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Research article

On the classical Gauss sums and their some new identities

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Abstract: In this paper, we use the analytic methods and the properties of the classical Gauss sums to study the calculating problems of some Gauss sums involving the character of order 12 modulo an odd prime p, and obtain several new and interesting identities for them.

Keywords: the character of order 12; the classical Gauss sums; analytic methods; identity; calculating formula
Mathematics Subject Classification: 11L10, 11L40

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1. Introduction

Let $q > 1$ be an integer. For any Dirichlet character χ modulo q, the classical Gauss sums $G(m, \chi; q)$ is defined as follow:

$$G(m, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma}{q}\right),$$

where m is any integer, $e(y) = e^{2\pi i y}$ and $i^2 = -1$.

For convenience, we write $\tau(\chi) = G(1, \chi; q)$. This sum plays a very important role in the study of elementary number theory and analytic number theory, many number theory problems are closely related to it. Because of this, many scholars have studied its various properties, and obtained a series of important results. Perhaps the most important properties of $G(m, \chi; q)$ are the following two:

(A) If $(m, q) = 1$, then we have the identity (see [1–3])

$$G(m, \chi; q) = \overline{\chi}(m) G(1, \chi; q) = \overline{\chi}(m) \tau(\chi).$$

(B) If χ is any primitive character modulo q, then one has also $G(m, \chi; q) = \overline{\chi}(m) \tau(\chi)$ and the identity $|\tau(\chi)| = \sqrt{q}$.

In addition, let $h > 1$ be any fixed positive integer, then for any prime p with $p \equiv 1 \bmod h$, there must be a Dirichlet character of order h. From now on, we fix χ_n to be a primitive character of order n

modulo p (i.e. $\chi_n^n = \chi_0$, the principal character modulo p, and $\chi_n^i \neq \chi_0$ for all $1 \leq i < n$) throughout the paper. W. P. Zhang and J. Y. Hu [4] (or B. C. Berndt and R. J. Evans [5]) studied the properties of some special Gauss sums, and obtained the following interesting results. That is, for any prime p with $p \equiv 1 \bmod 3$, one has the identity

$$\tau^3(\chi_3) + \tau^3(\overline{\chi}_3) = dp, \tag{1.1}$$

where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

Z. Y. Chen and W. P. Zhang [6] studied the case of the character of order four modulo p, and proved the identity

$$\tau^2(\chi_4) + \tau^2(\overline{\chi}_4) = 2\sqrt{p} \cdot \alpha, \tag{1.2}$$

where $\alpha = \frac{1}{2} \sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p} \right)$, and $\left(\frac{*}{p}\right) = \chi_2$ denotes the Legendre's symbol modulo p.

The constant $\alpha = \alpha(p)$ in (1.2) has a special meaning. In fact, we have the identity (for this see Theorems 4–11 in [7])

$$p = \alpha^2 + \beta^2 = \left(\frac{1}{2} \sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p} \right) \right)^2 + \left(\frac{1}{2} \sum_{a=1}^{p-1} \left(\frac{a + r\overline{a}}{p} \right) \right)^2, \tag{1.3}$$

where r is any quadratic non-residue modulo p. That is, $\chi_2(r) = -1$.

L. Chen [8] obtained another identity for the character of order six modulo p. That is, she proved the following conclusion:

Let p be a prime with $p \equiv 1 \bmod 6$, then one has the identity

$$\tau^3(\chi_6) + \tau^3(\overline{\chi}_6) = \begin{cases} p^{\frac{1}{2}} \cdot \left(d^2 - 2p\right), & \text{if } p \equiv 1 \bmod 12; \\ -i \cdot p^{\frac{1}{2}} \cdot \left(d^2 - 2p\right), & \text{if } p \equiv 7 \bmod 12, \end{cases} \tag{1.4}$$

where $i^2 = -1$, d is the same as defined in (1.1).

Some other results involving Gauss sums and character sums can also be found in [9–15], we will not list them all here.

It is not hard to see from [4, 6, 8] that the number of all such characters in formulaes (1.1), (1.2) and (1.4) is 2. That is, $\phi(3) = \phi(4) = \phi(6) = 2$. A natural thing to think about is: What about the case when the order n satisfies $\phi(n) > 2$? For example, the character of order 12 modulo p with $p \equiv 1 \bmod 12$. In this case, we have $\phi(12) = 4$, and all primitive characters of order 12 modulo p are $\chi_4\chi_3, \chi_4\overline{\chi}_3, \overline{\chi}_4\chi_3$ and $\overline{\chi}_4\overline{\chi}_3$.

