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Research article

Some new (p, q)-Dragomir–Agarwal and Iyengar type integral inequalities and their applications

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Abstract: The main objective of this paper is to derive some new post quantum analogues of Dragomir–Agarwal and Iyengar type integral inequalities essentially by using the strongly φ-preinvexity and strongly quasi φ-preinvexity properties of the mappings, respectively. We also discuss several new special cases which show that the results obtained are quite unifying. In order to illustrate the efficiency of our main results, some applications regarding (p, q)-differentiable mappings that are in absolute value bounded are given.

Keywords: Dragomir–Agarwal type inequality; Iyengar type inequality; (p, q)-integral; strongly φ-preinvex mapping; post quantum calculus; bounded functions
Mathematics Subject Classification: 26A33, 26A51, 26D07, 26D10, 26D15, 26D20

1. Introduction and preliminaries

Quantum calculus which is often known as q-calculus is the branch of mathematics in which we obtain q-analogues of the mathematical objects which can be recaptured by taking $q \to 1^-$. It is also known as calculus without limits and depends upon finite difference. In recent years it has emerged as a bridge between mathematics and physics. Due to its great many applications in different fields of applied sciences, it has attracted many researchers. Consequently a rapid developments in quantum calculus have been achieved. For example, Tariboon and Ntouyas [1] introduced the notions of q-derivatives and q-integrals on finite intervals. This idea attracted many researchers and resultantly many new q-analogues of classical mathematical objects have been obtained using their approach. They themselves have obtained the quantum analogues of Hermite–Hadamard's inequality, Hölders's inequality, trapezoidal inequality and Ostrowski type of inequalities etc. Alp *et al.* [2] obtained a new corrected version of q-Hermite–Hadamard's inequality. Noor *et al.* [3] and Sudsutad *et al.* [4] obtained independently some new quantum analogues of trapezoidal like inequalities. Liu and Zhuang [5] obtained quantum estimates of Hermite–Hadamard type of inequalities via twice q-differentiable convex functions. Noor *et al.* [6] obtained a new q-integral identity and obtained associated upper bounds by using the preinvexity property of the functions. Noor *et al.* [7] obtained new q-Ostrowski type of inequalities via first order q-differentiable convex functions. Deng *et al.* [8] obtained q-analogues of Simpson type of inequalities. Zhang *et al.* [9] obtained a very nice generalized q-integral identity and obtained associated bounds.

In recent years the classical concepts of quantum calculus have been modified in different directions, see [10–15]. One of the significant generalizations of q-calculus is the post quantum often known as (p,q)-calculus. The main idea is that in quantum calculus, we deal with a q-number with one base q, however, in (p,q)-calculus, we deal with two independent variables p and q. This idea was first considered in [16]. Tunç and Gov [17] recently introduced the concepts of (p,q)-derivatives and (p,q)-integrals on the finite intervals.

Definition 1.1 ([17]). *Let $\Psi : K \to \mathbb{R}$ be a continuous mapping and let $x \in K$ and $0 < \mathrm{q} < \mathrm{p} \leq 1$. Then the (p,q)-derivative on K of mapping Ψ at x is defined as*

$$_{\varpi_1}\mathrm{D}_{\mathrm{p},\mathrm{q}}\Psi(x) = \frac{\Psi(\mathrm{p}x + (1-\mathrm{p})\varpi_1) - \Psi(\mathrm{q}x + (1-\mathrm{q})\varpi_1)}{(\mathrm{p}-\mathrm{q})(x-\varpi_1)}, \quad x \neq \varpi_1.$$

Definition 1.2 ([17]). *Let $\Psi : K \subset \mathbb{R} \to \mathbb{R}$ be a continuous mapping. Then (p,q)-integral on K is defined as*

$$\int_{\varpi_1}^{x} \Psi(\lambda)\,{}_{\varpi_1}\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = (\mathrm{p}-\mathrm{q})(x-\varpi_1)\sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\Psi\left(\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}x + \left(1-\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\right)\varpi_1\right),$$

for $x \in K$.

For more details, see [18–20].

Let us recall some basic definitions that will be used in the sequel.

Definition 1.3 ([21]). *A mapping $\Psi : I \to \mathbb{R}$ is said to be convex, if*

$$\Psi(\lambda\varpi_1 + (1-\lambda)\varpi_2) \leq \lambda\Psi(\varpi_1) + (1-\lambda)\Psi(\varpi_2) \tag{1.1}$$

holds for all $\varpi_1, \varpi_2 \in I$ and $\lambda \in [0,1]$.

For more details, see [22–25].

Definition 1.4 ([26]). *A set $\mathscr{K}_\eta \subset \mathbb{R}$ is said to be invex with respect to $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, if*

$$\varpi_1 + \lambda\eta(\varpi_2, \varpi_1) \in \mathscr{K}_\eta, \quad \forall \varpi_1, \varpi_2 \in \mathscr{K}_\eta, \lambda \in [0, 1].$$

Definition 1.5 ([9,27]). *A mapping $\Psi : \mathscr{K}_\eta \to \mathbb{R}$ on the invex set is said to be preinvex, if*

$$\Psi(\varpi_1 + \lambda\eta(\varpi_2, \varpi_1)) \leq (1 - \lambda)\Psi(\varpi_1) + \lambda\Psi(\varpi_2) \tag{1.2}$$

holds for all $\varpi_1, \varpi_2 \in \mathscr{K}_\eta$ and $\lambda \in [0, 1]$.

We now introduce the class of strongly φ-preinvex mappings.

Definition 1.6. *Let $\varphi : (0, 1) \to \mathbb{R}$ be a real mapping. A mapping $\Psi : \mathscr{K}_\eta \to \mathbb{R}$ on the invex set is said to be strongly φ-preinvex, if*

$$\Psi(\varpi_1 + \lambda\eta(\varpi_2, \varpi_1)) \leq (1 - \lambda)\varphi(1 - \lambda)\Psi(\varpi_1) + \lambda\varphi(\lambda)\Psi(\varpi_2) - \sigma\lambda(1 - \lambda)\eta^2(\varpi_2, \varpi_1),$$

holds for all $\varpi_1, \varpi_2 \in \mathscr{K}_\eta$, $\lambda \in (0, 1)$ with $\eta(\varpi_2, \varpi_1) > 0$ and $\sigma > 0$.

Note that, if we take $\eta(\varpi_2, \varpi_1) = \varpi_2 - \varpi_1$ in Definition 1.6, then we have the class of strongly φ-convex functions which was introduced and studied in [28].

Remark 1.1. *Note that, if we take, respectively $\varphi(\mu) = 1$, μ^{-1}, μ^{s-1} and $\varphi(\mu) = 1 - \mu$ in Definition 1.6, then we recapture the classes of strongly preinvex [27], strongly P-preinvex [29], strongly s-preinvex [29] and strongly tgs-preinvex mappings, respectively. Moreover, if we choose $\sigma \to 0^+$, then all of these classes reduce to preinvex [26], P-preinvex [30], s-preinvex [30] and tgs-preinvex mappings [31], respectively. This shows that the class of strongly φ-preinvex mappings is quite unifying one as it relates several other unrelated classes.*

For the sake of completeness, let us now recall the Dragomir–Agarwal and Iyengar type of inequalities. Dragomir and Agarwal [41] obtained the following new integral identity and obtained associated inequalities essentially using the class of first order differentiable convex functions.

