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Research article

Some families of differential equations associated with the Gould-Hopper-Frobenius-Genocchi polynomials

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Abstract: The basic objective of this paper is to utilize the factorization technique method to derive several properties such as, shift operators, recurrence relation, differential, integro-differential, partial differential expressions for Gould-Hopper-Frobenius-Genocchi polynomials, which can be utilized to tackle some new issues in different areas of science and innovation.

Keywords: recurrence relation; shift operators; differential equation; integro-differential equation; partial differential equation
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1. Introduction

The investigation of factorization technique is seen as the similarity among Maxwell's and Dirac's equations. They are both linear systems of equations and each of them contains partial derivatives of the first order. Also, both Maxwell's and Dirac's equations are Lorentz invariant. It may be remarked that in the case of Maxwell's equations, the linearity may be an over-simplification which leads to the difficulties with infinite self-energies.

An operational procedure which provides answers to the questions about eigenvalue problems and which are of significant importance to physicists is known as the factorization method [10]. The basic idea is to consider a pair of first-order differential equations, which on operating gives an equivalent second-order differential equation. The manufacturing process is also used for the calculation of transition probabilities. The method is generalized so that it will handle perturbation problems.

Let $\{l_m(u)\}_{m=0}^{\infty}$ be a sequence of polynomials such that

$$\deg(l_m(u)) = m, \ (m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}).$$

The sequences of two differential operators η_m^- and η_m^+, for $m = 0, 1, 2, \cdots$ are defined as:

$$\eta_m^-(l_m(u)) = l_{m-1}(u) \tag{1.1}$$

and

$$\eta_m^+(l_m(u)) = l_{m+1}(u), \tag{1.2}$$

are called the derivative and multiplicative operators, respectively.

A significant property known as the differential equation

$$(\eta_{m+1}^-\eta_m^+)\{l_m(u)\} = l_m(u), \tag{1.3}$$

is obtained on using these above mentioned operators η_m^- and η_m^+. The technique used in obtaining differential equations via Eq (1.3) is known as the factorization method. The main idea behind the factorization method is to find the derivative operator η_m^- and multiplicative operator η_m^+ such that the Eq (1.3) holds.

The iterations of η_m^- and η_m^+ to $l_m(u)$ provide the following relations:

$$(\eta_{m+1}^-\eta_m^+)l_m(u) = l_m(u); \tag{1.4}$$

$$(\eta_{m-1}^+\eta_m^-)l_m(u) = l_m(u); \tag{1.5}$$

$$(\eta_1^-\eta_2^-\eta_3^- \cdots \eta_{m-1}^-\eta_m^-)l_m(u) = l_0(u); \tag{1.6}$$

$$(\eta_{m-1}^+\eta_{m-2}^+\eta_{m-3}^+ \cdots \eta_1^+\eta_0^+)l_0(u) = l_m(u). \tag{1.7}$$

We can derive higher order differential equations that are fulfilled by particular special polynomials using the operational relations outlined above. The second-order differential equations were studied using the classical factorization method presented in [10].

The Frobenius-Genocchi polynomials $\mathcal{G}_m(x|u)$ satisfying the following exponential generating expression were explored in great depth by Yilmaz and Özarslan [14]:

$$\sum_{m=0}^{\infty} \mathcal{G}_m(x|u)\frac{t^m}{m!} = \frac{(1-u)t}{e^t - u}\, e^{xt}, \quad \forall\ u \in \mathbb{C}; \quad u \neq 1. \tag{1.8}$$

For, $x = 0$, these Frobenius-Genocchi polynomials $\mathcal{G}_m(x|u)$ reduce to the Frobenius-Genocchi numbers $\mathcal{G}_m(u)$. Therefore, $\mathcal{G}_m(u) = \mathcal{G}_m(0|u)$ and thus are given by the following expression:

$$\sum_{m=0}^{\infty} \mathcal{G}_m(0|u)\frac{t^m}{m!} = \frac{(1-u)t}{e^t - u}, \quad \forall\ u \in \mathbb{C}; \quad u \neq 1. \tag{1.9}$$

