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Research article

Energy equality for the compressible Navier-Stokes-Korteweg equations

Hui Fang[1], **Yihan Fan**[1,2] **and Yanping Zhou**[1,2,*]

[1] College of Science, China Three Gorges University, Yichang 443002, China
[2] Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, China

* **Correspondence:** Email: zhyp5208@163.com.

Abstract: In this paper, we investigate the problem of energy equality of the two and three dimensional compressible Navier-Stokes-Korteweg equations with general pressure law. By using the commutator estimation to deal with the nonlinear terms, we obtain the sufficient conditions for the regularity of weak solutions to conserve the energy.

Keywords: Navier-Stokes-Korteweg equations; compressible fluids; energy equality
Mathematics Subject Classification: 35Q35, 76N10

1. Introduction

The compressible Navier-Stokes-Korteweg (N-S-K) equations have been studied extensively in various fields due to its physical importance, complexity, rich phenomena and mathematical challenges. In this paper, we consider the N-S-K equations with general pressure law in the form

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) = \mathrm{div}(\mu \mathbb{D} u) + \nabla(\lambda \mathrm{div} u) + \mathrm{div} K, \tag{1.1}$$

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{1.2}$$

with initial data

$$\rho|_{t=0} = \rho_0(x), \quad \rho u|_{t=0} = \rho_0(x) u_0(x). \tag{1.3}$$

Here ρ denotes the density, u the velocity, $p(\rho)$ the general pressure, which will be specified later. $\mathbb{D} = \frac{1}{2}[\nabla u + \nabla^T u]$ stands for the deformation tensor, where ∇u denotes the gradient matrix $(\partial_i u^j)$ of u and $\nabla^T u$ is transpose. The positive constants μ, λ stand for the viscosity coefficients. The Korteweg stress tensor K is given by

$$K = \left(\frac{1}{2}(\rho\kappa'(\rho) + \kappa(\rho))|\nabla\rho|^2 + \rho\kappa(\rho)\Delta\rho\right)\mathbb{I} - \kappa(\rho)\nabla\rho \otimes \nabla\rho,$$

where $\kappa = \kappa(\rho) > 0$ is the coefficient of capillarity. The capillarity coefficient is a regular function which describes the phase transition. $\mathbb{I}$ denotes the identity matrix and $\nabla\rho \otimes \nabla\rho$ stands for the tensor product $(\partial_j\rho\partial_k\rho)_{jk}$. For the sake of simplicity, we will consider the periodic interval $(0, T) \times \mathbb{T}^d$ for some fixed time $T > 0$ in the dimensions two and three.

This compressible fluid model endowed with internal capillarity (Korteweg type) was first theoretically proposed by Korteweg [1]. However, the rigorous mathematical analysis did not take place until its modern form strictly from thermodynamics derived by Dunn-Serrin [2] in 1990s. Systems of Korteweg type arise in the simulation of several physical phenomena, such as capillarity phenomena in fluids with diffusing interfaces, in which the density undergoes a steep but still smooth changes of value. Owing to its importance in mathematics and physics, there are numerous works dedicated to the study of systems (1.1) and (1.2), involving local and global classical solutions [3, 4], local strong solutions [5], global weak solutions [6, 7], global strong solutions [8], long-time behavior [9–14] and blow-up results [15–17]. On the other hand, it is worth mentioning that Dębiec et al. [18] obtained an Onsager-type sufficient regularity condition for the conservation of weak solutions of the compressible Euler-Korteweg systems by using the strategies of Constantin et al. [19] and Feireisl et al. [20].

When $\kappa = 0$, systems (1.1) and (1.2) reduces to the famous compressible Navier-Stokes equations. There is a huge literature on the existence, blow-up and large-time behavior of the solutions (see [21–27]). Concerning the energy equality of the incompressible or compressible N-S equations, there have been a few results in recent years. More precisely, in the context of the incompressible N-S equations, the pioneering work was done by Serrin [28]. He proved the energy equality for weak solutions under the condition $u \in L^s(0, T; L^q(\mathbb{T}^d))$ with $\frac{2}{s} + \frac{d}{q} \leq 1$, $q > d$ and d is the dimension of space. Later, Shinbrot [29] removed the dimensional dependence and improved the condition to $\frac{2}{s} + \frac{2}{q} \leq 1$, $q \geq 4$. For the compressible N-S equations, Yu [30] proved that the energy is conserved if the velocity u satisfies $L^p_t L^q_x$ condition and the density ρ is bounded, meanwhile $\sqrt{\rho} \in L^\infty(0, T; H^1)$. The results of Akramov et al. [31] further supplemented Yu's results [30] by assuming that ρ and u have some differential regularity in time. Recently, by using a different approach, Nguyen et al. [32] obtained the energy conservation under a different set of regularity conditions. The advantage of their approach is that the temporal regularity of density can be avoided and milder conditions can be obtained. In addition, it is worth pointing out that Liang [33] and Berselli-Chiodaroli [34] recently derived the energy conservation criteria via the regularity of velocity and its gradient.

