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Research article

UHML stability of a class of Δ-Hilfer FDEs via CRM

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Abstract: We apply CRM based on an alternative FPT to investigate the approximation of a Δ-Hilfer FDE. In comparison to the Picard method, we show that the CRM has a better error estimate and economic solution.

Keywords: UHML stability; Δ-Hilfer; FDE; FPT
Mathematics Subject Classification: 46L05, 47B47, 47H10, 46L57, 39B62

1. Introduction and preliminaries

The proof of the existence and uniqueness theorem is due to Picard (1856–1941), who used an iteration scheme that guarantees a solution under the conditions specified. In fact, Picard's iteration scheme was the first method to solve analytically nonlinear differential equations. The Picard procedure is actually a practical extension of the Banach fixed point theorem, which is applicable to continuous contractible functions. Since any differential equation involves an unbounded derivative operator, the fixed point theorem is not suitable for it. To bypass this obstacle, Picard suggested applying the (bounded) inverse operator D^{-1} to the first derivative D. Recall that the inverse D^{-1} called in the mathematical literature is the antiderivative.

In 1967, Diaz and Margolis proved the alternative fixed point theorem (FPT) for any contraction mapping T on a generalized complete metric space X. The conclusion of the theorem, speaking in general terms, asserts that: either all consecutive pairs of the sequence of successive approximations (starting from an element x_0 of X) are infinitely far apart, or the sequence of successive approximations, with initial element x_0 converges to a fixed point of T (what particular fixed point depends, in general, on the initial element x_0). The present theorem contains as special cases both Banach's contraction mapping theorem for complete metric spaces, and Luxemburg's contraction mapping theorem for generalized metric spaces.

In 2007, Cădariu and Radu, through the Cădariu-Radu method (CRM) derived from the Diaz–Margolis theorem, discussed the generalized Ulam-Hyers stability for functional equations in a single variable, including nonlinear functional equations, linear functional equations, and a generalization of functional equation for the square root spiral.

One of the applicable branches of mathematical analysis is Fractional Calculus which considers integrals and derivatives of arbitrary order [1] so is an extension of integer-order calculus i.e., it unifies and generalizes the notions of integer-order differentiation and n-fold integration; for more details see [2–4].

Wang and et al. [21, 24], studied the following Δ-Hilfer fractional differential equation (FDE)

$$\begin{cases} {}^{\mathscr{H}}\mathcal{D}_{0^+}^{\lambda_1,\lambda_2;\Delta}\Phi(\zeta) = \mu(\zeta, \Phi(\zeta), \Phi(\eta(\zeta))), & \zeta \in (0, p], \\ \mathcal{I}_{0^+}^{1-\lambda_3;\Delta}\Phi(0^+) = J_0, & J_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

in which ${}^{\mathscr{H}}\mathcal{D}_{0^+}^{\lambda_1,\lambda_2;\Delta}(.)$ is the Δ-Hilfer fractional derivative of order $0 < \lambda_1 \leq 1$ and type $0 \leq \lambda_2 \leq 1$, $\mathcal{I}_{0^+}^{1-\lambda_3;\Delta}(.)$ is the Riemann-Liouville fractional integral of order $1-\lambda_3$, $\lambda_3 = \lambda_1 + \lambda_2(1-\lambda_1)$ with respect to the function Δ ([22]), and $\mu : (0, p] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a given function.

Consider $[\ell, \jmath](0 \leq \ell < \jmath < \infty)$ and the continuous functions space $C[\ell, \jmath]$ which includes all continuous functions $\eta : [\ell, \jmath] \longrightarrow \mathbb{R}$ such that

$$\|\eta\|_{C[\ell,\jmath]} = \sup_{\ell \leq ð \leq \jmath} |\eta(ð)|.$$

The weighted space $C_{1-\lambda_3;\Delta}[\ell, \jmath]$ of continuous η on $(\ell, \jmath]$ is defined by (see [23])

$$C_{1-\lambda_3;\Delta}[\ell, \jmath] = \Big\{\eta : (\ell, \jmath] \longrightarrow \mathbb{R}; (\Delta(ð) - \Delta(\ell))^{1-\lambda_3}\eta(ð) \in C[\ell, \jmath]\Big\}, \quad 0 \leq \lambda_3 < 1$$

with norm

$$\|\eta\|_{C_{1-\lambda_3;\Delta}[\ell,\jmath]} = \sup_{ð \in [\ell,\jmath]} \Big|(\Delta(ð) - \Delta(\ell))^{1-\lambda_3}\eta(ð)\Big|.$$

