Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 4778–4792.
DOI: 10.3934/math.2022265
Received: 02 December 2021
Revised: 20 December 2021
Accepted: 21 December 2021
Published: 27 December 2021

Research article

Oscillatory and complex behaviour of Caputo-Fabrizio fractional order HIV-1 infection model

Shabir Ahmad[1,*], **Aman Ullah**[1], **Mohammad Partohaghighi**[2], **Sayed Saifullah**[1], **Ali Akgül**[3] **and Fahd Jarad**[4,5,6*]

[1] Department of Mathematics, University of Malakand, Chakdara, Dir Lower, Khyber Pakhtunkhwa, Pakistan

[2] Department of Mathematics, Clarkson University, New York, USA

[3] Art and Science Faculty, Department of Mathematics, Siirt University, TR-56100 Siirt, Turkey

[4] Department of Mathematics, Cankaya University, Etimesgut 06790, Ankara, Turkey

[5] King Abdulaziz University Jeddah, Saudi Arabia

[6] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: shabirahmad2232@gmail.com, fahd@cankaya.edu.tr.

Abstract: HIV-1 infection is a dangerous diseases like Cancer, AIDS, etc. Many mathematical models have been introduced in the literature, which are investigated with different approaches. In this article, we generalize the HIV-1 model through nonsingular fractional operator. The non-integer mathematical model of HIV-1 infection under the Caputo-Fabrizio derivative is presented in this paper. The concept of Picard-Lindelof and fixed-point theory are used to address the existence of a unique solution to the HIV-1 model under the suggested operator. Also, the stability of the suggested model is proved through the Picard iteration and fixed point theory approach. The model's approximate solution is constructed through three steps Adams-Bashforth numerical method. Numerical simulations are provided for different values of fractional-order to study the complex dynamics of the model. Lastly, we provide the oscillatory and chaotic behavior of the proposed model for various fractional orders.

Keywords: Picard iteration; fixed point theory; Caputo-Fabrizio derivative; chaotic behavior
Mathematics Subject Classification: 92B05, 92C10

1. Introduction

Human immunodeficiency virus (HIV) is a virus that affects cells that render a person more susceptible to other infections and diseases and helps the body fighting infection. A retrovirus that causes AIDS is the HIV [1]. HIV infects, destroys, and decreases $CD4^{+}$ T cells, thereby reducing immune system defense [2]. The body gets much highly responsive towards infections and steadily loses its defense. One of today's most severe and fatal diseases is AIDS. In 2019, 38 million individuals worldwide were living with HIV, 1.7 million people got newly infected with HIV, and 690 thousand people died from AIDS-related diseases, as per UNAIDS 2020 annual assessment. No vaccine for HIV has ever been found, despite significant success in handling the disease. Much effort has been made by researchers over the last two decades to develop mathematical models that have a significant rule in studying HIV-related disease control and prevention. The relationship between HIV viruses and uninfected $CD4^{+}$ cells and the impact of drug treatment on infected cells has usually described by most of these mathematical models. The simplest model is

$$\begin{cases} \dot{x} = & c - \beta x - \gamma xy, \\ \dot{y} = & \gamma xy - dy. \end{cases} \tag{1.1}$$

This model is influenced by Anderson's model and many other models [3, 4]. An updated model of Eq (1.1) has introduced by Tuckwell and Wan [5] with three categories: Uninfected cells x , infected $CD4^{+}$ T-cells y, and plasma virion density z. The proposed ODE-based model with three components is given by:

$$\begin{cases} \dot{x} = & s' - \mu x - \beta xz, \\ \dot{y} = & \beta xz - \varepsilon y, \\ \dot{z} = & cy - \varsigma z, \end{cases} \tag{1.2}$$

subject to the I.Cs $x(0) = k_1$,$y(0) = k_2$, and $z(0) =k_3$. The description of the parameters are given in Table 1. When drug treatment is not 100 percent effective, the rate of certain coefficients can vary. Infected cells that produce components of the virus are infected when the drug therapy starts. A part of the infected cells will improve if drug treatment is not successful, and the leftover cells will start developing a virus.

Table 1. Parametric values for the numerical simulation.

Parameters	Description	values
s'	"the rate of production or creation of $CD4+$ T-cells"	0.272
μ	"the rate of natural death"	0.00136
β	"the rate of infected $CD4^{+}$ cells from uninfected $CD4^{+}$ cells"	0.00027
ε	"the rate at which virus-producing cells multiply until they die"	0.33
c	"the rate at which infected cells produce virions viruses"	50
ς	"the rate at which virus particles die"	2

Differential equations in fractional order appear as mathematical modelling in biology and other areas of science. Because the DEs of the variable order save memory and has connected to fractals [16, 17]. The field of fractional calculus has earned interest among researchers during the last few decades. It is because fractional calculus can more effectively describe the persistence and inherited features of different components and procedures than ODE based models [6, 7]. Various operators have been introduced in fractional calculus concerned with different kernels. In recent decades, mathematicians have investigated the fractional operators from various point of view [8, 9].

Fractional operators have been used for modelling various infectious diseases. Shaikh et al. used fractional operator to study dynamical behaviour HIV/AIDS model [10]. Rahman et al. investigated time fractional Φ^4 equation under different fractional operators [11]. Various dynamical systems in economics field have also been studied through fractional calculus. For instance, in [12], the authors have investigated reliability index and option pricing formulas of the first-hitting time model based on the uncertain fractional-order differential equation with Caputo type. Many applications of the fractional calculus can be found in the literature. Different analytical and numerical methods have been used for solving nonlinear fractional DEs [13–15]. Most of the physical processes are modelled by nonlinear fractional order DEs. Solving nonlinear fractional DEs by analytical methods are very difficult. Therefore, researchers developed many numerical methods to solve fractional DEs numerically. Traditional fractional derivatives, on the other hand, possess a singular kernel that often creates problems with describing some properties. To resolve this, a new definition of fractional integral and derivative has introduced by Caputo and Fabrizio that includes an exponential kernel rather than a singular kernel [18]. Much consideration was also paid to these operators and proved to be better at adopting several real-world problems for mathematical models [19–21]. Saifullah et al. investigated Klein-Gordon Equation under nonsingular operators [23]. Ahmad et al. studied the Ambartsumian equation under the Caputo-Fabrizio fractional operator [24]. Moore et al. used the Caputo-Fabrizio fractional derivative to analyze the transmission of HIV disease [25]. Due to the success of this operator, we generalize the model (1.2) as follows

$$\begin{cases} {}^{CF}D_t^{\gamma}x(t) = & s^{'} - \mu x - \beta xz, \\ {}^{CF}D_t^{\gamma}y(t) = & \beta xz - \varepsilon y, \\ {}^{CF}D_t^{\gamma}z(t) = & cy - \varsigma z. \end{cases} \tag{1.3}$$

