Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 5752–5767.
DOI: 10.3934/math.2022318
Received: 08 October 2021
Revised: 06 December 2021
Accepted: 27 December 2021
Published: 11 January 2022

Research article

Mean square stability with general decay rate of nonlinear neutral stochastic function differential equations in the G-framework

Guangjie Li*

School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou 510006, China

* **Correspondence:** Email: scutliguangjie@163.com.

Abstract: Few results seem to be known about the stability with general decay rate of nonlinear neutral stochastic function differential equations driven by G-Brownain motion (G-NSFDEs in short). This paper focuses on the G-NSFDEs, and the coefficients of these considered G-NSFDEs can be allowed to be nonlinear. It is first proved the existence and uniqueness of the global solution of a G-NSFDE. It is then obtained the trivial solution of the G-NSFDE is mean square stable with general decay rate (including the trivial solution of the G-NSFDE is mean square exponentially stable and the trivial solution of the G-NSFDE is mean square polynomially stable) by G-Lyapunov functions technique. In this paper, auxiliary functions are used to dominate the Lyapunov function and the diffusion operator. Finally, an example is presented to illustrate the obtained theory.

Keywords: mean square stability; neutral stochastic functional differential equations; G-Brownain motion; general decay; λ-type function
Mathematics Subject Classification: 34K20, 60H10, 34K50

1. Introduction

Stochastic dynamical systems are often depicted by stochastic differential equations (SDEs in short), which have been widely used in many areas of science and engineering (see, e.g. [2, 16, 23, 31]). In reality, these stochastic dynamical systems depend not only on the present and past states but also on derivatives with functionals. For such systems, neutral stochastic functional differential equations (NSFDEs in short) are used to described them [12, 14]. In the study of stochastic dynamical systems, stability analysis is a hot topic and has attracted lots of attention, see [1, 13] and the references therein. So far, there are numerous literature on the stability of NSFDEs, we list [11, 15, 17, 22], for instance.

Motivated by describing measuring finance risk and volatility uncertainty, Peng [20] has developed a theoretical framework of G-expectation. Based on the framework of G-expectation, Peng [20, 21] introduced the G-Brownian motion and set up its Itô integral. Hu et al. [8] found that a weakly compact family of probability measures can be used to represent the G-expectation. Under the G-framework, many efforts have been made to study the stability of SDEs driven by G-Brownian motion (G-SDEs in short), see [3, 10, 26, 27, 29] and the references therein. Zhu et al. [30] derived the exponential stability and quasi sure exponential stability of the solutions to G-NSFDEs by variation-of-constants formula. Faizullah et al. [5] studied the mean square exponential stability of nonlinear G-NSFDEs. Pan et al. [18] derived the p-th moment exponential stability and quasi sure exponential stability of impulsive stochastic functional differential systems driven by G-Brownian motion using G-Lyapunov functions technique.

As we know, there are many results on the stability (e.g. moment exponential stability, almost sure exponential stability, almost sure polynomial stability) of SDEs. The concepts of these stability have been generalized to the stability with general decay rate, please refer to [9, 19, 24, 25, 28] and the references therein. Therein, Wu et al. [25] studied the almost sure stability of NSFDEs with infinite delay with general decay rate. Along this line, Hu et al. [9] investigated the stochastic stability of a class of unbounded delay neutral stochastic differential equations with general decay rate. Shen et al. [24] investigated the almost sure stability with general decay rate of neutral stochastic functional hybrid differential equations with Lévy noise. However, few results on the stability with general decay rate for G-SDEs, not to speak of the results on the stability with general decay rate for G-NSFDEs, which motivates the present research.

It is also known that the study of the stability is based on the existence and uniqueness of the global solution. In general, a unique global solution can be guaranteed by the local Lipschitz condition and the linear growth condition. But, in reality, the linear growth condition is somewhat restrictive. In this paper, we guarantee the existence and uniqueness of a global solution to the G-NSFDE by the local Lipschitz condition and a weaker condition. Moreover, a kind of λ-type function is defined in this paper. By applying G-Lyapunov functions technique, we can acquire a kind of mean square λ-type stability, including mean square exponential stability and mean square polynomial stability.

The rest of the paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, the existence and uniqueness of the global solution and the sufficient conditions for λ-type stability are shown, respectively. Finally, an example is presented to illustrate the obtained results.

2. Preliminaries

In this section, we briefly recall some preliminaries in G-framework. More relevant details can be found in [4, 7, 20, 21].

Let $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$. For $\forall a \in R^n$, denote by $|a| = \sqrt{a^T a}$. For $\forall a, b \in R$, $a \vee b$ and $a \wedge b$ represent the largest and smallest of a and b, respectively. For $\forall a \in R$, $a^+ = \frac{a+|a|}{2}$ and $a^- = \frac{|a|-a}{2}$. On a non-empty basic space Ω, one can define a linear space $\mathcal{H}$ of real-valued functions. We suppose that $\mathcal{H}$ satisfies $C \in \mathcal{H}$ for each constant C and if $X \in \mathcal{H}$, $|X| \in \mathcal{H}$. If $X_1, X_2, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(R^n)$, where $C_{l,Lip}(R^n)$ is the space of linear function $\varphi : R^n \to R$:

$$C_{l,Lip}(R^n) = \{\varphi | \ \exists C \in R^+, m \in N \text{ s.t. } |\varphi(b) - \varphi(c)| \leq C(1 + |b|^m + |c|^m)|b - c|\},$$

for $\forall b, c \in R^n$.

Definition 2.1. *[21] A functional $\hat{E}:\mathcal{H}\to R$ is called a sublinear expectation. If for all $X,Y\in\mathcal{H}$, $C\in R$ and $\lambda\geq 0$, it satisfies the following properties:*

(i) Monotonicity: If $X\geq Y$, then $\hat{E}[X]\geq\hat{E}[Y]$.
(ii) Constant preserving: $\hat{E}[C]=C$.
(iii) Sub-additivity: $\hat{E}[X+Y]\leq\hat{E}[X]+\hat{E}[Y]$.
(iv) Positive homogeneity: $\hat{E}[\lambda X]=\lambda\hat{E}[X]$.

Also, if $\hat{E}[X]=\hat{E}[-X]=0$, then $\hat{E}[C+\lambda X+Y]=C+\hat{E}[Y]$.