Definition 1.1 ([26])**.** Consider the interval $(\ell, \jmath)(-\infty \leq \ell < \jmath \leq +\infty)$ and $\lambda_1 > 0$. In addition, consider the increasing monotone map $\Delta(ð) > 0$ on $(\ell, \jmath]$ which has a continuous derivative $\Delta'(ð)$ on $(\ell, \jmath)$. For the mapping η, we define the fractional integrals w.r.t Δ, on $[\ell, \jmath]$ as

$$\mathcal{I}_{\ell^+}^{\lambda_1;\Delta}\eta(ð) = \frac{1}{\Gamma(\lambda_1)} \int_{\ell}^{ð} \Delta'(\zeta)(\Delta(ð) - \Delta(\zeta))^{\lambda_1-1}\eta(\zeta)d\zeta.$$

Definition 1.2 ([26])**.** Let $n-1 < \lambda_1 < n(n \in \mathbb{N})$, $0 \leq \lambda_2 \leq 1$ and $\mu, \Delta \in C^n[\ell, \jmath]$ be mappings such that Δ is non-decreasing and $\Delta'(ð) \neq 0$ for any $ð \in [\ell, \jmath]$. We define the Δ-Hilfer fractional derivative as

$${}^{\mathscr{H}}\mathcal{D}_{\ell^+}^{\lambda_1,\lambda_2;\Delta}\eta(ð) = \mathcal{I}_{\ell^+}^{\lambda_2(n-\lambda_1);\Delta}\Big(\frac{1}{\Delta'(ð)}\frac{d}{dð}\Big)^n \mathcal{I}_{\ell^+}^{(1-\lambda_2)(n-\lambda_1);\Delta}\eta(ð).$$

Theorem 1.3 ([26])**.** *Let $\eta \in C^1[0, \jmath]$, and $\lambda_2 \in [0, 1]$. Then*

$${}^{\mathscr{H}}\mathcal{D}_{0^+}^{\lambda_1,\lambda_2;\Delta}\mathcal{I}_{0^+}^{\lambda_1;\Delta}\eta(ð) = \eta(ð).$$

Theorem 1.4 ([26]). *Let $\eta \in C^1[0, \jmath]$ and $\lambda_2 \in [0,1]$. Then*

$$\mathcal{I}_{0^+}^{\lambda_1;\Delta}\,{}^{\mathscr{H}}\mathcal{D}_{0^+}^{\lambda_1,\lambda_2;\Delta}\eta(\eth) = \eta(\eth) - \frac{(\Delta(\eth) - \Delta'(0))^{\lambda_3 - 1}}{\Gamma(\lambda_3)}\mathcal{I}_{0^+}^{(1-\lambda_2)(1-\lambda_1);\Delta}\eta(0).$$

Lemma 1.5 ([26]). *Let $\alpha_1, \alpha_2 > 0$. If $f(\zeta) = (\Delta(\zeta) - \Delta(\ell))^{\alpha_2 - 1}$. Then*

$$\mathcal{I}_{\ell^+}^{\alpha_1;\Delta} f(x) = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2 + \alpha_1)}(\Delta(\zeta) - \Delta(\ell))^{\alpha_1 + \alpha_2 - 1}. \tag{1.2}$$

We study the FDE

$${}^{\mathscr{H}}\mathcal{D}_{0^+}^{\lambda_1,\lambda_2;\Delta}\Phi(\zeta) = \mu(\zeta, \Phi(\zeta), \Phi(\eta(\zeta))), \qquad \zeta \in (0, p], \tag{1.3}$$

$$\mathcal{I}_{0^+}^{1-\lambda_3;\Delta}\Phi(0^+) = J_0, \qquad J_0 \in \mathbb{R}, \tag{1.4}$$

where $\mu \in C((0, p] \times \mathbb{R}^2, \mathbb{R})$ and $\theta > 0$ and consider the inequality

$$\left|{}^{\mathscr{H}}\mathcal{D}_{0^+}^{\lambda_1,\lambda_2;\Delta}\Psi(\zeta) - \mu(\zeta, \Psi(\zeta), \Psi(\eta(\zeta)))\right| \leq \theta\, \Xi_{\lambda_1}((\Delta(\zeta) - \Delta(0))^{\lambda_1}), \quad \zeta \in (0, p], \tag{1.5}$$

where Ξ_{λ_1} is the Mittag-Leffler function (see [22]) defined by

$$\Xi_{\lambda_1}(\mathcal{Z}) := \sum_{n=0}^{\infty} \frac{\mathcal{Z}^n}{\Gamma(n\lambda_1 + 1)}, \quad \mathcal{Z} \in \mathbb{C},\ \Re(\lambda_1) > 0. \tag{1.6}$$

