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Research article

Refinements of Jensen's inequality and applications

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Abstract: The principal aim of this research work is to establish refinements of the integral Jensen's inequality. For the intended refinements, we mainly use the notion of convexity and the concept of majorization. We derive some inequalities for power and quasi–arithmetic means while utilizing the main results. Moreover, we acquire several refinements of Hölder inequality and also an improvement of Hermite–Hadamard inequality as consequences of obtained results. Furthermore, we secure several applications of the acquired results in information theory, which consist bounds for Shannon entropy, different divergences, Bhattacharyya coefficient, triangular discrimination and various distances.

Keywords: convex function; Jensen's inequality; majorization inequality; information theory
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1. Introduction

The field of mathematical inequalities and their applications has achieved a dynamic and exponential advancement within the last few decades with the dramatic affect on several areas of science [8, 9]. It is notable that, several inventive concepts about mathematical inequalities and their applications can be obtained with the help of convexity [1, 16, 21]. Among these mathematical inequalities, Jensen's inequality is one of the important inequality which has been made possible with the help of convexity [11, 17, 19]. This inequality has preserved some important structures and also there are a lot of inequalities which are the direct consequences of Jensen's inequality for example Hölder, Hermite–Hadamard, Ky Fan's and Young's inequalities etc [6, 16]. Jensen's inequality also performed a very significant role in statistics and many applications of this inequality have been observed involving estimations for different divergences [5, 12], several estimations for Zipf–Mandelbrot law [2, 3] and Shannon entropy [4].

In the following theorem, Jensen's integral inequality is stated:

Theorem 1.1. *Assume that I is an arbitrary interval in $\mathbb{R}$ and $\omega, \psi : [a, b] \to I$ are integrable functions such that $\omega > 0$. If $\Phi : I \to \mathbb{R}$ is a convex function and $\Phi \circ \psi$ is an integrable function on $[a, b]$, then*

$$\Phi\left(\frac{\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon}{\int_a^b \omega(\upsilon)d\upsilon}\right) \leq \frac{\int_a^b \omega(\upsilon)(\Phi \circ \psi)(\upsilon)d\upsilon}{\int_a^b \omega(\upsilon)d\upsilon}. \tag{1.1}$$

The inequality (1.1) *is valid in the opposite direction if the function Φ is concave on I.*

The concept of majorization is also an interesting topic for researchers since 1932 due to its important structure and properties [15]. Many researchers worked on this direction and a lot of results are devoted to this concept [16, 18]. Now, we are going to discuss some important literature about majorization. Let $\kappa \geq 2$ be a fixed natural number and

$$\mathbf{s}_1 = (\gamma_1, \gamma_2, \cdots, \gamma_\kappa), \quad \mathbf{s}_2 = (\delta_1, \delta_2, \cdots, \delta_\kappa)$$

be two $\kappa-$tuples with the real entries. Assume that

$$\gamma_{[1]} \geq \gamma_{[2]} \geq \cdots \geq \gamma_{[\kappa]}, \quad \delta_{[1]} \geq \delta_{[2]} \geq \cdots \geq \delta_{[\kappa]}$$

are the ordered components of the tuples $\mathbf{s}_1$ and $\mathbf{s}_2$ respectively.

Definition 1.2. The $\kappa-$tuple $\mathbf{s}_1$ is said to majorizes the $\kappa-$tuple $\mathbf{s}_2$ or $\mathbf{s}_2$ is said to be majorized by $\mathbf{s}_1$, if

$$\sum_{\ell=1}^{h} \gamma_{[\ell]} \geq \sum_{\ell=1}^{h} \delta_{[\ell]}, \quad h = 1, 2, \cdots, \kappa - 1$$

and

$$\sum_{\ell=1}^{\kappa} \gamma_\ell = \sum_{\ell=1}^{\kappa} \delta_\ell$$

are hold. In symbols, it is written as $\mathbf{s}_1 \succ \mathbf{s}_2$.

The following theorem is established for the majorized tuples while using convex function, which is famous in the literature as majorization theorem.

Theorem 1.3. *([15]) Let I be an interval in $\mathbb{R}$ and $\mathbf{s}_1, \mathbf{s}_2$ be two $\kappa-$tuples with entries in I. Then the inequality*

$$\sum_{j=1}^{\kappa} \Phi(\gamma_j) \geq \sum_{j=1}^{\kappa} \Phi(\delta_j) \tag{1.2}$$

is true for every continuous convex Φ on I if and only if $\mathbf{s}_1 \succ \mathbf{s}_2$.

The inequality (1.2) *is valid in the reverse direction if the function Φ is concave on I.*

For integrable functions, the definition of majorization can be stated as follows (see [18, p. 324]).

Definition 1.4. Assume that φ, $\psi : [a, b] \to \mathbb{R}$ are any two functions. Then the function φ is said to be majorized by the function ψ (abbreviated $\psi \succ \varphi$), if both φ, ψ are decreasing on $[a, b]$ and satisfy the conditions

$$\int_a^x \psi(\upsilon)d\upsilon \geq \int_a^x \varphi(\upsilon)d\upsilon, \qquad x \in [a, b] \tag{1.3}$$

and

$$\int_a^b \psi(\upsilon)d\upsilon \geq \int_a^b \varphi(\upsilon)d\upsilon. \tag{1.4}$$

The following theorem is the integral version of Theorem 1.3 (see [18, p. 325]).