In this article, we shall focus on this problem. We use the properties of the classical Gauss sums and analytic methods to prove the following results:

Theorem 1.1. *Let p be an odd prime with $p \equiv 1 \bmod 12$, then we have the identities*

$$\begin{aligned} &\tau^6(\chi_4\chi_3) + \tau^6(\chi_4\overline{\chi}_3) + \tau^6(\overline{\chi}_4\chi_3) + \tau^6(\overline{\chi}_4\overline{\chi}_3) \\ = \;& \chi_4(3) \cdot (-1)^{\frac{p-1}{4}} \cdot 2 \cdot \sqrt{p} \cdot \alpha \cdot \left(d^2 - 2p\right) \cdot \left(4\alpha^2 - 3p\right), \end{aligned}$$

where d is the same as defined in (1.1), and α is the same as defined as in (1.2).

Theorem 1.2. *Let p be an odd prime with $p \equiv 1 \bmod 12$, then we have the identity*

$$\left(\frac{\tau^3(\chi_4\chi_3) - \tau^3(\chi_4\overline{\chi}_3)}{\tau^3(\chi_4\chi_3) + \tau^3(\chi_4\overline{\chi}_3)}\right)^2 = -27 \cdot \frac{b^2}{d^2}.$$

Theorem 1.3. *Let p be an odd prime with $p \equiv 1 \bmod 24$, then we have the identity*

$$\left(\frac{\tau^6(\chi_8\chi_3)\,\tau^6\left(\chi_8^3\chi_3\right) - \tau^6(\chi_8\overline{\chi}_3)\,\tau^6\left(\chi_8^3\overline{\chi}_3\right)}{\tau^6(\chi_8\chi_3)\,\tau^6\left(\chi_8^3\chi_3\right) + \tau^6(\chi_8\overline{\chi}_3)\,\tau^6\left(\chi_8^3\overline{\chi}_3\right)}\right)^2 = -27 \cdot \frac{b^2d^2}{(d^2-2p)^2}.$$

For any prime p with $p \equiv 1 \bmod 12$ and any integer $n \geq 0$, we define

$$G_n(p) = \frac{\tau^{3n}(\chi_4\overline{\chi}_3)}{\tau^{3n}(\chi_4\chi_3)} + \frac{\tau^{3n}(\chi_4\chi_3)}{\tau^{3n}(\chi_4\overline{\chi}_3)} = \frac{\tau^{3n}(\overline{\chi}_4\overline{\chi}_3)}{\tau^{3n}(\overline{\chi}_4\chi_3)} + \frac{\tau^{3n}(\overline{\chi}_4\chi_3)}{\tau^{3n}(\overline{\chi}_4\overline{\chi}_3)}$$

and

$$H_n(p) = \frac{\tau^{n}(\overline{\chi}_4\chi_3)}{\tau^{n}(\chi_4\chi_3)} + \frac{\tau^{n}(\chi_4\chi_3)}{\tau^{n}(\overline{\chi}_4\chi_3)} = \frac{\tau^{n}(\overline{\chi}_4\overline{\chi}_3)}{\tau^{n}(\chi_4\overline{\chi}_3)} + \frac{\tau^{n}(\chi_4\overline{\chi}_3)}{\tau^{n}(\overline{\chi}_4\overline{\chi}_3)}.$$

Then we have the following second order recurrence formulas for $G_n(p)$ and $H_n(p)$. That is, we have:

Theorem 1.4. *For any prime p with $p \equiv 1 \bmod 12$, we have the second order recurrence formula*

$$G_{n+2}(p) = \frac{d^2-2p}{p} \cdot G_{n+1}(p) - G_n(p), \quad n \geq 0,$$

where the two initial values $G_0(p) = 2$ and $G_1(p) = \frac{d^2-2p}{p}$. Therefore,

$$G_n(p) = \left(\frac{d^2-2p+3\sqrt{3}idb}{2p}\right)^n + \left(\frac{d^2-2p-3\sqrt{3}idb}{2p}\right)^n, \; n \geq 0, \; i^2 = -1.$$

Theorem 1.5. *Let p be a prime. If $p \equiv 1 \bmod 24$, then we have the second order recurrence formula*

$$H_{n+2}(p) = \frac{2\alpha}{\sqrt{p}} \cdot H_{n+1}(p) - H_n(p), \quad n \geq 0,$$

where the two initial values $H_0(p) = 2$ and $H_1(p) = \frac{2\alpha}{\sqrt{p}}$. Therefore,

$$H_n(p) = \left(\frac{\alpha+i\beta}{\sqrt{p}}\right)^n + \left(\frac{\alpha-i\beta}{\sqrt{p}}\right)^n, \; n \geq 0, \; i^2 = -1.$$

If $p \equiv 13 \bmod 24$, then we have the second order recurrence formula

$$H_{n+2}(p) = -\frac{2\alpha}{\sqrt{p}} \cdot H_{n+1}(p) - H_n(p), \quad n \geq 0,$$

where the two initial values $H_0(p) = 2$ and $H_1(p) = -\frac{2\alpha}{\sqrt{p}}$. Therefore,

$$H_n(p) = \left(\frac{-\alpha + i\beta}{\sqrt{p}}\right)^n + \left(\frac{-\alpha - i\beta}{\sqrt{p}}\right)^n, \ n \geq 0, \ i^2 = -1,$$

and β is the same as defined in (1.3).