Definition 1.6 ([21]). Equations (1.3) and (1.4) have UHML stability with respect to $\Xi_{\lambda_1}((\Delta(\zeta) - \Delta(0))^{\lambda_1})$ if there exists $c > 0$ such that, for each $\theta > 0$ and each solution $\Psi \in C_{1-\lambda_3;\Delta}(0, p]$ to (1.5) and $\mathcal{I}_{0^+}^{1-\lambda_3;\Delta}\Psi(0^+) = J_0$, there exists a solution $\Phi \in C_{1-\lambda_3;\Delta}(0, p]$ to (1.3)-(1.4) with

$$|\Psi(\zeta) - \Phi(\zeta)| \leq c\,\theta\,\Xi_{\lambda_1}((\Delta(\zeta) - \Delta(0))^{\lambda_1}), \quad \zeta \in (0, p].$$

Remark 1.7. A mapping $\Psi \in C_{1-\lambda_3;\Delta}[0, p]$ is a solution of (1.5) iff we can find a mapping $\tilde{h}_\Psi \in C_{1-\lambda_3;\Delta}(0, p]$ such that

(*i*) $|\tilde{h}(\zeta)| \leq \theta\,\Xi_{\lambda_1}((\Delta(\zeta) - \Delta(0))^{\lambda_1}), \quad \zeta \in (0, p]$,

(*ii*) ${}^{\mathscr{H}}\mathcal{D}_{0^+}^{\lambda_1,\lambda_2;\Delta}\Psi(\zeta) = \mu(\zeta, \Psi(\zeta), \Psi(\eta(\zeta))) + \tilde{h}(\zeta), \quad \zeta \in (0, p]$.

Lemma 1.8 ([23]). *Consider the continuous map $\mu : (0, p] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$. Then* (1.3) *and* (1.4) *are equivalent to*

$$\Phi(\zeta) = \frac{(\Delta(\zeta) - \Delta(0))^{\lambda_3 - 1}}{\Gamma(\lambda_3)}J_0 + \frac{1}{\Gamma(\lambda_1)}\int_0^\zeta \Delta'(\xi)(\Delta(\zeta) - \Delta(\xi))^{\lambda_1 - 1}\mu(\xi, \Phi(\xi), \Phi(\eta(\xi)))d\xi. \tag{1.7}$$

Remark 1.9 ([24]). Let $\Psi \in C_{1-\lambda_3;\Delta}(0, p]$ be a solution of (1.5) and $\mathcal{I}_{0^+}^{1-\lambda_3;\Delta}\Psi(0^+) = J_0$. Then Ψ satisfies the following integral inequality:

$$\begin{aligned}&\left|\Psi(\zeta) - \frac{(\Delta(\zeta) - \Delta(0))^{\lambda_3 - 1}}{\Gamma(\lambda_3)}J_0 - \frac{1}{\Gamma(\lambda_1)}\int_0^\zeta \Delta'(\lambda_1)(\Delta(\zeta) - \Delta(\xi))^{\lambda_1 - 1}\mu(\xi, \Psi(\xi), \Psi(\eta(\xi)))d\xi\right| \\ \leq\ & \theta\,\Xi_{\lambda_1}((\Delta(\zeta) - \Delta(0))^{\lambda_1}).\end{aligned}$$

Remark 1.10. There are some slight mistakes and typos in [24]. In Definition 2.5 and Remark 2.6 in [24], the space $C[-h,d]$ should be $C[-h,0]\cap C_{1-\alpha;\psi}[0,d]$. For the precise definition needed in [24], see Definition 1.6 and Remark 1.9 above. We note that the statement (some typos) and proof of Theorem 3.1 in [24] is fine once $X = C[-h,d]$ in [24] is replaced by $C[-h,0]\cap C_{1-\alpha;\psi}[0,d]$ with the norm (there was accidently a typo in relation to the norm in [24]) $\max\{\|.\|_{C[-h,0]}, \|.\|_{C_{1-\alpha;\psi}[0,d]}\}$.

Theorem 1.11 ([9]). *Consider the complete* $[0,\infty]$*-valued metric space* $(\mathcal{Y}, d)$ *and* $\mathcal{L}:\mathcal{Y}\to\mathcal{Y}$ *satisfy*

$$d(\mathcal{L}y,\mathcal{L}t)\le \kappa d(t,y),\ \ \kappa\in(0,1).$$

For $y\in\mathcal{Y}$*, there are two ways*

$$(I)\qquad\qquad d(\mathcal{L}^m y,\mathcal{L}^{m+1}y)=\infty,\ \ \forall m\in\mathbb{N},$$

or

(II) there is $m_0\in\mathbb{N}$ *such that:*
(1) $d(\mathcal{L}^m y,\mathcal{L}^{m+1}y)<\infty,\qquad \forall m\ge m_0$;
(2) $\mathcal{L}^m y\to t^*$ *as fixed point of* $\mathcal{L}$;
(3) t^* *is unique in* $V=\{t\in\mathcal{Y}\mid d(\mathcal{L}^{m_0}y,t)<\infty\}$;
(4) $(1-\kappa)d(t,t^*)\le d(t,\mathcal{L}t)$ *for every* $t\in\mathcal{Y}$.

