n-1}, Sa_{n-1}, \frac{\alpha}{k})\right]}{1 + F_b(a_{n-1}, a_n, \frac{\alpha}{k})},\\
&\qquad \frac{\mathcal{F}_b(a_{n-1}, Sa_{n-1}, \frac{\alpha}{k})\left[2 + \mathcal{F}_b(a_{n-1}, Sa_n, \frac{\alpha}{k})\right]}{1 + \mathcal{F}_b(a_{n-1}, Sa_n, \frac{\alpha}{k}) + \mathcal{F}_b(a_n, Sa_{n-1}, \frac{\alpha}{k})}, F_b(a_{n-1}, a_n, \frac{\alpha}{k})\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\geq \min\Bigg\{\frac{F_b(a_{n-1}, a_n, \frac{\alpha}{k})\left[1 + F_b(a_n, a_{n+1}, \frac{\alpha}{k})\right]}{1 + F_b(a_n, a_{n+1}, \frac{\alpha}{k})}, \frac{F_b(a_n, a_{n+1}, \frac{\alpha}{k})\left[1 + F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right]}{1 + F_b(a_{n-1}, a_n, \frac{\alpha}{k})},\\
&\qquad \frac{F_b(a_{n-1}, a_n, \frac{\alpha}{k})\left[2 + F_b(a_{n-1}, a_{n+1}, \frac{\alpha}{k})\right]}{1 + F_b(a_{n-1}, a_{n+1}, \frac{\alpha}{k}) + F_b(a_n, a_n, \frac{\alpha}{k})}, F_b(a_{n-1}, a_n, \frac{\alpha}{k})\Bigg\}
\end{aligned}$$

$$F_b(a_n, a_{n+1}, \alpha) \geq \min\left\{F_b(a_n, a_{n+1}, \frac{\alpha}{k}), F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\}. \tag{3.8}$$

If

$$\min\left\{F_b(a_n, a_{n+1}, \frac{\alpha}{k}), F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\} = F_b(a_n, a_{n+1}, \frac{\alpha}{k}),$$

so (3.8) implies

$$F_b(a_n, a_{n+1}, \alpha) \geq F_b(a_n, a_{n+1}, \frac{\alpha}{k}).$$

Then nothing to prove by Lemma 3.2. While, if

$$\min\left\{F_b(a_n, a_{n+1}, \frac{\alpha}{k}), F_b(a_{n-1}, a_n, \frac{\alpha}{k})\right\} = F_b(a_{n-1}, a_n, \frac{\alpha}{k}),$$

then from (3.6) we have

$$F_b(a_n, a_{n+1}, \alpha) \geq F_b(a_{n-1}, a_n, \frac{\alpha}{k}) \geqslant \ldots \geqslant F_b(a_0, a_1, \frac{\alpha}{k^n}).$$

By adopting the same procedure as in Theorem 3.1 after inequality (3.2) we can complete the proof. □

The same result in fuzzy metric spaces is stated as follows.

Corollary 3.2. Let $(\Omega, F, *)$ be a G-complete FMS and $\Theta_{\mathcal{F}}$ be a Hausdorff FMS. Let $S : \Omega \to \hat{C}_0(\Omega)$ be a multivalued mapping satisfying

$$\Theta_{\mathcal{F}}(S\zeta, S\varrho, k\alpha) \geq \min\left\{\frac{\mathcal{F}(\zeta, S\zeta, \alpha)\left[1 + \mathcal{F}(\varrho, S\varrho, \alpha)\right]}{1 + \mathcal{F}(S\zeta, S\varrho, \alpha)}, \frac{\mathcal{F}(\varrho, S\varrho, \alpha)\left[1 + \mathcal{F}(\zeta, S_{\zeta}, \alpha)\right]}{1 + F(\zeta, \varrho, \alpha)}, \frac{\mathcal{F}(\zeta, S\zeta, \alpha)\left[2 + \mathcal{F}(\zeta, S\varrho, \alpha)\right]}{1 + \mathcal{F}(\zeta, S\varrho, \alpha) + \mathcal{F}(\varrho, S\zeta, \alpha)}, F(\zeta, \varrho, \alpha)\right\}$$

for all $\zeta, \varrho \in \Omega$, then S has a fixed point.

