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Research article

On the absence of global solutions to two-times-fractional differential inequalities involving Hadamard-Caputo and Caputo fractional derivatives

Ibtehal Alazman[1]**, Mohamed Jleli**[2] **and Bessem Samet**[2,*]

[1] Department of Mathematics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh 11566, Saudi Arabia

[2] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia

* **Correspondence:** Email: bsamet@ksu.edu.sa.

Abstract: In this paper, we consider a two-times nonlinear fractional differential inequality involving both Hadamard-Caputo and Caputo fractional derivatives of different orders, with a singular potential term. We obtain sufficient criteria depending on the parameters of the problem, for which a global solution does not exist. Some examples are provided to support our main results.

Keywords: two-times fractional differential inequality; global solution; nonexistence; Hadamard-Caputo fractional derivative; Caputo fractional derivative
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1. Introduction

Time-fractional differential equations arise in the mathematical modeling of a variety of real-world phenomena in many areas of sciences and engineering, such as elasticity, heat transfer, circuits systems, continuum mechanics, fluid mechanics, wave theory, etc. For more details, we refer the reader to [4, 6–8, 14, 15, 17, 24] and the references therein. Consequently, the study of time-fractional differential equations attracted much attention of many researchers (see e.g. [1,5,9,10,19,22,23] and the references therein).

Multi-time differential equations arise, for example, in analyzing frequency and amplitude modulation in oscillators, see Narayan and Roychowdhury [18]. Some methods for solving Multi-time differential equations can be found in [20, 21].

The study of blowing-up solutions to time-fractional differential equations was initiated by Kirane and his collaborators, see e.g. [3, 11–13]. In particular, Kirane et al. [11] considered the two-times fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}_{0|t}u(t,s) + {}^{C}D^{\beta}_{0|s}|u|^{m}(t,s) = |u|^{p}(t,s), & t,s>0, \\ \\ u(0,s) = u_0(s),\ u(t,0) = u_1(t), & t,s>0, \end{cases} \tag{1.1}$$

where $p, m > 1$, $0 < \alpha, \beta < 1$, ${}^{C}D^{\alpha}_{0|t}$ is the Caputo fractional derivative of order α with respect to the first time-variable t, and ${}^{C}D^{\beta}_{0|s}$ is the Caputo fractional derivative of order β with respect to the second time-variable s. Namely, the authors provided sufficient conditions for which any solution to (1.1) blows-up in a finite time. In the same reference, the authors extended their study to the case of systems.

In this paper, we investigate the nonexistence of global solutions to two-times-fractional differential inequalities of the form

$$\begin{cases} {}^{HC}D^{\alpha}_{a|t}u(t,s) + {}^{C}D^{\beta}_{a|s}|u|^{m}(t,s) \geq (s-a)^{\gamma}\left(\ln\frac{t}{a}\right)^{\sigma}|u|^{p}(t,s), & t,s>a, \\ \\ u(a,s) = u_0(s),\ u(t,a) = u_1(t), & t,s>a, \end{cases} \tag{1.2}$$

where $p > 1$, $m \geq 1$, $\gamma, \sigma \in \mathbb{R}$, $a > 0$, $0 < \alpha, \beta < 1$, ${}^{HC}D^{\alpha}_{a|t}$ is the Hadamard-Caputo fractional derivative of order α with respect to the first time-variable t, and ${}^{C}D^{\beta}_{a|s}$ is the Caputo fractional derivative of order β with respect to the second time-variable s. Using the test function method (see e.g. [16]) and a judicious choice of a test function, we establish sufficient conditions ensuring the nonexistence of global solutions to (1.2). Our obtained conditions depend on the parameters $\alpha, \beta, p, m, \gamma, \sigma$, and the initial values.

Our motivation for considering problems of type (1.2) is to study the combination effect of the two fractional derivatives of different nature ${}^{HC}D^{\alpha}_{a|t}$ and ${}^{C}D^{\beta}_{a|s}$ on the nonexistence of global solutions to (1.2). As far as we know, the study of nonexistence of global solutions for time fractional differential equations (or inequalities) involving both Hadamard-Caputo and Caputo fractional derivatives, was never considered in the literature.

The rest of the paper is organized as follows: In Section 2, we recall some concepts from fractional calculus and provide some useful lemmas. In Section 3, we state our main results and provide some examples. Section 4 is devoted to the proofs of our main results.

2. Some preliminaries

Let $a, T \in \mathbb{R}$ be such that $0 < a < T$. The left-sided and right-sided Riemann-Liouville fractional integrals of order $\theta > 0$ of a function $\vartheta \in L^1([a,T])$, are defined respectively by (see [10])

$$(I^{\theta}_{a}\vartheta)(t) = \frac{1}{\Gamma(\theta)}\int_a^t (t-\tau)^{\theta-1}\vartheta(\tau)\,d\tau$$

and

$$(I^{\theta}_{T}\vartheta)(t) = \frac{1}{\Gamma(\theta)}\int_t^T (\tau-t)^{\theta-1}\vartheta(\tau)\,d\tau,$$

for almost everywhere $t \in [a,T]$, where Γ is the Gamma function.