Regarding the study of energy equality for systems (1.1) and (1.2), only few results are available in the literature since several mathematical difficulties appear in the analysis. The strong nonlinearity in the higher order derivatives determined by the Korteweg term is the major difficulty. To get round this difficulty, several regularities of density inevitably need to be required. In the current article, we provide modest sufficient conditions on the regularity of weak solutions to ensure the energy conservation. Inspired by the works of Nguyen et al. [32], Liang [33] and Leslie-Shvydkoy [35], a suitable test function $\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}$ is used instead of u^ε, where the convolution is performed only in spatial variable. So time regularity of the density could be ignored. However, to compensate, vacuum must be excluded, or at least assume that the inverse density is inherently bounded. Our main result in this paper can be listed as follows.

Theorem 1.1. *Let $\Omega = \mathbb{T}^d (d = 2, 3)$ and (ρ, u) be a weak solution of N-S-K with initial data* (1.3). *Assume that*

$$\begin{cases} 0 < \alpha \le \rho(t, x) \le \beta < \infty, \nabla\rho, \Delta\rho(t, x) \in L^2((0, T) \times \mathbb{T}^d)), \\ p \in C^1(0, \infty), \kappa \in C^2(0, \infty), \\ u \in L^\infty((0, T);\ L^2(\mathbb{T}^d)) \cap L^2((0, T);\ H^1(\mathbb{T}^d)), u \in L^4((0, T) \times (\mathbb{T}^3)). \end{cases} \tag{1.4}$$

where α, β are positive constants. Then the energy equality holds, i.e.,

$$\begin{aligned} &\int_{\mathbb{T}^d} \left(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{1}{2}\kappa(\rho)|\nabla\rho|^2\right)(x, t)\mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} \mu|\mathbb{D}u|^2 \mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbb{T}^d} \lambda|\mathrm{div}u|^2 \mathrm{d}x\mathrm{d}s \\ &= \int_{\mathbb{T}^d} \left(\frac{1}{2}\rho_0|u_0|^2 + h(\rho_0) + \frac{1}{2}\kappa(\rho_0)|\nabla\rho_0|^2\right)\mathrm{d}x, \quad \forall\, t \in (0, T), \end{aligned} \tag{1.5}$$

where $h(\rho)$ is defined by

$$h(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2}\mathrm{d}r. \tag{1.6}$$

Remark 1.1. If we have

$$u \in L^p((0, T); L^q(\mathbb{T}^3)) \quad \text{with} \begin{cases} \frac{2}{p} + \frac{2}{q} = 1, \quad q \ge 4, \\ \\ \frac{1}{p} + \frac{3}{q} = 1, \quad 3 < q < 4, \end{cases}$$

then by interpolation, it follows that

$$\|u\|_{L^4((0,T)\times\mathbb{T}^3)} \le C\|u\|^{1-a}_{L^p((0,T);\, L^q(\mathbb{T}^3))}\|u\|^a_{L^\infty((0,T);\, L^2(\mathbb{T}^3))}$$

for some $a \in (0, 1)$. The result of Theorem 1.1 is also valid with the above assumption on the velocity.

Throughout the paper, C denotes generic constants, which may depend on d, T, $\|\rho\|_{L^\infty((0,T)\times\Omega)}$, $\|\frac{1}{\rho}\|_{L^\infty((0,T)\times\Omega)}$ and other scalar parameters.

The rest of the paper is organized as follows. In Section 2, we fix some symbols and give the definition of weak solutions of systems (1.1) and (1.2). Some useful estimates are collected for the proofs of our result. Section 3 is devoted to proving Theorem 1.1.

2. Preliminaries

Let η: $\mathbb{R}^d \to \mathbb{R}$ denote the standard mollifying kernel in $\mathbb{R}^d$. For any $\varepsilon > 0$, we set $\eta^\varepsilon(x) = \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right)$. For any function $f \in L^1_{loc}(\Omega)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta^\varepsilon)(x) = \int_{\mathbb{R}^d} f(x - y)\eta^\varepsilon(y)\mathrm{d}y, \quad x \in \Omega_\varepsilon,$$

where $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$.

The definition of weak solution for systems (1.1) and (1.2) is as follows.

Definition 1. (*weak solution*) We say that (ρ, u) is a weak solution to systems (1.1) and (1.2) with initial data given in (1.3), if it satisfies

(1)

$$\int_0^T \int_\Omega (\partial_t \varphi \cdot \rho + \rho u \cdot \nabla \varphi) \mathrm{d}x\mathrm{d}t = 0$$

for any test function $\varphi \in C_0^\infty(\Omega \times (0, T))$.