Proof. Taking $b = 1$ in Theorem 3.4, one can complete the proof. □

4. Consequences

This section is about the construction of some fixed point results involving integral inequalities as consequences of our results. Define a function $\tau\colon [0, +\infty) \to [0, +\infty)$ by

$$\tau(\alpha) = \int_0^{\alpha} \psi(\alpha) d\alpha \quad \text{for all } \alpha > 0, \tag{4.1}$$

where $\tau(\alpha)$ is a non-decreasing and continuous function. Moreover, $\psi(\alpha) > 0$ for $\alpha > 0$ and $\psi(\alpha) = 0$ if and only if $\alpha = 0$.

Theorem 4.1. *Let $(\Omega, F_b, *)$ be a complete fuzzy b-metric space and $\Theta_{\mathcal{F}_b}$ be a Hausdorff fuzzy b-metric space. Let $S : \Omega \to \hat{C}_0(\Omega)$ be a multivalued mapping satisfying*

$$\int_0^{\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha)} \psi(\alpha) d\alpha \geq \int_0^{F_b(\zeta, \varrho, \alpha)} \psi(\alpha) d\alpha \tag{4.2}$$

for all $\zeta, \varrho \in \Omega$, where $bk < 1$, then S has a fixed point.

Proof. Taking (4.1) in account, (4.2) implies that

$$\tau(\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha)) \geqslant \tau(F_b(\zeta, \varrho, \alpha)).$$

Since τ is continuous and non-decreasing, we have

$$\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha) \geqslant F_b(\zeta, \varrho, \alpha).$$

The rest of the proof follows immediately from Theorem 3.1. □

A more general form of Theorem 4.1 can be stated as an immediate consequence of Theorem 3.2.

Theorem 4.2. *Let $(\Omega, F_b, *)$ be a complete FBMS and $\Theta_{\mathcal{F}_b}$ be a Hausdorff FBMS. Let $S : \Omega \to \hat{C}_0(\Omega)$ be a multivalued mapping satisfying*

$$\int_0^{\Theta_{\mathcal{F}_b}(S\zeta, S\varrho, k\alpha)} \psi(\alpha) d\alpha \geq \int_0^{\beta(\zeta, \varrho, \alpha)} \psi(\alpha) d\alpha, \tag{4.3}$$

where

$$\beta(\zeta,\varrho,\alpha)=\min\left\{\frac{\mathcal{F}_b(\varrho,S\varrho,\alpha)\,[1+\mathcal{F}_b(\zeta,S\varrho,\alpha)]}{1+F_b(\zeta,\varrho,\alpha)},F_b(\zeta,\varrho,\alpha)\right\}$$

for all $\zeta,\varrho\in\Omega$*, where* $bk<1$*, then* S *has a fixed point.*

Proof. Taking (4.1) in account, (4.3) implies that

$$\tau(\Theta_{\mathcal{F}_b}(S\zeta,S\varrho,k\alpha))\geqslant\tau(\beta(\zeta,\varrho,\alpha).$$

Since τ is continuous and non-decreasing, we have

$$\Theta_{\mathcal{F}_b}(S\zeta,S\varrho,k\alpha)\geqslant\beta(\zeta,\varrho,\alpha).$$

The rest of the proof follows immediately from Theorem 3.2. □

Remark 4.1. By taking $b=1$ in Theorem 4.2, Theorem 3.1 of [25] can be obtained.

Theorem 4.3. *Let* $(\Omega,F_b,*)$ *be a complete FBMS and* $\Theta_{\mathcal{F}_b}$ *be a Hausdorff FBMS. Let* $S:\Omega\to\hat{C}_0(\Omega)$ *be a multivalued mapping satisfying*

$$\int_0^{\Theta_{\mathcal{F}_b}(S\zeta,S\varrho,k\alpha)}\psi(\alpha)d\alpha\geq\int_0^{\beta(\zeta,\varrho,\alpha)}\psi(\alpha)d\alpha,$$

where

$$\beta(\zeta,\varrho,\alpha)=\min\left\{\frac{\mathcal{F}_b(\varrho,S\varrho,\alpha)\,[1+\mathcal{F}_b(\zeta,S\zeta,\alpha)+\mathcal{F}_b(\varrho,S\zeta,\alpha)]}{2+F_b(\zeta,\varrho,\alpha)},F_b(\zeta,\varrho,\alpha)\right\}$$

for all $\zeta,\varrho\in\Omega$*, where* $bk<1$*, then* S *has a fixed point.*

Note that if $\beta(\zeta,\varrho,\alpha)=F_b(\zeta,\varrho,\alpha)$, then the above result follows from Theorem 4.1. Similar results on integral inequalities can be obtained as a consequence of Theorem 3.4.