Notice that, if $\vartheta \in C([a,T])$, then $I_a^\theta\vartheta, I_T^\theta\vartheta \in C([a,T])$ with

$$(I_a^\theta\vartheta)(a) = (I_T^\theta\vartheta)(T) = 0. \tag{2.1}$$

The Caputo fractional derivative of order $\theta \in (0,1)$ of a function $\vartheta \in AC([a,\infty))$, is defined by (see [10])

$${}^C D_a^\theta\vartheta(t) = (I_a^{1-\theta}\vartheta')(t) = \frac{1}{\Gamma(1-\theta)}\int_a^t (t-\tau)^{-\theta}\vartheta'(\tau)\,d\tau,$$

for almost everywhere $t \geq a$.

Lemma 2.1. *[see [10]] Let $\kappa > 0$, $p,q \geq 1$, and $\frac{1}{p}+\frac{1}{q} \leq 1+\kappa$ ($p \neq 1$, $q \neq 1$, in the case $\frac{1}{p}+\frac{1}{q} = 1+\kappa$). Let $\vartheta \in L^p([a,T]$ and $w \in L^q([a,T])$. Then*

$$\int_a^T (I_a^\kappa\vartheta)(t)w(t)\,dt = \int_a^T \vartheta(t)(I_T^\kappa w)(t)\,dt.$$

The left-sided and right-sided Hadamard fractional integrals of order $\theta > 0$ of a function $\vartheta \in L^1([a,T])$, are defined respectively by (see [10])

$$(J_a^\theta\vartheta)(t) = \frac{1}{\Gamma(\theta)}\int_a^t \left(\ln\frac{t}{\tau}\right)^{\theta-1}\vartheta(\tau)\frac{1}{\tau}\,d\tau$$

and

$$(J_T^\theta\vartheta)(t) = \frac{1}{\Gamma(\theta)}\int_t^T \left(\ln\frac{\tau}{t}\right)^{\theta-1}\vartheta(\tau)\frac{1}{\tau}\,d\tau,$$

for almost everywhere $t \in [a,T]$.

Notice that, if $\vartheta \in C([a,T])$, then $J_a^\theta\vartheta, J_T^\theta\vartheta \in C([a,T])$ with

$$(J_a^\theta\vartheta)(a) = (J_T^\theta\vartheta)(T) = 0. \tag{2.2}$$

The Hadamard-Caputo fractional derivative of order $\theta \in (0,1)$ of a function $\vartheta \in AC([a,\infty))$, is defined by (see [2])

$${}^{HC}D_a^\theta\vartheta(t) = (J_a^{1-\theta}\delta\vartheta)(t) = \frac{1}{\Gamma(1-\theta)}\int_a^t \left(\ln\frac{t}{\tau}\right)^{-\theta}\delta\vartheta(\tau)\frac{1}{\tau}\,d\tau,$$

for almost everywhere $t \geq a$, where

$$\delta\vartheta(t) = t\vartheta'(t).$$

We have the following integration by parts rule.

Lemma 2.2. *Let $\kappa > 0$, $p,q \geq 1$, and $\frac{1}{p}+\frac{1}{q} \leq 1+\kappa$ ($p \neq 1$, $q \neq 1$, in the case $\frac{1}{p}+\frac{1}{q} = 1+\kappa$). If $\vartheta \circ \exp \in L^p([\ln a, \ln T])$ and $w \circ \exp \in L^q([\ln a, \ln T])$, then*

$$\int_a^T (J_a^\kappa\vartheta)(t)w(t)\frac{1}{t}\,dt = \int_a^T \vartheta(t)(J_T^\kappa w)(t)\frac{1}{t}\,dt.$$

Proof. Using the change of variable $x = \ln\tau$, we obtain

$$\begin{aligned}(J_a^\kappa\vartheta)(t) &= \frac{1}{\Gamma(\kappa)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\kappa-1}\vartheta(\tau)\frac{1}{\tau}\,d\tau\\ &= \frac{1}{\Gamma(\kappa)}\int_{\ln a}^{\ln t}(\ln t - x)^{\kappa-1}\,(\vartheta\circ\exp)(x)\,dx,\end{aligned}$$

that is,

$$(J_a^\kappa\vartheta)(t) = \left(I_{\ln a}^\kappa\vartheta\circ\exp\right)(\ln t). \tag{2.3}$$

Similarly, we have

$$(J_T^\kappa w)(t) = \left(I_{\ln T}^\kappa w\circ\exp\right)(\ln t). \tag{2.4}$$

By (2.3), we obtain

$$\int_a^T (J_a^\kappa\vartheta)(t)w(t)\frac{1}{t}\,dt = \int_a^T\left(I_{\ln a}^\kappa\vartheta\circ\exp\right)(\ln t)w(t)\frac{1}{t}\,dt.$$

Using the change of variable $x = \ln t$, we get

$$\int_a^T (J_a^\kappa\vartheta)(t)w(t)\frac{1}{t}\,dt = \int_{\ln a}^{\ln T}\left(I_{\ln a}^\kappa\vartheta\circ\exp\right)(x)(w\circ\exp)(x)\,dx.$$

Since $\vartheta\circ\exp\in L^p([\ln a,\ln T])$ and $w\circ\exp\in L^q([\ln a,\ln T])$, by Lemma 2.1, we deduce that

$$\int_a^T (J_a^\kappa\vartheta)(t)w(t)\frac{1}{t}\,dt = \int_{\ln a}^{\ln T}(\vartheta\circ\exp)(x)\left(I_{\ln T}^\kappa w\circ\exp\right)(x)\,dx.$$

Using again the change of variable $x = \ln t$, there holds

$$\int_a^T (J_a^\kappa\vartheta)(t)w(t)\frac{1}{t}\,dt = \int_a^T\vartheta(t)\left(I_{\ln T}^\kappa w\circ\exp\right)(\ln t)\frac{1}{t}\,dt.$$

Then, by (2.4), the desired result follows. □

By elementary calculations, we obtain the following properties.