(2)

$$\int_0^T \int_\Omega (\rho u \cdot \partial_t \varphi + \rho u \otimes u : \nabla \varphi + p(\rho) \nabla \cdot \varphi$$
$$- \mu \mathbb{D} u : \nabla \varphi - \lambda (\nabla \cdot u)(\nabla \cdot \varphi) - K \cdot \nabla \varphi) \mathrm{d}x\mathrm{d}t = 0$$

for any test vector field $\varphi \in C_0^\infty(\Omega \times (0, T))^d$.

(3) $\rho(\cdot, t) \rightharpoonup \rho_0$ in $\mathcal{D}'(\Omega)$ as $t \to 0$, i.e.,

$$\lim_{t \to 0} \int_\Omega \rho(x, t) \varphi(x) \mathrm{d}x = \int_\Omega \rho_0(x) \varphi(x) \mathrm{d}x$$

for any test function $\varphi \in C_0^\infty(\Omega)$.

(4) $(\rho u)(\cdot, t) \rightharpoonup \rho_0 u_0$ in $\mathcal{D}'(\Omega)$ as $t \to 0$, i.e.,

$$\lim_{t \to 0} \int_\Omega (\rho u)(x, t) \varphi(x) \mathrm{d}x = \int_\Omega (\rho_0 u_0)(x) \varphi(x) \mathrm{d}x$$

for any test vector field $\varphi \in C_0^\infty(\Omega)^d$.

Next, we introduce three lemmas about the properties of mollifiers.

Lemma 2.1. [32] Let $2 \le d \in N$, $1 \le p, q \le \infty$ and $f : (0, T) \times \mathbb{T}^d \to \mathbb{R}$.

(1) Assume $f \in L^p(0, T; L^q(\mathbb{T}^d))$. Then for any $\varepsilon > 0$, there holds

$$\|f^\varepsilon\|_{L^p(0,T;L^\infty(\mathbb{T}^d)} \le C \varepsilon^{-\frac{d}{q}} \|f\|_{L^p(0,T;L^q(\mathbb{T}^d))},$$
$$\|\nabla f^\varepsilon\|_{L^p(0,T;L^\infty(\mathbb{T}^d)} \le C \varepsilon^{-1-\frac{d}{q}} \|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}.$$

(2) Assume $f \in L^p(0, T; L^q(\mathbb{T}^d))$. Then for any $\varepsilon > 0$, there holds

$$\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d)} \le C \varepsilon^{-1} \|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}.$$

Moreover, if $p, q < \infty$ then

$$\limsup_{\varepsilon \to 0} \varepsilon \|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d)} = 0.$$

(3) Assume $f \in L^p(0, T; L^q(\mathbb{T}^d))$ and $g : (0, T) \times \mathbb{T}^d \to R$ with $0 < c_1 \le g \le c_2 < \infty$. Then for any $\varepsilon > 0$, there holds

$$\left\| \nabla \frac{f^\varepsilon}{g^\varepsilon} \right\|_{L^p(0,T;L^q(\mathbb{T}^d))} \le C(c_1, c_2) \varepsilon^{-1} \|f\|_{L^p((0,T);L^q(\mathbb{T}^d))}.$$

Moreover, if $p, q < \infty$, then

$$\limsup_{\varepsilon\to 0} \varepsilon \left\| \nabla \frac{f^\varepsilon}{g^\varepsilon} \right\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0.$$

(4) Assume $f \in L^2(0,T;H^1(\mathbb{T}^2))$, then for any $\varepsilon > 0$, there holds

$$\|\nabla f^\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{T}^2))} \leq C\varepsilon^{-1}\|f\|_{L^2(0,T;H^1(\mathbb{T}^2))}.$$

Moreover, for any $r \in [1,2]$, it follows that

$$\limsup_{\varepsilon\to 0} \varepsilon \|\nabla f^\varepsilon\|_{L^r(0,T;L^\infty(\mathbb{T}^2))} = 0.$$

Lemma 2.2. [32] Let $p, p_1 \in [1,\infty)$ and $p_2 \in (1,\infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Assume $f \in L^{p_1}((0,T);\ W^{1,p_1}(\mathbb{T}^d))$ and $g \in L^{p_2}((0,T)\times(\mathbb{T}^d))$. Then for any $\varepsilon > 0$, there holds

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p((0,T)\times\mathbb{T}^d)} \leq C\varepsilon \|f\|_{L^{p_1}((0,T);\,W^{1,p_1}(\mathbb{T}^d))} \|g\|_{L^{p_2}((0,T)\times(\mathbb{T}^d))}.$$

Moreover, if $p < \infty$ then

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1} \|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p((0,T)\times\mathbb{T}^d)} = 0.$$