Theorem 4.4. *Let* $(\Omega,F_b,*)$ *be a complete FBMS and* $\Theta_{\mathcal{F}_b}$ *be a Hausdorff FBMS. Let* $S:\Omega\to\hat{C}_0(\Omega)$ *be a multivalued mapping satisfying*

$$\int_0^{\Theta_{\mathcal{F}_b}(S\zeta,S\varrho,k\alpha)}\psi(\alpha)d\alpha\geq\int_0^{\gamma(\zeta,\varrho,\alpha)}\psi(\alpha)d\alpha,$$

where

$$\gamma(\zeta,\varrho,\alpha)=\min\Bigg\{\frac{\mathcal{F}_b(\zeta,S\zeta,\alpha)\,[1+\mathcal{F}_b(\varrho,S\varrho,\alpha)]}{1+\mathcal{F}_b(S\zeta,S\varrho,\alpha)},\frac{\mathcal{F}_b(\varrho,S\varrho,\alpha)\,[1+\mathcal{F}_b(\zeta,S\zeta,\alpha)]}{1+\mathcal{F}_b(\zeta,\varrho,\alpha)},$$
$$\frac{\mathcal{F}_b(\zeta,S\zeta,\alpha)\,[2+\mathcal{F}_b(\zeta,S\varrho,\alpha)]}{1+\mathcal{F}_b(\zeta,S\varrho,\alpha)+\mathcal{F}_b(\varrho,S\zeta,\alpha)},F_b(\zeta,\varrho,\alpha)\Bigg\}$$

for all $\zeta,\varrho\in\Omega$*, where* $bk<1$*, then* S *has a fixed point.*

5. An application

Nonlinear integral equations arise in a variety of fields of physical science, engineering, biology, and applied mathematics [31, 32]. This theory in abstract spaces is a rapidly growing field with lot of applications in analysis as well as other branches of mathematics [33].

Fixed point theory is a valuable tool for the existence of a solution of different kinds of integral as well as differential inclusions, such as [33–35]. Many authors provided a solution of different integral inclusions in this context, for instance see [30,36–40]. In this section, a Volterra-type integral inclusion as an application of Theorem 3.1 is studied.

Consider $\Omega = C([0, 1], \mathbb{R})$ as the space of all continuous functions defined on $[0, 1]$ and define the G-complete fuzzy b-metric on Ω by

$$F_b(\xi, \varrho, \alpha) = e^{-\frac{\sup_{\varepsilon\in[0,1]} |\xi(\varepsilon) - \varrho(\varepsilon)|^2}{\alpha}}$$

for all $\alpha > 0$ and $\xi, \varrho \in \Omega$.

Consider the integral inclusion:

$$\xi(\varepsilon) \in \int_0^u G(\varepsilon, \sigma, \xi(\sigma))d\sigma + h(\varepsilon) \quad \text{for all } \ \varepsilon, \sigma \in [0, 1] \text{ and } h, \xi \in C([0, 1], \tag{5.1}$$

where $G\colon [0, 1] \times [0, 1] \times \mathbb{R} \to P_{cv}(\mathbb{R})$ is a multivalued continuous function.

For the above integral inclusion, we define a multivalued operator $S : \Omega \to \hat{C}_0(\Omega)$ by

$$S(\xi(\varepsilon)) = \left\{ w \in \Omega : w \in \int_0^u G(\varepsilon, \sigma, \xi(\sigma))d\sigma + h(\varepsilon), \quad \varepsilon \in [0, 1] \right\}.$$

The next result proves the existence of a solution of the integral inclusion (5.1).

Theorem 5.1. *Let $S : \Omega \to \hat{C}_0(\Omega)$ be the multivalued integral operator given by*

$$S(\xi(\varepsilon)) = \left\{ w \in \Omega : w \in \int_0^\varepsilon G(\varepsilon, \sigma, \xi(\sigma))d\sigma + h(\varepsilon), \quad \varepsilon \in [0, 1] \right\}.$$

Suppose the following conditions are satisfied:

1) *$G\colon [0, 1]\times[0, 1]\times\mathbb{R} \to P_{cv}(\mathbb{R})$ is such that $G(\varepsilon, \sigma, \xi(\sigma))$ is lower semi-continuous in $[0, 1]\times[0, 1]$;*
2) *For all $\varepsilon, \sigma \in [0, 1]$, $f(\varepsilon, \sigma) \in \Omega$ and for all $\xi, \varrho \in \Omega$, we have*

$$|G(\varepsilon, \sigma, \xi(\sigma)) - G(\varepsilon, \sigma, \varrho(\sigma))|^2 \leq f^2(\varepsilon, \sigma)|\xi(\sigma) - \varrho(\sigma)|^2,$$

   where $f\colon [0, 1] \to [0, +\infty)$ is continuous;
3) *There exists $0 < k < 1$ such that*

$$\sup_{\varepsilon\in[0,1]} \int_0^\varepsilon f^2(\varepsilon, \sigma)d\sigma \leq k.$$

Then the integral inclusion (5.1) has the solution in Ω.

Proof. For $G\colon [0,1]\times[0,1]\times\mathbb{R} \to P_{cv}(\mathbb{R})$, it follows from Michael's selection theorem that there exists a continuous operator

$$G_i\colon [0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$$

such that $G_i(\varepsilon,\sigma,\xi(\sigma)) \in G(\varepsilon,\sigma,\xi(\sigma))$ for all $\varepsilon,\sigma\in[0,1]$. It follows that

$$\xi(\varepsilon) \in \int_0^{\varepsilon} G_i(\varepsilon,\sigma,\xi(\sigma))d\sigma + h(\varepsilon)) \in S(\xi(\varepsilon))$$

hence $S(\xi(\varepsilon)) \neq \emptyset$ and closed. Moreover, since $h(\varepsilon)$ is continuous on $[0,1]$, and G is continuous, their ranges are bounded. This means that $S(\xi(\varepsilon))$ is bounded and $S(\xi(\varepsilon)) \in \hat{C}_0(\Omega)$. For $q, r\in\Omega$, there exist $q(\varepsilon)\in S(\xi(\varepsilon))$ and $r(\varepsilon)\in S(\varrho(\varepsilon))$ such that

$$q(\xi(\varepsilon)) = \left\{ w\in\Omega : w \in \int_0^{\varepsilon} G_i(\varepsilon,\sigma,\xi(\sigma))d\sigma + h(\sigma), \quad \varepsilon\in[0,1] \right\}$$

and

$$r(\varrho(u)) = \left\{ w\in\Omega : w \in \int_0^{\varepsilon} G_i(\varepsilon,\sigma,\varrho(\sigma))d\sigma + h(\varepsilon), \quad \varepsilon\in[0,1] \right\}.$$

It follows from item 5.1 that

$$|G_i(\varepsilon,\sigma,\xi(\sigma)) - G_i(\varepsilon,\sigma,\varrho(\sigma))|^2 \leq f^2(\varepsilon,\sigma)|\xi(\sigma)-\varrho(\sigma)|^2.$$

Now,

$$\begin{aligned}
e^{-\frac{\sup\limits_{t\in[0,1]}|q(\varepsilon)-r(\varepsilon))|^2}{k\alpha}} &\geq e^{-\frac{\sup\limits_{\varepsilon\in[0,1]}\int_0^{\varepsilon}|G_i(\varepsilon,\sigma,\xi(\sigma)) - G_i(\varepsilon,\sigma,\varrho(\sigma))|^2 d\sigma}{k\alpha}}\\
&\geq e^{-\frac{\sup\limits_{\varepsilon\in[0,1]}\int_0^{\varepsilon} f^2(\varepsilon,\sigma)|\xi(\sigma)-\varrho(\sigma)|^2 d\sigma}{k\alpha}}\\
&\geq e^{-\frac{|\xi(\sigma)-\varrho(\sigma)|^2 \sup\limits_{\varepsilon\in[0,1]}\int_0^{\varepsilon} f^2(\varepsilon,\sigma)d\sigma}{k\alpha}}\\
&\geq e^{-\frac{k|\xi(\sigma)-\varrho(\sigma)|^2}{k\alpha}}\\
&= e^{-\frac{|\xi(\sigma)-\varrho(\sigma)|^2}{\alpha}}\\
&\geq e^{-\frac{\sup\limits_{\sigma\in[0,1]}|\xi(\sigma)-\varrho(\sigma)|^2}{\alpha}}\\
&= F_b(\xi,\varrho,\alpha).
\end{aligned}$$

So, we have

$$F_b(q,r,k\alpha) \geq F_b(\xi,\varrho,\alpha).$$

By interchanging the roll of ξ and ϱ, we reach to

$$\Theta_{\mathcal{F}_b}(S\xi, S\varrho, k\alpha) \geq F_b(\xi, \varrho, \alpha).$$

Hence, S has a fixed point in Ω, which is a solution of the integral inclusion (5.1). □

6. Conclusions

In this article we proved certain fixed point results for Hausdorff fuzzy b-metric spaces. The main results are validated by an example. Theorem 3.2 generalizes the result of [25]. These results extend the theory of fixed points for multivalued mappings in a more general class of fuzzy b-metric spaces. For instance, some fixed point results can be obtained by taking $b = 1$ (corresponding to G-complete FMSs). An application for the existence of a solution for a Volterra type integral inclusion is also provided.

