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Research article

Pointwise convergence problem of free Benjamin-Ono-Burgers equation

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Abstract: In this paper, we show the almost everywhere pointwise convergence of free Benjamin-Ono-Burgers equation in $H^s(\mathbf{R})$ with $s > 0$ with the aid of the maximal function estimate.

Keywords: pointwise convergence; free Benjamin-Ono-Burgers equations; maximal function estimate
Mathematics Subject Classification: 42B15, 42B25

1. Introduction

In this paper, we investigate the pointwise convergence problem of the free Benjamin-Ono-Burgers equation

$$u_t - \partial_x^2 u + \mathcal{H}\partial_x^2 u = 0, \tag{1.1}$$

$$u(x, 0) = f(x). \tag{1.2}$$

Carleson [4] initiated the pointwise converge problem, more precisely, Carleson showed pointwise convergence problem of the one dimensional Schrödinger equation in $H^s(\mathbf{R})$ with $s \geq \frac{1}{4}$. Dahlberg and Kenig [8] showed that the pointwise convergence of the Schrödinger equation does not hold for $s < \frac{1}{4}$ in any dimension. The pointwise convergence problem of Schrödinger equations in higher dimension has been investigated by many authors, for example, see [1, 2, 6, 7, 9, 14, 16, 17, 20–25]. The results of Bourgain [3] and Du et al. [11, 12] showed that $s > \frac{n}{2(n+1)}$ is the necessary condition for the pointwise convergence problem of n dimensional Schrödinger equation. Moreover, Miao et al. [19] studied the the maximal inequality for 2D fractional order Schrödinger operators.

The Benjamin-Ono-Burgers equation (1.1) was obtained by Ewdin and Roberts [13] in the study of intense magnetic flux tubes of the solar atmosphere. The dissipative effects $\epsilon\partial_x^2 u$ in that literature are due to weak thermal conduction, where ϵ is a measure of the importance of thermal conduction and is assumed small. Some people have studied the Cauchy problem for the Benjamin-Ono-Burgers equation [5, 15, 18]. In this paper, motivated by [10, 25], we show the pointwise convergence of free Benjamin-Ono-Burgers equation in $H^s(\mathbf{R})$ with $s > 0$.

We present some notations before stating the main results. We always assume that $0 < \epsilon < 10^{-8}$. $|E|$ denotes by the Lebesgue measure of set E.

$$\mathcal{F}_x f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} f(x)dx, \mathcal{F}_x^{-1} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} f(x)dx,$$

$$U_1(t)u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi - it\xi|\xi| - t\xi^2} \mathcal{F}_x u_0(\xi)d\xi, \|f\|_{L_x^q L_t^p} = \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(x,t)|^p dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}.$$

$H^s(\mathbf{R}) = \left\{ f \in \mathcal{S}'(\mathbf{R}) : \|f\|_{H^s(\mathbf{R})} = \|\langle \xi \rangle^s \mathcal{F}_x f\|_{L^2_\xi(\mathbf{R})} < \infty \right\}$, where $\langle \xi \rangle^s = (1 + \xi^2)^{\frac{s}{2}}$.

The main result is as follows.

Theorem 1.1. *Let $f \in H^s(\mathbf{R})$ with $s > 0$. Then, we have*

$$U_1(t)f(x) \longrightarrow f(x) \tag{1.3}$$

almost everywhere as $t \longrightarrow 0$.

2. Preliminary

In this section, we present the proof of Lemma 2.1.

Lemma 2.1. *For $t \geq 0$, we have*

$$\left| \int_{\mathbf{R}} e^{ix\xi - it\xi|\xi| - t\xi|\xi|} \frac{d\xi}{|\xi|^\epsilon} \right| \leq C|x|^{-1+\epsilon}. \tag{2.1}$$

Proof. Let $x\xi = \eta$. Then, we have

$$\int_{\mathbf{R}} e^{ix\xi - it\xi|\xi| - t\xi|\xi|} \frac{d\xi}{|\xi|^\epsilon} = Cx^{-1}|x|^\epsilon \int_{\mathbf{R}} e^{i\eta - itx^{-1}|x|^{-1}\eta|\eta| - tx^{-2}\eta^2} \frac{d\eta}{|\eta|^\epsilon}. \tag{2.2}$$

We define

$$I := \int_{\mathbf{R}} e^{i\eta - tx^{-1}|x|^{-1}\eta|\eta| - tx^{-2}\eta^2} \frac{d\eta}{|\eta|^\epsilon}. \tag{2.3}$$