Lemma 2.3. *For sufficiently large λ, let*

$$\phi_1(t) = \left(\ln\frac{T}{a}\right)^{-\lambda}\left(\ln\frac{T}{t}\right)^{\lambda},\quad a\le t\le T. \tag{2.5}$$

Let $\kappa\in(0,1)$. Then

$$(J_T^\kappa\phi_1)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\kappa+\lambda+1)}\left(\ln\frac{T}{a}\right)^{-\lambda}\left(\ln\frac{T}{t}\right)^{\kappa+\lambda}, \tag{2.6}$$

$$(J_T^\kappa\phi_1)'(t) = -\frac{\Gamma(\lambda+1)}{\Gamma(\kappa+\lambda)}\left(\ln\frac{T}{a}\right)^{-\lambda}\left(\ln\frac{T}{t}\right)^{\kappa+\lambda-1}\frac{1}{t}. \tag{2.7}$$

Lemma 2.4. *For sufficiently large λ, let*

$$\phi_2(s) = (T-a)^{-\lambda}(T-s)^{\lambda},\quad a\le s\le T. \tag{2.8}$$

Let $\kappa\in(0,1)$. Then

$$(I_T^\kappa\phi_2)(s) = \frac{\Gamma(\lambda+1)}{\Gamma(\kappa+\lambda+1)}(T-a)^{-\lambda}(T-s)^{\kappa+\lambda}, \tag{2.9}$$

$$(I_T^\kappa\phi_2)'(s) = -\frac{\Gamma(\lambda+1)}{\Gamma(\kappa+\lambda)}(T-a)^{-\lambda}(T-s)^{\kappa+\lambda-1}. \tag{2.10}$$

3. Main results

First, let us define global solutions to (1.2). To do this, we need to introduce the functional space

$$X_a := \{u \in C([a,\infty)\times[a,\infty)) : \ u(\cdot,s) \in AC([a,\infty)),\ |u|^m(t,\cdot) \in AC([a,\infty))\}.$$

We say that u is a global solution to (1.2), if $u \in X_a$ and u satisfies the fractional differential inequality

$${}^{HC}D^{\alpha}_{a|t}u(t,s) + {}^{C}D^{\beta}_{a|s}|u|^m(t,s) \geq (s-a)^{\gamma}\left(\ln\frac{t}{a}\right)^{\sigma}|u|^p(t,s)$$

for almost everywhere $t, s \geq a$, as well as the initial conditions

$$u(a,s) = u_0(s),\ u(t,a) = u_1(t), \quad t,s > a.$$

Now, we state our main results.

Theorem 3.1. *Let $u_0 \in L^1([a,\infty))$, $u_1 \in L^m([a,\infty), \frac{1}{t}\,dt)$, and $u_1 \not\equiv 0$. Let*

$$0 < \beta < \frac{1}{m} \leq 1, \quad \gamma > \max\left\{\frac{m-1}{1-m\beta}, m(\sigma+1)-1\right\}\beta. \tag{3.1}$$

If

$$m\max\{\gamma+1,\sigma+1\} < p < 1+\frac{\gamma}{\beta}, \tag{3.2}$$

then, for all $\alpha \in (0,1)$, (1.2) *admits no global solution.*

Remark 3.1. *Notice that by* (3.1)*, the set of exponents p satisfying* (3.2) *is nonempty.*

Theorem 3.2. *Let $u_0 \in L^1([a,\infty))$, $u_1 \in L^m([a,\infty), \frac{1}{t}\,dt)$, and $u_1 \not\equiv 0$. Let*

$$0 < \beta < \frac{1}{m} \leq 1, \quad 1-\frac{1}{m} < \alpha < 1, \quad \sigma > \frac{(m-1)(1-\alpha)}{1-m\beta} - \alpha. \tag{3.3}$$

If

$$\beta\max\left\{\frac{m-1}{1-m\beta}, m(\sigma+1)-1\right\} < \gamma < \frac{(\sigma+\alpha)\beta}{1-\alpha} \tag{3.4}$$

and

$$p = 1+\frac{\gamma}{\beta}, \tag{3.5}$$

then (1.2) *admits no global solution.*

Remark 3.2. *Notice that by* (3.3)*, the set of real numbers γ satisfying* (3.4) *is nonempty.*

We illustrate our obtained results by the following examples.