Theorem 1.5. *Assume that φ, ψ are decreasing functions on $[a, b]$, then the inequality*

$$\int_a^b \Phi(\psi(\upsilon))d\upsilon \geq \int_a^b \Phi(\varphi(\upsilon))d\upsilon \tag{1.5}$$

holds for each continuous convex function $\Phi : [a, b] \to \mathbb{R}$ if and only if $\psi \succ \varphi$.

If Φ is concave on $[a, b]$, then (1.5) *holds in the opposite direction.*

In 1995, Maligranda et al. [14] presented the weighted version of (1.5), which is given in the following theorem.

Theorem 1.6. *Assume that $\Phi : I \to \mathbb{R}$ is convex function and $\varphi, \psi, \omega : [a, b] \to I$ are continuous functions such that $\omega(\upsilon) \geq 0$, for $\upsilon \in [a, b]$ and satisfying*

$$\int_a^x \omega(\upsilon)\psi(\upsilon)d\upsilon \geq \int_a^x \omega(\upsilon)\varphi(\upsilon)d\upsilon, \quad \textit{for} \quad x \in [a, b] \tag{1.6}$$

and

$$\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon = \int_a^b \omega(\upsilon)\varphi(\upsilon)d\upsilon. \tag{1.7}$$

If the function φ is decreasing on $[a, b]$, then

$$\int_a^b \omega(\upsilon)\Phi[\psi(\upsilon)]d\upsilon \geq \int_a^b \omega(\upsilon)\Phi[\varphi(\upsilon)]d\upsilon. \tag{1.8}$$

If the function Φ is concave on $[a, b]$, then (1.8) *is true in the reverse direction.*

In the following theorem, Dragomir [7] presented a weighted majorization inequality by taking certain integrable functions.

Theorem 1.7. *Assume that $\Phi : I \to \mathbb{R}$ is convex function and $\psi, \varphi, \omega : [a, b] \to I$ are continuous functions such that $\omega > 0$ on $[a, b]$. If φ and $\psi - \varphi$ are monotonic in the same direction and satisfying the condition* (1.7), *then* (1.8) *is true.*

In the following theorem, Dragomir [7] gave another majorization result under some relax conditions on ψ, φ and strict condition on Φ.

Theorem 1.8. *Assume that, the function $\Phi : I \to \mathbb{R}$ is non-decreasing and convex, $\psi, \varphi, \omega : [a, b] \to I$ are continuous functions such that $\omega > 0$ on $[a, b]$. Also, let φ and $\psi - \varphi$ be monotonic in the same direction. If*

$$\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon \geq \int_a^b \omega(\upsilon)\varphi(\upsilon)d\upsilon,$$

then (1.8) *is true.*

In the recent decades, the majorization become a very popular area for the researchers and obtained different generalizations [10, 13], extensions [1] and improvements [10, 23] of majorization type inequalities. Moreover, the majorization type inequalities have also been proved for other classes of convex functions [20, 22, 24].

The aim of the this research work is to obtained refinements of the celebrated Jensen inequality. The intended refinements are established by utilizing the theory of majorization and the notion of convexity. We present some fruitful consequences of the main results in the form of an improvement of Hermite–Hadamard inequality and refinements of Hölder inequality. Furthermore, we also presented some inequalities for quasi–arithmetic and power means as consequences of the obtained results. Moreover, we give several applications of the constructed results in information theory. These applications provide bounds for Csiszár divergence, Kullback–Leibler divergence, Shannon entropy, various distances, triangular discrimination and Bhattachayya coefficient.

2. Main results

This section of note is dedicated to the refinements of Jensen's inequality. The intended refinements will be made possible with the help of the notion of convexity and concept of majorization. We commence this section with the following result, in which a refinement is obtained for the Jensen inequality with the support of Theorem 1.6.

Theorem 2.1. *Assume that $\Phi : I \to \mathbb{R}$ is a convex function and $\varphi, \psi, \omega : [a, b] \to I$ are any integrable functions such that $\omega(\upsilon) > 0$, for all $\upsilon \in [a, b]$ with $\overline{\omega} = \int_a^b \omega(\upsilon)d\upsilon$. Also, assume that conditions* (1.6) *and* (1.7) *are valid and $\overline{\psi} = \frac{1}{\int_a^b \omega(\upsilon)d\upsilon} \int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon$. If $\lambda \in [0, 1]$ and the function φ is decreasing on $[a, b]$, then*

$$\Phi(\overline{\psi}) \leq \frac{\int_a^b \omega(\upsilon)\Phi\Big((1-\lambda)\overline{\psi} + \lambda\varphi(\upsilon)\Big)d\upsilon}{\overline{\omega}}$$

$$
\begin{aligned}
&\leq (1-\lambda)\Phi(\overline{\psi}) + \frac{\lambda}{\overline{\omega}}\int_a^b \omega(\upsilon)\Phi[\varphi(\upsilon)]d\upsilon \\
&\leq \frac{(1-\lambda)}{\overline{\omega}}\int_a^b \omega(\upsilon)\Phi[\psi(\upsilon)]d\upsilon + \frac{\lambda}{\overline{\omega}}\int_a^b \omega(\upsilon)\Phi[\varphi(\upsilon)]d\upsilon \\
&\leq \frac{1}{\overline{\omega}}\int_a^b \omega(\upsilon)\Phi[\psi(\upsilon)]d\upsilon. \qquad (2.1)
\end{aligned}
$$

The aforementioned inequality will be true in the reverse sense, if the function Φ *is concave.*

Proof. By utilizing the condition (1.7), we may write

$$
\Phi\left(\frac{1}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon\right)
= \Phi\left(\frac{1}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\left((1-\lambda)\frac{\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon}{\int_a^b \omega(\upsilon)d\upsilon} + \lambda\varphi(\upsilon)\right)d\upsilon\right). \qquad (2.2)
$$