Lemma 2.3. [33] Assume that $0 < \alpha \leq \rho(t,x) \leq \beta < \infty$ and $u \in W^{1,p}(\mathbb{T}^d)$ with $p \in [1,\infty]$. Then

$$\left\| \nabla \left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^p(\mathbb{T}^d)} \leq C\|\nabla u\|_{L^p(\mathbb{T}^d)}.$$

By the same proof as the Lemma 2.3 in [18], we have

Lemma 2.4. Let $1 \leq q \leq \infty$ and suppose $\upsilon \in L^q((0,T)\times\mathbb{T}^d)$ and $f \in C^1(0,\infty)$, if $\sup \left\| \frac{\partial f}{\partial \upsilon} \right\|_{L^\infty((0,T)\times\mathbb{T}^d)} < \infty$. Then there exists a constant $C > 0$ such that

$$\|f(\upsilon^\varepsilon) - f^\varepsilon(\upsilon)\|_{L^q((0,T)\times\mathbb{T}^d)} \leq C \sup f' \, \|\upsilon^\varepsilon(t,x) - \upsilon(t,x)\|_{L^q((0,T)\times\mathbb{T}^d)}.$$

Moreover, if $q < \infty$, then

$$\limsup_{\varepsilon\to 0} \|f(\upsilon^\varepsilon) - f^\varepsilon(\upsilon)\|_{L^q((0,T)\times\mathbb{T}^d)} = 0. \tag{2.1}$$

Proof. We observe by Taylor's theorem that

$$|f(\upsilon^\varepsilon(t,x)) - f(\upsilon(t,x))| \leq |f'(\upsilon(t,x))\,(\upsilon^\varepsilon(t,x) - \upsilon(t,x))|, \tag{2.2}$$

where the constant C can be chosen independently of x. Similarly,

$$|f(\upsilon(t,y)) - f(\upsilon(t,x))| \leq |f'(\upsilon(t,x))\,(\upsilon(t,y) - \upsilon(t,x))|. \tag{2.3}$$

Applying the convolution with respect to y to (2.3), and then invoking Jensen's inequality, we have

$$|f^\varepsilon(\upsilon(t,x)) - f(\upsilon(t,x))| \leq |f'(\upsilon(t,x))\,(\upsilon^\varepsilon(t,x) - \upsilon(t,x))|. \tag{2.4}$$

Summing up (2.2) and (2.4), using Minkowski and Hölder inequalities, we conclude that

$$\begin{aligned}
&\|f(\upsilon^\varepsilon(t,x)) - f^\varepsilon(\upsilon(t,x))\|_{L^q((0,T)\times\mathbb{T}^d)} \\
&\leq 2\left(\int_{\operatorname{supp}\eta^\varepsilon} \eta^\varepsilon(y) \int_{((0,T)\times\mathbb{T}^d)} |f'(\upsilon(t,x))(\upsilon(t,x-y) - \upsilon(t,x))|^q \mathrm{d}x\mathrm{d}t\mathrm{d}y \right)^{\frac{1}{q}} \\
&\leq C \sup f' \, \|\upsilon^\varepsilon(t,x) - \upsilon(t,x)\|_{L^q((0,T)\times\mathbb{T}^d)},
\end{aligned}$$

which implies (2.1) by density. This completes the proof of Lemma 2.4. □

3. Proof of Theorem 1.1

By smoothing the momentum Eq (1.1) in space, we obtain

$$\partial_t(\rho u)^\varepsilon + \mathrm{div}(\rho u\otimes u)^\varepsilon + \nabla p^\varepsilon(\rho) = \mathrm{div}(\mu\mathbb{D}u)^\varepsilon + \nabla(\lambda\mathrm{div}u)^\varepsilon + \mathrm{div}K^\varepsilon(\rho,\nabla\rho,\Delta\rho) \tag{3.1}$$

for any $0<\varepsilon<1$. Here

$$K(\rho,\nabla\rho,\Delta\rho) = \left(\frac{1}{2}(\rho\kappa'(\rho)+\kappa(\rho))|\nabla\rho|^2 + \rho\kappa(\rho)\Delta\rho\right)\mathbb{I} - \kappa(\rho)\nabla\rho\otimes\nabla\rho, \tag{3.2}$$

and

$$K^\varepsilon(\rho,\nabla\rho,\Delta\rho) = K(\rho,\nabla\rho,\Delta\rho)*\eta^\varepsilon.$$

Multiplying $\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}$ on both sides of (3.1) and then integrating on $(\tau,t)\times\mathbb{T}^d$, for $0<\tau<t<T$, we have

$$\begin{aligned}
0 =& \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\partial_t(\rho u)^\varepsilon \mathrm{d}x\mathrm{d}s + \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}(\rho u\otimes u)^\varepsilon \mathrm{d}x\mathrm{d}s + \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\nabla p^\varepsilon(\rho)\mathrm{d}x\mathrm{d}s\\
&- \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}(\mu\mathbb{D}u)^\varepsilon \mathrm{d}x\mathrm{d}s - \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\nabla(\lambda\mathrm{div}u)^\varepsilon\mathrm{d}x\mathrm{d}s\\
&- \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}K^\varepsilon(\rho,\nabla\rho,\Delta\rho)\mathrm{d}x\mathrm{d}s := A+B+D+E+F+G.
\end{aligned} \tag{3.3}$$

In what follows, we are going to estimate them one by one.