Example 3.1. *Consider the fractional differential inequality*

$$\begin{cases} {}^{HC}D^{\alpha}_{a|t}u(t,s) + {}^{C}D^{\frac{1}{4}}_{a|s}u^2(t,s) \geq (s-a)\left(\ln\frac{t}{a}\right)^{-1}|u|^p(t,s), \quad t,s>a, \\ u(a,s) = (1+s^2)^{-1},\ u(t,a) = \exp(-t), \quad t,s>a, \end{cases} \tag{3.6}$$

where $a > 0$ *and* $0 < \alpha < 1$. *Observe that* (3.6) *is a special case of* (1.2) *with*

$$\beta = \frac{1}{4},\ m = 2,\ \sigma = -1,\ \gamma = 1,\ u_0(s) = (1+s^2)^{-1},\ u_1(t) = \exp(-t).$$

Moreover, we have

$$0 < \beta = \frac{1}{4} < \frac{1}{2} = \frac{1}{m} < 1, \quad \max\left\{\frac{m-1}{1-m\beta}, m(\sigma+1)-1\right\}\beta = \frac{\max\{2,0\}}{4} = \frac{1}{2} < \gamma = 1,$$

and $u_0 \in L^1([a,\infty))$, $u_1 \in L^m([a,\infty), \frac{1}{t}\,dt)$. *Hence, condition* (3.1) *is satisfied. Then, by Theorem 3.1, we deduce that, if*

$$m \max\{\gamma+1, \sigma+1\} < p < 1 + \frac{\gamma}{\beta},$$

that is,

$$4 < p < 5,$$

then (3.6) *admits no global solution.*

Example 3.2. *Consider the fractional differential inequality*

$$\begin{cases} {}^{HC}D^{\frac{3}{4}}_{a|t} u(t,s) + {}^{C}D^{\frac{1}{2}}_{a|s} |u|(t,s) \geq (s-a)^{\frac{1}{4}} \left(\ln \frac{t}{a}\right)^{-\frac{1}{2}} |u|^{\frac{3}{2}}(t,s), \quad t, s > a, \\ u(a,s) = (1+s^2)^{-1},\ u(t,a) = \exp(-t), \quad t, s > a, \end{cases} \tag{3.7}$$

where $a > 0$. *Then* (3.7) *is a special case of* (1.2) *with*

$$\alpha = \frac{3}{4},\ \beta = \frac{1}{2},\ m = 1,\ \sigma = -\frac{1}{2},\ \gamma = \frac{1}{4},\ p = \frac{3}{2},\ u_0(s) = (1+s^2)^{-1},\ u_1(t) = \exp(-t).$$

On the other hand, we have

$$0 < \beta = \frac{1}{2} < 1 = \frac{1}{m}, \quad 1 - \frac{1}{m} = 0 < \alpha = \frac{3}{4} < 1, \quad \sigma = -\frac{1}{2} > -\frac{3}{4} = \frac{(m-1)(1-\alpha)}{1-m\beta} - \alpha,$$

which shows that condition (3.3) *is satisfied. Moreover, we have*

$$\beta \max\left\{\frac{m-1}{1-m\beta}, m(\sigma+1)-1\right\} = -\frac{1}{4} < \gamma = \frac{1}{4} < \frac{1}{2} = \frac{(\sigma+\alpha)\beta}{1-\alpha}, \quad p = \frac{3}{2} = 1 + \frac{\gamma}{\beta},$$

which shows that conditions (3.4) *and* (3.5) *are satisfied. Then, by Theorem 3.2, we deduce that* (3.7) *admits no global solution.*

4. Proofs of the main results

In this section, C denotes a positive constant independent on T, whose value may change from line to line.

Proof of Theorem 3.1. Suppose that $u \in X_a$ is a global solution to (1.2). For sufficiently large T and λ, let

$$\varphi(t,s) = \phi_1(t)\phi_2(s), \quad a \le t, s \le T,$$

where ϕ_1 and ϕ_2 are defined respectively by (2.5) and (2.8). Multiplying the inequality in (1.2) by $\frac{1}{t}\varphi$ and integrating over $\Omega_T := (a,T)\times(a,T)$, we obtain

$$\begin{aligned} &\int_{\Omega_T} (s-a)^\gamma \left(\ln\frac{t}{a}\right)^\sigma |u|^p \varphi(t,s)\frac{1}{t}\,dt\,ds \\ &\le \int_{\Omega_T} {}^{HC}D^\alpha_{a|t} u\,\varphi(t,s)\frac{1}{t}\,dt\,ds + \int_{\Omega_T} {}^{C}D^\beta_{a|s}|u|^m\,\varphi(t,s)\frac{1}{t}\,dt\,ds. \end{aligned} \tag{4.1}$$

On the other hand, using Lemma 2.2, integrating by parts, using the initial conditions, and taking in consideration (2.2), we obtain

$$\begin{aligned} &\int_a^T {}^{HC}D^\alpha_{a|t} u\,\varphi(t,s)\frac{1}{t}\,dt \\ &= \int_a^T \left(J^{1-\alpha}_{a|t} t\frac{\partial u}{\partial t}\right)(t,s)\varphi(t,s)\frac{1}{t}\,dt \\ &= \int_a^T \frac{\partial u}{\partial t}(t,s)\left(J^{1-\alpha}_{T|t}\varphi\right)(t,s)\,dt \\ &= \left[u(t,s)\left(J^{1-\alpha}_{T|t}\varphi\right)(t,s)\right]_{t=a}^T - \int_a^T u(t,s)\frac{\partial\left(J^{1-\alpha}_{T|t}\varphi\right)}{\partial t}(t,s)\,dt \\ &= -u_0(s)\left(J^{1-\alpha}_{T|t}\varphi\right)(a,s) - \int_a^T u(t,s)\frac{\partial\left(J^{1-\alpha}_{T|t}\varphi\right)}{\partial t}(t,s)\,dt. \end{aligned}$$