Applying Jensen's inequality to the right hand side of (2.2), we get

$$
\Phi\left(\frac{1}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\left(\frac{(1-\lambda)}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon + \lambda\varphi(\upsilon)\right)d\upsilon\right)
$$

$$
\leq \frac{1}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\left(\frac{(1-\lambda)}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon + \lambda\varphi(\upsilon)\right)d\upsilon. \qquad (2.3)
$$

Now, using the definition of convex function on the right of (2.3), we acquire

$$
\frac{1}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\left(\frac{(1-\lambda)}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon + \lambda\varphi(\upsilon)\right)d\upsilon
$$

$$\leq (1-\lambda)\Phi\left(\frac{1}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon\right) + \frac{\lambda}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\varphi(\upsilon)\Big)d\upsilon. \tag{2.4}$$

Applying Jensen's inequality to the first term on the right hand side of (2.4), we obtain

$$\begin{aligned}
&(1-\lambda)\Phi\left(\frac{1}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon\right) + \frac{\lambda}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\varphi(\upsilon)\Big)d\upsilon \\
&\quad\leq \frac{1-\lambda}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\psi(\upsilon)\Big)d\upsilon + \frac{\lambda}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\varphi(\upsilon)\Big)d\upsilon.
\end{aligned} \tag{2.5}$$

Now, utilizing Theorem 1.6 on the right hand side of (2.5), we get

$$\begin{aligned}
&\frac{1-\lambda}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\psi(\upsilon)\Big)d\upsilon + \frac{\lambda}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\varphi(\upsilon)\Big)d\upsilon \\
&\quad\leq \frac{1-\lambda}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\psi(\upsilon)\Big)d\upsilon + \frac{\lambda}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\psi(\upsilon)\Big)d\upsilon \\
&\quad= \frac{1}{\int_a^b \omega(\upsilon)d\upsilon}\int_a^b \omega(\upsilon)\Phi\Big(\psi(\upsilon)\Big)d\upsilon.
\end{aligned} \tag{2.6}$$

From (2.3) − (2.6), we obtain (2.1). □

Corollary 1. *Suppose that all the assumptions of Theorem 2.1 hold. Moreover, if $\varphi(\upsilon) = \psi(\upsilon)$ for all $\upsilon \in [a, b]$, then*

$$\begin{aligned}
\Phi(\overline{\psi}) &\leq \frac{1}{\overline{\omega}}\int_a^b \omega(\upsilon)\Phi\Big((1-\lambda)\overline{\psi} + \lambda\psi(\upsilon)\Big)d\upsilon \\
&\leq (1-\lambda)\Phi(\overline{\psi}) + \frac{\lambda}{\overline{\omega}}\int_a^b \omega(\upsilon)\Phi[\psi(\upsilon)]d\upsilon \\
&\leq \frac{1}{\overline{\omega}}\int_a^b \omega(\upsilon)\Phi[\psi(\upsilon)]d\upsilon.
\end{aligned} \tag{2.7}$$

If the function Φ is concave, then the inequality (2.7) *will become positive in opposite sense.*

Proof. Since, if we put $\varphi(\upsilon) = \psi(\upsilon)$ for $\upsilon \in [a, b]$, then all the conditions of Theorem 2.1 are satisfied. Therefore, using (2.1) by putting $\varphi(\upsilon) = \psi(\upsilon)$, we obtain (2.7). $\square$

In the following theorem, we obtain the inequalities given in (2.1) while using Theorem 1.7 instead of Theorem 1.6.

Theorem 2.2. *Assume that, the function $\Phi : I \to \mathbb{R}$ is convex and $\psi, \varphi, \omega : [a, b] \to I$ are continuous functions such that $\omega(\upsilon) > 0$ for all $\upsilon \in [a, b]$. Also, assume that φ and $\psi - \varphi$ are monotonic in the parallel direction and satisfying*

$$\int_a^b \omega(\upsilon)\psi(\upsilon)d\upsilon = \int_a^b \omega(\upsilon)\varphi(\upsilon)d\upsilon. \tag{2.8}$$

If $\lambda \in [0, 1]$, then the inequalities given in (2.1) *are valid.*

Proof. By adopting the idea used in the proof of Theorem 2.1 along with the result of Theorem 1.7, we acquire (2.1). $\square$

Remark 1. Inequalities given in (2.1) can also be obtained under the conditions discussed in Theorem 1.8.

In the following result, we obtain refinements of the Hölder inequality.

Corollary 2. *Suppose that, the functions ρ, g_1, g_2 are non-negative on the interval $[a, b]$. Also, let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\lambda \in [0, 1]$. Then the following statements are true:*

(*i*) *If $\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon > 0$, then*

$$\begin{aligned}
\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon &\le \left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}} \\
&\quad\times\left[\int_a^b \rho(\upsilon)g_2^q(\upsilon)\left((1-\lambda)\frac{\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon}{\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon} + \lambda g_1(\upsilon)g_2^{\frac{-q}{p}}(\upsilon)\right)^p\right]^{\frac{1}{p}} \\
&\le \Bigg[(1-\lambda)\left(\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon\right)^p \\
&\qquad+\lambda\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{p-1}\Bigg]^{\frac{1}{p}} \\
&\le \left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.9}$$