Lemma 1.1. *Let $\Psi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function, If $\Psi' \in L[\varpi_1, \varpi_2]$, then*

$$\frac{\Psi(\varpi_1) + \Psi(\varpi_2)}{2} - \frac{1}{\varpi_2 - \varpi_1}\int_{\varpi_1}^{\varpi_2} \Psi(x)\mathrm{d}x = \frac{\varpi_2 - \varpi_1}{2}\int_0^1 (1 - 2\mu)\Psi'(\mu\varpi_1 + (1 - \mu)\varpi_2)\mathrm{d}\mu.$$

The right side of Hermite–Hadamard inequality can be estimated by the inequality of Iyengar [42], which reads as

$$\left|\frac{\Psi(\varpi_1) + \Psi(\varpi_2)}{2} - \frac{1}{\varpi_2 - \varpi_1}\int_{\varpi_1}^{\varpi_2} \Psi(x)\mathrm{d}x\right| \leq \frac{M(\varpi_2 - \varpi_1)}{4} - \frac{1}{4M(\varpi_2 - \varpi_1)}(\Psi(\varpi_2) - \Psi(\varpi_1))^2,$$

where by M we denote the Lipschitz constant, that is, $M = \sup\left\{\left|\frac{\Psi(x)-\Psi(y)}{x-y}\right|; x \neq y\right\}$.

Integral inequalities are important to predict upper and lower bounds in various applied sciences, e.g. in probability theory, functional inequalities, interpolation spaces, Sobolev spaces and information theory. For some recent studies and applications of integral inequalities in these directions, see [32–40].

The main motivation of this article is to derive a new post-quantum integral identity using (p,q)-differentiable mappings. Using the identity as an auxiliary result, we will obtain some new variants of Dragomir–Agarwal and Iyengar type integral inequalities essentially via the class of strongly φ-preinvex mappings. We also discuss several new special cases which show that the results obtained are quite unifying. Finally, to support our results, we present some applications to (p,q)-differentiable mappings that are in absolute value bounded. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

2. Auxiliary results

In this section, we discuss auxiliary results. These results will be helpful in obtaining the main results of this paper.

Lemma 2.1. *Let* $\omega \in [0,1]$ *and* $\lambda \in [0,\infty)$*, then*

$$\int_0^{\omega} v^{\lambda} \mathrm{d}_{\mathrm{p},\mathrm{q}} v = (\mathrm{p}-\mathrm{q}) \sum_{n=0}^{\infty} \left(\frac{\omega}{\mathrm{p}}\right)^{\lambda+1} \left(\frac{\mathrm{q}}{\mathrm{p}}\right)^{(\lambda+1)n} = \frac{\omega^{\lambda+1}(\mathrm{p}-\mathrm{q})}{\mathrm{p}^{\lambda+1}-\mathrm{q}^{\lambda+1}},$$

and

$$\int_0^{\omega} (1-v)^{\lambda} \mathrm{d}_{\mathrm{p},\mathrm{q}} v = (\mathrm{p}-\mathrm{q})\omega \sum_{n=0}^{\infty} \frac{\mathrm{q}^n}{\mathrm{p}^{n+1}} \left(1 - \frac{\mathrm{q}^n \omega}{\mathrm{p}^{n+1}}\right)^{\lambda}.$$

Now we derive a new (p,q)-integral identity which will be used as an auxiliary result for obtaining next results of the paper.

Lemma 2.2. *Suppose that* $\Psi : [\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1)] \subseteq \mathbb{R} \to \mathbb{R}$ *be a* (p,q)*-differentiable mapping on* $(\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1))$ *with* $\eta(\varpi_2,\varpi_1) > 0$ *and* $0 < \mathrm{q} < \mathrm{p} \le 1$*. If* ${}_{\varpi_1}\mathrm{D}_{\mathrm{p},\mathrm{q}}\Psi$ *is a* (p,q)*-integrable mapping on* $[\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1)]$*, then*

$$\frac{1}{\mathrm{p}\eta(\varpi_2,\varpi_1)} \int_{\varpi_1}^{\varpi_1+\mathrm{p}\eta(\varpi_2,\varpi_1)} \Psi(x)\, {}_{\varpi_1}\mathrm{d}_{\mathrm{p},\mathrm{q}} x - \frac{\mathrm{q}\Psi(\varpi_1) + \mathrm{p}\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{\mathrm{p},\mathrm{q}}}$$
$$= \frac{\mathrm{q}\eta(\varpi_2,\varpi_1)}{2} \int_0^1 \int_0^1 (\mu-\lambda) \left[{}_{\varpi_1}\mathrm{D}_{\mathrm{p},\mathrm{q}}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1)) - {}_{\varpi_1}\mathrm{D}_{\mathrm{p},\mathrm{q}}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1)) \right] \mathrm{d}_{\mathrm{p},\mathrm{q}}\mu\, \mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda, \tag{2.1}$$

where $[n]_{\mathrm{p},\mathrm{q}}$ *is the well-known* (p,q)*-integer expressed as:*

$$[n]_{\mathrm{p},\mathrm{q}} = \frac{\mathrm{p}^n - \mathrm{q}^n}{\mathrm{p}-\mathrm{q}}.$$

Proof. Using Definitions 1.1 and 1.2, we have

$$\int_0^1\int_0^1(\mu-\lambda)\left[{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))-{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))\right]d_{p,q}\mu\, d_{p,q}\lambda$$

$$=\int_0^1\int_0^1(\mu-\lambda)\left[\frac{\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))}{(p-q)\eta(\varpi_2,\varpi_1)\lambda}\right.$$

$$\left.-\frac{\Psi(\varpi_1+p\mu\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\mu\eta(\varpi_2,\varpi_1))}{(p-q)\eta(\varpi_2,\varpi_1)\mu}\right]d_{p,q}\mu\, d_{p,q}\lambda$$

$$=\int_0^1\int_0^1\frac{\mu\left[\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)\lambda}d_{p,q}\mu\, d_{p,q}\lambda$$

$$-\int_0^1\int_0^1\frac{\left[\Psi(\varpi_1+p\mu\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\mu\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)}d_{p,q}\mu\, d_{p,q}\lambda$$

$$-\int_0^1\int_0^1\frac{\left[\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)}d_{p,q}\mu\, d_{p,q}\lambda$$

$$+\int_0^1\int_0^1\frac{\lambda\left[\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)\mu}d_{p,q}\mu\, d_{p,q}\lambda. \tag{2.2}$$

