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Research article

Existence, uniqueness and travelling waves to model an invasive specie interaction with heterogeneous reaction and non-linear diffusion

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Abstract: It is the objective to provide a mathematical treatment of a model to predict the behaviour of an invasive specie proliferating in a domain, but with a certain hostile zone. The behaviour of the invasive is modelled in the frame of a non-linear diffusion (of Porous Medium type) equation with non-Lipschitz and heterogeneous reaction. First of all, the paper examines the existence and uniqueness of solutions together with a comparison principle. Once the regularity principles are shown, the solutions are studied within the Travelling Waves (TW) domain together with stability analysis in the frame of the Geometric Perturbation Theory (GPT). As a remarkable finding, the obtained TW profile follows a potential law in the stable connection that converges to the stationary solution. Such potential law suggests that the pressure induced by the invasive over the hostile area increases over time. Nonetheless, the finite speed, induced by the non-linear diffusion, slows down a possible violent invasion.

Keywords: porous medium equation; travelling waves; geometric perturbation; non-linear diffusion; heterogeneous reaction
Mathematics Subject Classification: 35K55, 35K57, 35K59, 35K65

1. Introduction

Biological invasion has been precisely defined by the Convection of Biological Diversity [19] as those species capable of threaten other species living peacefully in their ecosystem.

The invaded-invasive interaction between species has been discussed with an advection, that elucidates a non-linear diffusion, in [21]. The system derived in such reference was intended to describe the haptotactic cell invasion in a model for melanoma. In addition, [14] examines the spectral stability of travelling waves of the haptotaxis model studied in cancer invasion. The model has been analyzed making use of Evans function to a linearised operator. In [11], an invasive propagating front, driven by

diffusion, is characterized together with the dynamics of the invaded specie. In these last cited cases, the proposed models interpreted the advection as part of the complete random movement (no preferred direction) induced by the haptotactic evolution, as well as (see [11]) a pure advection to model the dynamic spatial preferred direction of motion.

Note that even when invasive-invaded systems are not precisely speaking predator-prey models, it is worth mentioning that once and invasive occupies a domain (for example a biological organ) the invaded specie extinguishes. Such extinguishing may lead the invasive to vanish (or even die if the organ fails) as there is not further specie to invade. This a-priori dynamic constitutes a link between invasive-invaded dynamic and predator-prey. In [22], the authors study the existence of invasion waves of a diffusive predator–prey model with two preys and one predator using the Schauder's fixed-point theorem together with LaSalle's invariant theorem to prove the existence of solutions between two equilibrium conditions. Recently the Hopf bifurcation method has been employed in [3] to study the harvesting effect in the predator and the density-dependent mortality in the prey. Additionally, [15] proposes a Hopf bifurcation to study a delayed density-dependent predator-prey system with Beddington-DeAngelis functional response. Stability and bifurcation methods have been employed as well in [16, 20, 27] for different functional responses.

There exists a mathematical connection among the different bifurcation and stability methods mentioned and the geometric perturbation theory employed in this paper, as such theories are intended to search for stable solutions, for example in the form of Travelling Waves (TW), connecting stationary conditions.

As it will be shown, the non-linear diffusion drives the mathematical methods employed in this paper. Within the mathematical applications to biology, Keller and Segel [18] proposed a non-linear diffusion to study the cells movement by chemotaxis:

$$\begin{aligned} u_t &= \nabla \cdot (d(u)u - \chi(v)u\nabla u) \quad x \in \Omega,\ t > 0 \\ v_t &= d_v \Delta v - uv \qquad\qquad x \in \Omega,\ t > 0, \end{aligned} \tag{1.1}$$

where u represents the cell density and v the chemical concentration. Note that $d(u)$ is the media diffusivity and $\chi(v)$ the distribution of chemical agent to which the cells are sensitive. The Keller and Segel model has been extended to account for certain regular reaction-absorption dynamics [1,7,23,26].

In other research areas, the non-linear diffusion in the form of Porous Medium Equation has been used to model the coagulation effect in an electromagnetic blood flow with annular vessel geometries [5] or to simulate the effect of porosity in a peristaltic transport in a Jeffrey fluid [12].

2. Model description and methods

The analyzed problem P is:

$$\begin{gathered} v_t = \Delta v^m + |x|^\sigma v^q (1 - v), \\ m > 1,\ 0 < q < 1,\ 0 \le v_0(x) \in L^1_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \end{gathered} \tag{2.1}$$

The problem P justification is based on studying the invasive dynamic as a single specie and the mathematical properties introduced by the spatial operator and the heterogeneous reaction term. Typically, the behaviour of an invasive specie has been modelled together with the specie to be invaded. In [22], the authors study the existence of invasion waves to a linear diffusion predator–prey model

with two preys and one predator. The existence of travelling semi-fronts connecting invasion-free equilibrium are shown based on the Schauder's fixed-point theorem for certain travelling waves speeds above one critical. In [25], the authors are concerned with the existence and asymptotic behavior of invasion wave solutions and non-quasimonotone conditions to an upper solution. To this end, the Schauder's fixed point theorem is employed. Further, in [24], the invasion traveling waves are shown to state exponentially stable for an exponentially weighted space by using the weighted energy. Along the presented, the invasive specie is modeled along its conquering state, for this purpose:

Consider that for $|x| < \gamma \to 0^+$, the invasive growing rate $v_t \sim 0$, in other words, the small area given by $|x| < \gamma$ represents a hostile zone in which the invasive accounts for difficulties to get into. Nonetheless, the fact of having $v_t \sim 0$ with quasi-null initial data in $|x| < \gamma$ indicates the slow motion on the invasive in such spatial domain. This slow motion is associated with the property known as finite propagation in the Porous Medium Equation theory. Therefore, the intention is to model two different domains:

- $|x| < \gamma \to 0^+$: The invasive specie has difficulties to get into the hostile domain. Nonetheless, the quasi-null initial data indicates the existence of a finite propagation that is kept along the invasive excursion in such domain. The fact of modelling with a non-linear diffusion permits to avoid the positivity condition typical in the gaussian-fickian diffusion. In addition, the non-Lipschitz condition introduces the existence of a null minimal solution that represents the impossibility of the invasive to invade the hostile zone. Nonetheless and in case the invasive penetrates the hostile area, the initial population growth is relatively high in accordance with the derivative of $v^q,\ q < 1$.
- $|x| >> \gamma$: The invasive reproduction rate is high as the environment can feed the invasive until it reaches the maximum concentration or saturation at equilibrium established at $v = 1$. This behaviour is introduced by a weak Allee effect $(1 - v)$.

The methods employed along this article consist on showing exitence of solutions to an equivalent Lipschitz problem via maximal and minimal monotone sequences. In addition, uniqueness and comparison are shown based on a generalization to weak solutions and the definition of a test function $\phi \in C^\infty(\mathbb{R}^d)$ to account for the degenerate diffusivity in the non-linear diffusion. Finally, the Geometric Perturbation theory is employed to show the existence and to determine Travelling Waves profiles.

3. Existence and uniqueness of solutions

3.1. Initial data growing condition

The theory developed to solve P (2.1) holds for initial data $0 \leq v_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. A generalization to consider functions of the form:

$$0 \leq v_0(x) \in L^1_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \tag{3.1}$$

can be considered provided a condition is set for the growing behavior in the initial data in the spirit of [9, 10].