(*ii*) *If* $\int_a^b \rho(\upsilon) g_1^p(\upsilon) d\upsilon > 0$, *then*

$$\int_a^b \rho(\upsilon) g_1(\upsilon) g_2(\upsilon) d\upsilon \le \left(\int_a^b \rho(\upsilon) g_1^p(\upsilon) d\upsilon \right)^{\frac{1}{p}}$$

$$\times \left[\int_a^b \rho(\upsilon) g_1^p(\upsilon) \left((1-\lambda) \frac{\int_a^b \rho(\upsilon) g_1(\upsilon) g_2(\upsilon) d\upsilon}{\int_a^b \rho(\upsilon) g_1^p(\upsilon) d\upsilon} + \lambda g_2(\upsilon) g_1^{\frac{-p}{q}}(\upsilon) \right)^q \right]^{\frac{1}{q}}$$

$$\le \left[(1-\lambda) \left(\int_a^b \rho(\upsilon) g_1(\upsilon) g_2(\upsilon) d\upsilon \right)^q \right.$$

$$\left. + \lambda \left(\int_a^b \rho(\upsilon) g_2^q(\upsilon) d\upsilon \right) \left(\int_a^b \rho(\upsilon) g_1^p(\upsilon) d\upsilon \right)^{q-1} \right]^{\frac{1}{q}}$$

$$\le \left(\int_a^b \rho(\upsilon) g_1^p(\upsilon) d\upsilon \right)^{\frac{1}{p}} \left(\int_a^b \rho(\upsilon) g_2^q(\upsilon) d\upsilon \right)^{\frac{1}{q}}. \tag{2.10}$$

(*iii*) *In the case, when* $0 < p < 1$, $q = \frac{p}{p-1}$ *and* $\int_a^b \rho(\upsilon) g_2^q(\upsilon) d\upsilon > 0$, *then*

$$\left(\int_a^b \rho(\upsilon) g_1^p(\upsilon) d\upsilon \right)^{\frac{1}{p}} \left(\int_a^b \rho(\upsilon) g_2^q(\upsilon) d\upsilon \right)^{\frac{1}{q}}$$

$$\ge \int_a^b \rho(\upsilon) g_2^q(\upsilon) \left((1-\lambda) \frac{\int_a^b \rho(\upsilon) g_1^p(\upsilon) d\upsilon}{\int_a^b \rho(\upsilon) g_2^q(\upsilon) d\upsilon} + \lambda g_1^p(\upsilon) g_2^{(p-pq)}(\upsilon) \right)^{\frac{1}{p}} d\upsilon$$

$$\ge \left(\int_a^b \rho(\upsilon) g_2^q(\upsilon) d\upsilon \right)^{\frac{1}{q}} \left[(1-\lambda) \left(\int_a^b \rho(\upsilon) g_1^p(\upsilon) d\upsilon \right) \right.$$

$$\left. + \lambda \left(\int_a^b \rho(\upsilon) g_2^q(\upsilon) d\upsilon \right)^q \left(\int_a^b \rho(\upsilon) g_1 g_2(\upsilon) d\upsilon \right) \right]$$

$$\ge \int_a^b \rho(\upsilon) g_1 g_2(\upsilon) d\upsilon. \tag{2.11}$$

(iv) *In the case, when* $p < 0$ *and* $\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon > 0$, *then*

$$\left(\int_\epsilon^\eta \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}}$$

$$\geq \int_a^b \rho(\upsilon)g_1^p(\upsilon)\left((1-\lambda)\frac{\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon}{\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon} + \lambda g_2^q(\upsilon)g_1^{(p-pq)}(\upsilon)\right)^{\frac{1}{q}} d\upsilon$$

$$\geq (1-\lambda)\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_1g_2(\upsilon)d\upsilon\right)^{\frac{1}{q}}$$

$$+\lambda\left(\int_a^b \rho(\upsilon)g_1g_2(\upsilon)d\upsilon\right) \geq \int_a^b \rho(\upsilon)g_1g_2(\upsilon)d\upsilon. \tag{2.12}$$

Proof. First, we prove (i). Since the function $\Phi(x) = \upsilon^p$ is convex for $\upsilon > 0$. Therefore, applying inequality (2.7) by choosing $\Phi(\upsilon) = \upsilon^p$, $\omega(\upsilon) = \rho(\upsilon)g_2^q(\upsilon)$ and $\psi(\upsilon) = g_1(\upsilon)g_2^{-\frac{q}{p}}(\upsilon)$, we get (2.9).

Now, we prove case (ii). For this, applying inequality (2.7) while selecting $\Phi(\upsilon) = \upsilon^q$, $\upsilon > 0$, $\omega(\upsilon) = \rho(\upsilon)g_1^p(\upsilon)$ and $\psi(\upsilon) = g_2(\upsilon)g_1^{-\frac{p}{q}}(\upsilon)$, we get (2.10).

Instantly, we prove inequality (2.11). If $0 < p < 1$, then clearly, both $\frac{1}{p}$ and $(1-p)^{-1}$ are positive and their sum is one. Therefore using (2.9) by putting $p = \frac{1}{p}$, $q = (1-p)^{-1}$, $g_1 = (g_1g_2)^p$ and $g_2 = g_2^{-p}$, we get (2.11).

Now, we prove the last case. Since $p < 0$. Therefore $q = \frac{p}{p-1} \in (0,1)$. Thus, utilizing (2.10) by putting $p = (1-q)^{-1}$, $q = \frac{1}{q}$, $g_1 = g_1^{-q}$ and $g_2 = (g_1g_2)^q$, we obtain (2.12). □

In following corollary, we establish some more refinements of the Hölder inequality.