It is clear that

$$\mathcal{G}_0(0|u) = \mathcal{G}_0(u) = 0 \text{ and } \mathcal{G}_1(u) = 1. \tag{1.10}$$

Further more in [14], it is given that the numbers $g_k(u)$ related to Frobenius-Genocchi polynomials $\mathcal{G}_m(x|u)$ are given by

$$g_k(u) := \sum_{l=0}^{k} \frac{1}{2^l}\binom{k}{l}\mathcal{G}_{k-l}(\frac{1}{2}|u) \tag{1.11}$$

and by series manipulations, one can find

$$g_0 = 0, \;\; g_1 = \frac{1}{2}. \tag{1.12}$$

Also, the Frobenius-Genocchi polynomials $\mathcal{G}_m(x|u)$ are recursively represented in terms of the Frobenius-Genocchi numbers $\mathcal{G}_m(u)$ as follows:

$$\mathcal{G}_m(x|u) = \sum_{k=0}^{m} \binom{m}{k} \mathcal{G}_k(u) x^{m-k}, \qquad m \geq 0. \tag{1.13}$$

The classical Genocchi polynomials $G_m(x)$ are an analogue of the Frobenius-Genocchi polynomials $\mathcal{G}_m(x|u)$, represented by the following generating expression:

$$\sum_{m=0}^{\infty} G_m(x) \frac{t^m}{m!} = \frac{2t}{e^t + 1}\, e^{xt}. \tag{1.14}$$

Especially, the rational numbers $G_m = G_m(0)$ are called the classical Genocchi numbers. These numbers and polynomials play essential roles in many different areas of mathematics including number theory, combinatorics, special functions and analysis. Obviously, the Frobenius-Genocchi polynomials $\mathcal{G}_m(x|u)$ give the classical Genocchi polynomials $G_m(x)$ for $u = -1$ in Eq (1.8).

Gould-Hopper based Frobenius-Genocchi polynomials given in [2], ${}_{g^{(j)}}\mathcal{G}_m(x, y|u)$ are represented by the succeeding generating expression

$$\frac{(1-u)t}{(e^t - u)} \exp(xt + yt^j) = \sum_{m=0}^{\infty} {}_{g^{(j)}}\mathcal{G}_m(x, y|u) \frac{t^m}{m!}. \tag{1.15}$$

Using (1.9) and generating expression of Gould-Hopper polynomials [8]:

$$\exp(xt + yt^j) = \sum_{m=0}^{\infty} g_m^{(j)}(x, y) \frac{t^m}{m!}$$

in the l.h.s. of (1.15), it follows that

$$\frac{(1-u)t}{(e^t - u)} \exp(xt + yt^j) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{G}_k(u) g_m^{(j)}(x, y) \frac{t^k}{k!} \frac{t^m}{m!}$$

or

$$\frac{(1-u)t}{(e^t - u)} \exp(xt + yt^j) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{G}_k(u) g_m^{(j)}(x, y) \frac{t^{m+k}}{k!\, m!},$$

using Cauchy product rule and replacing m by $m - k$ in above equation, the following series representation is established:

$$ {}_{g^{(j)}}\mathcal{G}_m(x, y|u) = \sum_{k=0}^{m} \binom{m}{k} \mathcal{G}_k(u) g_{m-k}^{(j)}(x, y). \tag{1.16}$$

Similarly, using generating expression

$$\frac{(1-u)t}{(e^t-u)}\exp(xt)=\sum_{m=0}^{\infty}{}_{g^{(j)}}\mathcal{G}_m(x|u)\frac{t^m}{m!}$$

and expanding the term $\exp(yt^j)$ in the l.h.s. of Eq (1.15), we obtain the following series representation for Gould-Hopper based Frobenius-Genocchi polynomials ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$:

$$ {}_{g^{(j)}}\mathcal{G}_m(x,y|u)=m!\sum_{k=0}^{[\frac{m}{j}]}\frac{\mathcal{G}_{m-k}(x|u)\;y^k}{k!\;(m-jk)!}. \tag{1.17}$$