2. Existence, uniqueness and approximation

In this section, we consider existence, uniqueness and approximation of (1.3). Some linear fractional equations and nonlinear fractional integro-differential equations were studied in [5–8] and for results on Ulam stability of fractional integro-differential equations see [9, 11–17]. Note Ulam stability of differential equations was considered in [18–20, 25, 27–37]. Using a new method, CRM, based on Theorem 1.11, we get a better stability and approximation (which improves an estimate in [24]). We assume the following conditions hold:
($\mathcal{F}_1$) $\mu\in C((0,p]\times\mathbb{R}^2,\mathbb{R}), \eta\in C([0,p],[0,p]),\ \eta(\xi)\le\xi$ for every $\xi\ge 0$.
($\mathcal{F}_2$) There exists $0<\Theta<\frac{1}{2}$ such that

$$\Big|\mu(\zeta,\vartheta_1,\vartheta_2)-\mu(\zeta,\kappa_1,\kappa_2)\Big|\le\Theta\sum_{j=1}^{2}|\vartheta_j-\kappa_j|\ \ \text{for each}\ \ \zeta\in(0,p],\vartheta_j,\kappa_j\in\mathbb{R}, j=1,2.$$

Theorem 2.1. *Assume that* ($\mathcal{F}_1$) *and* ($\mathcal{F}_2$) *are satisfied and suppose* Ψ *in* $C_{1-\lambda_3;\Delta}(0,p]$ *satisfies* (1.5) *and* $\mathcal{I}_{0^+}^{1-\lambda_3;\Delta}\Psi(0^+)=J_0$. *Then, there exists a unique function* Φ *satisfying* (1.3), (1.4) *such that*

$$|\Phi(\zeta)-\Psi(\zeta)|\le\frac{\theta}{1-2\Theta}\Xi_{\lambda_1}((\Delta(\zeta)-\Delta(0))^{\lambda_1}),\tag{2.1}$$

for every $\zeta\in(0,p]$.

Proof. Set $\mathcal{B}=C_{1-\lambda_3;\Delta}(0,p]$ and let $\mathbf{A}:\mathcal{B}\times\mathcal{B}\to[0,+\infty]$ be given by

$$\mathbf{A}(\rho,\varrho)=\inf\Big\{\complement\ge 0:|\rho(\zeta)-\varrho(\zeta)|\le\complement\theta\Xi_{\lambda_1}((\Delta(\zeta)-\Delta(0))^{\lambda_1})\ \text{ for }\ \zeta\in(0,p]\Big\}.\tag{2.2}$$

First we prove $(\mathcal{B}, \mathbf{A})$ is a $[0, +\infty]$-valued metric space and then we prove the completeness. Let $\mathbf{A}(\rho, \varrho) > \mathbf{A}(\rho, \nu) + \mathbf{A}(\nu, \varrho)$, for some ρ, ϱ and $\nu \in \mathcal{B}$. Then, there exists $\zeta_\circ \in (0, p]$ such that

$$|\rho(\zeta_\circ) - \varrho(\zeta_\circ)| > (\mathbf{A}(\rho, \nu) + \mathbf{A}(\nu, \varrho))\theta \Xi_{\lambda_1}(\Delta(\zeta_\circ) - \Delta(0))^{\lambda_1}.$$

Thus

$$\begin{aligned} & |\rho(\zeta_\circ) - \varrho(\zeta_\circ)| \\ > \ & |\rho(\zeta_\circ) - \nu(\zeta_\circ)| + |(\nu(\zeta_\circ) - \varrho(\zeta_\circ)|, \end{aligned}$$

a contradiction. For the completeness of $(\mathcal{B}, \mathbf{A})$, consider the Cauchy sequence ω_n in $(\mathcal{B}, \mathbf{A})$. Then, for any $\epsilon > 0$ there exists an integer $\aleph_\epsilon$ such that $\mathbf{A}(\omega_m, \omega_n) \leq \epsilon$ for all $m, n \geq \aleph_\epsilon$. In view of (2.2), we have

$$|\omega_m(\zeta) - \omega_n(\zeta)| \leq \epsilon\, \theta\, \Xi_{\lambda_1}(\Delta(\zeta) - \Delta(0))^{\lambda_1} \text{ for every } \zeta \in (0, p]. \tag{2.3}$$

Let $\zeta \in (0, p]$ be arbitrary and fixed, from (2.3) we can conclude in the reals that the sequence $\{\omega_n(\zeta)\}$ is Cauchy and convergence. Then there is a $\omega \in \mathcal{B}$ such that

$$\omega(\zeta) := \lim_{n\to\infty} \omega_n(\zeta), \quad (\zeta \in (0, p]). \tag{2.4}$$