Corollary 3. *Assume that* $p, q > 1$ *such that* $\frac{1}{p}+\frac{1}{q} = 1$. *Also, assume that* ρ, g_1 *and* g_2 *are non-negative on* $[a,b]$ *such that* $\rho g_1^p, \rho g_2^q, \rho g_1g_2 \in L^1[a,b]$ *and* $\lambda \in [0,1]$, *then the following statements are valid:*

(i) *If* $\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon > 0$, *then*

$$\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}}$$

$$\geq \int_a^b \rho(\upsilon)g_2^q(\upsilon)\left((1-\lambda)\frac{\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon}{\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon} + \lambda g_1^p(\upsilon)g_2^{-q}(\upsilon)\right)^{\frac{1}{p}} d\upsilon$$

$$\begin{aligned}
&\geq (1-\lambda)\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}} \\
&\quad +\lambda\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon \\
&\geq \int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon.
\end{aligned} \tag{2.13}$$

(ii) If $\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon > 0$, *then*

$$\begin{aligned}
&\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}} \\
&\geq \int_a^b \rho(\upsilon)g_1^p(\upsilon)\left((1-\lambda)\frac{\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon}{\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon}+\lambda g_1^{-p}(\upsilon)g_2^q(\upsilon)\right)^{\frac{1}{p}}d\upsilon \\
&\geq (1-\lambda)\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}} \\
&\quad +\lambda\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon \\
&\geq \int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon.
\end{aligned} \tag{2.14}$$

(iii) In the situation, when $p \in (0,1)$ *and* $q = \frac{p}{p-1}$ *with* $\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon > 0$, *then*

$$\begin{aligned}
&\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon \leq \left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}} \\
&\times\left[\int_a^b \rho(\upsilon)g_2^q(\upsilon)\left((1-\lambda)\frac{\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon}{\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon}+\lambda g_1(\upsilon)g_2^{1-q}(\upsilon)\right)^p\right]^{\frac{1}{p}} \\
&\leq \left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}}\left[(1-\lambda)\left(\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon\right)^p\right.
\end{aligned}$$

$$\times\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{1-p}+\lambda\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)\Bigg]^{\frac{1}{p}}$$

$$\leq\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}}. \tag{2.15}$$

(iv) *In the case, when* $p<0$ *and* $\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon>0$, *then*

$$\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon\leq\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}$$

$$\times\left[\int_a^b \rho(\upsilon)g_1^p(\upsilon)\left((1-\lambda)\frac{\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon}{\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon}+\lambda g_1^{1-q}g_2(\upsilon)(\upsilon)\right)^{1-q}\right]^{\frac{1}{q}}$$

$$\leq\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\Bigg[(1-\lambda)\left(\int_a^b \rho(\upsilon)g_1(\upsilon)g_2(\upsilon)d\upsilon\right)^{q}$$

$$\times\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{1-q}+\lambda\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)\Bigg]^{\frac{1}{q}}$$

$$\leq\left(\int_a^b \rho(\upsilon)g_1^p(\upsilon)d\upsilon\right)^{\frac{1}{p}}\left(\int_a^b \rho(\upsilon)g_2^q(\upsilon)d\upsilon\right)^{\frac{1}{q}}. \tag{2.16}$$

Proof. First, we prove inequality (2.13). Since the function $\Phi(\upsilon)=\upsilon^{\frac{1}{p}}$ is concave on $(0,\infty)$. Therefore, applying inequality (2.7) for $\Phi(\upsilon)=\upsilon^{\frac{1}{p}}$, $\omega(\upsilon)=\rho(\upsilon)g_2^q(\upsilon)$ and $\psi(\upsilon)=g_1^p(\upsilon)g_2^{-q}(\upsilon)$, we obtain (2.13).

Instantly, we prove (2.14). For this, utilizing inequality (2.7) by choosing $\Phi(\upsilon)=\upsilon^{\frac{1}{q}}$, $\upsilon>0$, $\omega=\rho g_1^q$ and $\psi=g_1^{-p}g_2^q$.

For the prove of (2.15), utilizing (2.13) by putting $p=\frac{1}{p}$, $q=(1-p)^{-1}$, $g_1=(g_1g_2)^p$ and $g_2=g_2^{-p}$.

Now, prove the last inequality. Since $p<0$. Therefore $q=\frac{p}{q-1}\in(0,1)$. Clearly this case is the reflection of case (iii). Instantly, utilizing (2.14) while taking $(1-q)^{-1}$, q, g_1^{-p}, $(g_1g_2)^q$ as a substitute of p, q, g_1, g_2 respectively, we acquire (2.16). □

Now, we give definitions of power and quasi means.

Definition 2.3. Suppose that ω and g are positive and integrable functions on $[a, b]$. Then the power mean of order $p \in \mathbb{R}$ is defined as:

$$\mathcal{M}(\omega, g) = \begin{cases} \left(\dfrac{\int_a^b \omega(\upsilon) g^p(\upsilon) d\upsilon}{\int_a^b \omega(\upsilon) d\upsilon} \right)^{\frac{1}{p}}, & \text{if} \quad p \neq 0, \\ \exp\left(\dfrac{\int_a^b \omega(\upsilon) \log g(\upsilon) d\upsilon}{\int_a^b \omega(\upsilon) d\upsilon} \right), & \text{if} \quad p = 0. \end{cases} \tag{2.17}$$

Definition 2.4. Suppose that, the functions g, ω are positive and integrable on $[a, b]$. Also, suppose that the function h is continuous and strictly monotonic on $(0, \infty)$, then the quasi–arithmetic mean is defined by:

$$\mathcal{M}_h(\omega, g) = h^{-1} \left(\frac{\int_a^b \omega(\upsilon) h(g(\upsilon)) d\upsilon}{\int_a^b \omega(\upsilon) d\upsilon} \right). \tag{2.18}$$

In the following corollary, we obtain inequalities for the power mean with the help of Corollary 1.