Estimate of term A

We mollify the continuity Eq (1.2) as

$$\partial_t\rho^\varepsilon + \mathrm{div}(\rho u)^\varepsilon = 0. \tag{3.4}$$

Using (3.4) and integration by parts, we compute that

$$\begin{aligned}
A =& \frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\partial_t\left(\frac{|(\rho u)^\varepsilon|^2}{\rho^\varepsilon}\right)\mathrm{d}x\mathrm{d}s - \frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\mathrm{div}(\rho u)^\varepsilon\frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}\mathrm{d}x\mathrm{d}s\\
:=& A_1 + A_2.
\end{aligned}$$

We see that A_1 is the desired term while A_2 could be canceled with the term B_3 later.

Estimate of term B

$$\begin{aligned}
B =& \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}[(\rho u\otimes u)^\varepsilon - (\rho u)^\varepsilon\otimes u^\varepsilon]\mathrm{d}x\mathrm{d}s + \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}[(\rho u)^\varepsilon\otimes u^\varepsilon]\mathrm{d}x\mathrm{d}s\\
=& -\int_\tau^t\int_{\mathbb{T}^d}\nabla\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right)[(\rho u\otimes u)^\varepsilon - (\rho u)^\varepsilon\otimes u^\varepsilon]\mathrm{d}x\mathrm{d}s + \int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}[(\rho u)^\varepsilon\otimes u^\varepsilon]\mathrm{d}x\mathrm{d}s\\
:=& B_1 + \int_\tau^t\int_{\mathbb{T}^d}\mathrm{div}u^\varepsilon\frac{|(\rho u)^\varepsilon|^2}{\rho^\varepsilon}\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_\tau^t\int_{\mathbb{T}^d}\frac{u^\varepsilon}{\rho^\varepsilon}\nabla|(\rho u)^\varepsilon|^2\mathrm{d}x\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= B_1 + \int_\tau^t \int_{\mathbb{T}^d} \operatorname{div} u^\varepsilon \frac{|(\rho u)^\varepsilon|^2}{\rho^\varepsilon} \mathrm{d}x\mathrm{d}s - \frac{1}{2}\int_\tau^t \int_{\mathbb{T}^d} \operatorname{div}\left(\frac{u^\varepsilon}{\rho^\varepsilon}\right)|(\rho u)^\varepsilon|^2 \mathrm{d}x\mathrm{d}s \\
&= B_1 + \frac{1}{2}\int_\tau^t \int_{\mathbb{T}^d} \operatorname{div} u^\varepsilon \frac{|(\rho u)^\varepsilon|^2}{\rho^\varepsilon} \mathrm{d}x\mathrm{d}s - \frac{1}{2}\int_\tau^t \int_{\mathbb{T}^d} u^\varepsilon \nabla\left(\frac{1}{\rho^\varepsilon}\right)|(\rho u)^\varepsilon|^2 \mathrm{d}x\mathrm{d}s \\
&= B_1 + \frac{1}{2}\int_\tau^t \int_{\mathbb{T}^d} \operatorname{div}(\rho^\varepsilon u^\varepsilon) \frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2} \mathrm{d}x\mathrm{d}s \\
&= B_1 + \frac{1}{2}\int_\tau^t \int_{\mathbb{T}^d} \operatorname{div}[(\rho^\varepsilon u^\varepsilon) - (\rho u)^\varepsilon] \frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2} \mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_\tau^t \int_{\mathbb{T}^d} \operatorname{div}(\rho u)^\varepsilon \frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2} \mathrm{d}x\mathrm{d}s \\
&= B_1 - \frac{1}{2}\int_\tau^t \int_{\mathbb{T}^d} [(\rho^\varepsilon u^\varepsilon) - (\rho u)^\varepsilon] \nabla\left(\frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}\right) \mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_\tau^t \int_{\mathbb{T}^d} \operatorname{div}(\rho u)^\varepsilon \frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2} \mathrm{d}x\mathrm{d}s \\
&:= B_1 + B_2 + B_3.
\end{aligned}
$$

It is obvious that $A_2 + B_3 = 0$. Next, we show that

$$
\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0} |B_1| = 0, \text{ and } \limsup_{\varepsilon\to 0}\limsup_{\tau\to 0} |B_2| = 0.
$$