We can see that

$$\int_0^1\int_0^1\frac{\mu\left[\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)\lambda}d_{p,q}\mu\, d_{p,q}\lambda$$

$$=\int_0^1\mu d_{p,q}\mu\int_0^1\frac{\left[\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)\lambda}d_{p,q}\lambda$$

$$=\frac{1}{[2]_{p,q}\eta(\varpi_2,\varpi_1)}\left[\sum_{n=0}^{\infty}\Psi\left(\varpi_1+\frac{q^n}{p^n}\eta(\varpi_2,\varpi_1)\right)-\sum_{n=0}^{\infty}\Psi\left(\varpi_1+\frac{q^{n+1}}{p^{n+1}}\eta(\varpi_2,\varpi_1)\right)\right]$$

$$=\frac{1}{[2]_{p,q}\eta(\varpi_2,\varpi_1)}\left[\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1)\right], \tag{2.3}$$

and

$$\int_0^1\int_0^1\frac{\left[\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)}d_{p,q}\mu\, d_{p,q}\lambda$$

$$=\int_0^1 d_{p,q}\mu\int_0^1\frac{\left[\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)}d_{p,q}\lambda$$

$$
\begin{aligned}
&= \frac{1}{\eta(\varpi_2,\varpi_1)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\varpi_1+p\frac{q^n}{p^{n+1}}\eta(\varpi_2,\varpi_1)\right)-\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\varpi_1+\frac{q^{n+1}}{p^{n+1}}\eta(\varpi_2,\varpi_1)\right)\right]\\
&= \frac{1}{\eta(\varpi_2,\varpi_1)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\varpi_1+p\frac{q^n}{p^{n+1}}\eta(\varpi_2,\varpi_1)\right)-\frac{p}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}\Psi\left(\varpi_1+p\frac{q^{n+1}}{p^{n+2}}\eta(\varpi_2,\varpi_1)\right)\right]\\
&= \frac{1}{\eta(\varpi_2,\varpi_1)}\left[\frac{1}{p}\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))+\sum_{n=1}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\varpi_1+p\frac{q^n}{p^{n+1}}\eta(\varpi_2,\varpi_1)\right)\right.\\
&\quad\left.-\frac{p}{q}\sum_{n=1}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\varpi_1+p\frac{q^n}{p^{n+1}}\eta(\varpi_2,\varpi_1)\right)\right]\\
&= \frac{1}{\eta(\varpi_2,\varpi_1)}\left[\frac{1}{p}\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))+\left(1-\frac{p}{q}\right)\sum_{n=1}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\varpi_1+\frac{q^n}{p^{n+1}}\eta(\varpi_2,\varpi_1)\right)\right]\\
&= \frac{1}{\eta(\varpi_2,\varpi_1)}\left[\frac{1}{q}\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))-\left(\frac{p-q}{q}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\varpi_1+\frac{q^n}{p^{n+1}}\eta(\varpi_2,\varpi_1)\right)\right]\\
&= \frac{1}{\eta(\varpi_2,\varpi_1)}\left[\frac{1}{q}\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))-\frac{1}{pq\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x\right]. \qquad (2.4)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_0^1\int_0^1 \frac{\lambda\left[\Psi(\varpi_1+p\mu\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\mu\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)\mu}d_{p,q}\mu\, d_{p,q}\lambda\\
&= \frac{1}{[2]_{p,q}\eta(\varpi_2,\varpi_1)}\left[\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1)\right], \qquad (2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1\int_0^1 \frac{\left[\Psi(\varpi_1+p\lambda\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1+q\lambda\eta(\varpi_2,\varpi_1))\right]}{(p-q)\eta(\varpi_2,\varpi_1)}d_{p,q}\mu\, d_{p,q}\lambda\\
&= \frac{1}{\eta(\varpi_2,\varpi_1)}\left[\frac{1}{q}\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))-\frac{1}{pq\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x\right]. \qquad (2.6)
\end{aligned}
$$

Substituting equalities (2.3)–(2.6) in (2.2), we get

$$
\begin{aligned}
&\int_0^1\int_0^1 (\mu-\lambda)\left[{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))-{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))\right]d_{p,q}\mu\, d_{p,q}\lambda\\
&= \frac{1}{pq\eta^2(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{2}{q\eta^2(\varpi_2,\varpi_1)}\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))
\end{aligned}
$$

$$+\frac{2}{[2]_{p,q}\eta^2(\varpi_2,\varpi_1)}\left[\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))-\Psi(\varpi_1)\right]. \quad (2.7)$$

Multiplying both sides of (2.7) by $\frac{q\eta(\varpi_2,\varpi_1)}{2}$, we obtain the required result. □

Remark 2.1. *If we take* p = 1 *in Lemma 2.2, then we have the following new equality:*

$$\frac{1}{\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_q x-\frac{q\Psi(\varpi_1)+\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_q}$$
$$=\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1(\mu-\lambda)\left[{}_{\varpi_1}D_q\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))-{}_{\varpi_1}D_q\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))\right]d_q\mu\,d_q\lambda.$$

3. Main results

In this section, we discuss our main results.

3.1. (p, q)-*Dragomir–Agarwal type integral inequalities*

We now derive (p, q)-analogues of Dragomir–Agarwal type integral inequalities via strongly φ-preinvex functions.

Theorem 3.1. *For* $\eta(\varpi_2,\varpi_1)>0$*, let* $\Psi:[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]\to\mathbb{R}$ *be* (p, q)*-differentiable mapping on* $(\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1))$ *and* ${}_{\varpi_1}D_{p,q}\Psi$ *be integrable on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *with* $0<q<p\le 1$*. If* $|{}_{\varpi_1}D_{p,q}\Psi|$ *is strongly* φ*-preinvex mapping on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *with modulus* $\sigma>0$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\le q\eta(\varpi_2,\varpi_1)\left[|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right.$$
$$\left.+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\int_0^1\int_0^1\lambda\varphi(\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right],$$

where

$$\mathbb{M}_1:=\frac{2[2]_{p,q}^2([3]_{p,q}-[2]_{p,q})+[4]_{p,q}([2]_{p,q}^2-[3]_{p,q})}{[2]_{p,q}^2[3]_{p,q}[4]_{p,q}},$$
$$\mathbb{M}_2:=\frac{2[2]_{p,q}([3]_{p,q}-[4]_{p,q})+[5]_{p,q}([2]_{p,q}[3]_{p,q}-[4]_{p,q})}{[2]_{p,q}[3]_{p,q}[4]_{p,q}[5]_{p,q}},$$
$$\mathbb{M}_3:=\mathbb{M}_1-\mathbb{M}_2$$
$$=\frac{2[2]_{p,q}^2([3]_{p,q}-[2]_{p,q})+[4]_{p,q}([2]_{p,q}^2-[3]_{p,q})}{[2]_{p,q}^2[3]_{p,q}[4]_{p,q}}$$

$$- \frac{2[2]_{p,q}([3]_{p,q} - [4]_{p,q}) + [5]_{p,q}([2]_{p,q}[3]_{p,q} - [4]_{p,q})}{[2]_{p,q}[3]_{p,q}[4]_{p,q}[5]_{p,q}}. \tag{3.1}$$

Proof. Using Lemma 2.2, property of modulus and the strongly φ-preinvexity of $|{}_{\varpi_1}D_{p,q}\Psi|$, we have

$$\begin{aligned}
&\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x - \frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|\\
&\le \frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\, d_{p,q}\lambda\\
&\quad+\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\, d_{p,q}\lambda\\
&\le \frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda|\Big[(1-\lambda)\varphi(1-\lambda)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|+\lambda\varphi(\lambda)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\\
&\quad-\sigma\eta^2(\varpi_2,\varpi_1)\lambda(1-\lambda)\Big]\,d_{p,q}\mu\, d_{p,q}\lambda\\
&\quad+\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda|\Big[(1-\mu)\varphi(1-\mu)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|+\mu\varphi(\mu)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\\
&\quad-\sigma\eta^2(\varpi_2,\varpi_1)\mu(1-\mu)\Big]\,d_{p,q}\mu\, d_{p,q}\lambda\\
&= \frac{q\eta(\varpi_2,\varpi_1)}{2}\Bigg[|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\\
&\quad+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\int_0^1\int_0^1\lambda\varphi(\lambda)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\lambda(1-\lambda)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\Bigg]\\
&\quad+\frac{q\eta(\varpi_2,\varpi_1)}{2}\Bigg[|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|\int_0^1\int_0^1(1-\mu)\varphi(1-\mu)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\\
&\quad+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\int_0^1\int_0^1\mu\varphi(\mu)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\\
&\quad-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\mu(1-\mu)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\Bigg]\\
&= q\eta(\varpi_2,\varpi_1)\Bigg[|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda
\end{aligned}$$

$$+|{}_{\varpi_1}\mathrm{D}_{p,q}\Psi(\varpi_2)|\int_0^1\int_0^1 \lambda\varphi(\lambda)|\mu-\lambda|\mathrm{d}_{p,q}\mu\,\mathrm{d}_{p,q}\lambda - \mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\Bigg].$$

This completes the proof. □

We now discuss some special cases of Theorem 3.1.