Corollary 4. *Suppose that g, ω are positive and integrable functions on $[a, b]$ with $\overline{\omega} := \int_a^b \omega(\upsilon) d\upsilon$. If $\lambda \in [0, 1]$ and $s, t \in \mathbb{R} \setminus \{0\}$ such that $t \geq s$, then*

$$\begin{aligned} \mathcal{M}_s(\omega; g) &\leq \left[\frac{\int_a^b \left((1 - \lambda) \mathcal{M}_s^s(\omega; g) + \lambda g^s(\upsilon) \right)^{\frac{t}{s}} d\upsilon}{\overline{\omega}} \right]^{\frac{1}{t}} \\ &\leq \left((1 - \lambda) \mathcal{M}_s^t(\omega; g) + \lambda \mathcal{M}_t^t(\omega; g) \right)^{\frac{1}{t}} \\ &\leq \mathcal{M}_t(\omega; g), \qquad t \neq 0, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \mathcal{M}_s(\omega; g) &\leq \frac{1}{s} \int_a^b \log \left((1 - \lambda) \mathcal{M}_s^s(\omega; g) + \lambda g^s(\upsilon) \right) d\upsilon \\ &\leq (1 - \lambda) \log \mathcal{M}_s(\omega; g) + \lambda \log \mathcal{M}_t(\omega; g) \\ &\leq \mathcal{M}_t(\omega; g), \qquad t = 0, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \mathcal{M}_t(\omega; g) &\leq \left[\frac{\int_a^b \left((1 - \lambda) \mathcal{M}_t^t(\omega; g) + \lambda g^t(\upsilon) \right)^{\frac{s}{t}} d\upsilon}{\overline{\omega}} \right]^{\frac{1}{s}} \\ &\leq \left((1 - \lambda) \mathcal{M}_t^s(\omega; g) + \lambda \mathcal{M}_s^s(\omega; g) \right)^{\frac{1}{s}} \\ &\leq \mathcal{M}_s(\omega; g), \qquad s \neq 0, \end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathcal{M}_t(\omega; g) &\leq \frac{1}{t}\int_a^b \log\Big((1-\lambda)\mathcal{M}_t^t(\omega; g) + \lambda g^t(\upsilon)\Big)d\upsilon \\ &\leq (1-\lambda)\log\mathcal{M}_t(\omega; g) + \lambda\log\mathcal{M}_s(\omega; g) \\ &\leq \mathcal{M}_s(\omega; g), \qquad s = 0. \end{aligned} \tag{2.22}$$

Proof. Since $t, s \in \mathbb{R}$ and $s, t \neq 0$. Therefore, utilizing (2.7) for $\Phi(\upsilon) = \upsilon^{\frac{t}{s}}$ $(\upsilon > 0)$, $\psi = g^s$ and then taking power $\frac{1}{t}$, we get (2.19). For the case of $t = 0$, taking limits as $t \to 0$ of (2.19), we obtain (2.20).

Similarly, using (2.7) by choosing $\Phi(\upsilon) = \upsilon^{\frac{s}{t}}$, $\psi = g^t$ and then taking power $\frac{1}{s}$, we get (2.21). When $s = 0$, taking limits of (2.21) as $s \to 0$, we acquire (2.22). □

With the help of Corollary 1, we get inequalities for the quasi means, which are stated in the following corollary.

Corollary 5. *Suppose that $\omega, \psi : [a, b] \to I$ are integrable functions such that $\omega > 0$ and $\overline{\omega} := \int_a^b \omega(\upsilon)d\upsilon$ and h is strictly monotonic and continuous function on I. If $\Phi \circ h^{-1}$ is convex function and $\lambda \in [0, 1]$, then*

$$\begin{aligned}&\Phi\Big(\mathcal{M}_h(\omega, \psi)\Big) \\ \leq\ &\frac{\int_a^b \omega(\upsilon)\Phi\Bigg(h^{-1}\Bigg(\frac{(1-\lambda)\int_a^b \omega(\upsilon)h\big(\psi(\upsilon)\big)d\upsilon}{\overline{\omega}} + \lambda\, h(\psi(\upsilon))\Bigg)\Bigg)d\upsilon}{\overline{\omega}} \\ \leq\ &(1-\lambda)\Phi\Big(\mathcal{M}_h(\omega, \psi)\Big) + \frac{\int_a^b \omega(\upsilon)\Phi(\psi(\upsilon))d\upsilon}{\overline{\omega}} \\ \leq\ &\frac{\int_a^b \omega(\upsilon)\Phi(\psi(\upsilon))d\upsilon}{\overline{\omega}}. \end{aligned} \tag{2.23}$$

Proof. Substituting $\psi = h \circ \psi$ and $\Phi = \Phi \circ h^{-1}$ in (2.7), we obtain (2.23). □

In the following corollary, we obtain a refinement of the Hermite–Hadamard inequality with the support of Corollary 1.