The triple $(\Omega,\mathcal{H},\hat{E})$ is called a sublinear expectation space. If (i) and (ii) are satisfied, $\hat{E}[\cdot]$ is called a nonlinear expectation and the triple $(\Omega,\mathcal{H},\hat{E})$ is relevantly called a nonlinear expectation space.

For the details of G-normal distribution, G-expectation, G-conditional expectation and G-Brownian motion, please see Chapter 3 and Chapter 4 of Peng [21].

Denote by $\Omega_T=\{\omega(\cdot\wedge T):\omega\in\Omega\}$, $\forall T\geq 0$. For $T\in R^+$, a partition μ_T of $[0,T]$ is a finite ordered subset $\mu_T=\{t_0,t_1,\cdots,t_N\}$ such that $0=t_0<t_1<t_2<\cdots<t_N=T$, $\pi(\mu_T)=\max\{|t_{i+1}-t_i|:i=0,1,\cdots,N-1\}$. Let

$$L_{ip}(\Omega_T)=\{\varphi(\omega(t_1),\omega(t_2),\cdots,\omega(t_n)):t_1,t_2,\cdots,t_n\in[0,T],\varphi\in C_{l,Lip}(R^n)\}$$

and its countably many union $L_{ip}(\Omega)=\bigcup_{n=1}^{\infty}L_{ip}(\Omega_n)$. Denote by $L_G^p(\Omega_T)$ by the completion of $L_{ip}(\Omega_T)$ under the norm $\|X\|_p=(\hat{E}(|X|^p))^{1/p}$, for any $p\geq 1$. Besides, the space is defined by

$$M_G^{p,0}([0,T])=\left\{\eta_t=\sum_{j=0}^{N-1}\xi_j I_{[t_j,t_{j+1})}(t):\xi_j\in L_G^p(\Omega_{t_j})\right\},\ p\geq 1,$$

and its completion $M_G^p([0,T])$ equipped with the norm

$$\|\eta\|_{M_G^p([0,T])}=\left(\frac{1}{T}\int_0^T\hat{E}[|\eta_t|^p]dt\right)^{\frac{1}{p}},$$

where $\hat{E}$ stands for the G-expectation.

We now show the representation lemma of G-expectation as follow.

Lemma 2.1. *[6, 8] Let $\hat{E}$ be the G-expectation on $(\Omega,L_G^1(\Omega))$. Then there is a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega,\mathcal{B}(\Omega))$ such that $\hat{E}[X]=\sup_{P\in\mathcal{P}}E_P[X]$, $\forall X\in L_G^1(\Omega)$. Moreover, $\mathcal{P}$ is called a set that represents the G-expectation $\hat{E}$.*

From Lemma 2.1, the weakly compact family of probability $\mathcal{P}$ characterizes the degree of Knightian uncertainty. If $\mathcal{P}$ is singleton, that is $\{P\}$, then the model has no ambiguity, the G-expectation $\hat{E}$ is the classical expectation. Then define G-upper capacity $V(\cdot)$ and G-lower capacity $v(\cdot)$ by

$$V(A)=\sup_{P\in\mathcal{P}}P(A),\ \forall A\in\mathcal{B}(\Omega),$$
$$v(A)=\inf_{P\in\mathcal{P}}P(A),\ \forall A\in\mathcal{B}(\Omega).$$

Definition 2.2. *A set $A\in\mathcal{B}(\Omega)$ is called polar if $V(A)=0$. A property is said to hold quasi-surely (q.s.) if it is true outside a polar set.*

$\mathcal{P}$-q.s. means that it holds P-almost surely (P-a.s.) for each $P \in \mathcal{P}$. If an event A satisfies $V(A) = 1$, then we claim that the event A occurs V-a.s.

Let $(\omega(t))_{t\geq 0}$ be a 1-dimensional G-Brownian motion with $G(a) = \frac{1}{2}\hat{E}[a^2(1)] = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)(\forall a \in R)$, where $\hat{E}[\omega^2(1)] = \bar{\sigma}^2,\ -\hat{E}[-\omega^2(1)] = \underline{\sigma}^2,\ 0 \leq \underline{\sigma} \leq \bar{\sigma} < \infty$. And $\mathcal{F}_t$ represents a filtration generated by G-Brownian motion $(\omega(t))_{t\geq 0}$. In the following, we next carry out the stochastic integral with respect to the quadratic variation of G-Brownian motion.

Definition 2.3. *The stochastic integral with respect to the quadratic variation of G-Brownian motion* $(\langle\omega\rangle(t))_{t\geq 0}$ *is given by*

$$\int_0^t \eta_t d\langle\omega\rangle(t) = \sum_{j=0}^{N-1} \xi_j \left(\langle\omega\rangle(t_{j+1}) - \langle\omega\rangle(t_j)\right), \forall \eta_t \in M^{1,0}([0,T]),$$

where $\langle\omega\rangle(t) = \lim_{N\to\infty} \sum_{j=0}^{N-1} \left(\langle\omega\rangle(t_{j+1}^N) - \langle\omega\rangle(t_j^N)\right)^2 = \omega^2(t) - 2\int_0^t \omega(s)d\omega(s).$

3. Main results

In this section, mean square stability with general decay rate of nonlinear G-NSFDEs is provided. We further give the following notations. Let $\tau > 0$, denote by $C([-\tau,0];R)$ the family of continuous functions $\phi : [-\tau,0] \to R$ with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. For $\phi \in C([-\tau,0];R)$, define $D(\phi) = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta) - \phi(0)|$. Let $U([-\tau,0];R^+)$ be the family of all Borel measurable bounded nonnegative functions $\eta(\theta)$ defined on $-\tau \leq \theta \leq 0$ such that $\int_{-\tau}^0 \eta(\theta)d\theta = 1$. Denote by $L^1(R^+;R^+)$ the family of all continuous nonnegative functions $a(\cdot) : R^+ \to R^+$ such that $\int_0^\infty a(s)ds < \infty$. Let I_B be the indicator function of a set B. For $t \geq 0$, let $(\Omega, \mathcal{H}, \{\Omega_t\}_{t\geq 0}, \hat{E}, V)$ be a generalized sublinear expectation space. Let $(\omega(t))_{t\geq 0}$ be a 1-dimensional G-Brownian motion defined on the sublinear expectation space.