I. If $\varphi(\mu) = \varphi(\lambda) = 1$, then Theorem 3.1 reduces to the following result for the class of strongly preinvex mapping.

Corollary 3.1. *Under the assumptions of Theorem 3.1, if* $|{}_{\varpi_1}\mathrm{D}_{p,q}\Psi|$ *is strongly preinvex on* $[\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1)]$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}\mathrm{d}_{p,q}x - \frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\left[\mathbb{M}_4|{}_{\varpi_1}\mathrm{D}_{p,q}\Psi(\varpi_1)| + \mathbb{M}_2|{}_{\varpi_1}\mathrm{D}_{p,q}\Psi(\varpi_2)| - \mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right],$$

where

$$\mathbb{M}_4 := \frac{[2]_{p,q}^2([4]_{p,q}+2) - 2[2]_{p,q}([3]_{p,q}+[4]_{p,q}) + [3]_{p,q}[4]_{p,q}}{[2]_{p,q}^2[3]_{p,q}[4]_{p,q}}.$$

II. If $\varphi(\lambda) = \lambda^{-1}, \varphi(\mu) = \mu^{-1}$, then Theorem 3.1 reduces to the following result for the class of strongly P-preinvex mapping.

Corollary 3.2. *Under the assumptions of Theorem 3.1, if* $|{}_{\varpi_1}\mathrm{D}_{p,q}\Psi|$ *is strongly P-preinvex on* $[\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1)]$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}\mathrm{d}_{p,q}x - \frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\left[\mathbb{M}_5(|{}_{\varpi_1}\mathrm{D}_{p,q}\Psi(\varpi_1)| + |{}_{\varpi_1}\mathrm{D}_{p,q}\Psi(\varpi_2)|) - \mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right],$$

where

$$\mathbb{M}_5 := \frac{2([2]_{p,q}-1)}{[2]_{p,q}[3]_{p,q}}.$$

III. If $\varphi(\mu) = \mu^{s-1}, \varphi(\lambda) = \lambda^{s-1}$, then Theorem 3.1 reduces to the following result for the class of strongly s-preinvex mapping.

Corollary 3.3. *Under the assumptions of Theorem 3.1, if* $|{}_{\varpi_1}\mathrm{D}_{p,q}\Psi|$ *is strongly s-preinvex on* $[\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1)]$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}\mathrm{d}_{p,q}x - \frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$

$$\leq q\eta(\varpi_2,\varpi_1)\left[(2^{1-s}\mathbb{M}_5-\mathbb{M}_6)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|+\mathbb{M}_6|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right],$$

where

$$\mathbb{M}_6 := \frac{2[2]_{p,q}([s+2]_{p,q}-[s+1]_{p,q})+[s+3]_{p,q}([2]_{p,q}[s+1]_{p,q}-[s+2]_{p,q})}{[2]_{p,q}[s+1]_{p,q}[s+2]_{p,q}[s+3]_{p,q}}.$$

IV. If $\varphi(\mu)=1-\mu, \varphi(\lambda)=1-\lambda$, then Theorem 3.1 reduces to the following result for the class of strongly tgs-preinvex mapping.

Corollary 3.4. *Under the assumptions of Theorem 3.1, if* $|{}_{\varpi_1}D_{p,q}\Psi|$ *is strongly tgs-preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_3\left[|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|-\sigma\eta^2(\varpi_2,\varpi_1)\right].$$

Theorem 3.2. *For* $\eta(\varpi_2,\varpi_1)>0$*, let* $\Psi:[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]\to\mathbb{R}$ *be* (p,q)*-differentiable mapping on* $(\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1))$ *and* ${}_{\varpi_1}D_{p,q}\Psi$ *be integrable on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *with* $0<q<p\leq 1$*. If* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly* φ*-preinvex mapping on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *with modulus* $\sigma>0$ *for* $r_1>1$ *with* $r_1^{-1}+r_2^{-1}=1$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}$$
$$\times\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)d_{p,q}\mu\, d_{p,q}\lambda+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\int_0^1\int_0^1\lambda\varphi(\lambda)d_{p,q}\mu\, d_{p,q}\lambda\right.$$
$$\left.-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}},$$

where

$$\mathbb{M}_7 := \frac{(p-q)^2}{(q^{r_1+1}-p^{r_1+1})}\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(3+q^{r_1-n+1}-q^{n+1}-2q^{p+1}-q^{p+2})r_1(r_1-1)\ldots(r_1-n+1)}{n![2]_{p,q}^{r_1-n+1}(q^{r_1+1}-p^{r_1+1})}.$$

Proof. Using Lemma 2.2, Hölder's inequality, property of modulus and the strongly φ-preinvexity of $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$, we have

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$

$$\leq \frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\, d_{p,q}\lambda$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\, d_{p,q}\lambda$$

$$\leq \frac{q\eta(\varpi_2,\varpi_1)}{2}\left(\int_0^1\int_0^1 |\mu-\lambda|^{r_1}d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_1}}\left(\int_0^1\int_0^1 |{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|^{r_2}d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\left(\int_0^1\int_0^1 |\mu-\lambda|^{r_1}d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_1}}\left(\int_0^1\int_0^1 |{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|^{r_2}d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$
$$\leq \frac{q\eta(\varpi_2,\varpi_1)}{2}\mathbb{M}_7^{\frac{1}{r_1}}$$
$$\times\left[\left(\int_0^1\int_0^1\Big[(1-\lambda)\varphi(1-\lambda)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+\lambda\varphi(\lambda)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\sigma\eta^2(\varpi_2,\varpi_1)\lambda(1-\lambda)\Big]d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left(\int_0^1\int_0^1\Big[(1-\mu)\varphi(1-\mu)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+\mu\varphi(\mu)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\sigma\eta^2(\varpi_2,\varpi_1)\mu(1-\mu)\Big]d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right]$$
$$=\frac{q\eta(\varpi_2,\varpi_1)}{2}\mathbb{M}_7^{\frac{1}{r_1}}$$
$$\times\left[\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)d_{p,q}\mu\, d_{p,q}\lambda+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\int_0^1\int_0^1\lambda\varphi(\lambda)d_{p,q}\mu\, d_{p,q}\lambda\right.\right.$$
$$\left.-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\lambda(1-\lambda)d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$
$$+\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}\int_0^1\int_0^1(1-\mu)\varphi(1-\mu)d_{p,q}\mu\, d_{p,q}\lambda+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\int_0^1\int_0^1\mu\varphi(\mu)d_{p,q}\mu\, d_{p,q}\lambda\right.$$
$$\left.\left.-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\mu(1-\mu)d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right]$$
$$=q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}$$
$$\times\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)d_{p,q}\mu\, d_{p,q}\lambda+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\int_0^1\int_0^1\lambda\varphi(\lambda)d_{p,q}\mu\, d_{p,q}\lambda\right.$$

$$-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\Bigg)^{\frac{1}{r_2}}.$$

This completes the proof. □

We now discuss some special cases of Theorem 3.2.