Corollary 6. *Suppose that the function $\Phi : [a, b] \to \mathbb{R}$ is convex and $\lambda \in [0, 1]$, then*

$$\begin{aligned}\Phi\Big(\frac{a+b}{2}\Big) &\leq \frac{\int_a^b \Phi\Big((1-\lambda)(\frac{a+b}{2}) + \lambda\upsilon\Big)d\upsilon}{b-a} \\ &\leq (1-\lambda)\Phi\Big(\frac{a+b}{2}\Big) + \lambda\frac{\int_a^b \Phi(\upsilon)d\upsilon}{b-a} \\ &\leq \frac{\int_a^b \Phi(\upsilon)d\upsilon}{b-a}. \end{aligned} \tag{2.24}$$

Proof. Using Corollary 1, for $\omega(\upsilon) = 1$ and $\psi(\upsilon) = \upsilon$ for all $\upsilon \in [a, b]$, we obtain (2.24). □

3. Applications in information theory

In this section, we are going to present some important applications of our main results in the information theory. These applications consists the refinements of different divergences, distances, Bhattacharyya coefficient, triangular discrimination and Shannon entropy.

In order to go ahead first we give the definitions of Shannon entropy and different divergences.

Definition 3.1. Let $\Phi : (0, \infty) \to \mathbb{R}$ be a convex function and $\psi, \varphi : [a, b] \to (0, \infty)$ be two positive integrable functions, then the Csiszár divergence is defined as:

$$C_d(\psi, \varphi) = \int_a^b \varphi(\upsilon)\Phi\left(\frac{\psi(\upsilon)}{\varphi(\upsilon)}\right) d\upsilon.$$

Definition 3.2. Let $\psi : [a, b] \to (0, \infty)$ be a probability density function. Then the Shannon entropy is defined by:

$$SE(\psi) = -\int_a^b \psi(\upsilon) \log \psi(\upsilon) d\upsilon.$$

Definition 3.3. Let $\psi, \varphi : [a, b] \to (0, \infty)$ be two probability densities. Then the Kullback–Leibler divergence is given by:

$$KL_d(\psi, \varphi) = \int_a^b \psi(\upsilon) \log\left(\frac{\psi(\upsilon)}{\varphi(\upsilon)}\right) d\upsilon.$$

Theorem 3.4. *Suppose that, the function $\Psi : (0, \infty) \to \mathbb{R}$ is convex and $\phi_1, \phi_2 : [a, b] \to (0, \infty)$ are two integrable functions. If $\lambda \in [0, 1]$, then*

$$\begin{aligned}
&\Psi\left(\frac{\int_a^b \phi_1(\upsilon)d\upsilon}{\int_a^b \phi_2(\upsilon)d\upsilon}\right)\int_a^b \phi_2(\upsilon)d\upsilon \\
&\quad \leq \int_a^b \phi_2(\upsilon)\Psi\left((1-\lambda)\frac{\int_a^b \phi_1(\upsilon)d\upsilon}{\int_a^b \phi_2(\upsilon)d\upsilon} + \lambda\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}\right)d\upsilon \\
&\quad \leq (1-\lambda)\Psi\left(\frac{\int_a^b \phi_1(\upsilon)d\upsilon}{\int_a^b \phi_2(\upsilon)d\upsilon}\right)\int_a^b \phi_2(\upsilon)d\upsilon + \lambda C_d(\phi_1, \phi_2) \\
&\quad \leq C_d(\phi_1, \phi_2). \qquad (3.1)
\end{aligned}$$

Proof. Using Corollary 1 by putting $\omega = \phi_2$, $\psi = \frac{\phi_1}{\phi_2}$ and $\Phi = \Psi$, we get (3.1). $\square$

Corollary 7. *Suppose that* $\phi_1, \phi_2 : [a, b] \to (0, \infty)$ *are integrable functions on* $[a, b]$. *If* ϕ_2 *is probability density function and* $\lambda \in [0, 1]$, *then*

$$\begin{aligned}
\log\left(\int_a^b \phi_1(\upsilon)d\upsilon\right) &\geq \int_a^b \phi_2(\upsilon)\log\left((1-\lambda)\int_a^b \phi_1(\upsilon)d\upsilon + \lambda\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}\right)d\upsilon \\
&\geq (1-\lambda)\log\left(\int_a^b \phi_1(\upsilon)d\upsilon\right) \\
&\quad +\lambda\int_a^b \phi_2(\upsilon)\log\phi_1(\upsilon)d\upsilon + \lambda SE(\phi_2) \\
&\geq \int_a^b \phi_2(\upsilon)\log\phi_1(\upsilon)d\upsilon + SE(\phi_2). \qquad (3.2)
\end{aligned}$$

Proof. Choosing $\Psi(\upsilon) = -\log(\upsilon)$ in (3.1), we obtain (3.2). $\square$

Corollary 8. *Suppose that* $\phi_1, \phi_2 : [a, b] \to (0, \infty)$ *are probability density functions such that their integral exits on* $[a, b]$. *If* $\lambda \in [0, 1]$, *then*

$$\begin{aligned}
0 &\leq \int_a^b \phi_2(\upsilon)\left(1+\lambda\Big(\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}-1\Big)\right) \\
&\quad \times\log\left(1+\lambda\Big(\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}-1\Big)\right)d\upsilon \\
&\leq \lambda KL_d(\phi_1, \phi_2) \leq KL_d(\phi_1, \phi_2). \qquad (3.3)
\end{aligned}$$

Proof. Taking $\Psi(\upsilon) = \upsilon\log\upsilon$, $\upsilon > 0$ in (3.1), we acquire (3.3). $\square$

Now, we give the definitions of different distances.