This hybrid special polynomials family carry the properties of the parent polynomials. These polynomials are important because they possess important properties such as generating function, series definition, recurrence relations, differential equations, summation formulae, integral representations etc. In [9, 11–13], the differential equations for the Appell and related families of polynomials are derived by using factorization method [10]. This approach is further extended to derive the differential and integral equations for the hybrid, mixed type and 2D special polynomials related to the Appell family, see for example [1, 3–7]. This provides motivation to establish the differential equations for the Gould-Hopper-Frobenius-Euler polynomials. This gives inspiration to build up the differential equations for the ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$ polynomials.

The article is organized as follows: In Section 2, the recurrence relation and shift operators for the Gould-Hopper based Frobenius-Genocchi polynomials are derived. In Section 3, the differential, integro-differential and partial differential equations for this family are established.

2. Recurrence relations and shift operators

For the Gould-Hopper based Frobenius-Genocchi polynomials ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$, the recurrence relation and shift operators are derived in this section.

In order to derive the recurrence relation for the Gould-Hopper based Frobenius-Genocchi polynomials ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$, the following result is proved:

Theorem 2.1. *For the Gould-Hopper based Frobenius-Genocchi polynomials with two variables* ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$ *, the following recurrence relation holds true:*

$$\begin{aligned}&{}_{g^{(j)}}\mathcal{G}_{m+1}(x,y|u)\\ =&\frac{m+1}{m}\Big[\left(x-\frac{m+1}{2(1-u)}\right){}_{g^{(j)}}\mathcal{G}_m(x,y|u)-\frac{1}{1-u}\sum_{k=2}^{m+1}\binom{m+1}{k}g_k(u)\\ &\times{}_{g^{(j)}}\mathcal{G}_{m-k+1}(x,y|u)+jy\frac{m!}{(m-j+1)!}\,{}_{g^{(j)}}\mathcal{G}_{m-j+1}(x,y|u)\Big],\end{aligned}\tag{2.1}$$

where u is a parameter, j is a positive integer and the numerical coefficients $g_k(u)$ *are related to* $\mathcal{G}_m(x|u)$ *that is Frobenius-Genocchi polynomials given by the relations* (1.11) *and* (1.12).

Proof. Taking derivatives of expression (1.15) with respect t on both sides, it follows that

$$\begin{aligned}&\sum_{m=0}^{\infty} {}_{g^{(j)}}\mathcal{G}_m(x,y|u)\ m\ \frac{t^{m-1}}{m!}\\&=\left(x+jyt^{j-1}+\frac{1}{t}-\frac{e^t}{e^t-u}\right)\sum_{m=0}^{\infty} {}_{g^{(j)}}\mathcal{G}_m(x,y|u)\frac{t^m}{m!}.\end{aligned}$$

Rearranging and simplifying the terms, we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\ (m-1)_{g^{(j)}}\mathcal{G}_m(x,y|u)\frac{t^{m-1}}{m!}\\&=x\sum_{m=0}^{\infty} {}_{g^{(j)}}\mathcal{G}_m(x,y|u)\frac{t^m}{m!}+jy\sum_{m=0}^{\infty} {}_{g^{(j)}}\mathcal{G}_m(x,y|u)\frac{t^{m+j-1}}{m!}\\&-\frac{1}{1-u}\ \sum_{m=0}^{\infty}\ \sum_{k=0}^{\infty} g_k(u)_{g^{(j)}}\mathcal{G}_m(x,y|u)\frac{t^{m+k}}{m!\ k!}.\end{aligned}$$

Replacing m by m+1 in the l.h.s. and using well known Cauchy-product rule in the r.h.s. by replacing m by m-j+1 and m-k in second and third terms, it follows that