We consider $x \geq 0, x < 0$, respectively. When $x > 0$, we have

$$I := \int_{\mathbf{R}} e^{i\eta - iA\eta|\eta| - A\eta^2} \frac{d\eta}{|\eta|^\epsilon} =: I_1 + I_2,. \tag{2.4}$$

where $I_1 := \int_0^\infty e^{i\eta - iA\eta^2 - A\eta^2} \frac{d\eta}{\eta^\epsilon}, I_2 := \int_{-\infty}^0 e^{i\eta + iA\eta^2 - A\eta^2} \frac{d\eta}{(-\eta)^\epsilon}$. Here $A = tx^{-2}$. For $\frac{1}{2A} \leq 2$, then, we have $I_1 := I_{11} + I_{12}$. Here, $I_{11} := \int_0^4 e^{i\eta - iA\eta^2 - A\eta^2} \frac{d\eta}{\eta^\epsilon}, I_{12} := \int_4^\infty e^{i\eta - iA\eta^2 - A\eta^2} \frac{d\eta}{\eta^\epsilon}$. Obviously, we have

$$|I_{11}| \leq \left| \int_0^4 e^{i\eta - iA\eta^2 - A\eta^2} \frac{d\eta}{\eta^\epsilon} \right| \leq \int_0^4 \eta^{-\epsilon} = \frac{4^{1-\epsilon}}{1-\epsilon}. \tag{2.5}$$

By using the integration by parts, we have

$$\begin{aligned} I_{12} &:= \int_4^{\infty} e^{i\eta - iA\eta^2 - A\eta^2} \frac{d\eta}{\eta^{\epsilon}} = \int_4^{+\infty} \frac{1}{(1-2A\eta)\eta^{\epsilon} i + 2A\eta^{1+\epsilon}} d e^{i\eta - iA\eta^2 - A\eta^2} \\ &= \frac{1}{(1-8A)4^{\epsilon} i + 2A4^{1+\epsilon}} - \int_4^{+\infty} e^{i\eta - A\eta^2} \left(\frac{1}{(1-2A\eta)\eta^{\epsilon} i + 2A\eta^{1+\epsilon}} \right)' d\eta. \end{aligned} \tag{2.6}$$

Thus, from (2.6), we have

$$\left| \left(\frac{1}{u+iv} \right)' \right| \le \frac{2|u'|}{u^2+v^2} + \frac{2|v'|}{u^2+v^2}. \tag{2.7}$$

From (2.7), we have

$$\left(\frac{1}{(1-2A\eta)\eta^{\epsilon} i + 2A\eta^{1+\epsilon}} \right)' \le \frac{2+3\epsilon}{\eta^{1+\epsilon}((1-2A\eta)^2 + 4A^2\eta^2)} \le \frac{2+3\epsilon}{\eta^{1+\epsilon}} \le \frac{3}{\eta^{1+\epsilon}}. \tag{2.8}$$

Combining (2.5) with (2.8), we have

$$\begin{aligned} |I_{12}| &\le \left| \int_4^{\infty} e^{i\eta - iA\eta^2 - A\eta^2} \frac{d\eta}{\eta^{\epsilon}} \right| \\ &\le \left| \frac{1}{(1-8A)4^{\epsilon} i + 2A4^{1+\epsilon}} \right| + \int_4^{+\infty} \left| \left(\frac{1}{-2A\eta^{1+\epsilon} + \eta^{\epsilon} i} \right)' \right| d\eta \\ &\le \frac{|1-8A|4^{\epsilon}}{(1-8A)^2 4^{2\epsilon} + 4A^2 4^{2+2\epsilon}} + \frac{4A^2 4^{2+2\epsilon}}{(1-8A)^2 4^{2\epsilon} + 4A^2 4^{2+2\epsilon}} + 3 \int_4^{+\infty} \eta^{-1-\epsilon} d\eta \\ &\le \frac{3}{2} + \frac{3}{\epsilon} \le \frac{6}{\epsilon}. \end{aligned} \tag{2.9}$$

When $\frac{1}{2A} \ge 2$, we have

$$I_1 := \int_0^{\infty} e^{i\eta - iA\eta^2 - A\eta^2} \frac{d\eta}{\eta^{\epsilon}} = \int_0^1 + \int_1^{\frac{1}{4A}} + \int_{\frac{1}{4A}}^{\frac{1}{A}} + \int_{\frac{1}{A}}^{\infty}. \tag{2.10}$$

We have

$$\left| \int_0^1 e^{i\eta - iA\eta^2 - A\eta^2} \frac{d\eta}{\eta^{\epsilon}} \right| \le \int_0^1 \eta^{-\epsilon} d\eta \le \frac{1}{1-\epsilon}. \tag{2.11}$$