Integrating over (a,T), we get

$$\begin{aligned} &\int_{\Omega_T} {}^{HC}D^\alpha_{a|t} u\,\varphi(t,s)\frac{1}{t}\,dt\,ds \\ &= -\int_a^T u_0(s)\left(J^{1-\alpha}_{T|t}\varphi\right)(a,s)\,ds - \int_{\Omega_T} u(t,s)\frac{\partial\left(J^{1-\alpha}_{T|t}\varphi\right)}{\partial t}(t,s)\,dt\,ds. \end{aligned} \tag{4.2}$$

Similarly, using Lemma 2.1, integrating by parts, using the initial conditions, and taking in consideration (2.1), we obtain

$$\begin{aligned}
&\int_a^T {}^C D_{a|s}^{\beta} |u|^m \, \varphi(t,s)\, ds \\
&= \int_a^T \left(I_{a|s}^{1-\beta} \frac{\partial |u|^m}{\partial s}(t,s) \right) \varphi(t,s)\, ds \\
&= \int_a^T \frac{\partial |u|^m}{\partial s}(t,s) \left(I_{T|s}^{1-\beta} \varphi \right)(t,s)\, ds \\
&= \left[|u|^m(t,s) \left(I_{T|s}^{1-\beta} \varphi \right)(t,s) \right]_{s=a}^{T} - \int_a^T |u|^m(t,s) \frac{\partial \left(I_{T|s}^{1-\beta} \varphi \right)}{\partial s}(t,s)\, ds \\
&= -|u_1(t)|^m \left(I_{T|s}^{1-\beta} \varphi \right)(t,a) - \int_a^T |u|^m(t,s) \frac{\partial \left(I_{T|s}^{1-\beta} \varphi \right)}{\partial s}(t,s)\, ds.
\end{aligned}$$

Integrating over (a, T), there holds

$$\begin{aligned}
&\int_{\Omega_T} {}^C D_{a|s}^{\beta} |u|^m \, \varphi(t,s) \frac{1}{t}\, dt\, ds \\
&= -\int_a^T |u_1(t)|^m \left(I_{T|s}^{1-\beta} \varphi \right)(t,a) \frac{1}{t}\, dt - \int_{\Omega_T} |u|^m(t,s) \frac{\partial \left(I_{T|s}^{1-\beta} \varphi \right)}{\partial s}(t,s) \frac{1}{t}\, dt\ ds.
\end{aligned} \tag{4.3}$$

It follows from (4.1)–(4.3) that

$$\begin{aligned}
&\int_{\Omega_T} (s-a)^{\gamma} \left(\ln \frac{t}{a} \right)^{\sigma} |u|^p \varphi(t,s) \frac{1}{t}\, dt\, ds \\
&+ \int_a^T u_0(s) \left(J_{T|t}^{1-\alpha} \varphi \right)(a,s)\, ds + \int_a^T |u_1(t)|^m \left(I_{T|s}^{1-\beta} \varphi \right)(t,a) \frac{1}{t}\, dt \\
&\leq \int_{\Omega_T} |u| \left| \frac{\partial \left(J_{T|t}^{1-\alpha} \varphi \right)}{\partial t} \right| dt\, ds + \int_{\Omega_T} |u|^m \left| \frac{\partial \left(I_{T|s}^{1-\beta} \varphi \right)}{\partial s} \right| \frac{1}{t}\, dt\ ds.
\end{aligned} \tag{4.4}$$

On the other hand, by Young's inequality, we have

$$\begin{aligned}
&\int_{\Omega_T} |u| \left| \frac{\partial \left(J_{T|t}^{1-\alpha} \varphi \right)}{\partial t} \right| dt\, ds \\
&\leq \frac{1}{2} \int_{\Omega_T} (s-a)^{\gamma} \left(\ln \frac{t}{a} \right)^{\sigma} |u|^p \varphi(t,s) \frac{1}{t}\, dt\, ds \\
&+ C \int_{\Omega_T} t^{\frac{1}{p-1}} (s-a)^{\frac{-\gamma}{p-1}} \left(\ln \frac{t}{a} \right)^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}}(t,s) \left| \frac{\partial \left(J_{T|t}^{1-\alpha} \varphi \right)}{\partial t} \right|^{\frac{p}{p-1}} dt\, ds.
\end{aligned} \tag{4.5}$$