Using (2.3) and let m tend to ∞, and we get

$$|\omega(\zeta) - \omega_n(\zeta)| \leq \epsilon\, \theta\, \Xi_{\lambda_1}(\Delta(\zeta) - \Delta(0))^{\lambda_1}. \tag{2.5}$$

Considering (2.2), we get

$$\mathbf{A}(\omega, \omega_n) \leq \epsilon,$$

i.e., $\omega_n \to \omega$ in $(\mathcal{B}, \mathbf{A})$ and shows the completeness of $(\mathcal{B}, \mathbf{A})$.

Let $\rho \in \mathcal{B}$. Define $\mathbb{J} : \mathcal{B} \longrightarrow \mathcal{B}$ as

$$\begin{aligned} & \mathbb{J}(\rho(\zeta)) \\ = \ & \frac{(\Delta(\zeta) - \Delta(0))^{\lambda_3 - 1}}{\Gamma(\lambda_3)} J_0 + \frac{1}{\Gamma(\lambda_1)} \int_0^\zeta \Delta'(\xi)(\Delta(\zeta) - \Delta(\xi))^{\lambda_1 - 1} \mu(\xi, \rho(\xi), \rho(\eta(\xi)))d\xi, \end{aligned} \tag{2.6}$$

for each $\zeta \in (0, p]$. Thus

$$\begin{aligned} & |\mathbb{J}(\rho(\zeta)) - \mathbb{J}(\rho(\zeta_0))| \\ = \ & \Bigg| \frac{(\Delta(\zeta) - \Delta(0))^{\lambda_3 - 1}}{\Gamma(\lambda_3)} J_0 + \frac{1}{\Gamma(\lambda_1)} \int_0^\zeta \Delta'(\xi)(\Delta(\zeta) - \Delta(\xi))^{\lambda_1 - 1} \mu(\xi, \rho(\xi), \rho(\eta(\xi)))d\xi \\ - \ & \frac{(\Delta(\zeta_0) - \Delta(0))^{\lambda_3 - 1}}{\Gamma(\lambda_3)} J_0 - \frac{1}{\Gamma(\lambda_1)} \int_0^{\zeta_0} \Delta'(\xi)(\Delta(\zeta_0) - \Delta(\xi))^{\lambda_1 - 1} \mu(\xi, \rho(\xi), \rho(\eta(\xi)))d\xi \Bigg| \end{aligned}$$

so $\mathbb{J} : \mathcal{B} \longrightarrow \mathcal{B}$ is continuous.

Next, we show that $\mathbb{J}$ is a contraction. Suppose $\rho, \varrho \in C_{1-\lambda_3;\Delta}(0, p]$, $\Bbbk \in [0, +\infty]$ and $\mathbf{A}(\rho(\zeta), \varrho(\zeta)) \leq \Bbbk$. Thus

$$|\rho(\zeta) - \varrho(\zeta)| \leq \Bbbk\theta\, \Xi_{\lambda_1}((\Delta(\zeta) - \Delta(0))^{\lambda_1}).$$

Using Remark 1.9 (see Remark 2.10 of [24]), for all $\zeta \in (0, p]$, we have

$$\begin{aligned}
&\Big|\mathbb{J}(\rho(\zeta)) - \mathbb{J}(\varrho(\zeta))\Big| \\
\leq\ & \frac{1}{\Gamma(\lambda_1)}\int_0^\zeta \Delta'(\xi)(\Delta(\zeta)-\Delta(\xi))^{\lambda_1-1}\Big|\mu(\xi,\rho(\xi),\rho(\eta(\xi))) - \mu(\xi,\varrho(\xi),\varrho(\eta(\xi)))\Big|d\xi \\
\leq\ & \frac{\Theta}{\Gamma(\lambda_1)}\int_0^\zeta \Delta'(\xi)(\Delta(\zeta)-\Delta(\xi))^{\lambda_1-1}\Big[|\rho(\xi)-\varrho(\xi)| + |\rho(\eta(\xi))-\varrho(\eta(\xi))|\Big]d\xi \\
\leq\ & \frac{2\Theta\,[\Bbbk\theta]}{\Gamma(\lambda_1)}\int_0^\zeta \Delta'(\xi)(\Delta(\zeta)-\Delta(\xi))^{\lambda_1-1}\Xi_{\lambda_1}((\Delta(\xi)-\Delta(0))^{\lambda_1})d\xi \\
\leq\ & 2\Theta\Bbbk\theta\Xi_{\lambda_1}((\Delta(\zeta)-\Delta(0))^{\lambda_1}).
\end{aligned}$$

Then we have

$$\mathbf{A}(\mathbb{J}(\rho), \mathbb{J}(\varrho)) \leq 2\Theta\ \mathbf{A}(\rho, \varrho).$$

Now, $(\mathcal{F}_2)$ implies the contractively property of $\mathbb{J}$.

