Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 5920–5924.
DOI: 10.3934/math.2022329
Received: 15 July 2021
Revised: 14 December 2021
Accepted: 27 December 2021
Published: 12 January 2022

Research article

Remarks on the K_2 group of $\mathbb{Z}[\zeta_p]$

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Abstract: In this paper, our aim is to obtain the K_2 analogues of both the Herbrand-Ribet theorem and the Vandiver's conjecture.

Keywords: K_2 group; Herbrand-Ribet theorem; Vandiver's conjecture
Mathematics Subject Classification: 19C99, 19F15

1. Introduction

It is well known that the Herbrand-Ribet theorem is about the relation between the p-th class group of cyclotomic field $\mathbb{Q}(\zeta_p)$ and the Bernoulli number.

We introduce some notations. Let $F = \mathbb{Q}(\zeta_p)$ be the cyclotomic field, and

$$G = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) = \{\sigma_a : 1 \leq a \leq p-1\}$$

be the Galois group, where $\sigma_a(\zeta_p) = \zeta_p^a$. Let ω be the Teichmuller character of group $(\mathbb{Z}/p)^\times$, that is, a character $\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times$ such that for $a \in \mathbb{Z}, (a, p) = 1$. Then $\omega(a)^{p-1} = 1$ and $\omega(a) \equiv a \mod p$. For the group ring $\mathbb{Z}_p[G]$, where $\mathbb{Z}_p$ is the p-adic integer ring, the idempotents are

$$\varepsilon_i = \frac{1}{p-1}\sum_{a=1}^{p-1}\omega^i(a)\sigma_a^{-1},\ 0 \leq i \leq p-2.$$

Let A be the p-part of $Cl(F)$, which is the class group of F. Then $A = \bigoplus_{i=0}^{p-2} A_i$, where $A_i = \varepsilon_i A$.

The Herbrand theorem states that if p divides the numerator of the Bernoulli number B_{p-i}, then $\varepsilon_i A \neq 0$. In 1976, Ribet [7] proved the converse of the Herbrand's theorem. So the Herbrand-Ribet theorem is as follow.

Theorem 1.1. *Let i be an odd integer with $3 \leq i \leq p-2$. If p divides the numerator of the Bernoulli number B_{p-i}, then $\varepsilon_i A \neq 0$.*

The Herbrand theorem is obtained by the properties of the Stickelberger element and the p-adic L-function. In [8], the Herbrand-Ribet theorem for function fields was obtained. In addition, Coats and Sinnott [2] proved an analogue of Stickelberger's theorem for the K_2 groups.

Throughout this paper, inspired by the above results, we obtain respectively the K_2 analogue of Herbrand-Ribet theorem and the K_2 analogue of the Vandiver conjecture.

2. K_2 analogue of Herbrand-Ribet theorem

Let S be a finite set of places of $F = \mathbb{Q}(\zeta_p)$ including the archimedean ones. Let O_S denote the ring of S-integers in F, i.e., the ring of all $a \in F$ such that $v(a) \geq 0$ for each place $v \notin S$. Then

$$0 \to \ker d^S \to K_2F \xrightarrow{d^S} \coprod_{v \notin S} \kappa^*(v) \to 0.$$

By Quillen's localization sequence, we have the isomorphism $\ker d^S \simeq K_2(O_S)$, which is moreover a G-isomorphism if S is stable under G (see [9, P. 271]).

Let $K_2(\mathbb{Z}[\zeta_p])$ be the K_2 group of the ring of algebraic integers $\mathbb{Z}[\zeta_p]$, and let C be the p-part of $K_2(\mathbb{Z}[\zeta_p])$. Then we have $C = \bigoplus_{i=0}^{p-2} C_i$, $C_i = \varepsilon_i C$.

Lemma 2.1. *There exist G-isomorphisms:*

$$\varepsilon_j A/p \simeq \varepsilon_{j+1} C/p, \quad 0 \leq j \leq p-3.$$

Proof. We note an isomorphism [4]

$$\mu_p \otimes A \simeq C/p, \tag{2.1}$$

where G acts on $\mu_p \otimes A$ by the formula

$$(\zeta \otimes x)^\rho = \zeta^\rho \otimes x^\rho, \ \text{ for } \zeta \in \mu_p,\ \rho \in G,\ x \in A.$$