From these theorems we may immediately deduce the following several interesting corollaries:

Corollary 1.1. *Let p be an odd prime with $p \equiv 1 \bmod 12$, then we have*

$$\begin{aligned}&\left[\tau^6\left(\chi_4\chi_3\right) + \tau^6\left(\chi_4\overline{\chi}_3\right) + \tau^6\left(\overline{\chi}_4\chi_3\right) + \tau^6\left(\overline{\chi}_4\overline{\chi}_3\right)\right]^2\\ &= 4 \cdot p \cdot \alpha^2 \cdot \left(d^2 - 2p\right)^2 \cdot \left(4\alpha^2 - 3p\right)^2.\end{aligned}$$

Corollary 1.2. *Let p be an odd prime with $p \equiv 1 \bmod 24$, then we have*

$$\left|\frac{\tau^6\left(\chi_8\chi_3\right)\tau^6\left(\chi_8^3\chi_3\right) - \tau^6\left(\chi_8\overline{\chi}_3\right)\tau^6\left(\chi_8^3\overline{\chi}_3\right)}{\tau^6\left(\chi_8\chi_3\right)\tau^6\left(\chi_8^3\chi_3\right) + \tau^6\left(\chi_8\overline{\chi}_3\right)\tau^6\left(\chi_8^3\overline{\chi}_3\right)}\right| = 3\sqrt{3} \cdot \frac{|b| \cdot |d|}{|d^2 - 2p|}.$$

Corollary 1.3. *Let p be an odd prime with $p \equiv 1 \bmod 12$, then we have*

$$\frac{\tau^3\left(\chi_4\overline{\chi}_3\right)}{\tau^3\left(\chi_4\chi_3\right)} = \frac{d^2 - 2p}{2p} \pm \frac{3\sqrt{3}db}{2p} i.$$

Corollary 1.4. *Let p be an odd prime with $p \equiv 1 \bmod 12$, then we have*

$$\frac{\tau\left(\overline{\chi}_4\chi_3\right)}{\tau\left(\chi_4\chi_3\right)} = \chi_4(-1) \cdot \frac{\alpha}{\sqrt{p}} \pm \frac{\beta}{\sqrt{p}} \cdot i.$$

How to determine the plus or minus signs in Corollaries 1.3 and 1.4 is also a meaningful problem. Interested readers may consider it.

2. Several lemmas

In this section, we first give three simple lemmas. Of course, the proofs of some lemmas need the knowledge of character sums. They can be found in many number theory books, such as [1–3], here we do not need to list.

Lemma 2.1. *Let p be a prime with $p \equiv 1 \bmod 12$, then we have the identity*

$$\frac{\tau^3\left(\chi_4\overline{\chi}_3\right)}{\tau^3\left(\chi_4\chi_3\right)} + \frac{\tau^3\left(\chi_4\chi_3\right)}{\tau^3\left(\chi_4\overline{\chi}_3\right)} = \frac{d^2 - 2p}{p}.$$

Proof. Note that $\chi_3^2 = \overline{\chi}_3$ and $\chi_4^2 = \overline{\chi}_4^2 = \chi_2$, from the properties of the classical Gauss sums we have

$$
\begin{aligned}
&\sum_{a=0}^{p-1}\chi_4\chi_3\left(a^2-1\right)=\sum_{a=0}^{p-1}\chi_4\chi_3\left((a+1)^2-1\right)=\sum_{a=1}^{p-1}\chi_4\chi_3(a)\chi_4\chi_3\left(a+2\right)\\
=\ &\frac{1}{\tau\left(\overline{\chi}_4\overline{\chi}_3\right)}\sum_{b=1}^{p-1}\overline{\chi}_4\overline{\chi}_3(b)\sum_{a=1}^{p-1}\chi_4\chi_3(a)e\left(\frac{b(a+2)}{p}\right)\\
=\ &\frac{\tau\left(\chi_4\chi_3\right)}{\tau\left(\overline{\chi}_4\overline{\chi}_3\right)}\sum_{b=1}^{p-1}\overline{\chi}_4\overline{\chi}_3(b)\overline{\chi}_4\overline{\chi}_3(b)e\left(\frac{2b}{p}\right)=\frac{\tau\left(\chi_4\chi_3\right)}{\tau\left(\overline{\chi}_4\overline{\chi}_3\right)}\sum_{b=1}^{p-1}\chi_2\chi_3(b)e\left(\frac{2b}{p}\right)\\
=\ &\frac{\overline{\chi}_3(2)\chi_2(2)\cdot\tau\left(\chi_4\chi_3\right)\cdot\tau\left(\chi_2\chi_3\right)}{\tau\left(\overline{\chi}_4\overline{\chi}_3\right)}. \qquad (2.1)
\end{aligned}
$$