The baseline integrability condition requires to introduce the following Banach space:

$$E_0 = \left\{\phi \in L^1_{loc}(\mathbb{R}^d) : \|\phi\|_r < \infty\right\}, \tag{3.2}$$

where the norm $\|\phi\|_r (r \geq 1)$ is defined as:

$$\|\phi\|_r = \sup_{R \geq r} R^{-d-a_\sigma} \int_{\mathbb{R}^d} |\phi(x)|\, dx, \tag{3.3}$$

with

$$a_\sigma = \max\left\{\frac{\sigma}{1-q}, \frac{2}{m-1}\right\}, \tag{3.4}$$

$B_R = \{x \in \mathbb{R};\ |x| < R\}$ and $\|\phi\|_* = \lim_{r\to\infty} \|\phi\|_r$

The a_σ expression in (3.4) is given as per two separated problems with different growing conditions in $\mathbb{R}^d$. The first problem is the homogeneous $v_t = \Delta v^m$ which provides the following growing order in $\mathbb{R}^d$:

$$v \sim |x|^{\frac{2}{m-1}}, \quad |x| >> 1. \tag{3.5}$$

The mentioned second problem is related with the forcing term solved independently to the diffusion:

$$v_t = |x|^\sigma v^q (1-v) \leq |x|^\sigma v^q, \quad v \leq |x|^{\frac{\sigma}{1-p}}. \tag{3.6}$$

Any initial data $v_0(x) \in L^1_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ shall be weighted with the exponent a_σ to account for preliminary regular solutions.

3.2. The Lipschitz problem

As part of the strategy followed to show existence of solutions, firstly the following Lipschitz problem (P') is analyzed:

$$\begin{array}{c} v_t = \Delta v^m + f_{lip}(x,v) \ in\ Q_T = \mathbb{R}^d \times [0,T], \\ v(x,0) = v_0(x) \geq 0 \in E_0, \end{array} \tag{3.7}$$

where $0 < T \leq \infty$ and $f_{lip}(x,v)$ is a Lipschitz function $f_{lip}(v) : [0,\infty) \to [0,\infty)$ with constant L.

The following truncation is defined so that certain techniques in the spirit of [8–10] are applicable:

$$|x|^\sigma_\epsilon = \left[\begin{array}{c} |x|^\sigma \ when\ \ 0 \leq |x| < \epsilon \\ \epsilon^\sigma \ when\ \ |x| \geq \epsilon \end{array}\right]. \tag{3.8}$$

Then, the following problem P'_ϵ is defined accordingly:

$$\begin{array}{c} v_t = \Delta v^m + |x|^\sigma_\epsilon f(v) \leq \Delta v^m + \epsilon^\sigma f(v) \ in\ Q_{T_\epsilon} = \mathbb{R}^d \times [0,T_\epsilon], \\ v(x,0) = v_0(x) \geq 0 \in E_0. \end{array} \tag{3.9}$$

Based on the problem P'_ϵ, the following existence theorem holds.

Theorem 3.2.1. *For a given $\epsilon > 0$ and $v_0 \in E_0$, there exists a unique v^ϵ in Q_{T_ϵ} (existing for each ϵ) continuous weak solution to the problem P'_ϵ in a time interval $(0, T_\epsilon)$.*

Proof. Note that [9] has showed existence of solutions for the problem

$$v_t = (v^m)_{xx} + \lambda v^q,\ \lambda > 0,\ m > 1,\ q \in \mathbb{R}. \tag{3.10}$$

In P'_ϵ the truncated term $|x|^\sigma_\epsilon$, bounded by ϵ^σ, plays the role of the parameter λ and the independent term is $v^q(1-v) \sim v^q$ in the proximity of zero where the non-Lipschitz condition imposes non-regularity. Thus, the aim is to prove the existence of solutions for the problem P'_ϵ within the time interval $(0, T_\epsilon)$ to be determined and considering the loss of regularity due to the non-Lipschitz codition. For this purpose, define the truncation in the initial data for $v_0 \in E_0$ and $n \geq 1$ as:

$$v_{0n}(x) = \left\{ \begin{array}{c} v_0(x) \ when \ \ |x| \leq n, \ \ v_0(x) < n, \\ n \ when \ \ |x| \leq n, \ \ v_0(x) \geq n, \\ 0 \ when \ \ |x| > n \end{array} \right\}. \tag{3.11}$$

The problem P'_ϵ with a Lipschitz forcing term and with bounded initial (3.11) has existence and uniqueness of solutions (Theorem 3.1 in [6]).

The intention is, now, to have a global bound for a subsolution (w) of P'_ϵ so that a value for T_ϵ is obtained. For this purpose, the following change is defined inspired in [9]:

$$\begin{array}{c} x \to x, \ \ t \to \tau = \frac{e^{\epsilon^\sigma L(m-1)t}-1}{\epsilon^\sigma L(m-1)}, \\ v(x,t) \to w(x,\tau). \end{array} \tag{3.12}$$

Note that w is a subsolution given by an exponential decay:

$$\begin{array}{c} v_t = w_\tau \tau_t = w_\tau e^{\epsilon^\sigma L(m-1)t}, \quad e^{\epsilon^\sigma L(m-1)t} w_\tau = \Delta w^m, \\ w_\tau = e^{-\epsilon^\sigma L(m-1)t} \underbrace{\Delta w^m}_{\Delta u^m}. \end{array} \tag{3.13}$$

The temporal evolution w is driven by the decaying exponential term. Given a particular value for Δv^m, the evolution of the homogeneous v is given by $v_t = \Delta v^m$, while the evolution of w is:

$$w_\tau = e^{-\epsilon^\sigma L(m-1)t} \Delta v^m. \tag{3.14}$$

Thus, for a $T >> 1$, w is indeed a subsolution, i.e., $w_\tau \leq \Delta v^m$. Note that to recover the original solution for the non-linear diffusion (also called Porous Medium Equation or PME), it suffices to consider:

$$v(x,t) = e^{\epsilon^\sigma L(m-1)t} w(x, \tau(t)). \tag{3.15}$$

Having such exponential decrease rate in (3.14) allows the bound of w by an already known estimation [4]:

$$w(x,\tau) \leq cR^{2/(m-1)} \tau^{-\alpha} ||w(\cdot,0)||_r^{2\alpha/d}, \tag{3.16}$$

where,

$$\begin{array}{c} \alpha = \frac{d}{d(m-1)+2}, \ \ |x| < R, \ \ 1 \leq r \leq R, \\ 0 < \tau = \frac{e^{\epsilon^\sigma L(m-1)t}-1}{\epsilon^\sigma L(m-1)} \leq c||w(\cdot,0)||_r^{(1-m)}, \end{array} \tag{3.17}$$

and $c \in \mathbb{R}^+$.

Note that the intention is to determine the existence time T_ϵ based on the bound estimation in (3.16).

From now on, consider the solution for the PME as $v_n^\epsilon(x,t)$. Indeed such solution is obtained for a given n in the truncation of the initial data and for a given ϵ in the truncation for the term $|x|^\sigma$.

Based on the expression (3.16), the following estimation applies for $v_n^\epsilon(x,t)$ considering that for t sufficiently large:

$$v(x,t) \sim e^{\epsilon^\sigma Lmt} w(x,\tau(t)), \tag{3.18}$$

so that,

$$v_n^\epsilon(x,t) \leq cR^{2/(m-1)} e^{Lm\epsilon^\sigma t} \tau^{-\alpha} \|v_n(\cdot,0)\|_r^{2\alpha/d}, \tag{3.19}$$

which can be re-written as:

$$v_n^\epsilon(x,t) \leq cR^{2/(m-1)} e^{Lm\epsilon^\sigma t} \left(\frac{e^{L\epsilon^\sigma(m-1)t} - 1}{L\epsilon^\sigma(m-1)} \right)^{-\alpha} \|v_n(\cdot,0)\|_r^{2\alpha/d}, \tag{3.20}$$

where:

$$|x| < R,\ 1 \leq r \leq R,\ 0 < t \leq T_{r,\epsilon}. \tag{3.21}$$

The time $T_{r,\epsilon}$ can be obtained operating the expression:

$$\tau = \frac{e^{\epsilon^\sigma L(m-1)t} - 1}{\epsilon^\sigma L(m-1)} \leq c\|v_n(\cdot,0)\|_r^{(1-m)}. \tag{3.22}$$

And upon operation:

$$T_{r,\epsilon} = \frac{1}{L\epsilon^\sigma(m-1)} \log\left(1 + cL\epsilon^\sigma(m-1)\|v_{0,n}\|_r^{1-m}\right). \tag{3.23}$$

For a given ϵ, consider the limit $n \to \infty$ to account for the whole initial data that is controlled by the norm $\|v_{0,n}\|_r$. Thus:

$$0 < t \leq T_\epsilon = \frac{1}{L\epsilon^\sigma(m-1)} \log\left(1 + cL\epsilon^\sigma(m-1)\|v_0\|_*^{1-m}\right), \tag{3.24}$$

where $r \to \infty$.