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{m}{m+1}{}_{g^{(j)}}\mathcal{G}_{m+1}(x,y|u)\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}x\ {}_{g^{(j)}}\mathcal{G}_m(x,y|u)\frac{t^m}{m!}+jy\sum_{m=0}^{\infty}\frac{m!}{(m-j+1)!}\ {}_{g^{(j)}}\mathcal{G}_{m-j+1}(x,y|u)\frac{t^m}{m!}\\&-\frac{1}{1-u}\ \sum_{m=0}^{\infty}\ \sum_{k=0}^{m}\binom{m}{k}g_k(u)\ {}_{g^{(j)}}\mathcal{G}_{m-k}(x,y|u)\frac{t^m}{m!}.\end{aligned}\tag{2.2}$$

Using $g_0=0$ and $g_1=\frac{1}{2}$, the third term in r.h.s. becomes 0 and $-\frac{m}{2(1-u)}\ {}_{g^{(j)}}\mathcal{G}_{m-1}(x,y|u)$ for $k=0$ and 1, respectively. To make the term same as ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$ as in first term, we replace m by $m+1$. Therefore, combining it with first term and comparing the coefficients of like powers of t on the both sides of the resultant equation, assertion (2.1) is obtained. □

Next, we find the shift operators for the Gould-Hopper based Frobenius-Genocchi polynomials ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$ by proving the following result:

Theorem 2.2. *For the Gould-Hopper based Frobenius-Genocchi polynomials ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$, the following expressions for the shift operators holds true:*

$${}_x\pounds_m^- := \frac{1}{m}D_x,\tag{2.3}$$

$${}_y\pounds_m^- := \frac{1}{m}D_x^{1-j}D_y,\tag{2.4}$$

$${}_x\pounds_m^+ := \frac{m+1}{m}\Big[\Big(x-\frac{m+1}{2(1-u)}\Big)-\frac{m+1}{1-u}\sum_{k=2}^{m+1}D_x^{k-1}\ \frac{g_k(u)}{k!}+jy\ D_x^{j-1}\Big],\tag{2.5}$$

$$ {}_{y}£_{m}^{+} := \frac{m+1}{m}\Big[\Big(x-\frac{m+1}{2(1-u)}\Big)-\frac{m+1}{1-u}\sum_{k=2}^{m+1}\frac{g_k(u)}{k!}D_x^{(k-1)(1-j)}\,D_y^{k-1}\;+jy\,D_x^{-(j-1)^2}D_y^{j-1}\Big], \tag{2.6} $$

where

$$ D_x := \frac{\partial}{\partial x},\quad D_y := \frac{\partial}{\partial y},\quad \text{and}\quad D_x^{-1} := \int_0^x f(\xi)d\xi. \tag{2.7} $$

Proof. Differentiating generating relation (1.15) with respect to x and then equating the coefficients of same powers of t on both sides of the resultant equation, we have

$$ \frac{\partial}{\partial x}\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = m\,{}_{g^{(j)}}\mathcal{G}_{m-1}(x,y|u), \tag{2.8} $$

so that

$$ \frac{1}{m}\frac{\partial}{\partial x}\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = m\,{}_{g^{(j)}}\mathcal{G}_{m-1}(x,y|u). $$

Consequently, it follows that

$$ {}_{x}£_{m}^{-}\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = \frac{1}{m}\frac{\partial}{\partial x}\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = {}_{g^{(j)}}\mathcal{G}_{m-1}(x,y|u), \tag{2.9} $$

which proves assertion (2.3).