Consider the following G-NSFDE

$$\begin{aligned} d[x(t) - N(\rho_1(x_t,t),t)] =& f(x(t),\rho_2(x_t,t),t)dt + g(x(t),\rho_3(x_t,t),t)d\langle\omega\rangle(t) \\ &+ h(x(t),\rho_4(x_t,t),t)d\omega(t),\ t \geq 0 \end{aligned} \tag{3.1}$$

with the initial value

$$\begin{aligned} x_0 = \varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\}\ &is\ \mathcal{F}_0 - measurable,\ C([-\tau,0];R) - valued\ random\ variable \\ &such\ that\ \varphi \in M_G^2([-\tau,0];R). \end{aligned} \tag{3.2}$$

Here, $x(t)$ is the value of stochastic process at time t and $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$, ρ_1, ρ_2, ρ_3 and $\rho_4 : C([-\tau,0];R) \times R^+ \to R$, $N : R \times R^+ \to R$, $f, g, h : R \times R \times R^+ \to R$ are continuous functions.

In the following, the mean square stability with general decay rate is investigated. Next, the definition of λ-type function is first presented (see [24]).

Definition 3.1. *The function* $\lambda : R \to (0,\infty)$ *is said to be λ-type function if the function satisfies the following three conditions:*

- *It is continuous and nondecreasing in R and differentiable in R^+;*

- $\lambda(0) = 1$, $\lambda(\infty) = \infty$ *and* $\lambda^* = \sup_{t\geq 0}\left[\frac{\lambda'(t)}{\lambda(t)}\right] < \infty$;
- $\lambda(t) \leq \lambda(s)\lambda(t-s)$ *for any* $s, t \geq 0$.

Definition 3.2. *The trivial solution of the G-NSFDE* (3.1) *with the initial value* (3.2) *is said to be mean square stable with general decay rate* $\lambda(t)$ *of order* ε *if*

$$\limsup_{t\to\infty} \frac{\log \hat{E}|x(t)|^2}{\log \lambda(t)} \leq -\varepsilon. \tag{3.3}$$

Remark 3.1. *For any* $t \in R$, *when* $\lambda(t)$ *is* e^t, $1+t^+$, *respectively, it follows from Definition 3.2 that the definitions of the mean square exponential stability and the mean square polynomial stability can be obtained, respectively.*

In order to prove the mean square stability with general decay rate of the G-NSFDE (3.1) with any given initial value (3.2), the following conditions are imposed.

(C1) (Local Lipschitz condition) Assume that for each $k = 1, 2, \ldots$, there exists a positive constant l_k such that for any $t \geq 0$,

$$\begin{aligned}&|f(\varphi(0), \rho_2(\varphi, t), t) - f(\phi(0), \rho_2(\phi, t), t)| \vee |g(\varphi(0), \rho_3(\varphi, t), t) - g(\phi(0), \rho_3(\phi, t), t)|\\ &\vee |h(\varphi(0), \rho_4(\varphi, t), t) - h(\phi(0), \rho_4(\phi, t), t)| \leq l_k\|\varphi - \phi\|,\end{aligned}$$

for all $\varphi, \phi \in C([-\tau, 0]; R)$ and $\|\varphi\| \vee \|\phi\| \leq k$. To discuss the stability, we assume that $f(0, 0, t) = g(0, 0, t) = h(0, 0, t) = 0 (\forall t \geq 0)$.

(C2) Assume that for any $t \geq 0$ and each $i = 1, 2, 3, 4$, $\varphi, \phi \in C([-\tau, 0]; R)$ and $\|\varphi\| \vee \|\phi\| \leq k$, $|\rho_i(\varphi, t) - \rho_i(\phi, t)| \leq \|\varphi - \phi\|$ and $|\rho_i(\varphi, t) - \varphi(0)| \leq D(\varphi)$.

Remark 3.2. *From (C2), we can observe that for all* $t \geq 0$, $\rho_1(0, t) = \rho_2(0, t) = \rho_3(0, t) = \rho_4(0, t) = 0$.

(C3) (Contractility condition) For any $t \geq 0$, $N(0, t) = 0$ and there exists a constant $\delta \in [0, 1)$ such that for all $\phi \in C([-\tau, 0]; R)$,

$$|N(\rho_1(\phi, t), t)| \leq \delta\|\phi\|. \tag{3.4}$$

Under conditions (C1)–(C3), we cannot ensure that the G-NSFDE (3.1) with any given initial value (3.2) admits a unique global solution (see [6]). In order to further advance the work, another condition is also imposed and more notations are given. Define $\tilde{x}(t) = x(t) - N(\rho_1(x_t, t), t)$. Denote by $C^{2,1}(R\times R^+; R^+)$ the family of all functions $V(x, t)$ on $R\times R^+$ which are continuously twice differentiable in x and once in t. Given any $V \in C^{2,1}(R \times R^+; R^+)$, for G-NSFDE (3.1), we define the function $LV : R \times R \times R^+ \to R$,

$$LV(x, y, t) = V_t(\tilde{x}, t) + V_x(\tilde{x}, t)f(x, y, t) + G\left(2g(x, y, t)V_x(\tilde{x}, t) + V_{xx}(\tilde{x}, t)h^2(x, y, t)\right),$$

where

$$V_t(x, t) = \frac{\partial V(x, t)}{\partial t},\ V_x(x, t) = \frac{\partial V(x, t)}{\partial x},\ V_{xx}(x, t) = \frac{\partial^2 V(x, t)}{\partial x^2}.$$

Another condition can be stated as follow.

(C4) Assume that there are functions $V \in C^{2,1}(R \times R^+; R^+)$, $a \in L^1(R^+; R^+)$, $W_1 \in C(R \times R^+; R^+)$, $W_2 \in C(R \times [-\tau, \infty); R^+)$, $\eta \in U([-\tau, 0]; R^+)$ and positive constants γ_1, γ_2 such that for $(\varphi(0), \varphi, t) \in R \times C([-\tau, 0]; R) \times R^+$,

$$LV(\varphi(0), \varphi, t) \leq a(t) - \gamma_1 W_1(\varphi(0), t) + \gamma_2 \int_{-\tau}^{0} \eta(\theta) W_2(\varphi(\theta), t + \theta) d\theta, \tag{3.5}$$

where $\gamma_1 \geq \gamma_2$ and $W_1(x, t) \geq W_2(x, t)$. Moreover,

$$\lim_{|\tilde{x}| \to \infty} [\inf_{t \in R^+} V(\tilde{x}, t)] = \infty. \tag{3.6}$$

Theorem 3.1. *Let conditions (C1)–(C4) hold. Then, there exists a unique global solution $x(t)$ on $t \geq -\tau$ to the G-NSFDE* (3.1) *with any given initial value* (3.2).