Similarly, since $p > m$, we have

$$\begin{aligned}
&\int_{\Omega_T} |u|^m \left| \frac{\partial \left(I_{T|s}^{1-\beta} \varphi \right)}{\partial s} \right| \frac{1}{t}\, dt\ ds \\
&\le \frac{1}{2} \int_{\Omega_T} (s-a)^{\gamma} \left(\ln \frac{t}{a} \right)^{\sigma} |u|^p \varphi(t,s) \frac{1}{t}\, dt\, ds \\
&+ C \int_{\Omega_T} \frac{1}{t} (s-a)^{\frac{-\gamma m}{p-m}} \left(\ln \frac{t}{a} \right)^{\frac{-\sigma m}{p-m}} \varphi^{\frac{-m}{p-m}}(t,s) \left| \frac{\partial \left(I_{T|s}^{1-\beta} \varphi \right)}{\partial s} \right|^{\frac{p}{p-m}} dt\ ds.
\end{aligned} \tag{4.6}$$

Hence, combining (4.4)–(4.6), we deduce that

$$\int_a^T u_0(s) \left(J_{T|t}^{1-\alpha} \varphi \right)(a,s)\, ds + \int_a^T |u_1(t)|^m \left(I_{T|s}^{1-\beta} \varphi \right)(t,a) \frac{1}{t}\, dt \le C(K_1 + K_2), \tag{4.7}$$

where

$$K_1 = \int_{\Omega_T} t^{\frac{1}{p-1}} (s-a)^{\frac{-\gamma}{p-1}} \left(\ln \frac{t}{a} \right)^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}}(t,s) \left| \frac{\partial \left(J_{T|t}^{1-\alpha} \varphi \right)}{\partial t} \right|^{\frac{p}{p-1}} dt\, ds$$

and

$$K_2 = \int_{\Omega_T} \frac{1}{t} (s-a)^{\frac{-\gamma m}{p-m}} \left(\ln \frac{t}{a} \right)^{\frac{-\sigma m}{p-m}} \varphi^{\frac{-m}{p-m}}(t,s) \left| \frac{\partial \left(I_{T|s}^{1-\beta} \varphi \right)}{\partial s} \right|^{\frac{p}{p-m}} dt\ ds.$$

By the definition of the function φ, we have

$$\left(J_{T|t}^{1-\alpha} \varphi \right)(a,s) = \phi_2(s) \left(J_{T|t}^{1-\alpha} \phi_1 \right)(a).$$

Thus, using (2.6), we obtain

$$\left(J_{T|t}^{1-\alpha} \varphi \right)(a,s) = C \phi_2(s) \left(\ln \frac{T}{a} \right)^{1-\alpha}.$$

Integrating over (a, T), we get

$$\int_a^T u_0(s) \left(J_{T|t}^{1-\alpha} \varphi \right)(a,s)\, ds = C \left(\ln \frac{T}{a} \right)^{1-\alpha} \int_a^T u_0(s) (T-a)^{-\lambda} (T-s)^{\lambda}\, ds. \tag{4.8}$$

Similarly, by the definition of the function φ, we have

$$\left(I_{T|s}^{1-\beta} \varphi \right)(t,a) = \phi_1(t) \left(I_{T|s}^{1-\beta} \phi_2 \right)(a).$$

Thus, using (2.9), we obtain

$$\left(I_{T|s}^{1-\beta} \varphi \right)(t,a) = C \phi_1(t) (T-a)^{1-\beta}.$$

Integrating over (a, T), we get

$$\int_a^T |u_1(t)|^m \left(I_{T|s}^{1-\beta} \varphi \right)(t,a) \frac{1}{t}\, dt = C (T-a)^{1-\beta} \int_a^T |u_1(t)|^m \left(\ln \frac{T}{a} \right)^{-\lambda} \left(\ln \frac{T}{t} \right)^{\lambda} \frac{1}{t}\, dt. \tag{4.9}$$

Combining (4.8) with (4.9), there holds

$$\begin{aligned}&\int_a^T u_0(s)\left(J_{T|t}^{1-\alpha}\varphi\right)(a,s)\,ds+\int_a^T |u_1(t)|^m\left(I_{T|s}^{1-\beta}\varphi\right)(t,a)\frac{1}{t}\,dt\\&=C\left(\ln\frac{T}{a}\right)^{1-\alpha}\int_a^T u_0(s)(T-a)^{-\lambda}(T-s)^{\lambda}\,ds\\&+C(T-a)^{1-\beta}\int_a^T |u_1(t)|^m\left(\ln\frac{T}{a}\right)^{-\lambda}\left(\ln\frac{T}{t}\right)^{\lambda}\frac{1}{t}\,dt.\end{aligned}$$

Since $u_0\in L^1([a,\infty))$, $u_1\in L^m([a,\infty),\frac{1}{t}\,dt)$, and $u_1\not\equiv 0$, by the dominated convergence theorem, we deduce that for sufficiently large T,

$$\begin{aligned}&\int_a^T u_0(s)\left(J_{T|t}^{1-\alpha}\varphi\right)(a,s)\,ds+\int_a^T |u_1(t)|^m\left(I_{T|s}^{1-\beta}\varphi\right)(t,a)\frac{1}{t}\,dt\\&\geq C(T-a)^{1-\beta}\int_a^\infty |u_1(t)|^m\frac{1}{t}\,dt.\end{aligned}\tag{4.10}$$