Next, differentiating generating relation (1.15) with respect to y and then equating the coefficients of same powers of t on both sides of the resultant equation, we have

$$ \frac{\partial}{\partial y}\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = \frac{m!}{(m-j)!}\,{}_{g^{(j)}}\mathcal{G}_{m-j}(x,y|u), \tag{2.10} $$

which in view of Eq (2.8) can be written as

$$ \frac{\partial}{\partial y}\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = m\frac{\partial^{j-1}}{\partial x^{j-1}}\,{}_{g^{(j)}}\mathcal{G}_{m-1}(x,y|u), \tag{2.11} $$

so that

$$ {}_{y}£_{m}^{-}\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = \frac{1}{m}D_x^{1-j}D_y\,\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = {}_{g^{(j)}}\mathcal{G}_{m-1}(x,y|u). \tag{2.12} $$

Thus assertion (2.4) is proved.

Next, to find raising operator ${}_{x}£_{m}^{+}$, the following relation is used:

$$ {}_{g^{(j)}}\mathcal{G}_k(x,y|u) = ({}_{x}£_{k+1}^{-}\;{}_{x}£_{k+2}^{-}\dots{}_{x}£_{m-1}^{-}\;{}_{x}£_{m}^{-})\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\}, \tag{2.13} $$

which can be further simplified on using expression (2.9) as:

$$ {}_{g^{(j)}}\mathcal{G}_k(x,y|u) = \frac{k!}{m!}D_x^{m-k}\{_{g^{(j)}}\mathcal{G}_m(x,y|u)\}. \tag{2.14} $$

Inserting above expression in relation (2.1), we have

$$
\begin{aligned}
&{}_{g^{(j)}}\mathcal{G}_{m+1}(x,y|u)\\
&=\frac{m+1}{m}\bigg(\Big(x-\frac{m+1}{2(1-u)}\Big)-\frac{m+1}{1-u}\sum_{k=2}^{m+1}D_x^{k-1}\ \frac{g_k(u)}{k!}+jy\ D_x^{j-1}\bigg){}_{g^{(j)}}\mathcal{G}_m(x,y|u).
\end{aligned}
\tag{2.15}
$$

This yields expression for raising operator (2.5).

Further, to obtain the raising operator ${}_y\pounds_m^+$, the following relation is used:

$$
{}_{g^{(j)}}\mathcal{G}_k(x,y|u)=({}_y\pounds_{k+1}^-\ {}_y\pounds_{k+2}^-\dots{}_y\pounds_{m-1}^-\ {}_y\pounds_m^-)\{{}_{g^{(j)}}\mathcal{G}_m(x,y|u)\},
\tag{2.16}
$$

simplifying above equation in view of expression (2.12), it follows that

$$
{}_{g^{(j)}}\mathcal{G}_k(x,y|u)=\frac{k!}{m!}D_x^{(k-1)(1-j)}D_y^{k-1}\{{}_{g^{(j)}}\mathcal{G}_m(x,y|u)\}.
\tag{2.17}
$$

Inserting above expression in relation (2.1), it follows that

$$
\begin{aligned}
{}_{g^{(j)}}\mathcal{G}_{m+1}(x,y|u)=\frac{m+1}{m}\bigg(\Big(x-\frac{m+1}{2(1-u)}\Big)-\frac{m+1}{1-u}\sum_{k=2}^{m+1}D_x^{(k-1)(1-j)}\ D_y^{k-1}\ \frac{g_k(u)}{k!}\\
+jy\ D_x^{-(j-1)^2}D_y^{j-1}\bigg){}_{g^{(j)}}\mathcal{G}_m(x,y|u).
\end{aligned}
\tag{2.18}
$$

This yields expression for raising operator (2.6). □

3. Differential equations

In this segment utilizing the factorization method, the differential, integro-differential, and partial differential expressions for the Gould-Hopper based Frobenius-Genocchi polynomials ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$ of 2-variables are established.