Let $\Psi \in \mathcal{B}$. Since $\mathbb{J}(\Psi) \in \mathcal{B}$, we have that

$$\begin{aligned}
&\Big|\mathbb{J}(\Psi(\zeta)) - \Psi(\zeta)\Big| \\
\leq\ & \left|\Psi(\zeta) - \frac{(\Delta(\zeta)-\Delta(0))^{\lambda_3-1}}{\Gamma(\lambda_3)}J_0 - \frac{1}{\Gamma(\lambda_1)}\int_0^\zeta \Delta'(\xi)(\Delta(\zeta)-\Delta(0))^{\lambda_1-1}\mu(\xi,\Psi(\xi),\Psi(\eta(\xi)))d\xi\right| \\
\leq\ & \theta\,\Xi_{\lambda_1}(\Delta(\zeta)-\Delta(0))^{\lambda_1}
\end{aligned}$$

for $\zeta \in (0, p]$, which implies that

$$\mathbf{A}(\mathbb{J}(\Psi), \Psi) \leq 1. \tag{2.7}$$

and hence for every $n \in \mathbb{N}$ we have $\mathbf{A}(\mathbb{J}^n(\Psi), \mathbb{J}^{n+1}(\Psi)) < +\infty$. From Theorem 1.11, option two, $\Phi \in \{\tilde{\boldsymbol{\sigma}} \in \mathcal{B} : \mathbf{A}(\mathbb{J}\Psi, \tilde{\boldsymbol{\sigma}}) < \infty\}$ is unique and $\mathbb{J}\Phi = \Phi$. Then

$$\Phi(\zeta) = \frac{(\Delta(\zeta)-\Delta(0))^{\lambda_3-1}}{\Gamma(\lambda_3)}J_0 + \frac{1}{\Gamma(\lambda_1)}\int_0^\zeta \Delta'(\xi)(\Delta(\zeta)-\Delta(\xi))^{\lambda_1-1}\mu(\xi,\Phi(\xi),\Phi(\eta(\xi)))d\xi, \tag{2.8}$$

for every $\zeta \in (0, p]$, where $\mathcal{I}_{0^+}^{1-\lambda_3;\Delta}\Phi(0^+) = J_0 \in \mathbb{R}$.

Using Theorem 1.11 and (2.7), we have

$$\begin{aligned}
\mathbf{A}(\Phi, \Psi) &\leq \frac{1}{1-2\Theta}\mathbf{A}(\mathbb{J}(\Psi), \Psi) \\
&\leq \frac{1}{1-2\Theta},
\end{aligned}$$

which implies (2.1).

□

3. Application

Example 3.1. Consider the following fractional system

$$\begin{cases} {}^{\mathscr{H}}\mathcal{D}_{0^+}^{0.25,0.75;e^{\zeta}}\Phi(\zeta) = 0.25\arctan(\Phi(\zeta)) + 0.25\sin\left(\Phi\left(\frac{\zeta}{2}\right)\right), & \zeta \in (0,1], \\ \mathcal{I}_{0^+}^{0.8125;e^{\zeta}}\Phi(0^+) = \Phi_0, \end{cases} \tag{3.1}$$

and the inequality

$$\left|{}^{\mathscr{H}}\mathcal{D}_{0^+}^{0.25,0.75;e^{\zeta}}\Phi(\zeta) - \mu\left(\zeta, \Phi(\zeta), \Phi\left(\frac{\zeta}{2}\right)\right)\right| \leq \theta\, \Xi_{0.25}((e^{\zeta}-1)^{0.25})$$

Let $\lambda_1 = 0.25, \lambda_2 = 0.75$, Then $\lambda_3 = \lambda_1 + \lambda_2(1-\lambda_1) = 0.8125, p = 1, \Delta(\zeta) = e^{\zeta}, \eta(\zeta) = \frac{\zeta}{2}, \mu(\zeta, \kappa_1, \kappa_2) = 0.25\arctan(\kappa_1) + 0.25\sin(\kappa_2)$ for $\kappa_1, \kappa_2 \in \mathbb{R}$, and $\Theta = 0.25$.

Now $(\mathcal{F}_2)$ follows since

$$\begin{aligned} & \left|0.25\arctan(\kappa_1) + 0.25\sin(\kappa_2) - 0.25\arctan(\kappa_1') - 0.25\sin(\kappa_2')\right| \\ \leq\ & \left|0.25\arctan(\kappa_1) - 0.25\arctan(\kappa_1')\right| + \left|0.25\sin(\kappa_2) - 0.25\sin(\kappa_2')\right| \\ \leq\ & 0.25\left[\left|\kappa_1 - \kappa_1'\right| + \left|\kappa_2 - \kappa_2'\right|\right], \end{aligned}$$

for every $\kappa_1, \kappa_2, \kappa_1', \kappa_2' \in \mathbb{R}$.