For the term B_1, by Hölder inequality, we arrive at

$$
\begin{aligned}
|B_1| &= \left|\int_\tau^t \int_{\mathbb{T}^d} \nabla\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right)[(\rho u \otimes u)^\varepsilon - (\rho u)^\varepsilon \otimes u^\varepsilon] \mathrm{d}x\mathrm{d}s\right| \\
&\le \left\|\nabla \frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right\|_{L^4((0,T)\times\mathbb{T}^d)} \|(\rho u \otimes u)^\varepsilon - (\rho u)^\varepsilon \otimes u^\varepsilon\|_{L^{\frac{4}{3}}((0,T)\times\mathbb{T}^d)}.
\end{aligned}
$$

Let's consider the case $d = 2$ firstly. Owing to the Gagliardo-Nirenberg inequality, we infer that

$$
\|u\|_{L^4((0,T)\times\mathbb{T}^2)} \le C\|u\|^{\frac{1}{2}}_{L^2((0,T);\, W^{1,2}(\mathbb{T}^2))}\|u\|^{\frac{1}{2}}_{L^\infty((0,T);\, L^2(\mathbb{T}^2))}, \tag{3.5}
$$

which gives $u \in L^4((0,T)\times\mathbb{T}^2)$. Meanwhile, thanks to Lemma 2.1 (3) and Lemma 2.2, we deduce that

$$
\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0} |B_1| = 0. \tag{3.6}
$$

For the case $d = 3$, we have the assumption $u \in L^4((0,T)\times\mathbb{T}^3)$. In the same manner, we could also deduce (3.6).

For the term B_2, using Hölder inequality and Lemma 2.1 (3), we have

$$
\begin{aligned}
|B_2| &\le C\left\|\nabla \frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}\right\|_{L^2((0,T)\times\mathbb{T}^d)} \|\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon\|_{L^2((0,T)\times\mathbb{T}^d)} \\
&\le C\varepsilon^{-1}\|u\|_{L^4((0,T)\times\mathbb{T}^d)} \|(\rho u)^\varepsilon - \rho^\varepsilon u^\varepsilon\|_{L^2((0,T)\times\mathbb{T}^d)},
\end{aligned}
$$

which, together with Lemma 2.2, gives

$$
\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0} |B_2| = 0.
$$

Estimate of term D

First of all, by definition of $h(\rho)$, we get

$$p(\rho) = \rho h'(\rho) - h(\rho).$$

We compute D as

$$\begin{aligned}
D &= \int_\tau^t \int_{\mathbb{T}^d} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \nabla[p^\varepsilon(\rho) - p(\rho^\varepsilon)] \mathrm{d}x\mathrm{d}s + \int_\tau^t \int_{\mathbb{T}^d} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \nabla p(\rho^\varepsilon) \mathrm{d}x\mathrm{d}s \\
&= \int_\tau^t \int_{\mathbb{T}^d} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \nabla[p^\varepsilon(\rho) - p(\rho^\varepsilon)] \mathrm{d}x\mathrm{d}s + \int_\tau^t \int_{\mathbb{T}^d} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \nabla(\rho^\varepsilon h'(\rho^\varepsilon) - h(\rho^\varepsilon)) \mathrm{d}x\mathrm{d}s \\
&= -\int_\tau^t \int_{\mathbb{T}^2} \operatorname{div}\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right) [p^\varepsilon(\rho) - p(\rho^\varepsilon)] \mathrm{d}x\mathrm{d}s + \int_\tau^t \int_{\mathbb{T}^d} (\rho u)^\varepsilon \nabla(h'(\rho^\varepsilon)) \mathrm{d}x\mathrm{d}s \\
&:= D_1 + D_2.
\end{aligned}$$

We show that D_1 converges to 0, as $\varepsilon, \tau \to 0$ firstly. The term D_2 could be estimated together with G_2 later. For the term D_1, by means of Hölder inequality and Lemma 2.3, we have

$$\begin{aligned}
|D_1| &= \left| \int_\tau^t \int_{\mathbb{T}^2} \operatorname{div}\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right) [p^\varepsilon(\rho) - p(\rho^\varepsilon)] \mathrm{d}x\mathrm{d}s \right| \\
&\le C \|\nabla u\|_{L^2((0,T)\times\mathbb{T}^d)} \|p^\varepsilon(\rho) - p(\rho^\varepsilon)\|_{L^2((0,T)\times\mathbb{T}^d)}
\end{aligned}$$

due to $\rho \in L^\infty((0,T)\times\mathbb{T}^d)$. Since $p \in C^1(0,\infty)$, it yields from Lemma 2.4 with $f = p$ that