Theorem 3.1. *The Gould-Hopper based Frobenius-Genocchi polynomials* ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$ *satisfy the following expression for the differential equation:*

$$
\bigg(\Big(x-\frac{m+1}{2(1-u)}\Big)D_x-\frac{m+1}{1-u}\sum_{k=2}^{m+1}D_x^k\ \frac{g_k(u)}{k!}+jy\ D_x^j-(m-1)\bigg){}_{g^{(j)}}\mathcal{G}_m(x,y|u)=0.
\tag{3.1}
$$

Proof. Using the factorization relation given by

$$
{}_x\pounds_{m+1}^-\ {}_x\pounds_m^+\{{}_{g^{(j)}}\mathcal{G}_m(x,y|u)\}={}_{g^{(j)}}\mathcal{G}_m(x,y|u),
\tag{3.2}
$$

and inserting the expressions (2.3) and (2.5) in the l.h.s. of above equation gives assertion (3.1). □

Theorem 3.2. *The Gould-Hopper based Frobenius-Genocchi polynomials* ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$ *satisfy the following expressions for the integro-differential equations:*

$$\Big(\big(x - \frac{m+1}{1-u}\big)D_y - \frac{m+1}{1-u}\sum_{k=2}^{m+1}\frac{g_k(u)}{k!}D_x^{(k-1)(1-j)}D_y^k + j\,D_x^{-(j-1)^2}D_y^{j-1}$$

$$+jy\,D_x^{-(j-1)^2}D_y^j - mD_x^{j-1}\Big)\,{}_{g^{(j)}}\mathcal{G}_m(x,y|u) = 0 \tag{3.3}$$

and

$$\Big(\big(x - \frac{m+1}{1-u}\big)D_x - \frac{m+1}{1-u}\sum_{k=2}^{m+1}\frac{g_k(u)}{k!}D_x^{(k-1)(1-j)+1}D_y^{k-1}$$

$$+jy\,D_x^{-(j-1)^2+1}\,D_y^{j-1} - m - 1\Big)\,{}_{g^{(j)}}\mathcal{G}_m(x,y|u) = 0. \tag{3.4}$$

Proof. Consider the following factorization relation:

$${}_y\pounds_{m+1}^-\;{}_y\pounds_m^+\{{}_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = {}_{g^{(j)}}\mathcal{G}_m(x,y|u). \tag{3.5}$$

Using the shift operators (2.4) and (2.6) in the above relation, assertion (3.3) is obtained.

Combining (2.3) and (2.6) of the shift operators with the factorization relation

$${}_x\pounds_{m+1}^-\;{}_y\pounds_m^+\{{}_{g^{(j)}}\mathcal{G}_m(x,y|u)\} = {}_{g^{(j)}}\mathcal{G}_m(x,y|u) \tag{3.6}$$

yields assertion (3.4). □

Theorem 3.3. *The Gould-Hopper based Frobenius-Genocchi polynomials ${}_{g^{(j)}}\mathcal{G}_m(x,y|u)$ satisfy the following expressions for the partial differential equations:*

$$\Big(\big(x - \frac{m+1}{1-u}\big)D_x^{m(j-1)}D_y + m(j-1)D_yD_x^{m(j-1)-1} - \frac{m+1}{1-u}\sum_{k=2}^{m+1}\frac{g_k(u)}{k!}D_x^{(j-1)(m-k+1)}D_y^k$$

$$+j\,D_x^{(j-1)(m-j+1)}D_y^{j-1} + jy\,D_x^{(j-1)(m-j+1)}D_y^j - mD_x^{(m+1)(j-1)}\Big)\,{}_{g^{(j)}}\mathcal{G}_m(x,y|u) = 0 \tag{3.7}$$

and

$$\Big(\big(x - \frac{m+1}{1-u}\big)D_x^{m(j-1)+1} + m(j-1)D_x^{m(j-1)} - \frac{m+1}{1-u}\sum_{k=2}^{m+1}\frac{g_k(u)}{k!}D_x^{(j-1)(m-k+1)+1}D_y^{k-1}$$

$$+jy\,D_x^{(j-1)(m-j+1)+1}D_y^{(j-1)} - m - 1D_x^{m(j-1)}\Big)\,{}_{g^{(j)}}\mathcal{G}_m(x,y|u) = 0, \tag{3.8}$$

respectively.