I. If $\varphi(\mu)=\varphi(\lambda)=1$, then Theorem 3.2 reduces to the following result for the class of strongly preinvex mapping.

Corollary 3.5. *Under the assumptions of Theorem 3.2, if* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}\left(\frac{([2]_{p,q}-1)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}}{[2]_{p,q}}-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}}.$$

II. If $\varphi(\mu)=\mu^{-1},\varphi(\lambda)=\lambda^{-1}$, then Theorem 3.2 reduces to the following result for the class of strongly P-preinvex mapping.

Corollary 3.6. *Under the assumptions of Theorem 3.2, if* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly P-preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}}.$$

III. If $\varphi(\mu)=\mu^{s-1},\varphi(\lambda)=\lambda^{s-1}$, then Theorem 3.2 reduces to the following result for the class of strongly s-preinvex mapping.

Corollary 3.7. *Under the assumptions of Theorem 3.2, if* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly s-preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}\left(\frac{(2^{1-s}[s+1]_{p,q}-1)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}}{[s+1]_{p,q}}-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}}.$$

IV. If $\varphi(\mu)=1-\mu,\varphi(\lambda)=1-\lambda$, then Theorem 3.2 reduces to the following result for the class of strongly tgs-preinvex mapping.

Corollary 3.8. *Under the assumptions of Theorem 3.2, if* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly tgs-preinvex on* $[\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)]$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}\left(\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}}\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.$$

Theorem 3.3. *For* $\eta(\varpi_2, \varpi_1) > 0$*, let* $\Psi : [\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)] \to \mathbb{R}$ *be* (p, q)*-differentiable mapping on* $(\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1))$ *and* ${}_{\varpi_1}D_{p,q}\Psi$ *be integrable on* $[\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)]$ *with* $0 < q < p \leq 1$*. If* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly* φ*-preinvex on* $[\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)]$ *with modulus* $\sigma > 0$ *for* $r_2 \geq 1$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}$$
$$\times\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\int_0^1\int_0^1\lambda\varphi(\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right.$$
$$\left.-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}},$$

where $\mathbb{M}_3$ *is given as in Theorem 3.1 and* $\mathbb{M}_5$ *is defined as in Corollary 3.2.*

Proof. Using Lemma 2.2, the well-known power mean inequality, property of modulus and the strongly φ-preinvexity of $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$, we have

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\,d_{p,q}\lambda$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\,d_{p,q}\lambda$$
$$\leq\frac{q\eta(\varpi_2,\varpi_1)}{2}\left(\int_0^1\int_0^1|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right)^{1-\frac{1}{r_2}}\left(\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|^{r_2}d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\left(\int_0^1\int_0^1|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right)^{1-\frac{1}{r_2}}\left(\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|^{r_2}d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$

$$
\begin{aligned}
&\leq \frac{q\eta(\varpi_2,\varpi_1)}{2}\mathbb{M}_5^{1-\frac{1}{r_2}} \\
&\times\left[\left(\int_0^1\int_0^1\Big((1-\lambda)\varphi(1-\lambda)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+\lambda\varphi(\lambda)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\sigma\eta^2(\varpi_2,\varpi_1)\lambda(1-\lambda)\Big)\,d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left(\int_0^1\int_0^1\Big((1-\mu)\varphi(1-\mu)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+\mu\varphi(\mu)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\sigma\eta^2(\varpi_2,\varpi_1)\mu(1-\mu)\Big)\,d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right] \\
&= \frac{q\eta(\varpi_2,\varpi_1)}{2}\mathbb{M}_5^{1-\frac{1}{r_2}} \\
&\times\left[\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\int_0^1\int_0^1\lambda\varphi(\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right.\right. \\
&\left.-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\lambda(1-\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}} \\
&+\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}\int_0^1\int_0^1(1-\mu)\varphi(1-\mu)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\int_0^1\int_0^1\mu\varphi(\mu)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right. \\
&\left.\left.-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\mu(1-\mu)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right] \\
&= q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}} \\
&\times\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\int_0^1\int_0^1\lambda\varphi(\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right. \\
&\left.-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.
\end{aligned}
$$

This completes the proof. □

We now discuss some special cases of Theorem 3.3.

I. If $\varphi(\mu)=\varphi(\lambda)=1$, then Theorem 3.3 reduces to the following result for the class of strongly preinvex mapping.

Corollary 3.9. *Under the assumptions of Theorem 3.3, if* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$*, then*

$$
\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|
$$

$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}\left(\mathbb{M}_4|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+\mathbb{M}_2|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.$$

II. If $\varphi(\mu)=\mu^{-1},\varphi(\lambda)=\lambda^{-1}$, then Theorem 3.3 reduces to the following result for the class of strongly P-preinvex mapping.

Corollary 3.10. *Under the assumptions of Theorem 3.3, if* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly P-preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$, *then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}\left(\mathbb{M}_5(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2})-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.$$

III. If $\varphi(\mu)=\mu^{s-1},\varphi(\lambda)=\lambda^{s-1}$, then Theorem 3.3 reduces to the following result for the class of strongly s-preinvex mapping.

Corollary 3.11. *Under the assumptions of Theorem 3.3, if* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly s-preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$, *then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}\left((2^{1-s}\mathbb{M}_5-\mathbb{M}_6)|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+\mathbb{M}_6|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.$$

IV. If $\varphi(\mu)=1-\mu,\varphi(\lambda)=1-\lambda$, then Theorem 3.3 reduces to the following result for the class of strongly tgs-preinvex mapping.

Corollary 3.12. *Under the assumptions of Theorem 3.3, if* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly tgs-preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$, *then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}\mathbb{M}_3^{\frac{1}{r_2}}\left(|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}+|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}-\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.$$

3.2. (p, q)-*Iyengar type integral inequalities*

In this section, we derive new (p, q)-Iyengar type integral inequalities essentially by using the strongly quasi-preinvexity property of the mappings. For this, let us recall the following definition.

Definition 3.1. *A mapping* $\Psi:\mathscr{K}_\eta\to\mathbb{R}$ *on the invex set is said to be strongly quasi-preinvex, if*

$$\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))\leq\max\{\Psi(\varpi_1),\Psi(\varpi_2)\}-\sigma\lambda(1-\lambda)\eta^2(\varpi_2,\varpi_1),$$

holds for all $\varpi_1,\varpi_2\in\mathscr{K}_\eta$, $\lambda\in(0,1)$ *with* $\eta(\varpi_2,\varpi_1)>0$ *and* $\sigma>0$.