Finally, solutions exist for $0 < t \leq T_\epsilon$ as the problem is Lipschitz with spacial-bounded forcing term (3.8) and with initial data increasing rate controlled by the norm $\|v_0\|_*$ (see (3.3) together with the $\|\cdot\|_*$ definition) .

□

Based on (3.24), two cases of existence shall be distinguished:

- $\epsilon \to 0 \Rightarrow \frac{cL\epsilon^\sigma(m-1)\|v_0\|_*^{1-m}}{L\epsilon^\sigma(m-1)} = c\|v_0\|_*^{1-m}$.
  This case corresponds to a finite existence time given by the weighted norm in the initial data (3.3).
- $\epsilon \to \infty \Rightarrow \frac{\log(cL\epsilon^\sigma(m-1)\|v_0\|_*^{1-m})}{L\epsilon^\sigma(m-1)} \to 0$.
  This case does not provide information about the existence time due to the globally not bounded evolution of $|x|^\sigma$. In this case, given $|x| = \epsilon$, it is possible to ensure existence of solutions, as the expression (3.24) provides a finite value for T_ϵ. Then and at least, there exists local solutions for finite values of $|x|$ in the proximity of ϵ. Note that the fact of losing a existence criteria when $\epsilon \to \infty$ can be considered as a condition in which blow-up may be given. The blow-up behaviour is characterized in Section. 3.3.2.

3.3. The non-Lipschitz problem

Consider the following non-Lipschitz problem, named as P_ϵ:

$$\begin{gathered} v_t = \Delta v^m + |x|_\epsilon^\sigma v^q(1-v) \leq \Delta v^m + |x|_\epsilon^\sigma v^q \leq \Delta v^m + \epsilon^\sigma v^q, \\ Q_{T_\epsilon} = \mathbb{R}^d \times [0, T_\epsilon], \\ v(x,0) = v_0(x), \\ v_0(x) \geq 0 \in E_0, \\ q < 1;\ m > 1\ N \geq 1 \end{gathered} \tag{3.25}$$

The non-Lipschitz reaction term makes the problem non regular as it is not possible to show uniqueness for any value of v, particularly when $v = 0$ or when v increases from zero to positivity. Our effort is, hence, focused on determining the existence and characterizing two particular solutions, the maximal and the minimal, so that any other solution exists between them.

Theorem 3.3.1. *There exist one maximal solution (v^M) to P_ϵ and one minimal (v_m) existing in $[0, T_\epsilon]$ with $T_\epsilon(\epsilon, \|v_0\|_*)$ such that any solution existing between them $v_m \leq v^\epsilon \leq v^M$.*

Proof. With the objective of applying Theorem 3.2.1, the following Lipschitz function is defined:

$$f_\delta(s) = \begin{bmatrix} \epsilon^\sigma \delta^{(q-1)} s \ for\ 0 \leq s < \delta \\ \epsilon^\sigma s^q \ for\ s \geq \delta \end{bmatrix}, \tag{3.26}$$

so that in the limit for $\delta \to 0$, the original term v^q $(q < 1)$ is recovered.

For building the maximal solution, consider the following problem P_ϵ^M:

$$\begin{gathered} v_t = \Delta v^m + f_\delta(v)\ \ in\ \ Q_{T_{\epsilon,\delta}} = \mathbb{R}^d \times [0, T_{\epsilon,\delta}], \\ v(x,0) = v_0(x) + \nu\ for\ x \in \mathbb{R}^d\ ,\ \nu > 0. \end{gathered} \tag{3.27}$$

ν is selected such that:

$$f_\delta(v_0 + \nu) > f(v_0), \tag{3.28}$$

which gives:

$$\nu > |f_\delta^{-1} f(v_0) - v_0|. \tag{3.29}$$

The Lipschitz constant for the expression $f_\delta(s)$ is:

$$\epsilon^\sigma(s_1^q - s_2^q) \leq \epsilon^\sigma L(s_1 - s_2) \leq \epsilon^\sigma \delta^{(q-1)} L(s_1 - s_2).$$

With $q < 1$. Therefore the last inequality holds for $|\delta| < 1$. The Lipschitz constant is then $\epsilon^\sigma L \delta^{(q-1)}$.

The problem P_ϵ^M has a unique solution, in virtue of Theorem 3.2.1, existing for a time interval $T_{\epsilon,\delta}$ given by:

$$T_{\epsilon,\delta} \geq \frac{1}{L\delta^{(q-1)}\epsilon^\sigma(m-1)} \log\left(1 + Lc\delta^{(q-1)}\epsilon^\sigma(m-1)\|v_0 + \nu\|_*^{1-m}\right). \tag{3.30}$$

The problem has, now, three different parameters: ϵ used to bound the forcing term, δ used to approximate the non-Lipschitz problem by a Lipschitz one and the paremeter ν that shall be chosen to ensure the maximality of v^M.

For a given ϵ and $\delta \to 0$ the non-Lipschitz problem is recovered. Then, it is possible to determine the following condition for the existence time:

$$T_{\epsilon,\delta\to 0} \geq 0. \tag{3.31}$$

Or explicitly with δ:

$$T_{\epsilon,\delta\to 0} \geq \frac{1}{L\epsilon^{\sigma}(m-1)}\delta^{1-q}. \tag{3.32}$$

This condition means that the existence time is given in the proximity of any δ.

To recover the forcing term $|x|^{\sigma}$, make $\epsilon \to \infty$,while to recover the non-Lipschitz problem impose $\delta \to 0$. To account for both effects simultaneously, consider:

$$\epsilon = \frac{1}{\delta^{a}},\ a > 0. \tag{3.33}$$

Previously and first of all, make $\delta \to \infty$ and $\epsilon \to 0$. In this case, the result obtained in the Lipschitz case (with $\sigma > 0$) applies.

$$T_{\epsilon,\delta} \geq \frac{\delta^{(1-q)}\delta^{a\sigma}}{L(m-1)} \log\left(1 + c\frac{1}{\delta^{(1-q)}\delta^{a\sigma}}L(m-1)\|v_0 + \nu\|_{*}^{1-m}\right), \tag{3.34}$$

$$T_{\epsilon\to 0,\delta\to\infty} \geq c\|v_0 + \nu\|_{*}^{1-m}, \tag{3.35}$$

for any $a > 0$.

Even when the value for $T_{\epsilon,\delta}$ has been obtained in the limit for $\epsilon \to 0$, it can be applied for any positive ϵ and by extension for any local single point in $x \in \mathbb{R}^d$ as the function $|x|^{\sigma} \in \mathbb{L}^{\infty}_{loc}$.