On the other hand, for any integer b with $(b,p)=1$, from the identity

$$
\sum_{a=0}^{p-1}e\left(\frac{ba^2}{p}\right)=1+\sum_{a=1}^{p-1}\left(1+\chi_2(a)\right)e\left(\frac{ba}{p}\right)=\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ba}{p}\right)=\chi_2(b)\cdot\sqrt{p}
$$

and note that $\chi_3(-1)=1, \overline{\chi}_4\chi_2=\chi_4$, we also have

$$
\begin{aligned}
&\sum_{a=0}^{p-1}\chi_4\chi_3\left(a^2-1\right)=\frac{1}{\tau\left(\overline{\chi}_4\overline{\chi}_3\right)}\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}_4\overline{\chi}_3(b)e\left(\frac{b(a^2-1)}{p}\right)\\
=\ &\frac{1}{\tau\left(\overline{\chi}_4\overline{\chi}_3\right)}\sum_{b=1}^{p-1}\overline{\chi}_4\overline{\chi}_3(b)e\left(\frac{-b}{p}\right)\sum_{a=0}^{p-1}e\left(\frac{ba^2}{p}\right)=\frac{\sqrt{p}}{\tau\left(\overline{\chi}_4\overline{\chi}_3\right)}\sum_{b=1}^{p-1}\overline{\chi}_4\overline{\chi}_3(b)\chi_2(b)e\left(\frac{-b}{p}\right)\\
=\ &\frac{\chi_4(-1)\sqrt{p}\cdot\tau\left(\chi_4\overline{\chi}_3\right)}{\tau\left(\overline{\chi}_4\overline{\chi}_3\right)}. \qquad (2.2)
\end{aligned}
$$

From (2.1), (2.2) and note that $\chi_3^3(2)=\chi_2^2(2)=\chi_4^2(-1)=1$, we have the identity

$$
\tau\left(\chi_2\chi_3\right)=\chi_3(2)\chi_2(2)\chi_4(-1)\sqrt{p}\cdot\frac{\tau\left(\chi_4\overline{\chi}_3\right)}{\tau\left(\chi_4\chi_3\right)}
$$

or

$$
\tau^3\left(\chi_2\chi_3\right)=\chi_2(2)\cdot\chi_4(-1)\cdot p^{\frac{3}{2}}\cdot\frac{\tau^3\left(\chi_4\overline{\chi}_3\right)}{\tau^3\left(\chi_4\chi_3\right)}. \qquad (2.3)
$$

Since $\chi_2(2)=(-1)^{\frac{p^2-1}{8}}$ and $\chi_4(-1)=(-1)^{\frac{p-1}{4}}$, so we have

$$
\chi_2(2)\cdot\chi_4(-1)=(-1)^{\frac{p^2-1}{8}}\cdot(-1)^{\frac{p-1}{4}}=(-1)^{\frac{p-1}{2}}=1. \qquad (2.4)
$$

Combining (2.3), (2.4) and formula (1.4) we have the identity

$$
p^{\frac{3}{2}}\cdot\left(\frac{\tau^3\left(\chi_4\overline{\chi}_3\right)}{\tau^3\left(\chi_4\chi_3\right)}+\frac{\tau^3\left(\chi_4\chi_3\right)}{\tau^3\left(\chi_4\overline{\chi}_3\right)}\right)=\tau^3\left(\chi_2\chi_3\right)+\tau^3\left(\chi_2\overline{\chi}_3\right)=p^{\frac{1}{2}}\cdot\left(d^2-2p\right)
$$

or

$$\frac{\tau^3(\chi_4\overline{\chi}_3)}{\tau^3(\chi_4\chi_3)}+\frac{\tau^3(\chi_4\chi_3)}{\tau^3(\chi_4\overline{\chi}_3)}=\frac{d^2-2p}{p}.$$

This proves Lemma 2.1.