In this paper, we explore an existence theory for the system (1.3) using a fixed point theory to ensure that the proposed model has at least one solution. Also, we utilize Euler method to derive the general procedure of solution to the model (1.3) under the CF derivative. In the literature, the study of oscillatory and chaotic dynamics of the considered model was missing. The most important is: We present the oscillatory and chaotic behaviour of the HIV1 infection for different fractional operator.

2. Preliminaries

Here we give definitions of CF fractional operators and formula of Laplace transform of CF derivative. Let FI represent the fractional integral.

Definition 2.1. *[18] If $V(t) \in H^1[0,T], T > 0, \gamma \in (0,1]$, then the CF derivative of $V(t)$ is defined as:*

$$ {}^{CF}D_t^{\gamma}\left[V(t)\right] = \frac{M(\gamma)}{1-\gamma}\int_0^t V^{'}(\varrho)K(t,\chi)d\chi,$$

where $K(t,\chi) = \exp\left[-\gamma\frac{t-\chi}{1-\gamma}\right]$ and $M(\gamma)$ represent normalization function such that $M(1) = M(0) = 1$.

Definition 2.2. *[19] The FI of $V(t)$ in CF sense is given by:*

$$ {}^{CF}I_t^{\gamma}\left[V(t)\right] = \frac{1-\gamma}{M(\gamma)}V(t) + \frac{\gamma}{M(\gamma)}\int_0^t V(\chi)d\chi, \quad t \geq 0,\ \gamma \in (0,1]. \tag{2.1}$$

Definition 2.3. *[22] For $M(\gamma)=1$, the Laplace transform of $\left[{}^{CF}D_t^{\gamma}\left[V(t)\right]\right]$ is:*

$$\mathscr{L}\left\{{}^{CF}D_t^{\gamma+M}\left[V(t)\right]\right\} = \frac{1}{1-\gamma}\mathscr{L}\left[V^{(h+\gamma)}(t)\right]\mathscr{L}\left[\exp\left(\frac{-\gamma t}{1-\gamma}\right)\right] \tag{2.2}$$

$$= \frac{1}{s+\gamma(1-s)}\left[s^{h+1}\mathscr{L}\left[V(t)\right] + \sum_{i=0}^{h} s^{h-i}V^{(i)}(0)\right]. \tag{2.3}$$

One can be obtain the following results for $h=0,1$ respectively

$$\mathscr{L}\left[{}^{CF}D_t^{\gamma}\left[V(t)\right]\right] = \frac{s\mathscr{L}\left[V(t)\right]-V(0)}{s+\gamma(1-s)}, \tag{2.4}$$

$$\mathscr{L}\left[{}^{CF}D_t^{\gamma+1}\left[V(t)\right]\right] = \frac{s\mathscr{L}\left[V(t)\right]+sV(0)-V^{'}(0)}{s+\gamma(1-s)}. \tag{2.5}$$

3. Main work

Here, Picard-Lindelof and fixed-point theory have addressed the existence of a unique solution to the proposed model. Also, the stability of the suggested model has proven by using the Picard iteration and fixed point theory. The model's general solution is constructed through Adams-Bashforth method.

3.1. Existence and uniqueness results

Consider the right hand sides of (1.2) as

$$\begin{aligned} \Omega_1(t,x) &= s^{'} - \mu x - \beta xz, \\ \Omega_2(t,y) &= \beta xz - \varepsilon y, \\ \Omega_3(t,z) &= cy - \varsigma z. \end{aligned}$$

So the system (1.3) gets the form

$$\begin{cases} {}^{CF}D_t^{\gamma}x(t) = & \Omega_1(t,x), \\ {}^{CF}D_t^{\gamma}y(t) = & \Omega_2(t,y), \\ {}^{CF}D_t^{\gamma}z(t) = & \Omega_3(t,z), \end{cases} \tag{3.1}$$

let

$$\Delta = \sup_{C[d,b_n]} \left\|\Omega_n(t,.)\right\|, \qquad \text{for}\, n=1,2,3,$$

with

$$C\left[d,b_n\right] = \left[t-d,t+d\right]\times\left[u-c_n,u+c_k\right] = G\times G_n, \quad for\, n=1,2,3.$$

Assume a uniform norm on $C\left[d,b_n\right]$ as:

$$\|\mathscr{B}\|_\infty = \sup_{t\in[t-d,t+d]} \left|\mathscr{B}(t)\right|. \tag{3.2}$$

Applying ${}^{CF}I_t^{\gamma}$ to (3.1), one can achieve

$$\begin{cases} x(t)-x(0) = & {}^{CF}I_t^{\gamma}\Omega_1(t,x), \\ y(t)-y(0) = & {}^{CF}I_t^{\gamma}\Omega_2(t,y), \\ z(t)-z(0) = & {}^{CF}I_t^{\gamma}\Omega_3(t,z). \end{cases} \tag{3.3}$$

$$\begin{cases} x(t) = & x(0) + \frac{1-\gamma}{M(\gamma)}\left[\Omega_1(t,x) - \Omega_1(0,x(0))\right] + \frac{\gamma}{M(\gamma)}\int_0^t \Omega_1(\chi,x)\,d\chi, \\ y(t) = & y(0) + \frac{1-\gamma}{M(\gamma)}\left[\Omega_2(t,y) - \Omega_2(0,y(0))\right] + \frac{\gamma}{M(\gamma)}\int_0^t \Omega_2(\chi,y)\,d\chi, \\ z(t) = & z(0) + \frac{1-\gamma}{M(\gamma)}\left[\Omega_3(t,z) - \Omega_3(0,z(0))\right] + \frac{\gamma}{M(\gamma)}\int_0^t \Omega_3(\chi,z)\,d\chi. \end{cases} \tag{3.4}$$