Proof. For any given initial value (3.2), it follows from (C1)–(C3) that the G-NSFDE (3.1) admits a unique maximal local solution $x(t)$ on $t \in [-\tau, \rho_e]$, where ρ_e is the explosion time. If we can prove that $\rho_e = \infty$ q.s., then we can illustrate that solution $x(t)$ is global. Let m_0 be a sufficiently large integer such that $\|x_0\| = \|\varphi\| = \sup_{-\tau \leq s \leq 0} x(s) < m_0$. For each integer $m > m_0$, define the stopping time $\rho_m = \inf\{t \in [0, \rho_e) : |x(t)| \geq m\}$. As usual we set $\inf \emptyset = \infty$ with $\emptyset$ is an empty set. Obviously, the sequence ρ_m is increasing as $m \to \infty$, and $\rho_\infty = \lim_{m \to \infty} \rho_m \leq \rho_e$. By G-Itô's formula, we can get that for $\forall t > 0$,

$$\begin{aligned} V(x(t) - N(\rho_1(x_t, t), t), t) =& V(x(0) - N(\rho_1(\eta(-\tau), 0), 0), 0) + \int_0^t LV(x(s), x_s, s) ds \\ &+ \int_0^t V_x(x(s) - N(\rho_1(x_s, s), s), s) h(x(s), \rho_4(x_s, s), s) d\omega(s) \\ &+ \int_0^t V_x(x(s) - N(\rho_1(x_s, s), s), s) g(x(s), \rho_3(x_s, s), s) d\langle\omega\rangle(s) \\ &+ \frac{1}{2} \int_0^t V_{xx}(x(s) - N(\rho_1(x_s, s), s), s) h^2(x(s), \rho_4(x_s, s), s) d\langle\omega\rangle(s) \\ &- \int_0^t G(2g(x(s), \rho_3(x_s, s), s) V_x(x(s) - N(\rho_1(x_s, s), s), s) \\ &\qquad + h^2(x(s), \rho_4(x_s, s), s) V_{xx}(x(s) - N(\rho_1(x_s, s), s), s)) ds \\ =& V(x(0) - N(\rho_1(\eta(-\tau), 0), 0), 0) + \int_0^t LV(x(s), x_s, s) ds + G_t, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} G_t =& \int_0^t V_x(x(s) - N(\rho_1(x_s, s), s), s) h(x(s), \rho_4(x_s, s), s) d\omega(s) \\ &+ \int_0^t V_x(x(s) - N(\rho_1(x_s, s), s), s) g(x(s), \rho_3(x_s, s), s) d\langle\omega\rangle(s) \\ &+ \frac{1}{2} \int_0^t V_{xx}(x(s) - N(\rho_1(x_s, s), s), s) h^2(x(s), \rho_4(x_s, s), s) d\langle\omega\rangle(s) \\ &- \int_0^t G(2g(x(s), \rho_3(x_s, s), s) V_x(x(s) - N(\rho_1(x_s, s), s), s) \end{aligned}$$

$$+ h^2(x(s), \rho_4(x_s, s), s)V_{xx}(x(s) - N(\rho_1(x_s, s), s), s))ds$$

is a G-martingale (see [21]). Taking expectation on the both sides of (3.7), we can get

$$\hat{E}V(\tilde{x}(t \wedge \rho_m), t \wedge \rho_m) = \hat{E}V(\tilde{x}(0), 0) + \hat{E}\int_0^{t\wedge\rho_m} LV(x(s), x_s, s)ds. \tag{3.8}$$

By (3.5) we gain

$$\begin{aligned}
&\hat{E}V(\tilde{x}(t \wedge \rho_m), t \wedge \rho_m)\\
\leq&\hat{E}V(\tilde{x}(0), 0) + \int_0^t a(s)ds - \gamma_1\hat{E}\int_0^{t\wedge\rho_m} W_1(x(s), s)ds\\
&+ \gamma_2\hat{E}\int_0^{t\wedge\rho_m}\int_{-\tau}^0 \eta(\theta)W_2(x(s+\theta), s+\theta)d\theta ds.
\end{aligned} \tag{3.9}$$

Noting that via $\int_{-\tau}^0 \eta(\theta)d\theta = 1$, we obtain

$$\begin{aligned}
&\int_0^{t\wedge\rho_m}\int_{-\tau}^0 \eta(\theta)W_2(x(s+\theta), s+\theta)d\theta ds\\
=&\int_0^{t\wedge\rho_m}\int_{s-\tau}^s \eta(u-s)W_2(x(u), u)duds\\
=&\int_{-\tau}^{t\wedge\rho_m}\left(\int_{u\vee 0}^{(u+\tau)\wedge(t\wedge\rho_m)} \eta(u-s)ds\right)W_2(x(u), u)du\\
\leq&\int_{-\tau}^{t\wedge\rho_m}\left(\int_u^{u+\tau} \eta(u-s)ds\right)W_2(x(u), u)du\\
=&\int_{-\tau}^{t\wedge\rho_m}\left(\int_{-\tau}^0 \eta(\theta)d\theta\right)W_2(x(s), s)ds\\
=&\int_{-\tau}^{t\wedge\rho_m} W_2(x(s), s)ds.
\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9), we have

$$\begin{aligned}
\hat{E}V(\tilde{x}(t \wedge \rho_m), t \wedge \rho_m) \leq&\hat{E}V(\tilde{x}(0), 0) + \int_0^{t\wedge\rho_m} a(s)ds - \gamma_1\hat{E}\int_{-\tau}^0 W_1(x(s), s)ds\\
&+ \gamma_2\hat{E}\int_{-\tau}^0 W_2(x(s), s)ds + \gamma_2\hat{E}\int_0^{t\wedge\rho_m} W_2(x(s), s)ds.
\end{aligned} \tag{3.11}$$