Lemma 2.2. *Let p be an odd prime with $p \equiv 1 \bmod 24$, then we have the identity*

$$\tau^6(\overline{\chi}_4\chi_3)=p^3\cdot\frac{\tau^6\left(\chi_8^3\overline{\chi}_3\right)}{\tau^6(\chi_8\chi_3)} \quad\text{and}\quad \tau^6(\overline{\chi}_4\overline{\chi}_3)=p^3\cdot\frac{\tau^6\left(\chi_8^3\chi_3\right)}{\tau^6(\chi_8\overline{\chi}_3)}.$$

Proof. From the method of proving Lemma 2.1 we have

$$\begin{aligned}
&\sum_{a=0}^{p-1}\chi_8\chi_3\left(a^2-1\right)=\frac{1}{\tau(\overline{\chi}_8\overline{\chi}_3)}\sum_{b=1}^{p-1}\overline{\chi}_8\overline{\chi}_3(b)\sum_{a=1}^{p-1}\chi_8\chi_3(a)e\left(\frac{b(a+2)}{p}\right)\\
=\ &\frac{\tau(\chi_8\chi_3)}{\tau(\overline{\chi}_8\overline{\chi}_3)}\sum_{b=1}^{p-1}\overline{\chi}_8\overline{\chi}_3(b)\overline{\chi}_8\overline{\chi}_3(b)e\left(\frac{2b}{p}\right)=\frac{\tau(\chi_8\chi_3)}{\tau(\overline{\chi}_8\overline{\chi}_3)}\sum_{b=1}^{p-1}\overline{\chi}_4\chi_3(b)e\left(\frac{2b}{p}\right)\\
=\ &\frac{\overline{\chi}_3(2)\chi_4(2)\cdot\tau(\chi_8\chi_3)\cdot\tau(\overline{\chi}_4\chi_3)}{\tau(\overline{\chi}_8\overline{\chi}_3)}
\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}
&\sum_{a=0}^{p-1}\chi_8\chi_3\left(a^2-1\right)=\frac{1}{\tau(\overline{\chi}_8\overline{\chi}_3)}\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}_8\overline{\chi}_3(b)e\left(\frac{b(a^2-1)}{p}\right)\\
=\ &\frac{1}{\tau(\overline{\chi}_8\overline{\chi}_3)}\sum_{b=1}^{p-1}\overline{\chi}_8\overline{\chi}_3(b)e\left(\frac{-b}{p}\right)\sum_{a=0}^{p-1}e\left(\frac{ba^2}{p}\right)=\frac{\sqrt{p}}{\tau(\overline{\chi}_8\overline{\chi}_3)}\sum_{b=1}^{p-1}\overline{\chi}_8\overline{\chi}_3(b)\chi_2(b)e\left(\frac{-b}{p}\right)\\
=\ &\frac{\chi_8(-1)\sqrt{p}\cdot\tau\left(\chi_8^3\overline{\chi}_3\right)}{\tau(\overline{\chi}_8\overline{\chi}_3)}.
\end{aligned}\tag{2.6}$$

Note that $\chi_3^6(2)=1, \chi_8^2(-1)=1, \chi_4^2(2)=\chi_2(2)=1$, from (2.5) and (2.6) we have

$$\tau^6(\overline{\chi}_4\chi_3)=p^3\cdot\frac{\tau^6\left(\chi_8^3\overline{\chi}_3\right)}{\tau^6(\chi_8\chi_3)}.\tag{2.7}$$

Substituting $\overline{\chi}_3$ for χ_3 in (2.7) gives us the identity

$$\tau^6(\overline{\chi}_4\overline{\chi}_3)=p^3\cdot\frac{\tau^6\left(\chi_8^3\chi_3\right)}{\tau^6(\chi_8\overline{\chi}_3)}.\tag{2.8}$$

Now Lemma 2.2 follows from (2.7) and (2.8).

Lemma 2.3. *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any character χ modulo p, we have the identity*

$$\tau\left(\chi^3\right)=\frac{1}{p}\cdot\chi^3(3)\cdot\tau(\chi)\cdot\tau(\chi\chi_3)\cdot\tau(\chi\overline{\chi}_3),$$

where χ_3 is a character of order three modulo p.

Proof. For this see [16] or [17]. The general result can also be found in [18].

3. Proofs of the theorems

Now we shall complete the proofs of our all results. First we prove Theorem 1.1. Let p be an odd prime with $p \equiv 1 \bmod 12$, then note that $\chi_4(-1) = \overline{\chi}_4(-1) = (-1)^{\frac{p-1}{4}}$, $\chi_4^3 = \overline{\chi}_4$, $\tau(\chi_4)\cdot\tau(\overline{\chi}_4) = \chi_4(-1)\cdot p$. From Lemmas 2.1 and 2.3 we have

$$\begin{aligned}
&\tau^6(\chi_4\overline{\chi}_3) + \tau^6(\chi_4\chi_3) = \frac{d^2-2p}{p}\cdot\tau^3(\chi_4\overline{\chi}_3)\cdot\tau^3(\chi_4\chi_3)\\
=\ &\frac{d^2-2p}{p}\cdot\overline{\chi}_4(3)\cdot(-1)^{\frac{p-1}{4}}\cdot\tau^6(\overline{\chi}_4).
\end{aligned} \tag{3.1}$$