Define the Picard operator $\Phi : C(G, G_1, G_2, G_3) \to C(G, G_1, G_2, G_3)$ as

$$\Phi(\mathscr{B}(t)) = \mathscr{B}_0(t) + \left[\Psi(t,\mathscr{B}(t)) - \Psi_0(t)\right]\frac{1-\gamma}{M(\gamma)} + \frac{\gamma}{M(\gamma)}\int_0^t \Psi(\chi,\mathscr{B}(\chi))\,d\chi, \tag{3.5}$$

where

$$\mathscr{B}(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}, \quad \mathscr{B}_0(t) = \begin{cases} x(0) \\ y(0) \\ z(0) \end{cases},$$

$$\Psi(t,\mathscr{B}(t)) = \begin{cases} \Omega_1(t,x) \\ \Omega_2(t,y) \\ \Omega_3(t,z) \end{cases}, \quad \Psi_0(t) = \begin{cases} \Omega_1(0,x(0)) \\ \Omega_2(0,y(0)) \\ \Omega_3(0,z(0)). \end{cases}$$

Assume that the proposed model obeys:

$$\|\mathscr{B}(t)\|_\infty \le max\{d_1, d_2, d_3\}. \tag{3.6}$$

Let $\Delta = \max\{\Delta_1, \Delta_2, \Delta_3\}$ and there exits $t_0 = \max\{t \in D\}$ so that $t_0 \ge t$, one get

$$\begin{aligned} \|\Phi\mathscr{B}(t) - \mathscr{B}_0(t)\| &= \left\| \Psi(t,\mathscr{B}(t))\frac{1-\gamma}{M(\gamma)} + \frac{\gamma}{M(\gamma)}\int_0^t \Psi(\chi,\mathscr{B}(\chi))\,d\chi \right\| \\ &\le \frac{1-\gamma}{M(\gamma)}\|\Psi(t,\mathscr{B}(t))\| + \frac{\gamma}{M(\gamma)}\int_0^t \|\Psi(\chi,\mathscr{B}(\chi))\|\,d\chi \\ &\le \frac{1-\gamma}{M(\gamma)}\Delta + \frac{\gamma}{M(\gamma)}t\Delta \\ &\le d\Delta \le \max\{d_1,d_2,d_3\} = d^{'}, \end{aligned}$$

where $d = \frac{1+\gamma t_0}{M(\gamma)}$, and satisfies $d < \frac{d^{'}}{\Delta}$. Also to evaluate the following equality

$$\|\Phi\mathscr{B}_1 - \Phi\mathscr{B}_2\| = \sup_{t\in D}\left|\mathscr{B}_1(t) - \mathscr{B}_2(t)\right|. \tag{3.7}$$

Using definition of Picard operator, we proceed as

$$\begin{aligned} \|\Phi\mathscr{B}_1 - \Phi\mathscr{B}_2\| &= \left\| \frac{1-\gamma}{M(\gamma)}\left[\Psi(t,\mathscr{B}_1(t)) - \Psi(t,\mathscr{B}_2(t))\right] \right. \\ &\quad + \left. \frac{\gamma}{M(\gamma)}\int_0^t \left[\Psi(\chi,\mathscr{B}_1(\chi)) - \Psi(\chi,\mathscr{B}_2(\chi))\right]d\chi \right\| \\ &\le \frac{1-\gamma}{M(\gamma)}\vartheta\|\mathscr{B}_1(t) - \mathscr{B}_2(t)\| + \frac{\gamma\vartheta}{M(\gamma)}\int_0^t \|\mathscr{B}_1(\chi) - \mathscr{B}_2(\chi)\|\,d\chi \end{aligned}$$

$$
\begin{aligned}
&\leq \left\{\frac{1-\gamma}{M(\gamma)}\vartheta + \frac{\gamma\vartheta t_0}{M(\gamma)}\right\}\|\mathscr{B}_1(t)-\mathscr{B}_2(t)\| \\
&\leq d\vartheta\,\|\mathscr{B}_1(t)-\mathscr{B}_2(t)\|,
\end{aligned}
$$

with $\vartheta < 1$. For Φ to fulfill contraction condition we must have $d\vartheta < 1$. Thus the Picard operator Φ obeys the contraction condition. Therefore, the proposed model posses a unique solution.

3.2. Stability of the proposed model

Here, we will demonstrate the Picard type stability by using fixed point theory. Applying ${}^{CF}I_t^{\gamma}$ on (1.3), we obtain

$$
\begin{cases}
x(t)-k_1 &= \frac{1-\gamma}{M(\gamma)}\left[s^{'}-\mu x(t)-\beta x(t)z(t)\right]+\frac{\gamma}{M(\gamma)}\int_0^t\left[s^{'}-\mu x(\chi)-\beta x(\chi)z(\chi)\right]d\chi,\\
y(t)-k_2 &= \frac{1-\gamma}{M(\gamma)}\left[\beta x(t)z(t)-\varepsilon y(t)\right]+\frac{\gamma}{M(\gamma)}\int_0^t\left[\beta x(\chi)z(\chi)-\varepsilon y(\chi)\right]d\chi,\\
z(t)-k_3 &= \frac{1-\gamma}{M(\gamma)}\left[cy(t)-\varsigma z(t)\right]+\frac{\gamma}{M(\gamma)}\int_0^t\left[cy(\chi)-\varsigma z(\chi)\right]d\chi.
\end{cases}
\tag{3.8}
$$