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Conflict of interest

The authors declare no conflict of interest.

References

1. B. C. Dhage, Condensing mappings and applications to existence theorems for common solution of differential equations, *Bull. Korean Math. Soc.,* **36** (1999), 565–578.
2. H. Pathak, S. Mishra, A. Kalinde, Common fixed point theorems with applications to nonlinear integral equations, *Demonstr. Math.,* **32** (1999), 547–564. https://doi.org/10.1515/dema-1999-0310
3. S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrals, *Fund. Math.,* **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
4. I. A. Bakhtin, The contraction mapping principle in quasi metric spaces, *Funct. Anal. Unianowsk Gos. Ped. Inst.*, **30** (1989), 26–37.
5. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. U. Ostraviensis,* **1** (1993), 5–11.
6. H. Aydi, M. Aslam, D. Sagheer, S. Batul, R. Ali, E. Ameer, Kannan-type contractions on new extended b-metric spaces, *J. Funct. Space.,* **2021** (2021). https://doi.org/10.1155/2021/7613684
7. T. Kamran, M. Postolache, A. Ghiura, S. Batul, R. Ali, The Banach contraction principle in C^*-algebra-valued b-metric spaces with application, *Fix. Point Theor. Appl.,* **2016** (2016), 10. https://doi.org/10.1186/s13663-015-0486-z

8. D. Shehwar, T. Kamran, C^*-valued G-contractions and fixed points, *J. Inequal. Appl.,* **304** (2015), 304. https://doi.org/10.1186/s13660-015-0827-9

9. D. Shehwar, S. Batul, T. Kamran, A. Ghura, Caristi's fixed point theorem on C*-algebra valued metric spaces, *J. Nonlinear Sci.,* **9** (2016), 584–588. https://doi.org/10.22436/jnsa.009.02.22

10. L. A. Zadeh, Fuzzy sets, *Inf. Control,* **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

11. I. Kramosil, J. Michálek, Fuzzy metric and statistical metric spaces, *Kybernetica,* **11** (1975), 326–334.

12. M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Set. Syst.,* **27** (1988), 385–389. https://doi.org/10.1016/0165-0114(88)90064-4

13. A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Set. Syst.,* **64** (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7

14. K. Javed, H. Aydi, F. Uddin, M. Arshad, On orthogonal partial b-metric spaces with an application, *J. Math.,* **2021** (2021). https://doi.org/10.1155/2021/6692063

15. F. Mehmood, R. Ali, C. Ionescu, T. Kamran, Extended fuzzy *b*-metric Spaces, *J. Math. Anal.,* **8** (2017), 124–131.

16. D. Mehit, A Banach contraction theorem in fuzzy metric spaces, *Fuzzy Set. Syst.,* **144** (2004), 431–439. https://doi.org/10.1016/S0165-0114(03)00305-1

17. S. N. Mishra, S. N. Sharma, S. L. Singh, Common fixed point of maps on fuzzy metric spaces, *Int. J. Math. Sci.,* **17** (1994), 253–258. https://doi.org/10.1155/S0161171294000372

18. A. F. Roldán-López-de-Hierro, E. Karapinar, S. Manro, Some new fixed point theorems in fuzzy metric space, *J. Intell. Fuzzy Syst.,* **27** (2014), 2257–2264. https://doi.org/10.3233/IFS-141189

19. K. Javed, F. Uddin, H. Aydi, A. Mukheimer, M. Arshad, Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application, *J. Math.,* **2021** (2021). https://doi.org/10.1155/2021/6663707

20. C. Vetro, Fixed points in a weak non-Archemedean fuzzy metric spaces, *Fuzzy Set. Syst.,* **162** (2011), 84–90. https://doi.org/10.1016/j.fss.2010.09.018