Similarly, we also have

$$\tau^6(\overline{\chi}_4\overline{\chi}_3) + \tau^6(\overline{\chi}_4\chi_3) = \frac{d^2-2p}{p}\cdot\chi_4(3)\cdot(-1)^{\frac{p-1}{4}}\cdot\tau^6(\chi_4). \tag{3.2}$$

From (1.2) we have the identity

$$8\cdot p^{\frac{3}{2}}\cdot\alpha^3 = \left(\tau^2(\chi_4)+\tau^2(\overline{\chi}_4)\right)^3 = \tau^6(\chi_4)+\tau^6(\overline{\chi}_4)+3p^2\cdot 2\sqrt{p}\cdot\alpha$$

or

$$\tau^6(\chi_4)+\tau^6(\overline{\chi}_4) = 8p^{\frac{3}{2}}\cdot\alpha^3 - 6p^{\frac{5}{2}}\cdot\alpha = 2p^{\frac{3}{2}}\cdot\alpha\cdot\left(4\alpha^2-3p\right). \tag{3.3}$$

Since $p \equiv 1 \bmod 12$, so $\chi_4^2(3) = \chi_2(3) = \left(\frac{p}{3}\right) = \left(\frac{1}{3}\right) = 1$. Therefore, $\chi_4(3) = \overline{\chi}_4(3)$. Combining (3.1)–(3.3) we have

$$\begin{aligned}
&\tau^6(\chi_4\overline{\chi}_3)+\tau^6(\chi_4\chi_3)+\tau^6(\overline{\chi}_4\overline{\chi}_3)+\tau^6(\overline{\chi}_4\chi_3)\\
=\ &\frac{d^2-2p}{p}\cdot\chi_4(3)\cdot(-1)^{\frac{p-1}{4}}\cdot\left(\tau^6(\chi_4)+\tau^6(\overline{\chi}_4)\right)\\
=\ &\frac{d^2-2p}{p}\cdot\chi_4(3)\cdot(-1)^{\frac{p-1}{4}}\cdot 2p^{\frac{3}{2}}\cdot\alpha\cdot\left(4\alpha^2-3p\right)\\
=\ &\chi_4(3)\cdot(-1)^{\frac{p-1}{4}}\cdot 2\sqrt{p}\cdot\alpha\cdot\left(d^2-2p\right)\cdot\left(4\alpha^2-3p\right).
\end{aligned}$$

This proves Theorem 1.1.

Now we prove Theorem 1.2. From Lemma 2.1 we have

$$\tau^6(\chi_4\overline{\chi}_3)+\tau^6(\chi_4\chi_3) = \frac{d^2-2p}{p}\cdot\tau^3(\chi_4\overline{\chi}_3)\cdot\tau^3(\chi_4\chi_3) \tag{3.4}$$

and

$$\left(\tau^3(\chi_4\overline{\chi}_3)+\tau^3(\chi_4\chi_3)\right)^2 = \frac{d^2}{p}\cdot\tau^3(\chi_4\overline{\chi}_3)\cdot\tau^3(\chi_4\chi_3). \tag{3.5}$$

Note that $4p - d^2 = 27b^2$, from (3.5) we also have

$$\left(\tau^3(\chi_4\overline{\chi}_3)-\tau^3(\chi_4\chi_3)\right)^2 = \frac{d^2-4p}{p}\cdot\tau^3(\chi_4\overline{\chi}_3)\cdot\tau^3(\chi_4\chi_3) = -27\cdot\frac{b^2}{p}\cdot\tau^3(\chi_4\overline{\chi}_3)\cdot\tau^3(\chi_4\chi_3). \tag{3.6}$$

From (3.5) and (3.6) we may immediately deduce the identity

$$\left(\frac{\tau^3(\chi_4\overline{\chi}_3)-\tau^3(\chi_4\chi_3)}{\tau^3(\chi_4\overline{\chi}_3)+\tau^3(\chi_4\chi_3)}\right)^2=-27\cdot\frac{b^2}{d^2}.$$

This proves Theorem 1.2.