Proof. Differentiating $m(j-1)$ times with respect to x of integro-differential equations (3.3) and (3.4), partial differential equations (3.7) and (3.8) are obtained. □

4. Conclusions

Differential equations are used to explain issues in several fields of science and engineering, with special functions serving as solutions in the majority of cases. The differential and integral equations that these hybrid type special polynomials satisfy can be used to address new issues in a variety of fields. To study the combination of operational representations with the factorization method and their applications to the theory of differential equations for other hybrid type special polynomials and for their relatives will be taken in further investigation.

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Conflict of interest

The authors declare no conflict of interest.

References

1. S. Araci, M. Riyasat, S. A. Wani, S. Khan, Differential and integral equations for the 3-variable Hermite-Frobenius-Euler and Frobenius-Genocchi polynomials, *Appl. Math. Inf. Sci.*, **11** (2017), 1335–1346. http://dx.doi.org/10.18576/amis/110510
2. S. A. Wani, S. Khan, T. Nahid, Gould-Hopper based Frobenius-Genocchi polynomials and their generalized form, *Afr. Mat.*, **31** (2020), 1397–1408. https://doi.org/10.1007/s13370-020-00804-2
3. S. Khan, M. Riyasat, S. A. Wani, On some classes of diffrential equations and associated integral equations for the Laguerre-Appell polynomials, *Adv. Pure Appl. Math.*, **9** (2018), 185–194. https://doi.org/10.1515/apam-2017-0079
4. M. Riyasat, S. A. Wani, S. Khan, Differential and integral equations associated with some hybrid families of Legendre polynomials, *Tbilisi Math. J.*, **11** (2018), 127–139. https://doi.org/10.32513/tbilisi/1524276035
5. S. Khan, S. A. Wani, A note on differential and integral equations for the Laguerre-Hermite polynomials, Proceedings of the Second International Conference on Computer and Communication Technologies, IC3T 2017, *Adv. Intell. Syst. Comput.*, **381** (2016), 547–555.
6. S. Khan, S. A. Wani, A note on differential and integral equations for the Legendre-Hermite polynomials, *IJARSE*, **7** (2018), 514–520.
7. S. A. Wani, S. Khan, S. A. Naikoo, Differential and integral equations for the Laguerre-Gould-Hopper based Appell and related polynomials, *Bol. Soc. Mat. Mex.*, **26** (2020), 617–646. https://doi.org/10.1007/s40590-019-00239-1
8. H. W. Gould, A. T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.*, **29** (1962), 51–63. https://doi.org/10.1215/S0012-7094-62-02907-1
9. M. X. He, P. E. Ricci, Differential equation of Appell polynomials via the factorization method, *J. Comput. Appl. Math.*, **139** (2002), 231–237. https://doi.org/10.1016/S0377-0427(01)00423-X
10. L. Infeld, T. E. Hull, The factorization method, *Rev. Mod. Phys.*, **23** (1951), 21–68. https://doi.org/10.1103/RevModPhys.23.21
11. M. A. Özarslan, B. Yılmaz, A set of finite order differential equations for the Appell polynomials, *J. Comput. Appl. Math.*, **259** (2014), 108–116. https://doi.org/10.1016/j.cam.2013.08.006

12. H. M. Srivastava, M. A. Özarslan, B. Yılmaz, Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials, *Filomat*, **28** (2014), 695–708.

13. B. Yılmaz, M. A. Özarslan, Differential equations for the extended 2D Bernoulli and Euler polynomials, *Adv. Differ. Equ.*, **107** (2013), 107. https://doi.org/10.1186/1687-1847-2013-107

14. B. Yılmaz, M. A. Özarslan, Frobenius-Euler and Frobenius-Genocchi polynomials and their differential equations, *New Trends Math. Sci.*, **3** (2015), 172–180.