We claim that the above isomorphism is a G-isomorphism. Let S be a set of the places of $\mathbb{Q}(\zeta_p)$ consisting of the archimedean ones and the finite ones above p. Let S_c denote the set of complex places. Then there is a natural exact sequence (see [9, Theorem 6.2])

$$0 \to \mu_p \otimes Cl(O_S) \to K_2O_S/p \xrightarrow{h_1^S} (\coprod_{v \in S - S_c} \mu_p)_0 \to 0, \tag{2.2}$$

where $(\coprod \mu_p)_0$ denotes the subgroup of the direct sum consisting of the elements $z = (z_v)$ such that $\sum z_v = 0$. The map h_1^S is that induced by the l-th power norm residue symbols for $v \in S - S_c$. Since S is stable under G, the above exact sequence is sequence of G-modules with G-homomorphisms(see [9, P. 271]). By [11, Theorem 73], C and the p-part of $K_2(\mathbb{Z}[\zeta_p, 1/p])$ are equal to $H^2_{ét}(\mathbb{Z}[\zeta_p, 1/p], \mathbb{Z}_p(2))$. Since $p\mathbb{Z}[\zeta_p] = (1-\zeta_p)^{p-1}$, the p-part of $Cl(O_S)$ is equal to A. Moreover, the fourth term in (2.2) is 0 (see [11, Example 5]), we get that (2.1) is a G-isomorphism.

Then we consider the following homomorphism

$$\delta : A \to \mu_p \otimes A,\ x \mapsto \zeta_p \otimes x.$$

Here, δ is not a homomorphism of G-modules. The kernel of δ is pA, so we get an isomorphism

$$\delta : A/pA \cong \mu_p \otimes A. \tag{2.3}$$

Next we give the explicit description of δ under the Galois group action. For $z := \zeta_{p^n}$, we have $\sigma_a(z) = z^{\omega(a)}$ (see [1, Lemma 3.3]), so there is

$$\sigma_a(\delta x) = \sigma_a(\zeta_p) \otimes \sigma_a x = \zeta_p^{\omega(a)} \otimes \sigma_a x = \omega(a) \cdot \delta(\sigma_a(x)).$$

Therefore,

$$\begin{aligned}\zeta_p \otimes \varepsilon_j x &= \zeta_p \otimes \Big(\frac{1}{p-1}\sum_{a=1}^{p-1} \omega^j(a)\sigma_a^{-1}(x)\Big)\\ &= \frac{1}{p-1}\sum_{a=1}^{p-1} \omega^{(j+1)}(a)\sigma_a^{-1}(\zeta_p \otimes x)\\ &= \varepsilon_{j+1}(\zeta_p \otimes x).\end{aligned}$$

Hence

$$\delta(\varepsilon_j x) = \zeta_p \otimes \varepsilon_j x = \varepsilon_{j+1}(\zeta_p \otimes x) = \varepsilon_{j+1}\delta(x). \tag{2.4}$$

By (2.4), the action of idempotents ε_j on (2.3) leads to

$$\varepsilon_j(A/pA) \simeq \varepsilon_{j+1}(\mu_p \otimes A).$$

Since (2.1) is a G-isomorphism, combining with the above isomorphism, we obatin

$$\varepsilon_j A/p \simeq \varepsilon_{j+1} C/p, \quad 0 \leq j \leq p-3$$

as desired. □

Next, we give the K_2 analogue of the Herbrand-Ribet theorem of the field $\mathbb{Q}(\zeta_p)$ as follow.

Theorem 2.1. *Let i be even,* $4 \leq i \leq p-3$. *Then*

$$C_i \neq 0 \Longleftrightarrow p | B_{p+1-i}.$$

Proof. It is clearly that

$$C_i \neq 0 \Leftrightarrow \varepsilon_i C/p \neq 0,$$

$$A_{i-1} \neq 0 \Leftrightarrow \varepsilon_{i-1} A/p \neq 0.$$

From Lemma 2.1, we have $\varepsilon_i C/p \simeq \varepsilon_{i-1} A/p$. Utilizing Theorem 1.1, we get $C_i \neq 0 \Leftrightarrow p|B_{p+1-i}$, as required.

However, the proof of " $\Rightarrow$ " can also be obtained by the properties of the Stickelberger element without using Theorem 1.1 and Lemma 2.1, We sketch the proof as follow.

Considering the Stickelberger element for the cyclotomic field $\mathbb{Q}(\zeta_p)$

$$\theta_1 = \sum_{a=1}^{p-1} \zeta(\sigma_a, -1)\sigma_a^{-1},$$

where $\zeta(\sigma, s)$ is the partial zeta function, we can prove that $(c^2 - \omega^i(c))B_{2,\omega^{-i}}$ annihilates C_i, moreover, for $i = 4, 6, \cdots, p-3$, $B_{2,\omega^{-i}}$ annihilates C_i.

We now suppose $C_i \neq 0$. Then $B_{2,\omega^{-i}} \equiv 0 \pmod p$. Since

$$B_{2,\omega^n} \equiv \frac{B_{n+2}}{n+2} \pmod p,$$

we get

$$B_{2,\omega^{-i}} = B_{2,\omega^{p-1-i}} \equiv \frac{B_{p+1-i}}{p+1-i} \pmod p.$$

Therefore, $p|B_{p+1-i}$.

3. K_2 analogue of Vandiver's conjecture

The Vandiver's conjecture states that p does not divide the class number of $\mathbb{Q}(\zeta_p)^+$, where $\mathbb{Q}(\zeta_p)^+$ is the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$. Equivalently, the Vandiver's conjecture says that all the even part $\varepsilon_i A$ are trivial.