Nonetheless, to recover the original problem $\delta \to 0$ and $\epsilon \to \infty$. Operating with the term $\delta \to 0$:

$$T_{\epsilon,\delta} \geq \frac{\delta^{(1-q)}\delta^{a\sigma}}{L(m-1)} \log\left(c\frac{1}{\delta^{(1-q)}\delta^{a\sigma}}L(m-1)\|v_0 + \nu\|_{*}^{1-m}\right),\ \ T_{\epsilon,\delta} \geq 0. \tag{3.36}$$

Or explicitly with δ:

$$T_{\epsilon,\delta\to 0} \geq \frac{1}{L(m-1)}\delta^{1-q+a\sigma}. \tag{3.37}$$

for any $a > 0$.

Given a value for δ small and other one for ϵ large, a value for $T_{\epsilon,\delta}$ is determined so that a maximal local solution exists.

For building the minimal solution, consider the following problem P^{m}_{ϵ}:

$$\begin{gathered} v_t = \Delta v^m + f_{\delta}(v)\ Q_{T_{\epsilon,\delta}} = \mathbb{R}^d \times [0, T_{\epsilon,\delta}], \\ v(x,0) = v_0(x)\ for\ x \in \mathbb{R}^d,\ \delta > 0. \end{gathered} \tag{3.38}$$

The problem P^{m}_{ϵ} has a unique solution (Theorem 3.2.1) existing for a time interval $(0,\ T_{\epsilon,\delta})$. Any solution, $v_{m,\delta}$, to the problem P^{m}_{ϵ} is a subsolution to the problem P_{ϵ} and to the original problem P. In the proximity of v null the non-Lipschitz reaction satisfies

$$\begin{gathered} f_{\delta}(v) \leq \epsilon^{\sigma}v^{q}(1-v) \leq \epsilon^{\sigma}v^{q} \leq |x|^{\sigma}v^{q}, \\ v_{m,\delta} \leq v. \end{gathered} \tag{3.39}$$

Given $\delta_1 > \delta_2$, $f_{\delta_1}(v) < f_{\delta_2}(v)$. For an arbitrary decreasing sequence of δ's, $v_{m,\delta}$ is a non-decreasing sequence that satisfies $v_{m,\delta} \leq v$. Then, in the limit with $\delta \to 0$:

$$v_m = \lim_{\delta\to 0} v_{m,\delta}. \tag{3.40}$$

Each $v_{m,\delta}$ does exist (Theorem 3.2.1), therefore, v_m is a minimal solution to the problem P_{ϵ} and to the problem P as v_m has been obtained under the change of the reaction original term $v^{q}(1-v)$ by a Lipschitz function from below $f_{\delta}(v)$ in the proximity of v null. □

A sharp estimation of each maximal and minimal solution, with a classification in accordance with the problem data, is done in the following section.

3.3.1. Assessments on minimal and maximal solutions

Admit quasi-null initial data: $v_0 = 0$, a.e. in $B_R = \{|x - x_0| < R\}$, such that the non-Lipschitz imposes non-uniqueness. An elementary minimal solution to P is $v_m = 0$ while a maximal solution, influenced by the reaction term, adopts the general form:

$$v_T^M = |x|^{\frac{\sigma}{1-q}}(1-q)^{\frac{1}{1-q}}(t-T)^{\frac{1}{1-q}}, \tag{3.41}$$

$T > 0$. To show this, consider:

$$v_T^M = |x|^{\theta} A(t-T)^{\alpha}. \tag{3.42}$$

Replacing into P in the proximity of null v such that $v^q(1-v) \sim v^q$:

$$|x|^{\theta} A\alpha(t-T)^{\alpha-1} = m\theta(m\theta-1)|x|^{m\theta-2}A^m(t-T)^{m\alpha} + |x|^{q\theta+\sigma}A^q(t-T)^{q\alpha}. \tag{3.43}$$

With a predominant reaction:

$$\theta = \frac{\sigma}{1-q}, \;\; \alpha = \frac{1}{1-q}, \;\; A = (1-q)^{\frac{1}{1-q}}. \tag{3.44}$$

$$v_{T\to 0^+}^M = |x|^{\frac{\sigma}{1-q}}(1-q)^{\frac{1}{1-q}}(t)^{1/(1-q)}. \tag{3.45}$$

The reaction shall be relevant in the proximity of $v = 0$. For this purpose, $t \to T \to 0^+$ so that $q\alpha < m\alpha$, in the right hand term of (3.43), i.e., $q < m$, which leads to recover the original data $0 < q < m$.

The local time evolution provides a positive and growing solution departing from the positive set supplementary to B_R . The spatial term shall not contradict such evolution when the reaction predominates over the diffusion, then returning to (3.43):

$$q\theta + \sigma > m\theta - 2 \to \; m\sigma + 2(1-\sigma)q + \sigma < 2, \tag{3.46}$$

which shall be considered as a parameter condition to account for a maximal solution as expressed in (3.45). As a consequence of (3.46) the following behaviour is expected in accordance with the parameters data: If $m\sigma + 2(1-\sigma)q + \sigma \geq 2$, the diffusion influences further compare to the reaction and finite speed of propagation shall be considered whenever the solution is null in a certain ball B_R or in the proximity of $|x| << \gamma$. Nontheless, if $m\sigma + 2(1-\sigma)q + \sigma < 2$, the reaction predominates, and the non-Lipschtiz condition provides non-uniqueness so that the obtained minimal and maximal solutions obtained apply.

3.3.2. Blow up analysis

The next intention is to provide a condition to distinguish between the existence of global in time solutions and the explosion in finite time. For this purpose, a critical exponent q^* is proved to exist. If the solution blows-up, the invasive specie proliferates in the domain inducing pressure over the hostile region $|x| < \gamma \to 0^+$.

Theorem 3.3.2. *Consider the critical exponent* $0 \leq q^* < 1$ *defined as:*

$$q^* = sign_+\left(1 - \frac{\sigma(m-1)}{2}\right), \tag{3.47}$$

For:

$$q > q^*, \tag{3.48}$$

blow up or explosion in finite time exists, while for:

$$q \leq q^*, \tag{3.49}$$

a global solution exists.

Proof. Consider the self-similar profile:

$$G(x,t) = t^{-\alpha} f(|x|t^{\beta}), \;\; \xi = |x|t^{\beta}. \tag{3.50}$$

With $d = 1$ in the sake of simplicity. Upon substitution into P:

$$-\alpha t^{-\alpha-1} f + \beta \underbrace{|x|t^{\beta}}_{\xi} t^{-\alpha-1} f' = t^{-\alpha m} t^{2\beta} f_{xx}^m + \xi^{\sigma} t^{-\sigma\beta-\alpha q} f^q. \tag{3.51}$$

Making the following equalities in (3.51):

$$-\alpha - 1 = -\alpha m + 2\beta, \;\; \alpha m - 2\beta = \alpha q + \beta\sigma. \tag{3.52}$$

So that:

$$\alpha = \frac{\sigma+2}{\sigma(m-1)+2(q-1)}, \;\; \beta = \frac{m-p}{\sigma(m-1)+2(q-1)}. \tag{3.53}$$

As the time exponent in (3.50) is given by $-\alpha$, the previous α expression in (3.53) shall be positive for the existence of finite time blow up, then:

$$\sigma(m-1) + 2(q-1) > 0, \tag{3.54}$$

so that, the critical exponent is defined as:

$$q^* = sign_+\left(1 - \frac{\sigma(m-1)}{2}\right), \tag{3.55}$$

and blow up exists if $q > q^*$ while global solutions exists if $q \leq q^*$.

□

3.3.3. Uniqueness

Uniqueness of solutions leads to consider only positive initial data $v_0 \geq \phi > 0$, so that the reaction term, $R(x,v) = |x|^{\sigma} v^q (1-v)$, is Lipschitz in the interval $[\phi, \infty)$. The following lemma holds:

Theorem 3.3.3. *Consider:*

$$v_0 \geq \phi > 0, \tag{3.56}$$

such that the reaction term is Lipschitz with constant $\frac{q}{\phi^{1-q}}$*, then uniqueness of solutions holds in* $Q_T = \mathbb{R}^d \times [0, T]$.