Now, Theorem 2.1 implies that, problem (3.1) has a unique solution and also is UHML stable with

$$|\Psi(\zeta) - \Phi(\zeta)| \leq 2\theta\Xi_{0.25}((e^{\zeta}-1)^{0.25}), \quad \zeta \in (0,1].$$

4. Concluding remarks

We applied the CRM to investigate the UHML stability of a class of Δ-Hilfer FDEs. Our method is shorter and faster in comparison with the Picard method presented in [24] to obtain the existence and uniqueness of the mentioned Δ-Hilfer FDE.

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Conflict of interest

The authors declare no potential conflict of interest.

References

1. R. Almeida, D. Tavares, D. F. M. Torres, *The variable-order fractional calculus of variations*, Springer Briefs in Applied Sciences and Technology, Springer, Cham, 2019. https://doi.org/10.1007/978-3-319-94006-9

2. M. A. Almalahi, M. S. Abdo, S. K. Panchal, Existence and Ulam-Hyers-Mittag-Leffler stability results of Ψ-Hilfer nonlocal Cauchy problem, *Rendiconti del Circolo Matematico di Palermo Series 2*, **70** (2021), 57–77. https://doi.org/10.1007/s12215-020-00484-8

3. M. A. Almalahi, S. K. Panchal, E_α-Ulam-Hyers stability result for ψ-Hilfer Nonlocal Fractional Differential Equation, *Discontinuity Nonlinearity Complexity*, **3** (2014), 1–6.

4. M. A. Almalahi, M. S. Abdo, S. K. Panchal, ψ-Hilfer Fractional Functional Differential Equation by Picard Operator Method, *J. Appl. Nonlinear Dyn.*, **9** (2020), 685–702. https://doi.org/10.5890/JAND.2020.12.011

5. M. Janfada, G. Sadegh, Stability of the Volterra integrodifferential equation, *Folia Math.*, **18** (2013), 11–20.

6. E. Oliveira, J. Sousa, Ulam-Hyers-Rassias stability for a class of fractional integro-differential equations, *Result Math.*, **37** (2018), 111. https://doi.org/10.1007/s00025-018-0872-z

7. J. Sousa, O. Oliveira, On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the Ψ-Hilfer operator, *J. Fixed Point Theory Appl.*, **20** (2018), 1–21. https://doi.org/10.1007/s11784-018-0587-5

8. J. Wang, X. Li, A uniform method to Ulam-Hyers stability for some linear fractional equations, *Mediterr J. Math.*, **13** (2016), 625–635. https://doi.org/10.1007/s00009-015-0523-5

9. J. B. Diaz, B. Margolis, A fixed point theorem of the alternative, for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.*, **74** (1968), 305–309. https://doi.org/10.1090/S0002-9904-1968-11933-0

10. L. Cădariu, V. Radu, Fixed point methods for the generalized stability of functional equations in a single variable, *Fixed Point Theory Appl.,* **2008** (2008), 749392. https://doi.org/10.1155/2008/749392

11. J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theory Differ. Equ.,* **2011** (2011), 1–10. https://doi.org/10.14232/ejqtde.2011.1.63

12. M. Benchohra, J. E. Lazreg, Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative, *Stud. Univ. Babes-Bolyai Math.*, **62** (2017), 27–38. https://doi.org/10.24193/subbmath.2017.0003

13. C. J. Sousa, E. Oliveira, Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation, *Appl. Math. Lett.*, **81** (2018), 50–56. https://doi.org/10.1016/j.aml.2018.01.016

14. M. Benchohra, J. E. Lazreg, On stability for nonlinear implicit fractional differential equations, *Le Math.*, **70** (2015), 49–61.

15. D. Vivek, K. Kanagarajan, E. M. Elsayed, Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions, *Mediterr. J. Math.*, **15** (2018), 15. https://doi.org/10.1007/s00009-017-1061-0

16. S. Abbas, M. Benchohra, L. E. Lagreg, A. Alsaedi, Y. Zhou, Existence and Ulam stability for fractional differntial equations of Hilfer-Hadamard type, *Adv. Differ. Equ.*, **2017** (2017), 180. https://doi.org/10.1186/s13662-017-1231-1

17. Y. Zhou, Existence and uniqueness of solutions for a system of fractional differential equations, *J. Frac. Calc. Appl. Anal.*, **12** (2009), 195–204.