For $W_1(x(s), s) \geq W_2(x(s), s)$ and $\gamma_1 \geq \gamma_2$, we can further get

$$\begin{aligned}
&\hat{E}V(\tilde{x}(t \wedge \rho_m), t \wedge \rho_m)\\
\leq&\hat{E}V(\tilde{x}(0), 0) + \int_0^t a(s)ds + \gamma_2\hat{E}\int_{-\tau}^0 W_2(x(s), s)ds - (\gamma_1 - \gamma_2)\hat{E}\int_0^{t\wedge\rho_m} W_2(x(s), s)ds\\
\leq&\hat{E}V(\tilde{x}(0), 0) + \int_0^t a(s)ds + \gamma_2\hat{E}\int_{-\tau}^0 W_2(x(s), s)ds.
\end{aligned} \tag{3.12}$$

Let $V_m = \inf_{|\tilde{x}|\geq m, t\in R^+} V(\tilde{x}, t)$ with $m \geq m_0$. Thus, for each $P \in \mathcal{P}$,

$$P(\rho_m \leq t)V_m \leq \hat{E}V(\tilde{x}(\rho_m \wedge t), \rho_m \wedge t) \leq \hat{E}V(\tilde{x}(0), 0) + \int_0^t a(s)ds + \gamma_2 \hat{E}\int_{-\tau}^0 W_2(x(s), s)ds.$$

Let $m \to \infty$, we can obtain

$$\begin{aligned} P(\rho_\infty \leq t) &= \lim_{m\to\infty} P(\rho_m \leq t) \\ &= \lim_{m\to\infty} \frac{\hat{E}V(\tilde{x}(0), 0) + \int_0^t a(s)ds + \gamma_2 \hat{E}\int_{-\tau}^0 W_2(x(s), s)ds}{V_m} = 0. \end{aligned}$$

Namely,

$$P(\rho_\infty > t) = 1.$$

Then, for the arbitrariness of t, we can obtain $V(\rho_\infty = \infty) = \sup_{P\in\mathcal{P}} P(\rho_\infty = \infty) = 1$, which implies that $\rho_\infty = \infty$ q.s..

Next, we will present the result of the mean square stability with general decay rate.

Theorem 3.2. *Let (C1)–(C3) hold. Assume that there are functions* $V \in C^{2,1}(R \times R^+; R^+)$, $W_1 \in C(R \times R^+; R^+)$, $W_2 \in C(R \times [-\tau, \infty); R^+)$, $\eta \in U([-\tau, 0]; R^+)$ *and positive constants* α_1, α_2, α_3, β_1, β_2 *such that for* $(\varphi(0), \varphi, t) \in R \times C([-\tau, 0]; R) \times R^+$,

$$LV(\varphi(0), \varphi, t) \leq -\beta_1 W_1(\varphi(0), t) + \beta_2 \int_{-\tau}^0 \eta(\theta) W_2(\varphi(\theta), t + \theta)d\theta, \tag{3.13}$$

where $\beta_1 \geq \beta_2$ *and* $W_1(x, t) \geq W_2(x, t)$. *Moreover,*

$$\alpha_1|\varphi(0) - N(\rho_1(\varphi, t), t)|^2 \tag{3.14}$$
$$\leq V(\varphi(0) - N(\rho_1(\varphi, t), t), t) \tag{3.15}$$
$$\leq \alpha_2 W_1(\varphi(0), t) + \alpha_3 \int_{-\tau}^0 \eta(\theta) W_2(\varphi(\theta), t + \theta)d\theta. \tag{3.16}$$

Then for any initial value (3.2)*, the trivial solution of the G-NSFDE* (3.1) *is said to be mean square stable with general decay rate.*

Proof. Note that (3.13) and (3.14) satisfy (3.5) and (3.6), respectively. So (C4) is true. Together with conditions (C1)–(C3), then it follows from Theorem 3.1 that we can assert that the G-NSFDE (3.1) admits a unique global solution with the given initial value (3.2). Set $h_1(x) = \beta_1 - \lambda^*\alpha_2 x - \alpha_3\lambda^*\lambda^x(\tau)x - \beta_2\lambda^x(\tau)$ and $h_2(x) = 1 - \delta^2\lambda^x(\tau)$ for $\forall x \geq 0$, where $\lambda(\cdot)$ denotes the λ-type function and $\lambda^x(\tau)$ represents that $\lambda(\tau)$ raised to the power of x. Obviously, $h_1(x)$ and $h_2(x)$ are continuous in x. Since $h_1(0) = \beta_1 - \beta_2 > 0$ and $h_2(0) = 1 - \delta^2 > 0$, by the local sign preserving property of a continuous function, there is a sufficiently small positive number ε such that $h_1(\varepsilon) = \beta_1 - \varepsilon\lambda^*\alpha_2 - \varepsilon\alpha_3\lambda^*\lambda^\varepsilon(\tau) - \beta_2\lambda^\varepsilon(\tau) > 0$ and $h_2(\varepsilon) = 1 - \delta^2\lambda^\varepsilon(\tau) > 0$. For $\forall t > 0$, applying the Itô's formula to $\lambda^\varepsilon(t)V(\tilde{x}(t), t)$, we get

$$\lambda^\varepsilon(t)V(\tilde{x}(t), t) = V(\tilde{x}(0), 0) + \int_0^t \varepsilon\frac{\lambda'(s)}{\lambda(s)}\lambda^\varepsilon(s)V(\tilde{x}(s), s)ds + \int_0^t \lambda^\varepsilon(s)LV(x(s), x_s, s)ds + M_t, \tag{3.17}$$

where

$$
\begin{aligned}
M_t = &\int_0^t \lambda^{\varepsilon}(s)V_x(\tilde{x}(s),s)h(x(s),\rho_4(x_s,s),s)d\omega(s) + \int_0^t \lambda^{\varepsilon}(s)V_x(\tilde{x}(s),s)g(x(s),\rho_3(x_s,s),s)d\langle\omega\rangle(s)\\
&+ \frac{1}{2}\int_0^t \lambda^{\varepsilon}(s)V_{xx}(\tilde{x}(s),s)h^2(x(s),\rho_4(x_s,s),s)d\langle\omega\rangle(s)\\
&- \int_0^t \lambda^{\varepsilon}(s)G(2g(x(s),\rho_3(x_s,s),s)V_x(\tilde{x}(s),s) + h^2(x(s),\rho_4(x_s,s),s)V_{xx}(\tilde{x}(s),s))ds
\end{aligned}
$$

is also a G-martingale. By (3.14), we can get

$$
\begin{aligned}
\int_0^t \varepsilon\frac{\lambda'(s)}{\lambda(s)}\lambda^{\varepsilon}(s)V(\tilde{x}(s),s)ds &\le \varepsilon\lambda^* \int_0^t \lambda^{\varepsilon}(s)V(\tilde{x}(s),s)ds\\
&\le \varepsilon\lambda^*\alpha_2 \int_0^t \lambda^{\varepsilon}(s)W_1(x(s),s)ds\\
&\quad + \varepsilon\lambda^*\alpha_3 \int_0^t \lambda^{\varepsilon}(s)\left(\int_{-\tau}^0 \eta(\theta)W_2(x(s+\theta),s+\theta)d\theta\right)ds.
\end{aligned}
\tag{3.18}
$$