Let $x_0(t)=k_1$, $y_0(t)=k_2$ and $z_0(t)=k_3$; then the Picard iteration is defined as:

$$
\begin{cases}
x_{i+1}(t) &= \frac{1-\gamma}{M(\gamma)}\left[s^{'}-\mu x_i(t)-\beta x_i(t)z_i(t)\right]+\frac{\gamma}{M(\gamma)}\int_0^t\left[s^{'}-\mu x_i(\chi)-\beta x_i(\chi)z_i(\chi)\right]d\chi,\\
y_{i+1}(t) &= \frac{1-\gamma}{M(\gamma)}\left[\beta x_i(t)z_i(t)-\varepsilon y_i(t)\right]+\frac{\gamma}{M(\gamma)}\int_0^t\left[\beta x_i(\chi)z_i(\chi)-\varepsilon y_i(\chi)\right]d\chi,\\
z_{i+1}(t) &= \frac{1-\gamma}{M(\gamma)}\left[cy_i(t)-\varsigma z_i(t)\right]+\frac{\gamma}{M(\gamma)}\int_0^t\left[cy_i(\chi)-\varsigma z_i(\chi)\right]d\chi.
\end{cases}
\tag{3.9}
$$

Definition 3.1. *[26] Let* $(\mathfrak{B},\|.\|)$ *represents a Banach space and* Φ *be a self mapping of* $\mathfrak{B}$ *with the inequality:*

$$
\left\|\Phi_x-\Phi_y\right\|\leq L\|x-\Phi_x\|+l\|x-y\|,
$$

$\forall\ x,y\in\mathfrak{B}$,*where* $L\geq 0$ *and* $0\leq l\leq 1$. *Then* Φ *is Picard* Φ*-stable.*

Now, let us consider the recursive formula for the proposed model (1.3) as:

$$
\begin{cases}
x_{i+1}(t) &= x_i(t)+\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[s^{'}-\mu x_i(t)-\beta x_i(t)z_i(t)\right]\right],\\
y_{i+1}(t) &= y_i(t)+\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[\beta x_i(t)z_i(t)-\varepsilon y_i(t)\right]\right],\\
z_{i+1}(t) &= z_i(t)+\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[cy_i(t)-\varsigma z_i(t)\right]\right].
\end{cases}
\tag{3.10}
$$

Theorem 3.2. *If* Φ *be a self mapping such that*

$$
\begin{cases}
\Phi(x_i(t)) &= x_{i+1}(t)=x_i(t)+\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[s^{'}-\mu x_i(t)-\beta x_i(t)z_i(t)\right]\right],\\
\Phi(y_i(t)) &= y_{i+1}(t)=y_i(t)+\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[\beta x_i(t)z_i(t)-\varepsilon y_i(t)\right]\right],\\
\Phi(z_i(t)) &= z_{i+1}(t)=z_i(t)+\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[cy_i(t)-\varsigma z_i(t)\right]\right].
\end{cases}
\tag{3.11}
$$

Then the iteration (3.9) is Φ*-stable if*

$$
\begin{cases}
\{1-\mu\Upsilon_1-\beta C_2\Upsilon_2\} &< 1,\\
\{1+\beta C_1\Upsilon_3-\varepsilon\Upsilon_4\} &< 1,\\
\{1+c\Upsilon_5-\varsigma\Upsilon_6\} &< 1.
\end{cases}
\tag{3.12}
$$

Proof. First, we need to show that Φ has a fixed point. For this, we compute $\Phi(x_i(t)) - \Phi(x_j(t))$ for all $(i, j) \in N \times N$ as follows:

$$\begin{aligned}
&\Phi(x_i(t)) - \Phi(x_j(t)) = \\
= \;& x_i(t) - x_j(t) + \mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[s^{'} - \mu x_i(t) - \beta x_i(t) z_i(t)\right]\right] \\
& -\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[s^{'} - \mu x_j(t) - \beta x_j(t) z_j(t)\right]\right] \\
= \;& x_i(t) - x_j(t) + \mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[-\mu\left(x_i(t) - x_j(t)\right)\right.\right. \\
& \left.\left.\beta\left(x_i(t) z_i(t) - x_j(t) z_j(t)\right)\right]\right]. \qquad (3.13)
\end{aligned}$$

Now, applying norm to Eq (3.2), one can obtain

$$\begin{aligned}
& \left\|\Phi(x_i(t)) - \Phi(x_j(t))\right\| \\
\leq \;& \left\|x_i(t) - x_j(t)\right\| \\
& + \left\|\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[-\mu\left(x_i(t) - x_j(t)\right) - \beta\left(x_i(t) z_i(t) - x_j(t) z_j(t)\right)\right]\right]\right\| \\
\leq \;& \left\|x_i(t) - x_j(t)\right\| \\
& + \mathscr{L}^{-1}\left\{\frac{s+\gamma(1-s)}{s} + \mathscr{L}\left[-\mu\left\|x_i(t) - x_j(t)\right\|\right.\right. \\
& \left.\left. -\beta\left(\|x_i(t)\|\,\|z_i(t)\| - \left\|x_j(t) z_j(t)\right\|\right)\right]\right\}. \qquad (3.14)
\end{aligned}$$

Due to the same role of both solutions, we assume that

$$\left\|\Phi(x_i(t)) - \Phi(x_j(t))\right\| \cong \left\|\Phi(y_i(t)) - \Phi(y_j(t))\right\| \cong \left\|\Phi(z_i(t)) - \Phi(z_j(t))\right\|. \qquad (3.15)$$

From Eqs (3.14) and (3.15), we get

$$\begin{aligned}
\|\Phi(x_i(t)) - \Phi(x_j(t))\| \leq \left\|x_i(t) - x_j(t)\right\| + \mathscr{L}^{-1}&\left\{\frac{s+\gamma(1-s)}{s}\mathscr{L}\left[-\mu\left\|x_i(t) - x_j(t)\right\|\right.\right. \\
& \left.\left. -\beta\left\|z_j(t)\right\|\left\|x_i(t) - x_j(t)\right\|\right]\right\}.
\end{aligned}$$