Now we prove Theorem 1.3. From Lemmas 2.1 and 2.2 we have

$$\begin{aligned}\frac{\left(d^2-2p\right)^2}{p^2}&=\left(\frac{\tau^3(\overline{\chi}_4\overline{\chi}_3)}{\tau^3(\overline{\chi}_4\chi_3)}+\frac{\tau^3(\overline{\chi}_4\chi_3)}{\tau^3(\overline{\chi}_4\overline{\chi}_3)}\right)^2=\frac{\tau^6(\overline{\chi}_4\overline{\chi}_3)}{\tau^6(\overline{\chi}_4\chi_3)}+\frac{\tau^6(\overline{\chi}_4\chi_3)}{\tau^6(\overline{\chi}_4\overline{\chi}_3)}+2\\ &=\frac{\tau^6\left(\chi_8^3\chi_3\right)\cdot\tau^6(\chi_8\chi_3)}{\tau^6\left(\chi_8^3\overline{\chi}_3\right)\cdot\tau^6(\chi_8\overline{\chi}_3)}+\frac{\tau^6\left(\chi_8^3\overline{\chi}_3\right)\cdot\tau^6(\chi_8\overline{\chi}_3)}{\tau^6\left(\chi_8^3\chi_3\right)\cdot\tau^6(\chi_8\chi_3)}+2,\end{aligned}$$

$$\begin{aligned}&\left(\tau^6\left(\chi_8^3\chi_3\right)\cdot\tau^6(\chi_8\chi_3)+\tau^6\left(\chi_8^3\overline{\chi}_3\right)\cdot\tau^6(\chi_8\overline{\chi}_3)\right)^2\\ =&\ \frac{\left(d^2-2p\right)^2}{p^2}\cdot\tau^6\left(\chi_8^3\chi_3\right)\cdot\tau^6(\chi_8\chi_3)\cdot\tau^6\left(\chi_8^3\overline{\chi}_3\right)\cdot\tau^6(\chi_8\overline{\chi}_3)\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}&\left(\tau^6\left(\chi_8^3\chi_3\right)\cdot\tau^6(\chi_8\chi_3)-\tau^6\left(\chi_8^3\overline{\chi}_3\right)\cdot\tau^6(\chi_8\overline{\chi}_3)\right)^2\\ =&\ \frac{-27b^2d^2}{p^2}\cdot\tau^6\left(\chi_8^3\chi_3\right)\cdot\tau^6(\chi_8\chi_3)\cdot\tau^6\left(\chi_8^3\overline{\chi}_3\right)\cdot\tau^6(\chi_8\overline{\chi}_3).\end{aligned}\tag{3.8}$$

Combining (3.7) and (3.8) we have the identity

$$\left(\frac{\tau^6(\chi_8\chi_3)\,\tau^6\left(\chi_8^3\chi_3\right)-\tau^6(\chi_8\overline{\chi}_3)\,\tau^6\left(\chi_8^3\overline{\chi}_3\right)}{\tau^6(\chi_8\chi_3)\,\tau^6\left(\chi_8^3\chi_3\right)+\tau^6(\chi_8\overline{\chi}_3)\,\tau^6\left(\chi_8^3\overline{\chi}_3\right)}\right)^2=-27\cdot\frac{b^2d^2}{(d^2-2p)^2}.$$

This proves Theorem 1.3.

From Lemma 2.1 we have $G_0(p)=2$ and $G_1(p)=\frac{d^2-2p}{p}$. For any integer $n\geq 0$, from Lemma 2.1 and the definition of $G_n(p)$ we have

$$\begin{aligned}\frac{d^2-2p}{p}\cdot G_{n+1}(p)&=G_1(p)\cdot G_{n+1}(p)\\ &=\left(\frac{\tau^3(\chi_4\overline{\chi}_3)}{\tau^3(\chi_4\chi_3)}+\frac{\tau^3(\chi_4\chi_3)}{\tau^3(\chi_4\overline{\chi}_3)}\right)\cdot\left(\frac{\tau^{3n+3}(\chi_4\overline{\chi}_3)}{\tau^{3n+3}(\chi_4\chi_3)}+\frac{\tau^{3n+3}(\chi_4\chi_3)}{\tau^{3n+3}(\chi_4\overline{\chi}_3)}\right)\\ &=\frac{\tau^{3n+6}(\chi_4\overline{\chi}_3)}{\tau^{3n+6}(\chi_4\chi_3)}+\frac{\tau^{3n+6}(\chi_4\chi_3)}{\tau^{3n+6}(\chi_4\overline{\chi}_3)}+\frac{\tau^{3n}(\chi_4\overline{\chi}_3)}{\tau^{3n}(\chi_4\chi_3)}+\frac{\tau^{3n}(\chi_4\chi_3)}{\tau^{3n}(\chi_4\overline{\chi}_3)}\\ &=G_{n+2}(p)+G_n(p),\end{aligned}$$

which implies the second order recurrence formula

$$G_{n+2}(p)=\frac{d^2-2p}{p}\cdot G_{n+1}(p)-G_n(p),\ n\geq 0.$$

Let x_1 and x_2 denote two roots of the quadratic equation $x^2 - \frac{d^2-2p}{p} \cdot x + 1 = 0$. Then note that $4p = d^2 + 27b^2$, we have

$$x_1 = \frac{d^2 - 2p + 3\sqrt{3} \cdot idb}{2p} \quad \text{and} \quad x_2 = \frac{d^2 - 2p - 3\sqrt{3} \cdot idb}{2p}.$$

From the properties of the second order recurrence formula and the initial conditions $G_0(p) = 2$, $G_1(p) = \frac{d^2-2p}{p}$, we may immediately deduce the general term

$$G_n(p) = \left(\frac{d^2 - 2p + 3\sqrt{3} \cdot idb}{2p}\right)^n + \left(\frac{d^2 - 2p - 3\sqrt{3} \cdot idb}{2p}\right)^n.$$

This proves Theorem 1.4.