18. J. Huang, Y. Li, Hyers-Ulam stability of delay differential equations of first order, *Math. Nachr.*, **289** (2016), 60–66. https://doi.org/10.1002/mana.201400298

19. L. D. Long, H. T. K. Van, H. T. K. Binh, R. Saadati, On backward problem for fractional spherically symmetric diffusion equation with observation data of nonlocal type, *Adv. Differ. Equ.*, **2021** (2021), 445. https://doi.org/10.1186/s13662-021-03603-6

20. R. Chaharpashlou, A. Atangana, R. Saadati, On the fuzzy stability results for fractional stochastic volterra integral equation, *Discrete Contin. Dyn. Syst. Ser. S*, **14** (2021), 3529–3539. https://doi.org/10.3934/dcdss.2020432

21. J. Wang, Y. Zhang, Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations, *Optimization*, **63** (2014), 1181–1190. https://doi.org/10.1080/02331934.2014.906597

22. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Equations*, Elsevier, Amsterdam, 2006.

23. E. Capelas de Olivera, C. D. Vanterler, J. Sousa, Ulam-Hyers-Rassias stability for a class of fractional integro-differential equations, *Results Math.*, **73** (2018), 111. https://doi.org/10.1007/s00025-018-0872-z

24. K. Liu, J. Wang, D. O'Regan, Ulam-Hyers-Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations, *Adv. Differ. Equ.*, **2019** (2019), 50. https://doi.org/10.1186/s13662-019-1997-4

25. S. Rezaei Aderyani, R. Saadati, M. Feckan, The Cadariu–Radu Method for Existence, Uniqueness and Gauss Hypergeometric Stability of W-Hilfer Fractional Differential Equations, *Mathematics*, **9** (2021), 1408. https://doi.org/10.3390/math9121408

26. C. D. Vanterler, J. Sousa, E. Capelas de Olivera, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.*, **60** (2018), 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005

27. E. Graily, S. M. Vaezpour, R. Saadati, Y. J. Cho, Generalization of fixed point theorems in ordered metric spaces concerning generalized distance, *Fixed Point Theory Appl.*, **2011** (2011), 30. https://doi.org/10.1186/1687-1812-2011-30

28. S. Shakeri, L. J. B. Ciric, R. Saadati, Common fixed point theorem in partially ordered *L*-fuzzy metric spaces, *Fixed Point Theory Appl.*, **2010** (2010), 125082. https://doi.org/10.1155/2010/125082

29. L. Ciric, M. Abbas, B. Damjanovic, Common fuzzy fixed point theorems in ordered metric spaces, *Math. Comput. Modell.*, **53** (2011), 1737–1741. https://doi.org/10.1016/j.mcm.2010.12.050

30. Y. J. Cho, R. Saadati, Lattictic non-Archimedean random stability of ACQ functional equation, *Adv. Differ. Equ.*, **2011** (2011), 31. https://doi.org/10.1186/1687-1847-2011-31

31. D. Mihet, R. Saadati, S. M. Vaezpour, The stability of an additive functional equation in Menger probabilistic ϕ-normed spaces, *Math. Slovaca*, **61** (2011), 817–826. https://doi.org/10.2478/s12175-011-0049-7

32. Y. J. Cho, C. Park, T. M. Rassias, R. Saadati, *Stability of functional equations in Banach algebras*, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-18708-2

33. D. Baleanu, R. Saadati, J. Sousa, The stability of the fractional Volterra integro-differential equation by means of Ψ-Hilfer operator revisited, *Math. Methods Appl. Sci.*, **44** (2021), 10905–10911. https://doi.org/10.1002/mma.7348

34. S. Rezaei Aderyani, R. Saadati, Best approximations of the ϕ-Hadamard fractional Volterra integro-differential equation by matrix valued fuzzy control functions, *Adv. Differ. Equ.*, **2021** (2021), 154. https://doi.org/10.1186/s13662-021-03305-z

35. R. Chaharpashlou, R. Saadati, Best approximation of a nonlinear fractional Volterra integro-differential equation in matrix MB-space, *Adv. Differ. Equ.*, **2021** (2021), 118. https://doi.org/10.1186/s13662-021-03275-2

36. R. Chaharpashlou, R. Saadati, A. Atangana, Ulam-Hyers-Rassias stability for nonlinear Ψ-Hilfer stochastic fractional differential equation with uncertainty, *Adv. Differ. Equ.*, **2020** (2020), 339. https://doi.org/10.1186/s13662-020-02797-5

37. N. Celik, A. R. Seadawy, Y. Saglam Ozkan, E. Yasar, A model of solitary waves in a nonlinear elastic circular rod: abundant different type exact solutions and conservation laws, *Chaos Solitons Fractals*, **143** (2021), 110486. https://doi.org/10.1016/j.chaos.2020.110486