Since x_i, x_j,z_i and z_j are convergent sequences, there exists constants C_1, C_2,C_3 and C_4 for all t such that

$$\|x_i\| \leq C_1, \quad \left\|x_j\right\| \leq C_2, \quad \|z_i\| \leq C_3, \quad \left\|z_j\right\| \leq C_4.$$

Thus, Eq (3.14) becomes

$$\left\|\Phi(x_i(t)) - \Phi(x_j(t))\right\| \;\leq\; \{1 - \mu\Upsilon_1 - \beta C_2\Upsilon_2\}\left\|x_i(t) - x_j(t)\right\|. \qquad (3.16)$$

Similarly, we have

$$\left\|\Phi(y_i(t)) - \Phi(y_j(t))\right\| \;\leq\; \{1 + \beta C_1\Upsilon_3 - \varepsilon\Upsilon_4\}\left\|y_i(t) - y_j(t)\right\|, \qquad (3.17)$$

$$\left\|\Phi(z_i(t)) - \Phi(z_j(t))\right\| \leq \{1 + c\Upsilon_5 - \varsigma\Upsilon_6\}\left\|z_i(t) - z_j(t)\right\|, \tag{3.18}$$

where Υ_m for $m = 1, 2, \cdots, 6$, are functions obtained from $\mathscr{L}^{-1}\left[\frac{s+\gamma(1-s)}{s}\mathscr{L}[*]\right]$. Now under the condition

$$\begin{cases}\{1 - \mu\Upsilon_1 - \beta C_2\Upsilon_2\} & < 1,\\ \{1 + \beta C_1\Upsilon_3 - \varepsilon\Upsilon_4\} & < 1,\\ \{1 + c\Upsilon_5 - \varsigma\Upsilon_6\} & < 1.\end{cases} \tag{3.19}$$

The operator Φ fulfills the condition of contraction mapping, so the operator Φ must have a fixed point. Now, we prove that Φ fulfills the theorem (1) conditions. To do so, we assume that

$$L = (0, 0, 0), \quad l = \begin{cases}\{1 - \mu\Upsilon_1 - \beta C_2\Upsilon_2\},\\ \{1 + \beta C_1\Upsilon_3 - \varepsilon\Upsilon_4\},\\ \{1 + c\Upsilon_5 - \varsigma\Upsilon_6\}.\end{cases}$$

Then all conditions of theorem (1) are satisfied. Hence, Φ is the Picard Φ-stable. □

4. Numerical method

Here we solve the considered model numerically using three step Adam-Bashforth technique. For the sake of convenience we consider the model (1.3) as

$${}^{\mathcal{CF}}D_0^{\gamma}\Lambda(t) = \Xi(t, \Lambda(t)), \quad \Lambda(0) = \Lambda_0, \quad 0 \leq t \leq T_1 < \infty, \tag{4.1}$$

where $\Lambda = (x, y, z) \in R_+^3$, $\Lambda_0 = (x_0, y_0, z_0)$ are the initial values. Using the definition of CF derivative the above Eq (4.1) becomes

$$\frac{M(\gamma)}{1-\gamma} + \int_0^t \Lambda'(\chi)exp\left[-\gamma\frac{t-\chi}{1-\gamma}\right]d\chi = \Xi(t, \Lambda(t)). \tag{4.2}$$

Now, Eq (4.2) implies that

$$\Lambda(t) - \Lambda(0) = \frac{1-\gamma}{M(\gamma)}\Xi(t, \Lambda(t)) + \frac{\gamma}{M(\gamma)}\int_0^t \Xi(\chi, \Lambda(\chi))exp\left[-\gamma\frac{t-\chi}{1-\gamma}\right]d\chi, \tag{4.3}$$

so that

$$\Lambda(t_{n+1}) - \Lambda(0) = \frac{1-\gamma}{M(\gamma)}\Xi(t_n, \Lambda(t_n)) + \frac{\gamma}{M(\gamma)}\int_0^{t_{n+1}} \Xi(\chi, \Lambda(\chi))exp\left[-\gamma\frac{t-\chi}{1-\gamma}\right]d\chi, \tag{4.4}$$

also we have

$$\Lambda(t_n) - \Lambda(0) = \frac{1-\gamma}{M(\gamma)}\Xi(t_{n-1}, \Lambda(t_{n-1})) + \frac{\gamma}{M(\gamma)}\int_0^{t_n} \Xi(\chi, \Lambda(\chi))exp\left[-\gamma\frac{t-\chi}{1-\gamma}\right]d\chi, \tag{4.5}$$

on subtraction of Eq (4.4) from Eq (4.5), we obtain

$$\Lambda(t_{n+1}) - \Lambda(t_n) = \frac{1-\gamma}{M(\gamma)}[\Xi(t_n, \Lambda_n) - \Xi(t_{n-1}, \Lambda_{n-1})]$$

$$+\frac{\gamma}{M(\gamma)}\int_{t_n}^{t_{n+1}}\Xi(t,\Lambda(t))dt, \tag{4.6}$$

in previous equation, the integral $\int_{t_n}^{t_{n+1}}\Xi(t,\Lambda(t))dt$ is given by

$$\begin{aligned}\int_{t_n}^{t_{n+1}}\Xi(t,\Lambda(t))dt &= \int_{t_n}^{t_{n+1}}\left[\frac{\Xi(t_n,\Lambda_n)}{h}(t-t_n)-\frac{\Xi(t_{n-1},\Lambda_{n-1})}{h}(t-t_{n-1})\right.\\ &\quad\left.+\frac{\Xi(t_{n-2},\Lambda_{n-2})}{h}(t-t_n)\right]\\ &= \frac{h}{12}\left[23\Xi(t_n,\Lambda_n)-16\Xi(t_{n-1},\Lambda_{n-1})+5\Xi(t_{n-2},\Lambda_{n-2})\right]. \end{aligned}\tag{4.7}$$