Proof. The non-linear diffusion term is associated to a degenerate diffusivity ($D(v) = mv^{m-1}$). In case $v \to 0$, for example if $\phi \to 0$, the degeneracy does not lead to positivity, and thus, solutions cannot be classical locally in time as $v(x, t \to 0) \to 0$. Then, uniqueness is shown for weak solutions defined in accordance with a test function $\psi(x, t) \in C^\infty(Q_T)$. For this purpose, consider the existence of two solutions $v_1(x, t)$ and $v_2(x, t)$. By initial assumption, and without loss of generality, consider that $v_1 \geq v_2$. Both solutions have the same initial positive data:

$$v_1(x, 0) = v_2(x, 0) = v_0(x) \geq \phi > 0. \tag{3.57}$$

A weak solutions is defined as:

$$\int_{\mathbb{R}^d} v_1(t)\,\psi(t)dx = \int_{\mathbb{R}^d} v(0)\,\psi(0)dx + \int_0^t \int_{\mathbb{R}^d} [v_1\,\psi_t + v_1^m\,\Delta\psi + |x|^\sigma v_1^q(1 - v_1)\,\psi]dxds, \tag{3.58}$$

$$\int_{\mathbb{R}^d} v_2(t)\,\psi(t)dx = \int_{\mathbb{R}^d} v(0)\,\psi(0)dx + \int_0^t \int_{\mathbb{R}^d} [v_2\,\psi_t + v_2^m\,\Delta\psi + |x|^\sigma v_2^q(1 - v_2)\,\psi]dxds, \tag{3.59}$$

and making the substraction:

$$\int_{\mathbb{R}^d} (v_1 - v_2)(t)\,\psi(t)dx = \int_0^t \int_{\mathbb{R}^s} [(v_1 - v_2)\,\psi_t + (v_1^m - v_2^m)\,\Delta\psi + |x|^\sigma (v_1^q(1 - v_1) - v_2^q(1 - v_2)\,\psi]dxds. \tag{3.60}$$

Under the Lipschitz condition and $v_1 \geq v_2$:

$$(v_1^q(1 - v_1) - v_2^q(1 - v_2)) \leq |v_1^q - v_2^q| \leq \frac{q}{\phi^{1-q}}|v_1 - v_2|. \tag{3.61}$$

Where the Lipschitz constant is obtained as $K_l = qv^{q-1} = q\frac{1}{v^{1-q}} \leq \frac{q}{\phi^{1-q}}$. Now:

$$(v_1^m - v_2^m) \leq mv_1^{m-1}|v_1 - v_2| \leq \kappa^{m-1}|v_1 - v_2|. \tag{3.62}$$

Where: $\kappa = \max_{(x,t)\in Q_T}\{v_1\}$.

Consider the test function $\psi(x, t) \in C^\infty(Q_T)$:

$$\psi(x, s) = \frac{e^{-ls}}{(1 + |x|^2)^\gamma}, \tag{3.63}$$

where γ is such that:

$$e^{ls} \int_{\mathbb{R}^d} |x|^\sigma \psi(x, s)dx < \infty. \tag{3.64}$$

Consider for simplicity and without loss of generality:

$$e^{ls} \int_{\mathbb{R}^d} |x|^\sigma \psi(x, s)dx = 1. \tag{3.65}$$

For this purpose, the mass shall be null when $|x| \to \infty$. This can be expressed considering that for $R >> 1$:

$$\int_{|(|x|-R)|\to\infty} |x|^{\sigma}\psi(x,s)dx = 0. \tag{3.66}$$

In the asymptotic $|x| \to \infty$:

$$|x|^{-2\gamma}\,|x|^{\sigma}\,|x|^{d} \to 0. \tag{3.67}$$

Then:

$$-2\gamma + \sigma + d < 0,\ \gamma > \frac{\sigma + d}{2} > 0. \tag{3.68}$$

Define:

$$\int_{\mathbb{R}^d} \psi(x,s)dx = e^{-ls}\int_{\mathbb{R}^d} \frac{1}{(1+|x|^2)^{\gamma}}dx = e^{-ls}\Psi(x), \tag{3.69}$$

where:

$$\Psi(x) = \int_{x\to\infty} \frac{1}{(1+|\beta|^2)^{\gamma}}d\beta. \tag{3.70}$$

The integral (3.70) is finite in virtue of the condition for γ in (3.68).

In the same way:

$$\int_{\mathbb{R}^d} \Delta\psi(x,s)dx \le \int_{\mathbb{R}^d} K_1(\gamma)\psi(x,s)dx = K_1(\gamma)e^{-ls}\Psi(x). \tag{3.71}$$

Where $K_1(\gamma)$ is obtained as:

$$\begin{aligned} e^{-ls}\Delta\psi &= e^{-ls}\gamma(\gamma+1)4x^2\frac{1}{(1+|x|^2)^{\gamma+2}} - 2\gamma\frac{1}{(1+|x|^2)^{\gamma+1}} \\ &\le e^{-ls}\gamma(\gamma+1)4x^2\frac{1}{(1+|x|^2)^{\gamma+2}}. \end{aligned} \tag{3.72}$$

In the limit with $|x| \to \infty$:

$$\begin{aligned} e^{-ls}\gamma(\gamma+1)4x^2\frac{1}{(1+|x|^2)^{\gamma+2}} &\sim e^{-ls}\frac{\gamma(\gamma+1)4}{(1+|x|^2)^{\gamma}}\frac{1}{|x|^2} \\ &\le e^{-ls}\frac{\gamma(\gamma+1)4}{(1+|x|^2)^{\gamma}} = \gamma(\gamma+1)4\psi(x). \end{aligned} \tag{3.73}$$

Then $K_1 = \gamma(\gamma+1)4$. Assessing each integral in (3.60):

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}^d}(v_1 - v_2)\,\psi_t dxds = \int_0^t\int_{\mathbb{R}^d} -l(v_1 - v_2)\,\psi dxds \\ &= \int_0^t\int_{\mathbb{R}^d} l(v_2 - v_1)\,\psi dxds \le l\sup|v_2 - v_1|\Psi(x)\int_0^t e^{-ls}ds \\ &= l\sup|v_2 - v_1|\Psi(x)\left(\frac{1}{l}\right)(1 - e^{-lt}). \end{aligned} \tag{3.74}$$

$$\begin{aligned} \int_0^t\int_{\mathbb{R}^d}(v_1^m - v_2^m)\,\Delta\psi dxds &\le \sup|v_1 - v_2|\int_0^t \kappa^{m-1}K_1(\gamma)\int_{\mathbb{R}}\psi(x,s)dxds \\ &= \sup|v_1 - v_2|\int_0^t \kappa^{m-1}K_1(\gamma)e^{-ls}\Psi(x)\sup|v_1 - v_2|\kappa^{m-1}K_1(\gamma)(1 - e^{-lt}). \end{aligned} \tag{3.75}$$

$$\int_0^t \int_{\mathbb{R}^d} |x|^\sigma (v_1^q(1-v_1) - v_2^q(1-v_2))\,\psi(x,s)dxds \leq \frac{q}{\phi^{1-q}} \sup|v_1 - v_2| \int_0^t e^{-ls} \int_{\mathbb{R}} e^{ls}|x|^\sigma \psi(x,s)dxds$$
$$= \frac{q}{\phi^{1-q}} \sup|v_1 - v_2|(1 - e^{-lt}). \tag{3.76}$$