Similarly, we can also deduce Theorem 1.5. In fact, note that $\chi_4(3) \cdot \chi_4(-1) = 1$ and $\tau(\chi_4)\tau(\overline{\chi}_4) = \chi_4(-1) \cdot p = \tau(\chi_4\overline{\chi}_3)\tau(\overline{\chi}_4\chi_3)$. From Lemma 2.3 we have

$$\tau(\overline{\chi}_4) = \frac{1}{p} \cdot \overline{\chi}_4(3) \cdot \tau(\chi_4) \cdot \tau(\chi_4\chi_3) \cdot \tau(\chi_4\overline{\chi}_3)$$

or

$$\tau^2(\overline{\chi}_4) = \tau(\chi_4\chi_3) \cdot \tau(\chi_4\overline{\chi}_3) = p \cdot \chi_4(-1) \cdot \frac{\tau(\chi_4\chi_3)}{\tau(\overline{\chi}_4\chi_3)} \tag{3.9}$$

and

$$\tau^2(\chi_4) = \tau(\overline{\chi}_4\chi_3) \cdot \tau(\overline{\chi}_4\overline{\chi}_3) = p \cdot \chi_4(-1) \cdot \frac{\tau(\overline{\chi}_4\chi_3)}{\tau(\chi_4\chi_3)}. \tag{3.10}$$

Combining (1.2), (3.9) and (3.10) we have the identity

$$\frac{\tau(\chi_4\chi_3)}{\tau(\overline{\chi}_4\chi_3)} + \frac{\tau(\overline{\chi}_4\chi_3)}{\tau(\chi_4\chi_3)} = \chi_4(-1) \cdot \frac{2\alpha}{\sqrt{p}}. \tag{3.11}$$

Now let us divide p into two cases:

If $p \equiv 1 \bmod 24$, then $\chi_4(-1) = 1$. From (3.11) and the method of proving Theorem 1.4 we have $H_0(p) = 2$, $H_1(p) = \frac{2\alpha}{\sqrt{p}}$ and $H_{n+2}(p) = \frac{2\alpha}{\sqrt{p}} \cdot H_{n+1}(p) - H_n(p)$ for all $n \geq 0$. The general term of $H_n(p)$ is

$$H_n(p) = \left(\frac{\alpha + i\beta}{\sqrt{p}}\right)^n + \left(\frac{\alpha - i\beta}{\sqrt{p}}\right)^n, \ n \geq 0, \ i^2 = -1,$$

where β is defined as in (1.3).

If $p \equiv 13 \bmod 24$, then $\chi_4(-1) = -1$. From (3.11) and the method of proving Theorem 1.4 we have $H_0(p) = 2$, $H_1(p) = -\frac{2\alpha}{\sqrt{p}}$ and $H_{n+2}(p) = -\frac{2\alpha}{\sqrt{p}} \cdot H_{n+1}(p) - H_n(p)$ for all $n \geq 0$. The general term of $H_n(p)$ is

$$H_n(p) = \left(\frac{-\alpha + i\beta}{\sqrt{p}}\right)^n + \left(\frac{-\alpha - i\beta}{\sqrt{p}}\right)^n, \ n \geq 0, \ i^2 = -1,$$

These complete the proofs of our all results.

4. Conclusions

The main results of this paper are to prove some new identities for the classical Gauss sums. For example, if p is a prime with $p \equiv 1 \bmod 12$, then for any character χ_4 of order four and character χ_3 of order three modulo p, we have the identity

$$\begin{aligned}
& \tau^6(\chi_4\chi_3) + \tau^6(\chi_4\overline{\chi}_3) + \tau^6(\overline{\chi}_4\chi_3) + \tau^6(\overline{\chi}_4\overline{\chi}_3) \\
= \; & \chi_4(3) \cdot (-1)^{\frac{p-1}{4}} \cdot 2 \cdot \sqrt{p} \cdot \alpha \cdot \left(d^2 - 2p\right) \cdot \left(4\alpha^2 - 3p\right).
\end{aligned}$$

These results not only give the exact values of some special Gauss sums, and they are also some new contribution to research in related fields.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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