Thus,

$$\begin{aligned}\Lambda(t_{n+1})-\Lambda(t_n) &= \frac{1-\gamma}{M(\gamma)}[\Xi(t_n,\Lambda_n)-\Xi(t_{n-1},\Lambda_{n-1})]\\ &\quad+\frac{\gamma h}{12M(\gamma)}[23\Xi(t_n,\Lambda_n)-16\Xi(t_{n-1},\Lambda_{n-1})\\ &\quad+5\Xi(t_{n-2},\Lambda_{n-2})]. \end{aligned}\tag{4.8}$$

Equation (4.8) implies that

$$\begin{aligned}\Lambda(t_{n+1})-\Lambda(t_n) &= \left(\frac{1-\gamma}{M(\gamma)}+\frac{23\gamma h}{12M(\gamma)}\right)\Xi(t_n,\Lambda_n)\\ &\quad-\left(\frac{1-\gamma}{M(\gamma)}+\frac{16\gamma h}{12M(\gamma)}\right)\Xi(t_{n-1},\Lambda_{n-1})\\ &\quad+\frac{5\gamma h}{12M(\gamma)}\Xi(t_{n-2},\Lambda_{n-2})]. \end{aligned}\tag{4.9}$$

Hence we have,

$$\begin{aligned}\Lambda_{n+1} &= \Lambda(t_n)+\left(\frac{1-\gamma}{M(\gamma)}+\frac{23\gamma h}{12M(\gamma)}\right)\Xi(t_n,\Lambda_n)\\ &\quad-\left(\frac{1-\gamma}{M(\gamma)}+\frac{16\gamma h}{12M(\gamma)}\right)\Xi(t_{n-1},\Lambda_{n-1})\\ &\quad+\frac{5\gamma h}{12M(\gamma)}\Xi(t_{n-2},\Lambda_{n-2})]+\mathrm{R}_n^{\gamma}(t), \end{aligned}\tag{4.10}$$

which is the required obtained numerical solution using three step ABM scheme. In Eq (4.10), we have

$$\mathrm{R}_t^{\gamma}(t) = \frac{\gamma}{M(\gamma)}\int_0^t \frac{3}{8}\Xi^4(\chi)h^3 d\chi$$

$$\|\mathrm{R}_t^{\gamma}(t)\|_\infty = \frac{\gamma}{M(\gamma)}\left\|\int_0^t \frac{3}{8}\Xi^4(\chi)h^3\right\|_\infty d\chi \tag{4.11}$$

$$\leq \frac{3\gamma h^3}{8M(\gamma)} \int_0^t \|\Xi^4(\chi)\|_\infty d\chi$$
$$\leq \frac{3\gamma h^3}{8M(\gamma)} t_{max}(\mathcal{P}),$$

where $\mathcal{P} = max_{\chi\in[0,t]} \|\Xi^4(\chi)\|_\infty$.

5. Numerical simulations and discussion

Now, we use the stated numerical scheme as presented in the previous section to get the approximate solutions of the considered system as proposed in the current investigation using the fractional Caputo-Fabrizio operator.

We take I.Cs as (100,0,1) for the simulation in Figures 1–3. In this section, we have presented the three compartments of the proposed model graphically via Matlab at fractional-order $\gamma = 0.85, 0.9, 0.95, 1$. From the figures, we conclude that when the uninfected cells $x(t)$ going on decreasing, then the infected CD4$^+$ T-cells $y(t)$ and plasma virion density $z(t)$ is going to increase. Also, we see that smaller the fractional-order, faster the decay and growth process, and when the fractional order tends to 1, the fractional-order curve goes to the integer-order curve. Figures 4–5 represent the complex behaviour of the proposed model. We have used the following parameters values for studying oscillatory and chaotic behaviour

$$s^{'} = 0.0272;\ \mu = 2.00136;\ \beta = 0.00027;\ \epsilon = 3.8;\ c = 1.5;\ \zeta = 2.9.$$

The oscillatory and chaos behaviour is presented in the Figures 4 and 5, respectively. Thus fractional-order model extends the model defined by integer order operator. So from the above discussion, we reach to decide that mathematical modelling of real phenomena under Caputo-Fabrizio derivative is better for modelling the infectious diseases.

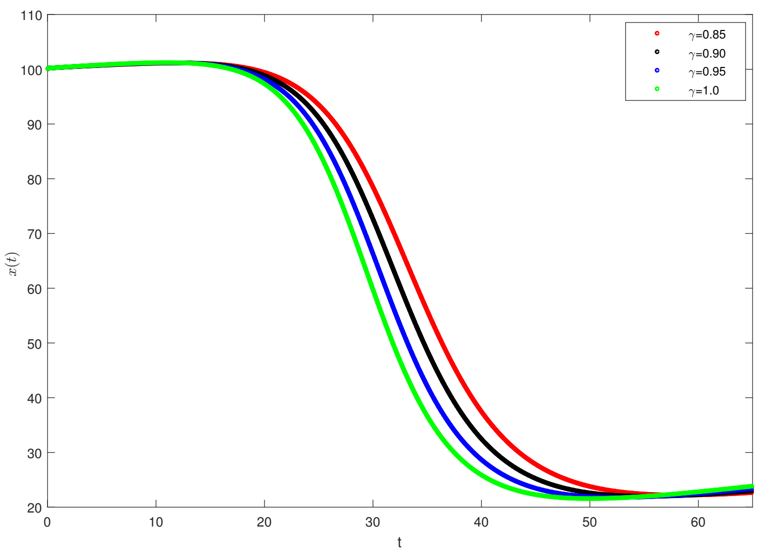

Figure 1. Graphical representation of $x(t)$ under Caputo-Fabrizio derivative at different fractional order.