Lemma 3.1. *For any irregular prime p,* $A_{2i} = 0$, *where* $1 \leq i \leq 14$.

Proof. From [10] (Tables §1 Bernoulli numbers), for $i = 1, 2, 3, 4, 5, 7$, we have $p \nmid B_{2i}$. So from Theorem 1.1, we have $A_{p-2i} = 0, i = 1, 2, 3, 4, 5, 7$. By the reflection theorem (see [10, Theorem 10.9])

$$p\text{-rank}A_{2i} \leq p\text{-rank}A_{p-2i},$$

we get $A_{2i} = 0$.

Let P_n denote the maximal prime factor of B_n if B_n has a prime factor. For $i = 6, 8, 9, 10, 11, 12, 13, 14$, from [10] (Tables §1 Bernoulli numbers) we have

$$P_{12} = 691,\ P_{16} = 3617,\ P_{18} = 43867,\ P_{20} = 617,$$

$$P_{22} = 593,\ P_{24} = 2294797,\ P_{26} = 657931,\ P_{28} = 362903.$$

These primes are all less than 12,000,000. But it is well know that the Vandiver conjecture has been checked to be true for all irregular primes less than 12,000,000. So we get $A_{2i} = 0$ for $i = 6, 8, 9, 10, 11, 12, 13, 14$. □

Now we can make a K_2-analogue of Vandiver's conjecture as follow.

Conjecture 3.1. *For odd i,* $\varepsilon_i C = 0$, *where C is the p-part of* $K_2(\mathbb{Z}[\zeta_p])$.

It has been proved that $\varepsilon_{p-3}A$ always vanishes (see [5]) and that if the prime $p \equiv 3 \pmod 4$, then $\varepsilon_{(p+1)/2}A$ is trivial (see [3, 6]). Combining these results with Lemmas 2.1 and 3.1, we get the following result, which checks some cases of Conjecture 3.1.

Theorem 3.1. *For any irregular prime p,* $C_{2i+1} = 0$ $(1 \leq i \leq 14)$, $C_{p-2} = 0$ *and* $C_{(p+3)/2} = 0$ *if* $p \equiv 3 \pmod 4$.

4. Conclusions

We gave the K_2 analogue of Herbrand-Ribet theorem and prove the case. The K_2 analogue of Vandiver's conjecture was also obtained, but this case is hard to prove. However, we just check some special circumstances of it.

Acknowledgments

The authors are thankful for the careful reviews of referees and the editor. The first author was supported by the National Natural Science Foundation of China (No. 11901079), China Postdoctoral Science Foundation (No. 2021M700751) and the Scientific and Technological Research Program Foundation of Jilin Province (No. JJKH20190690KJ; No. 20200401085GX; No. JJKH20220091KJ). The second author was supported by the National Natural Science Foundation of China (No. 11601211).

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. J. Browkin, On the p-rank of the tame kernel of algebraic number fields, *J. Reine Angew. Math.*, **432** (1992), 135–149. https://doi.org/10.1515/crll.1992.432.135
2. J. Coates, W. Sinnott, An analogue of Stickelberger's theorem for the higher K-groups, *Invent. Math.*, **24** (1974), 149–161. https://doi.org/10.1007/BF01404303
3. H. R. Qin, Q. Q. Shen, A density theorem and its applicition, *Sci. China Math.*, **58** (2015), 1621–1626. https://doi.org/10.1007/s11425-015-4999-z
4. F. Keune, On the structures of the K_2 of the ring of integers in a number field, *K-Theory*, **2** (1989), 625–645. https://doi.org/10.1007/BF00535049
5. M. Kurihara, Some remarks on conjectures about cyclotomic fields and K-groups of $\mathbb{Z}$, *Comopositio Math.*, **81** (1992), 223–236.
6. R. Osburn, Vanishing of eigenspace and cyclotomic fields, *Int. Math. Res. Notices*, **2005** (2005), 1195–1202. https://doi.org/10.1155/IMRN.2005.1195
7. K. A. Ribet, A modular construction of unramified p-extensions of $\mathbb{Q}(\zeta_p)$, *Invent. Math.*, **34** (1976), 151–162. https://doi.org/10.1007/BF01403065
8. L. Taelman, A Herbrand-Ribet theorem for function fields, *Invent. Math.*, **188** (2012), 253–275. https://doi.org/10.1007/s00222-011-0346-3
9. J. Tate, Relations between K_2 and Galois cohomology, *Invent. Math.*, **36** (1976), 257–274. https://doi.org/10.1007/BF01390012
10. L. C. Washington, *Introduction to cyclotomic fields*, 2 Eds., New York: Springer, 1997.
11. C. Weibel, Algebraic K-theory of rings of integers in local and gobal fields, In: E. M. Friedlander, D. R. Grayson, *Handbook of K-theory*, Berlin, Heidelberg: Springer, 2005.