After compilation in (3.60):

$$\int_{\mathbb{R}^d} (v_1 - v_2)(t)\,\psi(t)dx \leq \sup|v_2 - v_1|\Psi(x)(1 - e^{-lt}) + \sup|v_1 - v_2|\kappa^{m-1}K_1(\gamma)(1 - e^{-lt})$$
$$+ \frac{q}{\phi^{1-q}} \sup|v_1 - v_2|(1 - e^{-lt}). \tag{3.77}$$

For a given $t > 0$ such that $\sup|v_1 - v_2| \to 0$ and knowing that $\phi > 0$, $|\Psi(x)| < \infty$:

$$\int_{\mathbb{R}^d} (v_1 - v_2)(t)\,\psi(t)dx \leq 0 \to v_1(t) \leq v_2(t). \tag{3.78}$$

Initially, it was assumed $u_1(t) \geq u_2(t)$, the only compatible result is to consider:

$$v_1(t) = v_2(t), \tag{3.79}$$

showing, then, the uniqueness of solutions. □

3.3.4. Comparison of solutions

Theorem 3.3.4. *Let u and v be two solutions to the problem P in Q_T, such that $0 < u_0 \leq v_0$ in $\mathbb{R}^d$ and $u_0,\ v_0 \in E_0$, then the following comparison principle holds:*

$$0 < u \leq v \ \ in \ \ Q_T \tag{3.80}$$

Proof. Considering the definition of a weak solution with a test function $\phi(x,t) \in C^\infty(Q_T)$ and for $0 \leq \tau < t < T$:

$$\int_{\mathbb{R}^d} u(t)\,\phi(t)dx = \int_{\mathbb{R}^d} u_0(\tau)\,\phi(\tau)dx + \int_\tau^t \int_{\mathbb{R}^d} [(u)\,\phi_t + (u^m)\,\Delta\phi + |x|^\sigma\, u^q(1-u)\,\phi]dxds. \tag{3.81}$$

$$\int_{\mathbb{R}^d} v(t)\,\phi(t)dx = \int_{\mathbb{R}^d} v_0(\tau)\,\phi(\tau)dx + \int_\tau^t \int_{\mathbb{R}^d} [(v)\,\phi_t + (v^m)\,\Delta\phi + |x|^\sigma\, v^q(1-v)\,\phi]dxds, \tag{3.82}$$

where $u_0(\tau)$ and $v_0(\tau)$ are the initial data time translation in τ.

After subtraction:

$$\int_{\mathbb{R}^d} (u - v)(t)\,\phi(t)dx = \int_{\mathbb{R}^d} (u_0 - v_0)(\tau)\,\phi(\tau)dx$$
$$+ \int_\tau^t \int_{\mathbb{R}^d} [(u - v)\,\phi_t + (u^m - v^m)\,\Delta\phi + |x|^\sigma\,(u^q(1-u) - v^q(1-v))\,\phi]dxds. \tag{3.83}$$

Note that the functions u^q, v^q are positive by initial assumption.

The intention is to assess each of the integrals involved in (3.83), making use of the norm defined in (3.3) and using the same test function structure than in (3.63), but probably with a different exponent γ, namely:

$$\phi(x,s) = \frac{e^{-ls}}{(1+|x|^2)^{\gamma}}. \tag{3.84}$$

Then:

$$\int_{\mathbb{R}^d} (u_0 - v_0)(\tau)\,\phi(\tau)dx \leq \|u_0 - v_0\|_* \cdot \|\phi\|_*, \tag{3.85}$$

where:

$$\begin{aligned} \|\phi\|_* &= \lim_{R\to\infty} R^{-d-a_\sigma} \int_{\mathbb{R}^d} |\phi(x)|\,dx \\ &\sim |x|^{-d-a_\sigma} \int_{\mathbb{R}^d} \frac{e^{-ls}}{(1+|x|^2)^{\gamma}} dx. \end{aligned} \tag{3.86}$$

Assume $|x| \to \infty$ and γ selected as:

$$|x|^{-d-a_\sigma} \int_{\mathbb{R}^d;|x|\to\infty} \frac{dx}{(1+|x|^2)^{\gamma}} = 0. \tag{3.87}$$

For this purpose:

$$|x|^{-d-a_\sigma}|x|^{-2\gamma}|x|^d = 0, \tag{3.88}$$

when $|x| \to \infty$.

This condition implies that:

$$-d - a_\sigma - 2\gamma + d < 0, \tag{3.89}$$

for which it suffices to consider:

$$\gamma > -\frac{a_\sigma}{2}, \tag{3.90}$$

where $a_\sigma > 0$ as shown in (3.4).

With the intention of preserving the decreasing behaviour with $|x|$ in (3.84), it is required $\gamma > 0$, so that (3.90) is satisfied as well.

As the function $\phi(x)$ is monotone decreasing with $|x|$, the maximum value for ϕ corresponds to $|x| = 0$, $\max_{x\in\mathbb{R}^d} \phi(x,s) = e^{-ls}$. And returning to the integral (3.85):

$$\begin{aligned} \int_{\mathbb{R}^d} (u_0 - v_0)(\tau)\,\phi(\tau)dx &\leq \|u_0 - v_0\|_* \cdot \|\phi\|_* = \|u_0 - v_0\|_* \lim_{R\to\infty} R^{-d-a_\sigma} \int_{\mathbb{R}^d} |\phi(x)|\,dx \\ &\leq \|u_0 - v_0\|_* \lim_{R\to\infty} R^{-d-a_\sigma} \max|\phi(x)|R^d = \|u_0 - v_0\|_* e^{-l\tau} \lim_{R\to\infty} R^{-a_\sigma}, \end{aligned} \tag{3.91}$$

where $a_\sigma > 0$.

Assessing the rest of the integrals involved on (3.83):

$$\begin{aligned} \int_\tau^t \int_{\mathbb{R}^d} [(u-v)\,\phi_t dx ds = \int_\tau^t \int_{\mathbb{R}^d} (u-v)\,(-l)\,\phi \leq \int_\tau^t |-l|\|u-v\|_* \cdot \|\phi\|_* ds \\ \leq \|u-v\|_* (e^{-l\tau} - e^{-lt}) \lim_{R\to\infty} R^{-a_\sigma}. \end{aligned} \tag{3.92}$$

The assessment in the integral associated to the diffusion term is based on (3.75):

$$\begin{aligned}\int_\tau^t \int_{\mathbb{R}^d} (u^m - v^m)\,\Delta\phi \le \int_\tau^t \kappa^{m-1}\|u-v\|_* K_1(\gamma)\|\phi\|_* ds \\ \le \int_\tau^t \kappa^{m-1}\|u-v\|_* K_1(\gamma) e^{-ls} ds \lim_{R\to\infty} R^{-a_\sigma} \\ = \kappa^{m-1}\|u-v\|_* K_1(\gamma)\frac{1}{l}(e^{-l\tau}-e^{-lt}) \lim_{R\to\infty} R^{-a_\sigma}.\end{aligned} \tag{3.93}$$

Before proceeding with the reaction term integral, the following shall be considered:

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^\sigma \phi(x) dx \le \|\phi\|_* \int_{\mathbb{R}^d} |x|^\sigma dx = \lim_{R\to\infty} R^{-d-a_\sigma} \int_{\mathbb{R}^d} |\phi(x)|\,dx \int_{\mathbb{R}^d} |x|^\sigma dx \\ \sim \lim_{R\to\infty} R^{-d-a_\sigma} e^{-ls} \int_{\mathbb{R}^d} |x|^{-2\gamma} dx \int_{\mathbb{R}^d} |x|^\sigma dx \sim e^{-ls} \frac{1}{\sigma+1}\frac{1}{(-2\gamma+1)} \lim_{R\to\infty} R^{-d-a_\sigma} R^{-2\gamma+1} R^{\sigma+1}.\end{aligned} \tag{3.94}$$