By the property of $\lambda(t)$, we can obtain

$$
\begin{aligned}
&\int_0^t \lambda^{\varepsilon}(s)\left(\int_{-\tau}^0 \eta(\theta)W_2(x(s+\theta),s+\theta)d\theta\right)ds\\
&= \int_{-\tau}^0 \eta(\theta)d\theta \int_0^t \lambda^{\varepsilon}(s)W_2(x(s+\theta),s+\theta)ds\\
&= \int_{-\tau}^0 \eta(\theta)d\theta \int_{\theta}^{t+\theta} \lambda^{\varepsilon}(u-\theta)W_2(x(u),u)du\\
&\le \int_{-\tau}^0 \eta(\theta)d\theta \int_{-\tau}^t \lambda^{\varepsilon}(u)\lambda^{\varepsilon}(-\theta)W_2(x(u),u)du\\
&\le \lambda^{\varepsilon}(\tau)\int_{-\tau}^t \lambda^{\varepsilon}(s)W_2(x(s),s)ds\\
&\le \lambda^{\varepsilon}(\tau)\int_{-\tau}^0 \lambda^{\varepsilon}(s)W_2(x(s),s)ds + \lambda^{\varepsilon}(\tau)\int_0^t \lambda^{\varepsilon}(s)W_2(x(s),s)ds.
\end{aligned}
\tag{3.19}
$$

Substituting (3.19) into (3.18), one can gain

$$
\begin{aligned}
&\int_0^t \varepsilon\frac{\lambda'(s)}{\lambda(s)}\lambda^{\varepsilon}(s)V(\tilde{x}(s),s)ds\\
&\le \varepsilon\lambda^*\alpha_2 \int_0^t \lambda^{\varepsilon}(s)W_1(x(s),s)ds\\
&\quad + \varepsilon\lambda^*\alpha_3\lambda^{\varepsilon}(\tau)\int_{-\tau}^0 \lambda^{\varepsilon}(s)W_2(x(s),s)ds + \varepsilon\lambda^*\alpha_3\lambda^{\varepsilon}(\tau)\int_0^t \lambda^{\varepsilon}(s)W_2(x(s),s)ds.
\end{aligned}
\tag{3.20}
$$

Applying (3.13), one can also compute

$$\int_0^t \lambda^\varepsilon(s)LV(x(s),x_s,s)ds$$
$$\leq -\beta_1\int_0^t \lambda^\varepsilon(s)W_1(x(s),s)ds+\beta_2\int_0^t \lambda^\varepsilon(s)\left(\int_{-\tau}^0 \eta(\theta)W_2(x(s+\theta),s+\theta)d\theta\right)ds. \tag{3.21}$$

Employing (3.19), we gain

$$\begin{aligned}&\int_0^t \lambda^\varepsilon(s)LV(x(s),x_s,s)ds\\ &\leq -\beta_1\int_0^t \lambda^\varepsilon(s)W_1(x(s),s)ds+\beta_2\lambda^\varepsilon(\tau)\int_{-\tau}^0 \lambda^\varepsilon(s)W_2(x(s),s)ds\\ &\quad+\beta_2\lambda^\varepsilon(\tau)\int_0^t \lambda^\varepsilon(s)W_2(x(s),s)ds.\end{aligned} \tag{3.22}$$

Substituting (3.20) and (3.22) into (3.17), and by $W_1(x,t)\geq W_2(x,t)$, we have

$$\begin{aligned}&\lambda^\varepsilon(t)V(\tilde{x}(t),t)\\ \leq&V(\tilde{x}(0),0)+\varepsilon\lambda^*\alpha_2\int_0^t \lambda^\varepsilon(s)W_1(x(s),s)ds\\ &+\varepsilon\lambda^*\alpha_3\lambda^\varepsilon(\tau)\int_{-\tau}^0 \lambda^\varepsilon(s)W_2(x(s),s)ds+\varepsilon\lambda^*\alpha_3\lambda^\varepsilon(\tau)\int_0^t \lambda^\varepsilon(s)W_2(x(s),s)ds\\ &-\beta_1\int_0^t \lambda^\varepsilon(s)W_1(x(s),s)ds+\beta_2\lambda^\varepsilon(\tau)\int_{-\tau}^0 \lambda^\varepsilon(s)W_2(x(s),s)ds\\ &+\beta_2\lambda^\varepsilon(\tau)\int_0^t \lambda^\varepsilon(s)W_2(x(s),s)ds+M_t\\ \leq&V(\tilde{x}(0),0)+[\varepsilon\lambda^*\alpha_3\lambda^\varepsilon(\tau)+\beta_2\lambda^\varepsilon(\tau)]\int_{-\tau}^0 \lambda^\varepsilon(s)W_2(x(s),s)ds\\ &-[\beta_1-\varepsilon\lambda^*\alpha_2-\varepsilon\lambda^*\alpha_3\lambda^\varepsilon(\tau)-\beta_2\lambda^\varepsilon(\tau)]\int_0^t \lambda^\varepsilon(s)W_2(x(s),s)ds+M_t.\end{aligned}$$

Due to $h_1(\varepsilon)=\beta_1-\varepsilon\lambda^*\alpha_2-\varepsilon\lambda^*\alpha_3\lambda^\varepsilon(\tau)-\beta_2\lambda^\varepsilon(\tau)>0$. Thus,

$$\begin{aligned}&\lambda^\varepsilon(t)V(\tilde{x}(t),t)\\ &\leq V(\tilde{x}(0),0)+[\varepsilon\lambda^*\alpha_3\lambda^\varepsilon(\tau)+\beta_2\lambda^\varepsilon(\tau)]\int_{-\tau}^0 \lambda^\varepsilon(s)W_2(x(s),s)ds+M_t\\ &=C_1+M_t,\end{aligned} \tag{3.23}$$

where $C_1=V(\tilde{x}(0),0)+[\varepsilon\lambda^*\alpha_3\lambda^\varepsilon(\tau)+\beta_2\lambda^\varepsilon(\tau)]\int_{-\tau}^0 \lambda^\varepsilon(s)W_2(x(s),s)ds$. Further, we can compute for any constant $\gamma\in(0,1)$,

$$\lambda^\varepsilon(t)|x(t)|^2\leq\frac{1}{1-\gamma}\lambda^\varepsilon(t)|\tilde{x}(t)|^2+\frac{\delta^2}{\gamma}\lambda^\varepsilon(t)\|x_t\|^2. \tag{3.24}$$