$$\limsup_{\varepsilon\to 0} \limsup_{\tau\to 0} |D_1| = 0.$$

Estimate of term E

$$\begin{aligned}
E &= -\int_\tau^t \int_{\mathbb{T}^d} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \operatorname{div}(\mu \mathbb{D} u)^\varepsilon \mathrm{d}x\mathrm{d}s \\
&= -\int_\tau^t \int_{\mathbb{T}^d} \frac{(\rho u)^\varepsilon - \rho^\varepsilon u^\varepsilon}{\rho^\varepsilon} \operatorname{div}(\mu \mathbb{D} u)^\varepsilon \mathrm{d}x\mathrm{d}s - \int_\tau^t \int_{\mathbb{T}^d} \operatorname{div}(\mu \mathbb{D} u)^\varepsilon u^\varepsilon \mathrm{d}x\mathrm{d}s \\
&= -\int_\tau^t \int_{\mathbb{T}^d} \frac{(\rho u)^\varepsilon - \rho^\varepsilon u^\varepsilon}{\rho^\varepsilon} \operatorname{div}(\mu \mathbb{D} u)^\varepsilon \mathrm{d}x\mathrm{d}s + \int_\tau^t \int_{\mathbb{T}^d} \mu |\mathbb{D} u^\varepsilon|^2 \mathrm{d}x\mathrm{d}s \\
&:= E_1 + E_2.
\end{aligned}$$

E_2 is our expected term. By Hölder inequality and Lemma 2.1 (2), E_1 could be estimated as

$$\begin{aligned}
|E_1| &\le C \|\operatorname{div}(\mu \mathbb{D} u)^\varepsilon\|_{L^2((0,T)\times\mathbb{T}^d)} \left\| \frac{(\rho u)^\varepsilon - \rho^\varepsilon u^\varepsilon}{\rho^\varepsilon} \right\|_{L^2((0,T)\times\mathbb{T}^d)} \\
&\le C\varepsilon^{-1} \|\nabla u\|_{L^2((0,T)\times\mathbb{T}^d)} \|(\rho u)^\varepsilon - \rho^\varepsilon u^\varepsilon\|_{L^2((0,T)\times\mathbb{T}^d)}.
\end{aligned}$$

Noting that $\nabla u \in L^2((0,T)\times\mathbb{T}^d)$ and $\rho \in L^\infty((0,T)\times\mathbb{T}^d)$ and using Lemma 2.2, we have

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|E_1| = 0.$$

Estimate of term F

$$\begin{aligned}
F &= -\int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\nabla(\lambda\mathrm{div}u)^\varepsilon\mathrm{d}x\mathrm{d}s\\
&= -\int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\nabla(\lambda\mathrm{div}u)^\varepsilon\mathrm{d}x\mathrm{d}s-\int_\tau^t\int_{\mathbb{T}^d}\nabla(\lambda\mathrm{div}u)^\varepsilon u^\varepsilon\mathrm{d}x\mathrm{d}s\\
&= -\int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\nabla(\lambda\mathrm{div}u)^\varepsilon\mathrm{d}x\mathrm{d}s+\int_\tau^t\int_{\mathbb{T}^d}\lambda|\mathrm{div}u^\varepsilon|^2\mathrm{d}x\mathrm{d}s\\
&:= F_1+F_2.
\end{aligned}$$

F_2 is the desired term. Similar to the estimate of E_1, we check that

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|F_1| = 0.$$

Estimate of term G

Applying div to (3.2), one obtains

$$\mathrm{div}K(\rho,\nabla\rho,\Delta\rho) = -\rho\nabla\left(\frac{1}{2}\kappa'(\rho)|\nabla\rho|^2-\mathrm{div}(\kappa(\rho)\nabla\rho)\right), \tag{3.7}$$

where we have used the fact $\mathrm{div}\mathbb{I}=\nabla$. According to (3.7), the term G can be written as

$$\begin{aligned}
G &= -\int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}K^\varepsilon(\rho,\nabla\rho,\Delta\rho)\mathrm{d}x\mathrm{d}s\\
&= -\int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}[K^\varepsilon(\rho,\nabla\rho,\Delta\rho)-K(\rho^\varepsilon,\nabla\rho^\varepsilon,\Delta\rho^\varepsilon)]\mathrm{d}x\mathrm{d}s\\
&\quad -\int_\tau^t\int_{\mathbb{T}^d}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}K(\rho^\varepsilon,\nabla\rho^\varepsilon,\Delta\rho^\varepsilon)\mathrm{d}x\mathrm{d}s\\
&= \int_\tau^t\int_{\mathbb{T}^2}\nabla\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right)[K^\varepsilon(\rho,\nabla\rho,\Delta\rho)-K(\rho^\varepsilon,\nabla\rho^\varepsilon,\Delta\rho^\varepsilon)]\mathrm{d}x\mathrm{d}s\\
&\quad +\int_\tau^t\int_{\mathbb{T}^d}(\rho u)^\varepsilon\nabla\left(\frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2-\mathrm{div}(\kappa(\rho^\varepsilon)\nabla\rho^\varepsilon)\right)\mathrm{d}x\mathrm{d}s\\
&:= G_1+G_2.
\end{aligned}$$