Therefore, γ needs to satisfy the following inequality for convergence:

$$-d - a_\sigma + \sigma + 2 - 2\gamma < 0, \tag{3.95}$$

so that,

$$\gamma > \frac{-d - a_\sigma + \sigma + 2}{2}. \tag{3.96}$$

Aiming a single value of γ and considering (3.90), (3.96):

$$\gamma > \max\left\{\frac{-d - a_\sigma + \sigma + 2}{2},\ -\frac{a_\sigma}{2},\ 0\right\}. \tag{3.97}$$

The assessment in the reaction terms leads to (with K_l the Lipschitz constant (3.61)):

$$\begin{aligned}\int_\tau^t \int_{\mathbb{R}^d} (u^q(1-u) - v^q(1-v))\,\phi dx ds \le K_l \int_\tau^t \|u-v\|_* \int_{\mathbb{R}^d} |x|^\sigma \phi ds \\ \le K_l \|u-v\|_* \frac{1}{\sigma+1}\frac{1}{(-2\gamma+1)} \lim_{R\to\infty} R^{-d-a_\sigma} R^{-2\gamma+1} R^{\sigma+1} \int_\tau^t e^{-ls} ds \\ \le K_l \|u-v\|_* \frac{1}{l}\frac{1}{\sigma+1}\frac{1}{|-2\gamma+1|} \lim_{R\to\infty} R^{-d-a_\sigma} R^{-2\gamma+1} R^{\sigma+1}(e^{-l\tau}-e^{-lt}).\end{aligned} \tag{3.98}$$

Finally, and after compilation:

$$\begin{aligned}\int_{\mathbb{R}^d} (u-v)(t)\,\phi(t) dx \le \|u_0 - v_0\|_* e^{-l\tau} \lim_{R\to\infty} R^{-a_\sigma} + \|u-v\|_* (e^{-l\tau}-e^{-lt}) \lim_{R\to\infty} R^{-a_\sigma} \\ + \kappa^{m-1}\|u-v\|_* K_1(\gamma)\frac{1}{l}(e^{-l\tau}-e^{-lt}) \lim_{R\to\infty} R^{-a_\sigma} \\ + K_l \|u-v\|_* \frac{1}{l}\frac{1}{\sigma+1}\frac{1}{|-2\gamma+1|} \lim_{R\to\infty} R^{-d-a_\sigma} R^{-2\gamma+1} R^{\sigma+1}(e^{-l\tau}-e^{-lt}).\end{aligned} \tag{3.99}$$

Note that $\lim_{R\to\infty} R^{-a_\sigma} = 0$ and $\lim_{R\to\infty} R^{-d-a_\sigma} R^{-2\gamma+1} R^{\sigma+1} = 0$. Where γ is as per expression (3.97).

In terms of the time variable with $\tau \to \infty$ and knowing that $\tau \to \infty < s < t$, then $s\ ,\ t \to \infty$. Under this condition in the time variables:

$$\int_{\mathbb{R}^d} (u-v)(t)\,\phi(t)dx \leq 0 \ \ \rightarrow \ \ u(t) \leq v(t),\ \ t \to \infty. \tag{3.100}$$

Given a solution with positive initial data, the solution is unique (see Theorem 3.3.3). The ordered properties expressed at the initial conditions are preserved upon evolution with $t \to \infty$ (3.100), hence for any finite time $u(t) \leq v(t),\ \ t > 0,\ \ in\ \ Q_T$ □

4. Travelling waves existence and regularity

The TW profiles are expressed as $v(x,t) = f(\xi)$, $\xi = x \cdot n_d - at\ \in \mathbb{R}$, where n_d is a unitary vector in $\mathbb{R}^d$ that defines the TW-propagation direction. a is the TW-speed and $f : \mathbb{R} \to (0,\infty)$ belongs to $L^\infty(\mathbb{R}^d)$. Note that two TW are equivalent under translation $\xi \to \xi + \xi_0$ and symmetry $\xi \to -\xi$. For the sake of simplicity, the vector n_d is $n_d = (1,0,...,0)$, then $v(x,t) = f(\xi),\quad \xi = x - at\ \in \mathbb{R}$.

Consider $v(x,t) = f(\xi)$, then the problem P (2.1) in the TW domain:

$$\begin{gathered} -af' = (f^m)' + |\xi + at|^\sigma f^q(1-f), \\ f \in L^\infty(\mathbb{R}),\ \ f'(\xi) > 0,\ \ f(\infty) = 1. \end{gathered} \tag{4.1}$$

Working with the density and flux variables

$$X = f\ ,\ Y = -(f^m)', \tag{4.2}$$

the following system holds:

$$\begin{aligned} X' &= \tfrac{1}{m} X^{1-m} Y, \\ Y' &= \tfrac{a}{m} X^{1-m} Y + |\xi + at|^\sigma X^q (1-X), \end{aligned} \tag{4.3}$$

with the critical point $(1,0)$ that represents a situation in which the invasive reaches the maximum concentration at $v = f = X = 1$ in the given domain. The analysis of the TW features in the proximity of the critical point permits to ennunciate:

Lemma 1. *The critical point* $(1,0)$ *is a degenerate node with:*

- *One null eigenvalue,*
- *One real eigenvalue related to the TW speed a.*

Proof. In the proximity of the critical point, the system (4.3) is rewritten in the compact form:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ 0 & \frac{a}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{4.4}$$

with eigenvalues $\left(0, \frac{a}{m}\right)$. This shows the existence of monotone stable TW solutions. The null eigenvalue corresponds to the equilibrium solution $X = 1,\ \ Y = 0$.

Based on the computation of the associated monotone eigenvectors, it is possible to obtain the following leading front behaviour in the proximity of the critical point:

$$X = c_1 - c_2\, e^{\frac{a}{m}\xi}, \quad c_1 > 0,\ c_2 > 0. \tag{4.5}$$

or equivalently $\xi \to -\xi$ which shows the regularity towards convergence in the TW solutions approaching the critical point. □

4.1. Geometric perturbation theory

The geometric perturbation theory permits to show the asymptotic evolution of a hyperbolic manifold defined to determine a TW profile. For this purpose, consider the manifold:

$$M_0 = \{X, Y \ / \ X' = \frac{1}{m}X^{1-m}Y\,;\ Y' = \frac{a}{m}X^{1-m}Y + |\xi + at|^{\sigma}X^q(1-X)\}, \tag{4.6}$$

so that the stationary condition $(1, 0)$ holds.

The perturbed manifold M_ϵ close to M_0 is defined as:

$$M_\epsilon = \{X \sim 1, Y \ / \ X' = \frac{1}{m}Y\,;\ Y' = \frac{a}{m}Y\}. \tag{4.7}$$

The intention is to use the Fenichel invariant manifold theorem [13] as formulated in [2] and [17]. Then, the manifold M_ϵ shall be proved to be a normally hyperbolic manifold, i.e., the eigenvalues of M_ϵ close to the critical point, and transversal to the tangent space shall have non-zero real part. This is shown based on the following system for M_ϵ:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ 0 & \frac{a}{m} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}, \tag{4.8}$$

with eigenvalues $(0\,, a/m)$. For $\lambda = 0$, the eigenvector is $[1, 0]$ tangent to M_ϵ. Therefore, M_ϵ is a hyperbolic manifold. The next intention is to show that the manifold M_ϵ is locally invariant under the flow given by the set of equations (4.3). For this purpose, it is required [17] that for all $R > 0$, for all open interval J with $a \in J$ and for any value of $i \in \mathbb{N}$, there exists a δ such that for $\epsilon \in (0, \delta)$ the manifold M_ϵ is invariant. Hence, consider $i \geq 1$ and the functions:

$$\begin{aligned} \phi_1^{M_0} &= \tfrac{1}{m}X^{1-m}Y, \\ \phi_2^{M_0} &= \tfrac{a}{m}X^{1-m}Y + |\xi + at|^{\sigma}X^q(1-X), \\ \phi_1^{M_\epsilon} &= \tfrac{1}{m}Y, \\ \phi_2^{M_\epsilon} &= \tfrac{a}{m}Y, \end{aligned} \tag{4.9}$$

which are $C^i(\overline{B_R(0)} \times \overline{I} \times [0, \delta])$ in the proximity of the critical point $(1, 0)$.