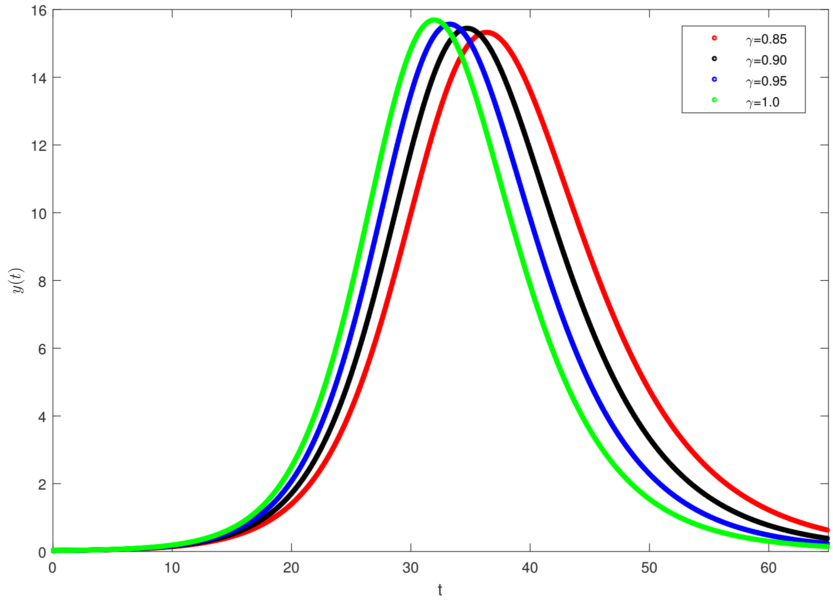

Figure 2. Graphical representation of $y(t)$ under Caputo-Fabrizio derivative at different fractional order.

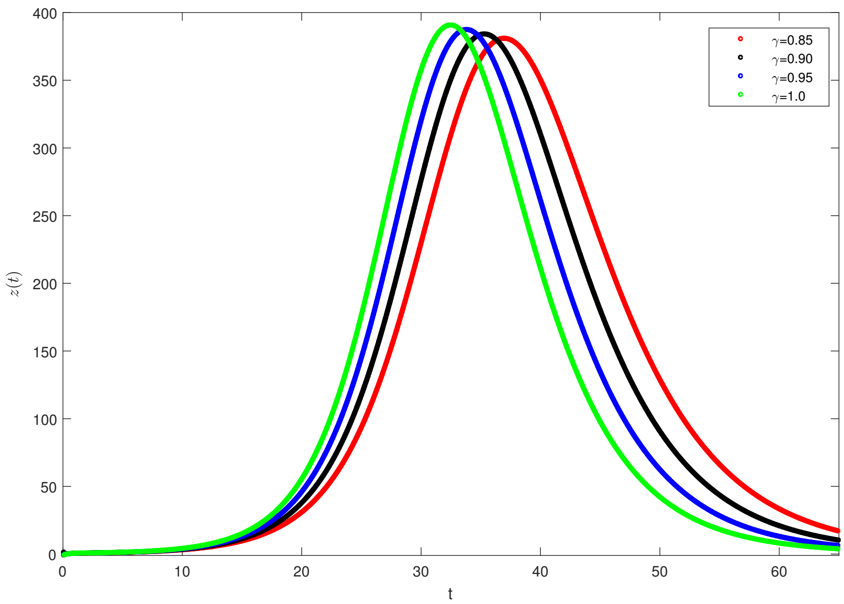

Figure 3. Graphical representation of $z(t)$ under Caputo-Fabrizio derivative at different fractional order.

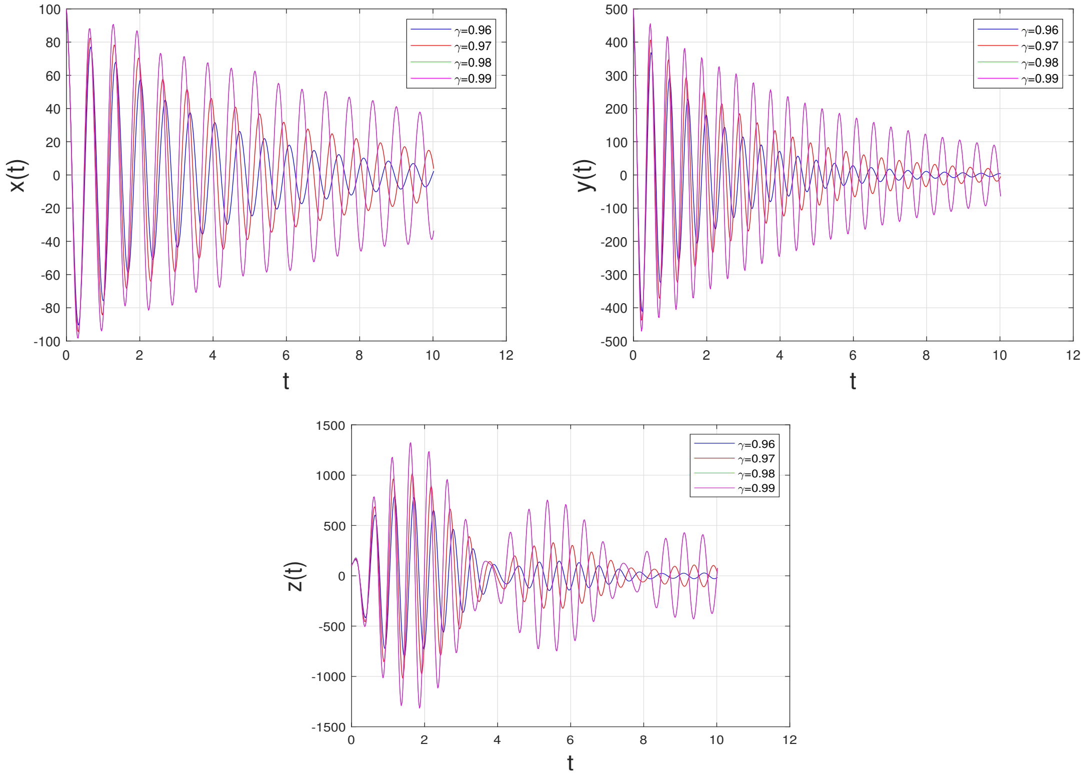

Figure 4. Oscillatory behaviour of the different class for different values of γ.