Theorem 3.4. *For* $\eta(\varpi_2,\varpi_1) > 0$, *let* $\Psi : [\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1)] \to \mathbb{R}$ *be* (p,q)*-differentiable mapping on* $(\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1))$ *and* ${}_{\varpi_1}D_{p,q}\Psi$ *be integrable on* $[\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1)]$ *with* $0 < q < p \leq 1$. *If* $|{}_{\varpi_1}D_{p,q}\Psi|$ *is strongly quasi-preinvex mapping on* $[\varpi_1, \varpi_1 + \eta(\varpi_2,\varpi_1)]$ *with modulus* $\sigma > 0$, *then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x - \frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\left[\mathbb{M}_5\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\} - \mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right],$$

where $\mathbb{M}_3$ *is given as in Theorem 3.1 and* $\mathbb{M}_5$ *is defined as in Corollary 3.2.*

Proof. Using Lemma 2.2, property of modulus and the strongly quasi-preinvexity of $|{}_{\varpi_1}D_{p,q}\Psi|$, we have

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x - \frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq \frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\, d_{p,q}\lambda$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\, d_{p,q}\lambda$$
$$\leq \frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda|\left[\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\} - \sigma\eta^2(\varpi_2,\varpi_1)\lambda(1-\lambda)\right]d_{p,q}\mu\, d_{p,q}\lambda$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1 |\mu-\lambda|\left[\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\} - \sigma\eta^2(\varpi_2,\varpi_1)\mu(1-\mu)\right]d_{p,q}\mu\, d_{p,q}\lambda$$
$$=\frac{q\eta(\varpi_2,\varpi_1)}{2}\left[\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\}\int_0^1\int_0^1|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\right.$$
$$\left.-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\lambda(1-\lambda)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\right]$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\left[\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\}\int_0^1\int_0^1|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\right.$$
$$\left.-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\mu(1-\mu)|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\right]$$
$$= q\eta(\varpi_2,\varpi_1)\left[\mathbb{M}_5\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|\} - \mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right].$$

This completes the proof. □

Theorem 3.5. *For* $\eta(\varpi_2,\varpi_1)>0$*, let* $\Psi:[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]\to\mathbb{R}$ *be* (p, q)*-differentiable mapping on* $(\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1))$ *and* ${}_{\varpi_1}D_{p,q}\Psi$ *be integrable on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *with* $0<q<p\le 1$*. If* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly quasi-preinvex mapping on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *with modulus* $\sigma>0$ *for* $r_1>1$ *with* $r_1^{-1}+r_2^{-1}=1$*, then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\le q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}\left(\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2},|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}},$$

where $\mathbb{M}_7$ *is given as in Theorem 3.2.*

Proof. Using Lemma 2.2, Hölder's inequality, property of modulus and the strongly quasi-preinvexity of $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$, we have

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\le\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\,d_{p,q}\lambda$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\,d_{p,q}\lambda$$
$$\le\frac{q\eta(\varpi_2,\varpi_1)}{2}\left(\int_0^1\int_0^1|\mu-\lambda|^{r_1}d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_1}}\left(\int_0^1\int_0^1|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|^{r_2}d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\left(\int_0^1\int_0^1|\mu-\lambda|^{r_1}d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_1}}\left(\int_0^1\int_0^1|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|^{r_2}d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$
$$\le\frac{q\eta(\varpi_2,\varpi_1)}{2}\mathbb{M}_7^{\frac{1}{r_1}}\left[\left(\int_0^1\int_0^1\left[\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2},|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}-\sigma\eta^2(\varpi_2,\varpi_1)\lambda(1-\lambda)\right]d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left(\int_0^1\int_0^1\left[\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2},|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}-\sigma\eta^2(\varpi_2,\varpi_1)\mu(1-\mu)\right]d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right]$$
$$=\frac{q\eta(\varpi_2,\varpi_1)}{2}\mathbb{M}_7^{\frac{1}{r_1}}$$
$$\times\left[\left(\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2},|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\lambda(1-\lambda)d_{p,q}\mu\,d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right.$$

$$+\left(\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2},|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1\mu(1-\mu)d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}\Bigg]$$

$$=q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}\left(\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2},|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}}.$$

This completes the proof. □

Theorem 3.6. *For* $\eta(\varpi_2,\varpi_1)>0$, *let* $\Psi:[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]\to\mathbb{R}$ *be* (p,q)*-differentiable mapping on* $(\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1))$ *and* ${}_{\varpi_1}D_{p,q}\Psi$ *be integrable on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *with* $0<q<p\leq 1$. *If* $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$ *is strongly quasi-preinvex mapping on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *for* $r_2\geq 1$, *then*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}\left(\mathbb{M}_5\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2},|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}},$$

where $\mathbb{M}_3$ *is given as in Theorem 3.1 and* $\mathbb{M}_5$ *is defined as in Corollary 3.2.*

Proof. Using Lemma 2.2, the well-known power mean inequality, property of modulus and the strongly quasi-preinvexity of $|{}_{\varpi_1}D_{p,q}\Psi|^{r_2}$, we have

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\, d_{p,q}\lambda$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|d_{p,q}\mu\, d_{p,q}\lambda$$
$$\leq\frac{q\eta(\varpi_2,\varpi_1)}{2}\left(\int_0^1\int_0^1|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\right)^{1-\frac{1}{r_2}}\left(\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\lambda\eta(\varpi_2,\varpi_1))|^{r_2}d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$
$$+\frac{q\eta(\varpi_2,\varpi_1)}{2}\left(\int_0^1\int_0^1|\mu-\lambda|d_{p,q}\mu\, d_{p,q}\lambda\right)^{1-\frac{1}{r_2}}\left(\int_0^1\int_0^1|\mu-\lambda||{}_{\varpi_1}D_{p,q}\Psi(\varpi_1+\mu\eta(\varpi_2,\varpi_1))|^{r_2}d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}$$
$$\leq\frac{q\eta(\varpi_2,\varpi_1)}{2}\mathbb{M}_5^{1-\frac{1}{r_2}}$$
$$\times\left[\left(\int_0^1\int_0^1|\mu-\lambda|\left(\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2},|{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}-\sigma\eta^2(\varpi_2,\varpi_1)\lambda(1-\lambda)\right)d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}\right.$$

$$+\left(\int_0^1\int_0^1 |\mu-\lambda|\left(\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\} - \sigma\eta^2(\varpi_2,\varpi_1)\mu(1-\mu)\right) d_{p,q}\mu\, d_{p,q}\lambda\right)^{\frac{1}{r_2}}\Bigg]$$

$$= \frac{q\eta(\varpi_2,\varpi_1)}{2}\mathbb{M}_5^{1-\frac{1}{r_2}}$$

$$\times\Bigg[\Bigg(\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}\int_0^1\int_0^1 |\mu-\lambda| d_{p,q}\mu\, d_{p,q}\lambda$$

$$-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1 \lambda(1-\lambda)|\mu-\lambda| d_{p,q}\mu\, d_{p,q}\lambda\Bigg)^{\frac{1}{r_2}}$$

$$+\Bigg(\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\}\int_0^1\int_0^1 |\mu-\lambda| d_{p,q}\mu\, d_{p,q}\lambda$$

$$-\sigma\eta^2(\varpi_2,\varpi_1)\int_0^1\int_0^1 \mu(1-\mu)|\mu-\lambda| d_{p,q}\mu\, d_{p,q}\lambda\Bigg)^{\frac{1}{r_2}}\Bigg]$$

$$= q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}\left(\mathbb{M}_5\max\{|{}_{\varpi_1}D_{p,q}\Psi(\varpi_1)|^{r_2}, |{}_{\varpi_1}D_{p,q}\Psi(\varpi_2)|^{r_2}\} - \mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.$$

This completes the proof. □

Remark 3.1. *If we choose* $p = 1$ *in our main results, we can get new special cases regarding quantum analogues of Dragomir–Agarwal and Iyengar type of integral inequalities essentially by using the strongly* φ*-preinvexity property of the mappings. We omit here their proofs and the details are left to the interested reader.*

4. Application to bounded functions

We suppose that the following condition is satisfied:

$$|{}_{\varpi_1}D_{p,q}\Psi| \le \Omega, \tag{4.1}$$

which means that the (p,q)-differentiable mapping Ψ is in absolute value bounded from the positive real number Ω. Applying the above condition, we are in a position to derive some new interesting inequalities using our main results.