Since $|1-2A\eta| \ge \frac{1}{2}$, we have

$$\begin{aligned} &\left| \int_1^{\frac{1}{4A}} e^{i\eta - iA\eta^2 - A\eta^2} \left(\frac{1}{(1-2A\eta)\eta^{\epsilon} i + 2A\eta^{1+\epsilon}} \right)' \right| \\ &\le \left| \int_1^{\frac{1}{4A}} \frac{2+3\epsilon}{\eta^{1+\epsilon}((1-2A\eta)^2 + 4A^2\eta^2)} d\eta \right| \le 12 \int_1^{\frac{1}{4A}} \frac{1}{\eta^{1+\epsilon}} d\eta \le \frac{12}{\epsilon}. \end{aligned} \tag{2.12}$$

Since $4A^2\eta^2 \ge \frac{1}{4}$, we have

$$\left| \int_{\frac{1}{4A}}^{\frac{1}{A}} e^{i\eta - iA\eta^2 - A\eta^2} \left(\frac{1}{(1-2A\eta)\eta^{\epsilon} i + 2A\eta^{1+\epsilon}} \right)' \right|$$

$$\leq \left| \int_{\frac{1}{4A}}^{\frac{1}{A}} \frac{2+3\epsilon}{\eta^{1+\epsilon}((1-2A\eta)^2+4A^2\eta^2)} d\eta \right| \leq 12 \int_{1}^{\frac{1}{4A}} \frac{1}{\eta^{1+\epsilon}} d\eta \leq \frac{12}{\epsilon}. \tag{2.13}$$

Since $A\eta \geq 1$, we have

$$\begin{aligned} &\left| \int_{\frac{1}{A}}^{\infty} e^{i\eta - iA\eta^2 - A\eta^2} \left(\frac{1}{(1-2A\eta)\eta^{\epsilon} i + 2A\eta^{1+\epsilon}} \right)' \right| \\ &\leq \left| \int_{\frac{1}{A}}^{\infty} \frac{2+3\epsilon}{\eta^{1+\epsilon}((1-2A\eta)^2+4A^2\eta^2)} d\eta \right| \leq \int_{\frac{1}{A}}^{\infty} \frac{1}{\eta^{1+\epsilon}} d\eta \leq \frac{1}{\epsilon}. \end{aligned} \tag{2.14}$$

For I_2, let $y = -\eta$, we have

$$I_2 := \int_0^{\infty} e^{-iy + iAy^2 - Ay^2} \frac{dy}{y^{\epsilon}}. \tag{2.15}$$

Thus, I_2 can be proved similarly to I_1. When $x < 0$, we have

$$I := \int_{\mathbf{R}} e^{i\eta + A\eta|\eta|} \frac{d\eta}{|\eta|^{\epsilon}} =: I_3 + I_4, \tag{2.16}$$

where $I_3 := \int_0^{\infty} e^{i\eta + A\eta^2} \frac{d\eta}{\eta^{\epsilon}}, I_4 := \int_{-\infty}^{0} e^{i\eta - A\eta^2} \frac{d\eta}{(-\eta)^{\epsilon}}$. Thus, I_3, I_4 can be proved similarly to I_1.

This completes the proof of Lemma 2.1.

3. Maximal function estimate

In this section, we establish the maximal function estimate.

Lemma 3.1. *For $f \in H^{\epsilon}(\mathbf{R})$, we have*

$$\|U_1(t)f\|_{L_x^1 L_t^{\infty}} \leq C \|f\|_{\dot{H}^{\epsilon}(\mathbf{R})}. \tag{3.1}$$

Proof. To prove Lemma 3.1, it suffices to prove

$$\left| \int_{-1}^{1} U_1(t(x)) f(x) dx \right| \leq C(\epsilon) \|f\|_{\dot{H}^{\epsilon}(\mathbf{R})} \tag{3.2}$$

for all measurable functions $t : \mathbf{R} \longrightarrow \mathbf{R}$. By using the Fubini's theorem and the Cauchy-Schwarz inequality as well as Lemma 2.1, we have

$$\begin{aligned} &\int_{-1}^{1} \int_{\mathbf{R}} e^{ix\xi - it(x)\xi|\xi| - t(x)\xi^2} \mathcal{F}_x f(\xi) d\xi dx \leq C \int_{\mathbf{R}} \mathcal{F}_x f(\xi) \left[\int_{-1}^{1} \int_{\mathbf{R}} e^{ix\xi - it(x)\xi|\xi| - t(x)\xi^2} dx \right] d\xi \\ &\leq C \left| \int_{\mathbf{R}} |\mathcal{F}_x f(\xi)|^2 |\xi|^{\epsilon} d\xi \right|^{\frac{1}{2}} \int_{\mathbf{R}} \left| \int_{-1}^{1} e^{ix\xi - it(x)\xi|\xi| - t(x)\xi^2} dx \right|^2 \frac{d\xi}{|\xi|^{\epsilon}} \\ &\leq C \|f\|_{\dot{H}^{\epsilon}(\mathbf{R})} \int_{-1}^{1} \int_{-1}^{1} \int_{\mathbf{R}} e^{i(x-y)\xi - it(x)\xi|\xi| - (t(x)+t(y))\xi^2} \frac{d\xi}{|\xi|^{\epsilon}} dx dy \\ &\leq C \|f\|_{\dot{H}^{\epsilon}(\mathbf{R})} \int_{-1}^{1} \int_{-1}^{1} |x-y|^{-1+\epsilon} dx dy \leq \frac{C}{\epsilon} \|f\|_{\dot{H}^{\epsilon}(\mathbf{R})}. \end{aligned} \tag{3.3}$$

This completes the proof of Lemma 3.1.