21. S. Nădăban, Fuzzy *b*-metric Spaces, *Int. J. Comput. Commun.,* **11** (2016), 273–281. https://doi.org/10.15837/ijccc.2016.2.2443

22. D. Rakić, A. Mukheimer, T. Dosenovic, Z. D. Mitrović, S. Radenović, On some new fixed point results in fuzzy b-metric spaces, *J. Inequal. Appl.,* **2020** (2020). https://doi.org/10.1186/s13660-020-02371-3

23. D. Rakić, T. Dosenović, Z. Mitrović, M. De la Sen, Some fixed point theorems of Cirić type in fuzzy metric spaces, *Mathematics,* **8** (2020), 297. https://doi.org/10.3390/math8020297

24. J. Rodriguez-Lopez, S. Romaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Set. Syst.,* **147** (2004), 273–283. https://doi.org/10.1016/j.fss.2003.09.007

25. A. Shahzad, A. Shoaib, Q. Mahmood, Fixed point results for the multivalued mapping in Hausdorff fuzzy metric space, *J. Fix. Point Theor. A.,* **2017** (2017).

26. B. Schweizer, A. Sklar, Statistical metric spaces, *Pac. J. Math.,* **10** (1960), 314–334. https://doi.org/10.2140/pjm.1960.10.313
27. N. Hussain, P. Salimi, V. Parvaneh, Fixed point results for various contractions in parametric and fuzzy b-metric spaces, *J. Nonlinear Sci. Appl.,* **8** (2015), 719–739. https://doi.org/10.22436/jnsa.008.05.24
28. Q. Zheyong, H. Shihuang, Coupled fixed points for multivalued mappings in fuzzy metric spaces, *J. Fix. Point Theor. A.,* **2013** (2013), 162. https://doi.org/10.1186/1687-1812-2013-162
29. V. Gupta, N. Mani, A. Saini, *Fixed point theorems and its applications in fuzzy metric spaces,* Proceedings of the conference AEMDS-2013, 2013. https://doi.org/10.1186/1687-1812-2013-133
30. N. Hussain, M. A. Taoudi, Fixed point theorems for multivalued mappings in ordered Banach spaces with application to integral inclusions, *Fix. Point Theor. A.*, **2016** (2016), 65. https://doi.org/10.1186/s13663-016-0555-y
31. P. Diamond, Theory and applications of fuzzy Volterra integral equations, *IEEE T. Fuzzy Syst.,* **10** (2002), 97–102. https://doi.org/10.1109/91.983284
32. A. Jerri, *Introduction to integral equations with applications,* John Wiley & Sons, 1999.
33. D. J. Guo, V. Lakshmikantham, X. Z. Liu, *Nonlinear integral equations in abstract spaces,* Kluwer Academic Publishers, Dordrecht, 1996.
34. L. Lishan, C. Wu, F. Guo, Existence theorems of global solutions of initial value problems for nonlinear integro-differential equations of mixed type in Banach spaces and applications, *Comput. Math. Appl.,* **47** (2004), 13–22. https://doi.org/10.1016/S0898-1221(04)90002-8
35. Z. Peiguo, X. Hao, Existence and uniqueness of solutions for a class of nonlinear integro-differential equations on unbounded domains in Banach spaces, *Adv. Differ. Equ.,* **2018** (2018), 247. https://doi.org/10.1186/s13662-018-1681-0
36. M. U. Ali, T. Kamran, M. Postolache, Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem, *Nonlinear Anal.-Model.,* **22** (2017), 17–30. https://doi.org/10.15388/NA.2017.1.2
37. S. S. Alshoraify, A. Shoaib, M. Arshad, New types of F-Contraction for multivalued mappings and related fixed point results in abstract spaces, *J. Funct. Space.,* **2019** (2019). https://doi.org/10.1155/2019/1812461
38. M. Anwar, R. Ali, D. Sagheer, N. Hussain, Wardowski type α-F-contractive approach for nonself multivalued mappings, *U. P. B. Sci. Bull.,* **82** (2020), 69–78.
39. M. S. Ashraf, R. Ali, N. Hussain, New fuzzy fixed point results in generalized fuzzy metric spaces with application to integral equations, *IEEE Access*, **8** (2020), 91653–91660. https://doi.org/10.1109/ACCESS.2020.2994130
40. F. Mehmood, R. Ali, N. Hussain, Contractions in fuzzy rectangular b-metric spaces with application, *J. Intell. Fuzzy Syst.,* **37** (2019), 1275–1285. https://doi.org/10.3233/JIFS-182719