A value for $R > 0$ can be chosen considering that $M_\epsilon \cap B_R(0)$ is large enough so as to study the complete TW evolution along the domain. The determination of δ is based on assessing the distance between the flows in M_0 and M_ϵ. For this purpose, assume that the involved functions in such flows are measurable a.e. in $B_R(0)$:

$$\|\phi_1^{M_0} - \phi_1^{M_\epsilon}\| \leq \frac{1}{m}\|Y\|\,\|X^{1-m} - 1\| \leq \delta_1\|X^{1-m} - 1\|, \tag{4.10}$$

In addition,

$$\|\phi_2^{M_0} - \phi_2^{M_\epsilon}\| \leq \frac{a}{m}\|Y\| \, \|X^{1-m} - 1\| + |\xi + at|^\sigma |X^q|(1 - X) \leq \delta_2 \|X^{1-m} - 1\|, \tag{4.11}$$

which keeps the normal hyperbolic condition for $X \nearrow 1$ for $\delta = \max\{\delta_1, \delta_2\} \in (0, \infty)$.

Once M_ϵ has been shown to remain invariant with regards to the M_0 under the flow (4.3), the TW profiles can be obtained operating in M_ϵ.

5. Travelling waves profiles and finite propagation

Based on the normal hyperbolic condition of M_ϵ under the flow (4.3), asymptotic TW profiles are obtained:

$$Y' = \frac{a}{m} Y \to (f^m)'' = \frac{a}{m}(f^m)' \tag{5.1}$$

The last equation can be solved with standard methods:

$$(f^m)' - \frac{a}{m}(f^m) = K, \tag{5.2}$$

where K can be solved in the stationary $f = 1$, so that $K = 1 - \frac{a}{m}$. Hence:

$$f(\xi) = \left(\frac{1 - m}{m} \xi + \|v_0\|^{1-m} \right)^{\frac{1}{m-1}} + B, \tag{5.3}$$

where $B = \left(1 - \frac{m}{a}\right)^{\frac{1}{m}}$ and the norm in v_0 is defined as per (3.2). The positivity evolution for f permits to conclude on some regularity results in the quasilinear parabolic operator (see Section 3). Nonetheless, whenever:

$$f \to \epsilon \to 0^+ \quad in \quad B_R^T = B_R(x_0.R) \times [T - \epsilon, T + \epsilon], \tag{5.4}$$

for $T > 0$, the non-linear diffusion elucidates the existence of a finite propagation speed. Cases of $f \to \epsilon \to 0^+$ may happen for null initial conditions or in the borders of the hostile zone. The characterization of such finite propagation speed is the purpose of the following theorem:

Theorem 5.0.1. *There exists finite propagation speed when*

$$v \to \epsilon \to 0^+ \quad in \quad B_R^T = B_R(x_0.R) \times [T - \epsilon, T + \epsilon], \tag{5.5}$$

$T >> 1$*, where finite propagation refers to the existence of a positive convergent tail approaching the null solution.*

Proof. Consider the pressure variable w:

$$w = \frac{m}{m - 1} v^{m-1}, \tag{5.6}$$

so that the equation v in P reads:

$$w_t = (m - 1) w \Delta w + |\nabla w|^2 + \mu |x|^\sigma w^\delta, \tag{5.7}$$

where $\delta = \frac{q+m-2}{m-1}$ and $\mu = m\left(\frac{m-1}{m}\right)^{\delta}$. Note that $w \to 0$ then:

$$w_t \sim |\nabla w|^2 + \mu |x|^{\sigma} w^{\delta}. \tag{5.8}$$

A solution to a similar equation has been provided in [8]. For this purpose, define the following solution:

$$W(x,t) = a\left(bt + r - \frac{1}{n}\right)_+, \ r = |x|, \ \ n \in \mathbb{N}, \tag{5.9}$$

where each of the coefficients shall be assessed to ensure W is a maximal solution. The determination of each of the constant involved follows a similar approach to that in [11] but with the required modification. For $0 \le \tau \le 1$, impose $b\tau = \frac{1}{2n}$. Under this condition:

$$W(x,t) \equiv 0 \quad for \quad r < \frac{1}{2n} \quad and \quad 0 \le t \le \tau. \tag{5.10}$$

Any solution to the equation (5.8) is bounded when $0 < v < 1$, then:

$$v(x,t) \le K_1 < 1 \ for \ x \in \mathbb{R}, \ 0 \le t \le \tau \ and \ K_1(p, \|u_0\|_{\infty}). \tag{5.11}$$

$W(x,t)$ is required to be a maximal solution:

$$W(x,t) \ge v(x,t), \tag{5.12}$$

then

$$a\left(bt + r - \frac{1}{n}\right)_+ \ge K_1. \tag{5.13}$$

For $r > \frac{1}{n}$, consider $r = \frac{2}{n}$ and for $t = 0$:

$$a\left(\frac{2}{n} - \frac{1}{n}\right)_+ \ge K_1, \quad a \ge nK_1. \tag{5.14}$$

Note that:

$$W(x,t) \ge v(x,t), \tag{5.15}$$

in $r = \frac{2}{n}$ and $0 \le t \le \tau$. A condition for b is obtained considering that $W(x,t)$ is a supersolution in $0 < r < \frac{2}{n}, \ \ 0 \le t \le \tau$:

$$W_t \ge \frac{m-1}{m}|\nabla W|^2 + c \cdot \nabla W \frac{1}{m-1}. \tag{5.16}$$

In addition:

$$W_t = ab; \ W_r = a, \tag{5.17}$$

then:

$$b \ge \frac{m-1}{m}a + c\frac{1}{m-1} \tag{5.18}$$

For the values of a and b in expressions (5.14) and (5.18) respectively, the function $W(x,t)$ is a supersolution locally:

$$W(x,t) \ge w(x,t), \quad 0 < |x| < \frac{2}{n}, \quad 0 \le t \le \tau. \tag{5.19}$$

The inequality (5.19) reflects the null condition of W in B_R^T, then, any minimal solution $w(x,t)$ satisfies such null condition and, hence, it exhibits finite propagation in B_R^T.

□

The expression (5.3) provides the characteristic profile in relation with the diffusion front (note the parameter m). The invasive TW profile follows a potential law $\sim \xi^{\frac{1}{m-1}} (m > 1)$ in the convergence to the stationary condition as a result of the invasive proliferation. Such potential law reflects the behaviour of the invasive in the proximity of the hostile zone $\xi \to 0^+$ where the derivative is not bounded. The invasive pressure increases in the hostile zone, nonetheless the finite propagation feature in the border of such hostile area due to the non-linear diffusion slows down the invasive pressure and preserves the hostile area.

6. Conclusions

The proposed problem P (2.1) has been discussed stressing existence, uniqueness, comparison of solutions and Travelling Waves supported by the Geometric Perturbation Theory. In addition, the finite speed of propagation, induced by the porous medium diffusion, has been shown and a characterization of such property has been explored. The potential law in the TW profile suggests that the pressure induced by the invasive over the hostile area increases along the evolution. Nonetheless, the finite speed induced by the non-linear diffusion avoids a violent invasion of the specie in such hostile zone.

Conflict of interest

The author declares no conflict of interest.

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