Proposition 4.1. *Under the conditions of Theorem 3.1, the following inequality holds:*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x - \frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$

$$\le q\eta(\varpi_2,\varpi_1)\Bigg[\Omega\Bigg\{\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)|\mu-\lambda| d_{p,q}\mu\, d_{p,q}\lambda + \int_0^1\int_0^1 \lambda\varphi(\lambda)|\mu-\lambda| d_{p,q}\mu\, d_{p,q}\lambda\Bigg\}$$

$$-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\Big].$$

Proof. Applying inequality (4.1) in Theorem 3.1, we have the desired result. □

Proposition 4.2. *Under the conditions of Theorem 3.2, the following inequality holds:*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$

$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}$$

$$\times\left(\Omega^{r_2}\left\{\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)d_{p,q}\mu\,d_{p,q}\lambda+\int_0^1\int_0^1\lambda\varphi(\lambda)d_{p,q}\mu\,d_{p,q}\lambda\right\}-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}}.$$

Proof. Using inequality (4.1) in Theorem 3.2, we get the desired result. □

Proposition 4.3. *Under the conditions of Theorem 3.3, the following inequality holds:*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$

$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}$$

$$\times\left(\Omega^{r_2}\left\{\int_0^1\int_0^1(1-\lambda)\varphi(1-\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda+\int_0^1\int_0^1\lambda\varphi(\lambda)|\mu-\lambda|d_{p,q}\mu\,d_{p,q}\lambda\right\}\right.$$

$$\left.-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.$$

Proof. Applying inequality (4.1) in Theorem 3.3, we obtain the desired result. □

Proposition 4.4. *Under the conditions of Theorem 3.4, the following inequality holds:*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$

$$\leq q\eta(\varpi_2,\varpi_1)\left[\mathbb{M}_5\Omega-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right].$$

Proof. Using inequality (4.1) in Theorem 3.4, we have the desired result. □

Proposition 4.5. *Under the conditions of Theorem 3.5, the following inequality holds:*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$

$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_7^{\frac{1}{r_1}}\left(\Omega^{r_2}-\frac{\sigma\eta^2(\varpi_2,\varpi_1)([3]_{p,q}-[2]_{p,q})}{[2]_{p,q}[3]_{p,q}}\right)^{\frac{1}{r_2}}.$$

Proof. Applying inequality (4.1) in Theorem 3.5, we get the desired result. □

Proposition 4.6. *Under the conditions of Theorem 3.6, the following inequality holds:*

$$\left|\frac{1}{p\eta(\varpi_2,\varpi_1)}\int_{\varpi_1}^{\varpi_1+p\eta(\varpi_2,\varpi_1)}\Psi(x)\,{}_{\varpi_1}d_{p,q}x-\frac{q\Psi(\varpi_1)+p\Psi(\varpi_1+\eta(\varpi_2,\varpi_1))}{[2]_{p,q}}\right|$$
$$\leq q\eta(\varpi_2,\varpi_1)\mathbb{M}_5^{1-\frac{1}{r_2}}\left(\mathbb{M}_5\Omega^{r_2}-\mathbb{M}_3\sigma\eta^2(\varpi_2,\varpi_1)\right)^{\frac{1}{r_2}}.$$

Proof. Using inequality (4.1) in Theorem 3.6, we obtain the desired result. □

5. Conclusions

In this paper, we have established a new post-quantum integral identity using (p, q)-differentiable mappings. From the applied identity as an auxiliary result, we have obtained some new variants of Dragomir–Agarwal and Iyengar type integral inequalities essentially pertaining to the class of strongly φ-preinvex and strongly quasi φ-preinvex mappings, respectively. We also discuss several new special cases which show that the results obtained are quite unifying. In order to illustrate the efficiency of our main results, some applications regarding (p, q)-differentiable mappings that are in absolute value bounded are provided as well. To the best of our knowledge, these results are new in the literature. Since the class of strongly φ-preinvex mappings have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special mappings, quantum mechanics, related optimization theory, and mathematical inequalities and may stimulate further research in different areas of pure and applied sciences. Studies relating convexity, partial convexity, and preinvex mappings (as contractive operators) may have useful applications in complex interdisciplinary studies, such as maximizing the likelihood from multiple linear regressions involving Gauss–Laplace distribution. For more details, please see [43, 44].

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. J. Tariboon, S. K. Ntouyas, Quantum integral inequalities on finite intervals, *J. Inequal. Appl.*, **2014** (2014), 121. https://doi.org/10.1186/1029-242X-2014-121
2. N. Alp, M. Z. Sarikaya, M. Kunt, İ. İşcan, q-Hermite–Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, *J. King Saud Univ. Sci.*, **30** (2018), 193–203. https://doi.org/10.1016/j.jksus.2016.09.007

3. M. A. Noor, K. I. Noor, M. U. Awan, Some quantum estimates for Hermite–Hadamard inequalities, *Appl. Math. Comput.*, **251** (2015), 675–679. https://doi.org/10.1016/j.amc.2014.11.090

4. W. Sudsutad, S. K. Ntouyas, J. Tariboon, Quantum integral inequalities for convex functions, *J. Math. Inequal.*, **9** (2015), 781–793. https://doi.org/10.7153/jmi-09-64

5. W. J. Liu, H. F. Zhuang, Some quantum estimates of Hermite–Hadamard inequalities for convex functions, *J. Appl. Anal. Comput.*, **7** (2017), 501–522. https://doi.org/10.11948/2017031

6. M. A. Noor, K. I. Noor, M. U. Awan, Some quantum integral inequalities via preinvex functions, *Appl. Math. Comput.*, **269** (2015), 242–251. https://doi.org/10.1016/j.amc.2015.07.078

7. M. A. Noor, M. U. Awan, K. I. Noor, Quantum Ostrowski inequalities for q-differentiable convex functions, *J. Math. Inequal.*, **10** (2016), 1013–1018. https://doi.org/10.7153/jmi-10-81

8. Y. P. Deng, M. U. Awan, S. H. Wu, Quantum integral inequalities of Simpson-type for strongly preinvex functions, *Mathematics*, **7** (2019), 751. https://doi.org/10.3390/math7080751

9. Y. Zhang, T. S. Du, H. Wang, Y. J. Shen, Different types of quantum integral inequalities via (α, m)–convexity, *J. Inequal. Appl.*, **2018** (2018), 264. https://doi.org/10.1186/s13660-018-1860-2

10. C. Promsakon, M. A. Ali, H. Budak, M. Abbas, F. Muhammad, T. Sitthiwirattham, On generalizations of quantum Simpson's and quantum Newton's inequalities with some parameters, *AIMS Math.*, **6** (2021), 13954–13975. https://doi.org/10.3934/math.2021807

11. X. You, M. A. Ali, S. Erden, H. Budak, Y. M. Chu, On some new midpoint inequalities for the functions of two variables via quantum calculus, *J. Inequal. Appl.*, **2021** (2021), 142. https://doi.org/10.1186/s13660-021-02678-9

12. M. A. Ali, N. Alp, H. Budak, Y. M. Chu, Z. Zhang, On some new quantum midpoint-type inequalities for twice quantum differentiable convex functions, *Open Math.*, **19** (2021), 427–439. https://doi.org/10.1515/math-2021-0015