Then, for any $T > \tau$, we obtain

$$\begin{aligned}
&\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|x(t)|^2\\
&\le \frac{1}{1-\gamma}\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|\tilde{x}(t)|^2 + \frac{\delta^2}{\gamma}\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|x(t+\theta)|^2\\
&\le \frac{1}{1-\gamma}\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|\tilde{x}(t)|^2 + \frac{\delta^2}{\gamma}\lambda^{\varepsilon}(-\theta)[\|\eta\|^2 + \sup_{0\le t\le T} \lambda^{\varepsilon}(t)|x(t)|^2]\\
&\le \frac{1}{1-\gamma}\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|\tilde{x}(t)|^2 + \frac{\delta^2}{\gamma}\lambda^{\varepsilon}(\tau)\|\eta\|^2 + \frac{\delta^2}{\gamma}\lambda^{\varepsilon}(\tau)\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|x(t)|^2.
\end{aligned}$$

For $h_2(\varepsilon) > 0$, let $\gamma \in (\delta^2\lambda^{\varepsilon}(\tau), 1)$, we can get

$$\left(1 - \frac{\delta^2}{\gamma}\lambda^{\varepsilon}(\tau)\right)\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|x(t)|^2 \le \frac{1}{1-\gamma}\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|\tilde{x}(t)|^2 + \frac{\delta^2}{\gamma}\lambda^{\varepsilon}(\tau)\|\eta\|^2.$$

So

$$\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|x(t)|^2 \le \frac{\gamma}{(1-\gamma)(\gamma-\delta^2\lambda^{\varepsilon}(\tau))}\sup_{0\le t\le T} \lambda^{\varepsilon}(t)|\tilde{x}(t)|^2 + \frac{\delta^2\lambda^{\varepsilon}(\tau)\|\eta\|^2}{\gamma-\delta^2\lambda^{\varepsilon}(\tau)}. \tag{3.25}$$

For any $T > \tau$, (3.25) is true. Therefore, by (3.14) and (3.23), we have

$$\lambda^{\varepsilon}(t)|x(t)|^2 \le \frac{\gamma}{(1-\gamma)(\gamma-\delta^2\lambda^{\varepsilon}(\tau))}\frac{C_1+M_t}{\alpha_1} + \frac{\delta^2\lambda^{\varepsilon}(\tau)\|\eta\|^2}{\gamma-\delta^2\lambda^{\varepsilon}(\tau)} \tag{3.26}$$

for any $t > 0$. Taking the expectation of both sides of (3.26), we gain

$$\lambda^{\varepsilon}(t)\hat{E}|x(t)|^2 \le \frac{\gamma C_1}{\alpha_1(1-\gamma)(\gamma-\delta^2\lambda^{\varepsilon}(\tau))} + \frac{\delta^2\lambda^{\varepsilon}(\tau)\|\eta\|^2}{\gamma-\delta^2\lambda^{\varepsilon}(\tau)} = C_2,$$

where $C_2 = \frac{\gamma C_1}{\alpha_1(1-\gamma)(\gamma-\delta^2\lambda^{\varepsilon}(\tau))} + \frac{\delta^2\lambda^{\varepsilon}(\tau)\|\eta\|^2}{\gamma-\delta^2\lambda^{\varepsilon}(\tau)}$. Then it follows that

$$\limsup_{t\to\infty}\frac{\log \hat{E}|x(t)|^2}{\log|\lambda(t)|} \le -\varepsilon. \tag{3.27}$$

And then according to Definition 3.2, we can obtain the trivial solution of the G-NSFDE (3.1) is mean square stable with general decay rate $\lambda(t)$ of order ε. The proof is completed.

Corollary 3.1. *Let (C1)–(C3) hold. Denote $\tilde{x} = x(t) - N(\rho_1(x_t,t),t)$. Assume that there exist constant $\tilde{\delta} \in [0,1)$, positive constants ϱ_1, ϱ_2, and constants η_1, η_2 such that for $\forall(x,y,t) \in R\times R\times R^+$,*

$$|N(\rho_1(y,t),t)| \le \tilde{\delta}\int_{-\tau}^{0}|x(t+\theta)|d\theta \tag{3.28}$$

with $N(0,t) = 0$,

$$\tilde{x}f(x,y,t) \le -\varrho_1|x(t)|^2 + \varrho_2\int_{-\tau}^{0}|x(t+\theta)|^2 d\theta \tag{3.29}$$

and

$$2\tilde{x}g(x,y,t)+h^2(x,y,t)\le \eta_1|x(t)|^2+\eta_2\int_{-\tau}^{0}|x(t+\theta)|^2d\theta. \tag{3.30}$$

If $\varrho_1-\frac{1}{2}\bar{\sigma}^2\eta_1^+>\varrho_2+\frac{1}{2}\bar{\sigma}^2\eta_2^+$. *Then for any initial value* (3.2)*, the trivial solution of the G-NSFDE* (3.1) *is said to be mean square stable with general decay rate.*

Proof. Let $V(\tilde{x}(t),t)=|\tilde{x}(t)|^2$. By (3.28), we can get

$$|\tilde{x}(t)|^2=|x(t)-N(\rho_1(x_t,t),t)|^2\le 2|x(t)|^2+2|N(\rho_1(x_t,t),t)|^2\le 2|x(t)|^2+2\tilde{\delta}^2\tau\int_{-\tau}^{0}|x(t+\theta)|^2d\theta. \tag{3.31}$$