Applying Lemma 2.3, the term G_1 can be estimated similarly to D_1, and thus we get

$$|G_1| = \left|\int_\tau^t\int_{\mathbb{T}^2}\nabla\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right)[K^\varepsilon(\rho,\nabla\rho,\Delta\rho)-K(\rho^\varepsilon,\nabla\rho^\varepsilon,\Delta\rho^\varepsilon)]\mathrm{d}x\mathrm{d}s\right|$$

$$\leq C\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^d)}\|K^\varepsilon(\rho,\nabla\rho,\Delta\rho)-K(\rho^\varepsilon,\nabla\rho^\varepsilon,\Delta\rho^\varepsilon)\|_{L^2((0,T)\times\mathbb{T}^d)}$$

thanks to $\rho\in L^\infty((0,T)\times\mathbb{T}^d)$, $\nabla\rho,\Delta\rho\in L^2((0,T)\times\mathbb{T}^d)$). Since $\kappa\in C^2(0,\infty)$, it yields from Lemma 2.4 with $f=K$ that

$$\limsup_{\varepsilon\to0}\limsup_{\tau\to0}|G_1|=0.$$

It remains to estimate the term G_2. To this end, combining with D_2, we obtain

$$\begin{aligned}
D_2+G_2&=\int_\tau^t\int_{\mathbb{T}^d}(\rho u)^\varepsilon\nabla\left(h'(\rho^\varepsilon)+\frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2-\operatorname{div}(\kappa(\rho^\varepsilon)\nabla\rho^\varepsilon)\right)\mathrm{d}x\mathrm{d}s\\
&=-\int_\tau^t\int_{\mathbb{T}^d}\operatorname{div}(\rho u)^\varepsilon\left(h'(\rho^\varepsilon)+\frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2-\operatorname{div}(\kappa(\rho^\varepsilon)\nabla\rho^\varepsilon)\right)\mathrm{d}x\mathrm{d}s\\
&=\int_\tau^t\int_{\mathbb{T}^d}\partial_t\rho^\varepsilon\left(h'(\rho^\varepsilon)+\frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2-\operatorname{div}(\kappa(\rho^\varepsilon)\nabla\rho^\varepsilon)\right)\mathrm{d}x\mathrm{d}s\\
&=\int_\tau^t\int_{\mathbb{T}^d}\partial_t\left(h(\rho^\varepsilon)+\frac{1}{2}\kappa(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2\right)\mathrm{d}x\mathrm{d}s-\int_\tau^t\int_{\mathbb{T}^d}\operatorname{div}\left(\kappa(\rho^\varepsilon)\nabla\rho^\varepsilon\partial_t\rho^\varepsilon\right)\mathrm{d}x\mathrm{d}s\\
&=\int_\tau^t\int_{\mathbb{T}^d}\partial_t\left(h(\rho^\varepsilon)+\frac{1}{2}\kappa(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2\right)\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Collecting the above estimates A_1,E_2,F_2,D_2,G_2 and putting them into (3.3), we eventually deduce that

$$\begin{aligned}
\limsup_{\varepsilon\to0}\limsup_{\tau\to0}\Bigg|&\int_\tau^t\int_{\mathbb{T}^d}\partial_t\left(\frac{1}{2}\frac{|(\rho u)^\varepsilon|^2}{\rho^\varepsilon}+h(\rho^\varepsilon)+\frac{1}{2}\kappa(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2\right)\mathrm{d}x\mathrm{d}s\\
&+\int_\tau^t\int_{\mathbb{T}^d}\mu|\mathbb{D}u^\varepsilon|^2\mathrm{d}x\mathrm{d}s+\int_\tau^t\int_{\mathbb{T}^d}\lambda|\operatorname{div}u^\varepsilon|^2\mathrm{d}x\mathrm{d}s\Bigg|=0.
\end{aligned}$$

This completes the proof of Theorem 1.1.

4. Conclusions

In this paper, we study the energy conservation of the compressible Navier-Stokes-Korteweg equations with general pressure law in a periodic domain $\mathbb{T}^d$ with $d=2,3$. By using the commutator estimation to deal with the nonlinear terms, we obtain the sufficient conditions for the regularity of weak solutions to conserve the energy. We extend the results of Nguyen et al. [32] and Liang [33] from the compressible N-S equations to the N-S-Korteweg equations.

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Conflict of interest

The authors declare that they have no conflict of interest.

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