13. H. Budak, S. Khan, M. A. Ali, Y. M. Chu, Refinements of quantum Hermite–Hadamard-type inequalities, *Open Math.*, **19** (2021), 724–734. https://doi.org/10.1515/math-2021-0029

14. P. Siricharuanun, S. Erden, M. A. Ali, H. Budak, S. Chasreechai, T. Sitthiwirattham, Some new Simpson's and Newton's formulas type inequalities for convex functions in quantum calculus, *Mathematics*, **9** (2021), 1992. https://doi.org/10.3390/math9161992

15. M. Vivas-Cortez, M. U. Awan, S. Talib, A. Kashuri, M. A. Noor, Some new post–quantum integral inequalities involving twice (p, q)–differentiable ψ–preinvex functions and applications, *Axioms,* **10** (2021), 283. https://doi.org/10.3390/axioms10040283

16. R. Chakrabarti, R. Jagannathan, A (p, q)-oscillator realization of two-parameter quantum algebras, *J. Phys. A.*, **24** (1991). https://doi.org/10.1088/0305-4470/24/13/002

17. M. Tunç, E. Göv, Some integral inequalities via (p, q)-calculus on finite intervals, *RGMIA Res. Rep. Collect.*, **19** (2016), 95.

18. I. B. Sial, M. A. Ali, G. Murtaza, S. K. Ntouyas, J. Soontharanon, T. Sitthiwirattham, On some new inequalities of Hermite–Hadamard midpoint and trapezoid type for preinvex functions in (p, q)-calculus, *Symmetry*, **13** (2021), 1864. https://doi.org/10.3390/sym13101864

19. T. Sitthiwirattham, G. Murtaza, M. A. Ali, S. K. Ntouyas, M. Adeel, J. Soontharanon, On some new trapezoidal type inequalities for twice (p, q)-differentiable convex functions in post-quantum calculus, *Symmetry*, **13** (2021), 1605. https://doi.org/10.3390/sym13091605

20. H. Kalsoom, M. A. Ali, M. Idrees, P. Agarwal, M. Arif, New post quantum analogues of Hermite–Hadamard type inequalities for interval-valued convex functions, *Math. Probl. Eng.*, **2021** (2021), 5529650. https://doi.org/10.1155/2021/5529650

21. M. Kadakal, İ. İşcan, Exponential type convexity and some related inequalities, *J. Inequal. Appl.*, **2020** (2020), 82. https://doi.org/10.1186/s13660-020-02349-1

22. A. Ekinci, Inequalities for convex functions on time scales, *TWMS J. Appl. Eng. Math.*, **9** (2019), 64–72.

23. A. A. El-Deeb, S. Rashid, On some new double dynamic inequalities associated with Leibniz integral rule on time scales, *Adv. Differ. Equ.*, **2021** (2021), 125. https://doi.org/10.1186/s13662-021-03282-3

24. A. O. Akdemir, S. I. Butt, M. Nadeem, M. A. Ragusa, New general variants of Chebyshev type inequalities via generalized fractional integral operators, *Mathematics*, **9** (2021), 122. https://doi.org/10.3390/math9020122

25. S. I. Butt, M. Nadeem, G. Farid, On Caputo fractional derivatives via exponential s-convex functions, *Turk. J. Sci.*, **5** (2020), 140–146.

26. T. Weir, B. Mond, Preinvex functions in multiple objective optimization, *J. Math. Anal. Appl.*, **136** (1988), 29–38. https://doi.org/10.1016/0022-247X(88)90113-8

27. M. A. Noor, K. I. Noor, Some characterizations of strongly preinvex functions, *J. Math. Anal. Appl.*, **316** (2006), 697–706. https://doi.org/10.1016/j.jmaa.2005.05.014

28. S. S. Dragomir, Inequalities of Hermite–Hadamard type for φ–convex functions, *RGMIA Res. Rep. Collect.*, **16** (2013), 87.

29. H. Angulo, J. Gimenez, A. M. Moros, K. Nikodem, On strongly h–convex functions, *Ann. Funct. Anal.*, **2** (2011), 85–91. https://doi.org/10.15352/afa/1399900197

30. M. A. Noor, K. I. Noor, M. U. Awan, J. Li, On Hermite–Hadamard inequalities for h–preinvex functions, *Filomat*, **28** (2014), 1463–1474. https://doi.org/10.2298/FIL1407463N

31. M. U. Awan, S. Talib, M. A. Noor, K. I. Noor, On γ-preinvex functions, *Filomat*, **34** (2020), 4137–4159. https://doi.org/10.2298/FIL2012137A

32. S. I. Butt, M. K. Bakula, D. Pecarič, J. Pecarič, Jensen–Grüss inequality and its applications for the Zipf–Mandelbrot law, *Math. Meth. Appl. Sci.*, **44** (2021), 1664–1673. https://doi.org/10.1002/mma.6869

33. S. I. Butt, J. Pecarič, I. Perič, Refinement of integral inequalities for monotone functions, *J. Inequal. Appl.*, **2012** (2012), 301. https://doi.org/10.1186/1029-242X-2012-301

34. S. I. Butt, J. Pecarič, A. Vukelič, Generalization of Popoviciu-type inequalities via Fink's identity, *Mediterr. J. Math.*, **13** (2016), 1495–1511. https://doi.org/10.1007/s00009-015-0573-8

35. S. I. Butt, S. Yousaf, A. O. Akdemir, M. A. Dokuyucu, New Hadamard–type integral inequalities via a general form of fractional integral operators, *Chaos Solitons Fractals*, **148** (2021). https://doi.org/10.1016/j.chaos.2021.111025

36. S. I. Butt, S. Yousaf, A. Asghar, K. A. Khan, H. R. Moradi, New fractional Hermite–Hadamard–Mercer inequalities for harmonically convex function, *J. Funct. Spaces*, **2021** (2021), 5868326. https://doi.org/10.1155/2021/5868326

37. T. S. Du, C. Y. Luo, B. Yu, Certain quantum estimates on the parameterized integral inequalities and their applications, *J. Math. Inequal.*, **15** (2021), 201–228. https://doi.org/10.7153/jmi-2021-15-16

38. I. Slimane, Z. Dahmani, Normalized fractional inequalities for continuous random variables, *J. Interdiscip. Math.,* **2021** (2021), 1–15. https://doi.org/10.1080/09720502.2021.1932300

39. S. Rashid, S. I. Butt, S. Kanwal, H. Ahmad, M. K. Wang, Quantum integral inequalities with respect to Raina's function via coordinated generalized Ψ–convex functions with applications, *J. Funct. Spaces*, **2021** (2021), 6631474. https://doi.org/10.1155/2021/6631474

40. M. Alomari, q-Bernoulli inequality, *Turk. J. Sci.*, **3** (2018), 32–39.

41. S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, **11** (1998), 91–95. https://doi.org/10.1016/S0893-9659(98)00086-X

42. F. Qi, P. Cerone, S. S. Dragomir, Some new Iyengar type inequalities, *Rocky Mt. J. Math.*, **35** (2005), 997–1015. https://doi.org/10.1216/rmjm/1181069718

43. V. Kak, P. Cheung, *Quantum Calculus*, Springer Verlag, 2002.

44. J. E. Pecarič, F. Proschan, Y. L. Tong, *Convex Functions, Partial Ordering and Statistical Applications*, Academic Press, New York, 1991.