According to the definition of LV and $G(a)=\frac{1}{2}\hat{E}[a\omega^2(1)]=\frac{1}{2}(\bar{\sigma}^2a^+-\underline{\sigma}^2a^-)(\forall a\in R)$, we can compute

$$\begin{aligned}
LV(x(t),x_t,t)=&V_t(\tilde{x}(t),t)+V_x(\tilde{x}(t),t)f(x(t),x_t,t)\\
&+G\left(2g(x(t),x_t,t)V_x(\tilde{x}(t),t)+V_{xx}(\tilde{x}(t),t)h^2(x(t),x_t,t)\right)\\
=&2\tilde{x}(t)f(x(t),x_t,t)+G\left(4\tilde{x}(t)g(x(t),x_t,t)+2h^2(x(t),x_t,t)\right)\\
\le&-2\varrho_1|x(t)|^2+2\varrho_2\int_{-\tau}^{0}|x(t+\theta)|^2d\theta+2G\left(\eta_1|x(t)|^2+\eta_2\int_{-\tau}^{0}|x(t+\theta)|^2d\theta\right)\\
\le&-2\varrho_1|x(t)|^2+2\varrho_2\int_{-\tau}^{0}|x(t+\theta)|^2d\theta+2\cdot\frac{1}{2}\bar{\sigma}^2\left(\eta_1|x(t)|^2+\eta_2\int_{-\tau}^{0}|x(t+\theta)|^2d\theta\right)^+\\
&-2\cdot\frac{1}{2}\underline{\sigma}^2\left(\eta_1|x(t)|^2+\eta_2\int_{-\tau}^{0}|x(t+\theta)|^2d\theta\right)^-\\
\le&-2\varrho_1|x(t)|^2+2\varrho_2\int_{-\tau}^{0}|x(t+\theta)|^2d\theta+2\cdot\frac{1}{2}\bar{\sigma}^2\left(\eta_1^+|x(t)|^2+\eta_2^+\int_{-\tau}^{0}|x(t+\theta)|^2d\theta\right)\\
\le&-2\left(\varrho_1-\frac{1}{2}\bar{\sigma}^2\eta_1^+\right)|x(t)|^2+2\left(\varrho_2+\frac{1}{2}\bar{\sigma}^2\eta_2^+\right)\int_{-\tau}^{0}|x(t+\theta)|^2d\theta.
\end{aligned}\tag{3.32}$$

Thus, according to $\varrho_1-\frac{1}{2}\bar{\sigma}^2\eta_1^+>\varrho_2+\frac{1}{2}\bar{\sigma}^2\eta_2^+$, we can acquire that (3.31) and (3.32) satisfy (3.13) and (3.14), respectively. Therefore, it follows from Theorem 3.2 that we can get the trivial solution of the G-NSFDE (3.1) is mean square stable with general decay rate.

4. An example

In this section, an example is given to illustrate the obtained results. Let $\omega(t)$ be a 1-dimensional G-Brownian motion with $\omega(1)\sim N(0;[1/2,1])$. Let $\tau=1$ and $\eta(\theta)\equiv 1$ for $\theta\in[-1,0]$.

Example 4.1. Consider the following scalar nonlinear G-NSFDE:

$$\begin{aligned}
d\left[x(t)-\frac{1}{2}\int_{-1}^{0}|x(t+\theta)|d\theta\right]=&\left(-2x(t)+\int_{-1}^{0}|x(t+\theta)|d\theta\right)dt\\
&-\frac{1}{2}x(t)d\langle\omega\rangle(t)+\sin(x(t))d\omega(t),\ t\ge 0.
\end{aligned}\tag{4.1}$$

Take $V(x,t) = x^2$, it is easy to check that $\delta = \frac{1}{2}$. We also can compute

$$\begin{aligned} V(\tilde{x},t) = |\tilde{x}|^2 &= \left| x(t) - \frac{1}{2}\int_{-1}^{0} |x(t+\theta)| d\theta \right|^2 \\ &\le 2|x(t)|^2 + \frac{1}{2}\int_{-1}^{0} |x(t+\theta)|^2 d\theta, \end{aligned} \tag{4.2}$$

$$\begin{aligned} V_x(\tilde{x},t) f(x,y,t) &= 2\left(x(t) - \frac{1}{2}\int_{-1}^{0} |x(t+\theta)| d\theta \right)\left(-2x(t) + \int_{-1}^{0} |x(t+\theta)| d\theta \right) \\ &\le -2x^2(t) + \left(\int_{-1}^{0} |x(t+\theta)| d\theta \right)^2 \\ &\le -2x^2(t) + \int_{-1}^{0} |x(t+\theta)|^2 d\theta \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &2V_x(\tilde{x},t) g(x,y,t) + V_{xx}(\tilde{x},t) h^2(x,y,t) \\ &= 4\left(-\frac{1}{2}x(t) \right)\left(x(t) - \frac{1}{2}\int_{-1}^{0} |x(t+\theta)| d\theta \right) + 2\sin^2(x(t)) \\ &= -2x(t)\left(x(t) - \frac{1}{2}\int_{-1}^{0} |x(t+\theta)| d\theta \right) + 2\sin^2(x(t)) \\ &\le \frac{1}{2}x^2(t) + \frac{1}{2}\left(\int_{-1}^{0} |x(t+\theta)| d\theta \right)^2 \\ &\le \frac{1}{2}x^2(t) + \frac{1}{2}\int_{-1}^{0} |x(t+\theta)|^2 d\theta. \end{aligned} \tag{4.4}$$

It follows from (4.3) and (4.4) that we gain

$$LV(x(t), x_t, t) \le -\frac{7}{4}x^2(t) + \frac{5}{4}\int_{-1}^{0} |x(t+\theta)|^2 d\theta. \tag{4.5}$$

Therefore, $W_1(x,t) = W_2(x,t) = x^2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 1/2$, $\beta_1 = \frac{7}{4}$ and $\beta_2 = \frac{5}{4}$. According to Theorem 3.2 (or Corollary 3.1), we can assert that the trivial solution of (4.1) is λ-type mean square stable.

5. Conclusions

As we known, few results seem to be known about the stability with general decay rate of G-NSFDEs, which motivates this paper to study this problem. This paper first proves that the existence and uniqueness of the global solution for this kind of equations by the local Lipschitz condition and a weaker condition. A kind of λ-type function is also given in this paper. By applying G-Lyapunov function technique, a kind of mean square λ-type stability, including mean square exponential stability and mean square polynomial stability are obtained. Results in this paper enrich the conclusions on the stability of G-NSFDEs.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (No. 11901398).

Conflict of interest

The author declares no conflict of interest.

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