Definition 3.5. Let $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ be positive probability densities on $[a, b]$. Then the variational distance is defined by:

$$V_d(\phi_1, \phi_2) = \int_a^b |\phi_1(\upsilon) - \phi_2(\upsilon)|d\upsilon.$$

Definition 3.6. Let $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ be two positive probability densities. Then the Jeffrey's distance is defined by:

$$J_d(\phi_1, \phi_2) = \int_a^b (\phi_1(\upsilon) - \phi_2(\upsilon))\log\left(\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}\right)d\upsilon.$$

Definition 3.7. Let $\phi_1, \phi_2 : [a,b] \to (0,\infty)$ be two probability densities. Then the Hellinger distance is defined by:

$$H_d(\phi_1,\phi_2) = \int_a^b \left(\sqrt{\phi_1(\upsilon)} - \sqrt{\phi_2(\upsilon)}\right)^2 d\upsilon.$$

Corollary 9. *Suppose that $\phi_1, \phi_2 : [a,b] \to (0,\infty)$ are probability densities such that their integral exist on $[a,b]$. If $\lambda \in [0,1]$, then*

$$0 \leq \lambda \int_a^b \left|\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)} - 1\right| d\upsilon \leq \lambda V_d(\phi_1,\phi_2) \leq V_d(\phi_1,\phi_2). \tag{3.4}$$

Proof. Utilizing (3.1) by choosing $\Psi(\upsilon) = |\upsilon - 1|$, $\upsilon \in (0,\infty)$, we obtain (3.4). □

Corollary 10. *Suppose that, all the hypotheses of Corollary 9 hold, then*

$$\begin{aligned} 0 &\leq \int_a^b \left(\phi_1(\upsilon) - \phi_2(\upsilon)\right) \log\left(1 + \lambda\left(\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)} 1\right)\right) d\upsilon \\ &\leq \lambda J_d(\phi_1,\phi_2) \leq J_d(\phi_1,\phi_2). \end{aligned} \tag{3.5}$$

Proof. Using (3.1) for $\Psi(\upsilon) = (\upsilon - 1)\log \upsilon$, $\upsilon \in (0,\infty)$, we get (3.5). □

Corollary 11. *Suppose that, all the assumptions of Corollary 9 hold, then*

$$\begin{aligned} 0 &\leq \int_a^b \phi_2(\upsilon) \left(\sqrt{1 + \lambda\left(\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}\right)} - 1\right)^2 d\upsilon \\ &\leq \lambda H_d(\phi_1,\phi_2) \leq H_d(\phi_1,\phi_2). \end{aligned} \tag{3.6}$$

Proof. Applying Theorem 3.4 by choosing $\Psi(\upsilon) = (\sqrt{\upsilon} - 1)^2$, $\upsilon > 0$, we obtain (3.6). □

Now, we define the Bhattachayya coefficient and Triangular discrimination.

Definition 3.8. Let $\phi_1, \phi_2 : [a,b] \to (0,\infty)$ be any probabilities densities. Then the Bhattacharyya coefficient is defined by:

$$B_d(\phi_1,\phi_2) = \int_a^b \sqrt{\phi_1(\upsilon) - \phi_2(\upsilon)} d\upsilon.$$

Definition 3.9. Let ϕ_1, ϕ_2 be any positive probability densities functions on $[a,b]$. Then the triangular discrimination is defined by:

$$T_d(\phi_1,\phi_2) = \int_a^b \frac{\left(\phi_1(\upsilon) - \phi_2(\upsilon)\right)^2}{\phi_1(\upsilon) - \phi_2(\upsilon)} d\upsilon.$$

Corollary 12. *Suppose that, all the conditions of Corollary 9 are valid, then*

$$\begin{aligned} 1 &\geq \int_a^b \phi_2(\upsilon)\sqrt{1+\lambda\Big(\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}-1\Big)}d\upsilon \\ &\geq 1+\lambda\Big(B_d(\phi_1,\phi_2)-1\Big) \geq B_d(\phi_1,\phi_2). \end{aligned} \tag{3.7}$$

Proof. Using $\Psi(\upsilon) = -\sqrt{\upsilon}$, $\upsilon > 0$ in (3.1), we get (3.7). □

Corollary 13. *Suppose that, all the assumptions of Corollary 9 hold, then*

$$0 \leq \lambda^2 \int_a^b \frac{\Big(\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}-1\Big)^2}{2+\lambda\Big(\frac{\phi_1(\upsilon)}{\phi_2(\upsilon)}-1\Big)}d\upsilon \leq \lambda T_d(\phi_1,\phi_2) \leq T_d(\phi_1,\phi_2). \tag{3.8}$$

Proof. Applying Theorem 3.4 by choosing $\Psi(\upsilon) = \frac{(\upsilon-1)^2}{\upsilon+1}$, $\upsilon > 0$, we obtain (3.8). □

4. Conclusions

The Jensen inequality has recorded an exponential growth in the last few decades due to its remarkable properties. Several important inequalities such like Hölder, Hermite–Hadmard and Ky Fan's inequalities etc can easily be deduced from this inequality. This article is devoted to refinements of Jensen's inequality and its applications. We acquired the refinements of this inequality with the help of majorization results and convex functions. We utilized some certain functions with majorization conditions and obtained new refinements of Jensen's inequality. Furthermore, we utilized the obtained refinements and gave inequalities for the power as well as quasi–arithmetic means. Moreover, we also acquired refinements of Hölder and Hermite–Hadamard inequalities with the help of obtained refinements. In addition to this, we also presented some applications of the obtained refinements in the information theory. These applications includes, bounds for the different divergences, Shannon entropy, Bhattacharyya coefficient, various distances and triangular discrimination. The results given in the present article will give an addition to the mathematical inequalities and especially to Jensen's inequality.

Conflict of interest

The authors declare that they have no competing interest.

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