Now, we shall estimate the terms K_i, $i=1,2$. By the definition of the function φ, the term K_1 can be written as

$$K_1=\left(\int_a^T (s-a)^{\frac{-\gamma}{p-1}}\phi_2(s)\,ds\right)\left(\int_a^T t^{\frac{1}{p-1}}\left(\ln\frac{t}{a}\right)^{\frac{-\sigma}{p-1}}\phi_1^{\frac{-1}{p-1}}(t)\left|\left(J_{T|t}^{1-\alpha}\phi_1\right)'(t)\right|^{\frac{p}{p-1}}\,dt\right).\tag{4.11}$$

Next, by (2.8), we obtain

$$\begin{aligned}\int_a^T (s-a)^{\frac{-\gamma}{p-1}}\phi_2(s)\,ds &= (T-a)^{-\lambda}\int_a^T (s-a)^{\frac{-\gamma}{p-1}}(T-s)^{\lambda}\,ds\\&\leq \int_a^T (s-a)^{\frac{-\gamma}{p-1}}\,ds.\end{aligned}$$

On the other hand, by (3.1) and (3.2), it is clear that $\gamma<p-1$. Thus, we deduce that

$$\int_a^T (s-a)^{\frac{-\gamma}{p-1}}\phi_2(s)\,ds\leq C(T-a)^{1-\frac{\gamma}{p-1}}.\tag{4.12}$$

By (2.5) and (2.7), we have

$$\begin{aligned}&\int_a^T t^{\frac{1}{p-1}}\left(\ln\frac{t}{a}\right)^{\frac{-\sigma}{p-1}}\phi_1^{\frac{-1}{p-1}}(t)\left|\left(J_{T|t}^{1-\alpha}\phi_1\right)'(t)\right|^{\frac{p}{p-1}}\,dt\\&=\left(\ln\frac{T}{a}\right)^{-\lambda}\int_a^T\left(\ln\frac{T}{t}\right)^{\lambda-\frac{\alpha p}{p-1}}\left(\ln\frac{t}{a}\right)^{\frac{-\sigma}{p-1}}\frac{1}{t}\,dt\\&\leq\left(\ln\frac{T}{a}\right)^{-\frac{\alpha p}{p-1}}\int_a^T\left(\ln\frac{t}{a}\right)^{\frac{-\sigma}{p-1}}\frac{1}{t}\,dt.\end{aligned}$$

Notice that by (3.1) and (3.2), we have $\sigma<p-1$. Thus, we get

$$\begin{aligned}&\int_a^T t^{\frac{1}{p-1}}\left(\ln\frac{t}{a}\right)^{\frac{-\sigma}{p-1}}\phi_1^{\frac{-1}{p-1}}(t)\left|\left(J_{T|t}^{1-\alpha}\phi_1\right)'(t)\right|^{\frac{p}{p-1}}\,dt\\&\leq C\left(\ln\frac{T}{a}\right)^{1-\frac{\alpha p+\sigma}{p-1}}.\end{aligned}\tag{4.13}$$

Hence, it follows from (4.11)–(4.13) that

$$K_1 \le C(T-a)^{1-\frac{\gamma}{p-1}} \left(\ln \frac{T}{a}\right)^{1-\frac{\alpha p+\sigma}{p-1}}. \tag{4.14}$$

Similarly, we can write the term K_2 as

$$K_2 = \left(\int_a^T \frac{1}{t}\left(\ln \frac{t}{a}\right)^{\frac{-\sigma m}{p-m}} \phi_1(t)\, dt\right)\left(\int_a^T (s-a)^{\frac{-\gamma m}{p-m}} \phi_2^{\frac{-m}{p-m}}(s) \left|\left(I_{T|s}^{1-\beta}\phi_2\right)'(s)\right|^{\frac{p}{p-m}}\, ds\right). \tag{4.15}$$

By (2.5), we have

$$\begin{aligned}\int_a^T \frac{1}{t}\left(\ln \frac{t}{a}\right)^{\frac{-\sigma m}{p-m}} \phi_1(t)\, dt &= \left(\ln \frac{T}{a}\right)^{-\lambda} \int_a^T \left(\ln \frac{t}{a}\right)^{\frac{-\sigma m}{p-m}} \left(\ln \frac{T}{t}\right)^{\lambda} \frac{1}{t}\, dt \\ &\le \int_a^T \left(\ln \frac{t}{a}\right)^{\frac{-\sigma m}{p-m}} \frac{1}{t}\, dt.\end{aligned}$$

Notice that by (3.2), we have $\sigma m < p - m$. Thus, we get

$$\int_a^T \frac{1}{t}\left(\ln \frac{t}{a}\right)^{\frac{-\sigma m}{p-m}} \phi_1(t)\, dt \le C\left(\ln \frac{T}{a}\right)^{1-\frac{\sigma m}{p-m}}. \tag{4.16}$$

On the other hand, by (2.8) and (2.10), we have

$$\begin{aligned}&\int_a^T (s-a)^{\frac{-\gamma m}{p-m}} \phi_2^{\frac{-m}{p-m}}(s) \left|\left(I_{T|s}^{1-\beta}\phi_2\right)'(s)\right|^{\frac{p}{p-m}}\, ds \\ &= (T-a)^{-\lambda} \int_a^T (T-s)^{\lambda-\frac{\beta p}{p-m}} (s-a)^{\frac{-\gamma m}{p-m}}\, ds \\ &\le (T-a)^{-\frac{\beta p}{p-m}} \int_a^T (s-a)^{\frac{-\gamma m}{p-m}}\, ds.\end{aligned}$$