4. Proof of Theorem 1.1

In this section, we apply Lemma 2.1 and the density theorem to prove Theorem 1.1.

4.1. Proof of Theorem 1.1.

If f is rapidly decreasing function, $\forall\epsilon > 0$, then, we have

$$|U_1(t)f - f| \le |U_1(t)f - U_2(t)f| + |U_2(t)f - f| \le C|t| \int_{\mathbf{R}} |\xi|^2 \, |\mathcal{F}_x f(\xi)| \, d\xi \longrightarrow 0 \tag{4.1}$$

as $t \longrightarrow 0$. Here, $U_2(t) = \frac{1}{2}\int_{\mathbf{R}} e^{ix\xi} e^{-it|\xi|\xi} \mathcal{F}_x f(\xi) d\xi$.
When $f \in H^s(\mathbf{R})(s \ge \epsilon)$, by using the density theorem which can be seen in Lemma 2.2 of [10], there exists a rapidly decreasing function g such that $f = g + h$, where $\|h\|_{H^s(\mathbf{R})} < \epsilon(s \ge \epsilon)$. Thus, we have

$$\lim_{t\longrightarrow 0} |U_1(t)f - f| \le \lim_{t\longrightarrow 0} |U_1(t)g - g| + \lim_{t\longrightarrow 0} |U_1(t)h - h| \,. \tag{4.2}$$

We define $\mathrm{E}_\alpha = \left\{ x \in \mathbf{R} : \lim_{t\longrightarrow 0} |U_1(t)f - f| > \alpha \right\}$. Obviously, $E_\alpha \subset E_{1\alpha} \cup E_{2\alpha}$,

$$\mathrm{E}_{1\alpha} = \left\{ x \in \mathbf{R} : \lim_{t\longrightarrow 0} |U_1(t)g - g| > \frac{\alpha}{2} \right\}, \mathrm{E}_{2\alpha} = \left\{ x \in \mathbf{R} : \lim_{t\longrightarrow 0} |U_1(t)h - h| > \frac{\alpha}{2} \right\}. \tag{4.3}$$

Obviously,

$$E_\alpha \subset E_{1\alpha} \cup E_{2\alpha}. \tag{4.4}$$

From (4.1), we have

$$|E_{1\alpha}| = 0. \tag{4.5}$$

Obviously,

$$E_{2\alpha} \subset E_{21\alpha} \cup E_{22\alpha}, \tag{4.6}$$

where

$$E_{21\alpha} = \left\{ x \in \mathbf{R} : \sup_{t>0} |U_1(t)h| > \frac{\alpha}{4} \right\}, E_{22\alpha} = \left\{ x \in \mathbf{R} : |h| > \frac{\alpha}{4} \right\}. \tag{4.7}$$

Thus, from Lemma 3.1, we have

$$|E_{21\alpha}| = \int_{E_{21\alpha}} dx \le C \int_{E_{21\alpha}} \frac{\sup_{t>0} |U_1(t)h|}{\alpha} dx \le C\alpha^{-1} \|h\|_{L_x^1 L_t^\infty} \le C\alpha^{-1} \|f\|_{\dot{H}^\epsilon(\mathbf{R})} \,. \tag{4.8}$$

Obviously, we have

$$|E_{22\alpha}| \le \int_{E_{22\alpha}} dx \le C \int_{E_{22\alpha}} \frac{|h|^2}{\alpha^2} dx \le C\alpha^{-2} \|h\|_{L^2}^2. \tag{4.9}$$

From (4.5), (4.8) and (4.9), we have

$$|E_\alpha| \leq |E_{1\alpha}| + |E_{2\alpha}| \leq |E_{1\alpha}| + |E_{21\alpha}| + |E_{22\alpha}| \leq \frac{C\epsilon}{\alpha} + \frac{C\epsilon^2}{\alpha^2}. \tag{4.10}$$

Thus, for any $\alpha > 0$, from (4.10) we have

$$|E_\alpha| = 0. \tag{4.11}$$

Thus, we have

$$U_1(t)f - f \longrightarrow 0 \tag{4.12}$$

almost everywhere as t goes to zero.

This completes the proof of Theorem 1.1.

Conflict of interest

The authors declare there are no conflicts of interest.

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