Notice that by (3.2), we have $p > m(\gamma + 1)$. Therefore, we obtain

$$\int_a^T (s-a)^{\frac{-\gamma m}{p-m}} \phi_2^{\frac{-m}{p-m}}(s) \left|\left(I_{T|s}^{1-\beta}\phi_2\right)'(s)\right|^{\frac{p}{p-m}}\, ds \le C(T-a)^{1-\frac{\gamma m+\beta p}{p-m}}. \tag{4.17}$$

Combining (4.16) with (4.17), there holds

$$K_2 \le C\left(\ln \frac{T}{a}\right)^{1-\frac{\sigma m}{p-m}} (T-a)^{1-\frac{\gamma m+\beta p}{p-m}}. \tag{4.18}$$

Hence, it follows from (4.14) and (4.18) that

$$K_1 + K_2 \le C\left[\left(\ln \frac{T}{a}\right)^{1-\frac{\alpha p+\sigma}{p-1}} (T-a)^{1-\frac{\gamma}{p-1}} + \left(\ln \frac{T}{a}\right)^{1-\frac{\sigma m}{p-m}} (T-a)^{1-\frac{\gamma m+\beta p}{p-m}}\right]. \tag{4.19}$$

Thus, by (4.7), (4.10), and (4.19), we deduce that

$$\begin{aligned}&\int_a^\infty |u_1(t)|^m \frac{1}{t}\, dt \\ &\le C\left[\left(\ln \frac{T}{a}\right)^{1-\frac{\alpha p+\sigma}{p-1}} (T-a)^{\beta-\frac{\gamma}{p-1}} + \left(\ln \frac{T}{a}\right)^{1-\frac{\sigma m}{p-m}} (T-a)^{\beta-\frac{\gamma m+\beta p}{p-m}}\right].\end{aligned} \tag{4.20}$$

Notice that by (3.1) and (3.2), we have

$$\beta - \frac{\gamma}{p-1} < 0, \quad \beta - \frac{\gamma m + \beta p}{p-m} < 0.$$

Hence, passing to the limit as $T \to \infty$ in (4.20), we obtain a contradiction with $u_1 \not\equiv 0$. Consequently, (1.2) admits no global solution. The proof is completed. □

Proof of Theorem 3.2. Suppose that $u \in X_a$ is a global solution to (1.2). Notice that in the proof of Theorem 3.1, to obtain (4.20), we used that

$$p > m \geq 1, \quad p > \sigma + 1, \quad p > m(\sigma + 1), \quad p > m(\gamma + 1).$$

On the other hand, by (3.3)–(3.5), it can be easily seen that the above conditions are satisfied. Thus, (4.20) holds. Hence, taking $p = 1 + \frac{\gamma}{\beta}$ in (4.20), we obtain

$$\int_a^\infty |u_1(t)|^m \frac{1}{t}\, dt \leq C \left[\left(\ln \frac{T}{a} \right)^{1 - \frac{\alpha p + \sigma}{p-1}} + \left(\ln \frac{T}{a} \right)^{1 - \frac{\sigma m}{p-m}} (T-a)^{\beta - \frac{\gamma m + \beta p}{p-m}} \right]. \tag{4.21}$$

On the other hand, by (3.3)–(3.5), we have

$$1 - \frac{\alpha p + \sigma}{p-1} < 0, \quad \beta - \frac{\gamma m + \beta p}{p-m} < 0.$$

Hence, passing to the limit as $T \to \infty$ in (4.21), we obtain a contradiction with $u_1 \not\equiv 0$. This shows that (1.2) admits no global solution. The proof is completed. □

5. Conclusions

The two-times fractional differential inequality (1.2) is investigated. Namely, using the test function method and a judicious choice of a test function, sufficient conditions ensuring the nonexistence of global solutions to (1.2) are obtained. Two cases are discussed separately: $1 < p < 1 + \frac{\gamma}{\beta}$ (see Theorem 3.1) and $p = 1 + \frac{\gamma}{\beta}$ (see Theorem 3.2). In the first case, no assumption is imposed on the fractional order $\alpha \in (0, 1)$ of the Hadamard-Caputo fractional derivative, while in the second case, it is supposed that $\alpha > 1 - \frac{1}{m}$. About the initial conditions, in both cases, it is assumed that $u_0 \in L^1([a, \infty))$, $u_1 \in L^m([a, \infty), \frac{1}{t}\, dt)$, and $u_1 \not\equiv 0$.

Finally, it would be interesting to extend this study to two-times fractional evolution equations. For instance, the tow-times fractional semi-linear heat equation

$$\begin{cases} {}^{HC}D^{\alpha}_{a|t} u(t,s,x) + {}^{C}D^{\beta}_{a|s} |u|^m(t,s,x) \geq (s-a)^\gamma \left(\ln \frac{t}{a} \right)^\sigma |u|^p(t,s,x), & t, s > a,\ x \in \mathbb{R}^N, \\ u(a,s,x) = u_0(s,x),\ u(t,a,x) = u_1(t,x), & t, s > a,\ x \in \mathbb{R}^N, \end{cases}$$

deserves to be studied.

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Conflict of interest

The authors declare that they have no competing interests.

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