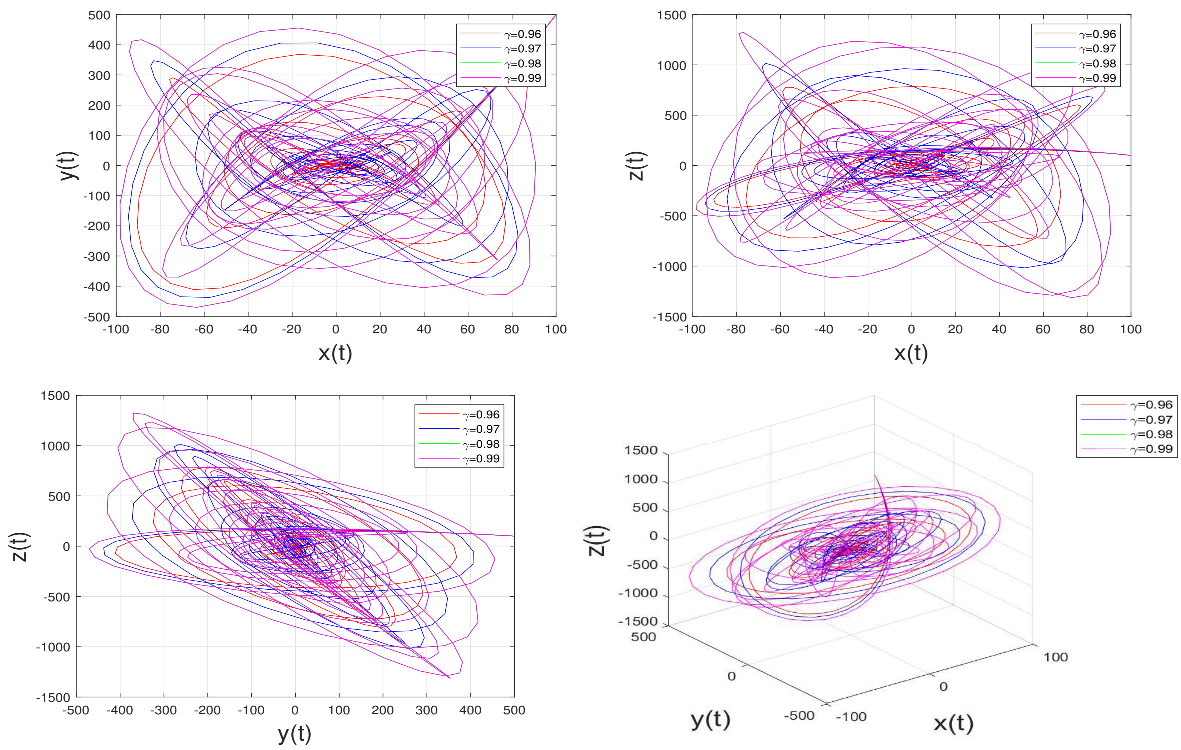

Figure 5. Chaos behaviour of the model for different γ values.

6. Conclusions

In this paper, we looked at the Caputo-Fabrizio fractional model of HIV-1 infection and how antiviral medication therapy affected it. The existence theory of the suggested model was built using a fixed point technique. We have presented the Picard stability of the suggested model through fixed point theory. In order to obtain the necessary numerical scheme for the model considered under CF operator, we have used Adams-Bashforth numerical method. We have depicted the results graphically to study the dynamics of the different classes for various fractional orders. Through graphical representation, we have presented the complex behavior of the model for different fractional orders. In the last, we have studied the limit cycle oscillations and chaos behavior of different compartments of the suggested model. In future, one can study the HIV model with control strategies under generalized operators.

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. R. Weiss, How does HIV cause AIDS? *Science*, **260** (1993), 1273–1279. https://doi.org/10.1126/science.8493571
2. D. C. Douek, M. Roederer, R. A. Koup, Emerging concepts in the immunopathogenesis of AIDS, *Annu. Rev. Med.*, **60** (2009), 471–484. https://doi.org/10.1146/annurev.med.60.041807.123549
3. R. M. Anderson, R. M. May, Population biology of infectious diseases: Part I, *Nature*, **280** (1979), 361–367.
4. R. M. May, R. M. Anderson, Population biology of infectious diseases: Part II, *Nature,* **280** (1979), 455–461. https://doi.org/10.1038/280455a0
5. H. C. Tuckwell, F. Y. M. Wan, Nature of equilibria and effects of drug treatments in some simple viral population dynamical models, *IMA J. Math. Control I.,* **17** (2000), 311–327. https://doi.org/10.1093/imamci/17.4.311
6. R. T. Alqahtani, S. Ahmad, A. Akgül, Dynamical analysis of bio-ethanol production model under generalized nonlocal operator in Caputo sense, *Mathematics*, **9** (2021), 2370. https://doi.org/10.3390/math9192370
7. P. A. Naik, J. Zu, K. M. Owolabi, Global dynamics of a fractional order model for the transmission of HIV epidemic with optimal control, *Chaos Soliton. Fract.*, **138** (2020), 109826. https://doi.org/10.1016/j.chaos.2020.109826
8. S. Ali, S. Mubeen, R. S. Ali, G. Rahman, A. Morsy, K. S. Nisar, et al., Dynamical significance of generalized fractional integral inequalities via convexity, *AIMS Math.,* **6** (2021), 9705–9730. http://dx.doi.org/10.3934/math.2021565
9. S. Aljoudi, B. Ahmad, A. Alsaedi, Existence and uniqueness results for a coupled system of Caputo-Hadamard fractional differential equations with nonlocal Hadamard type integral boundary conditions, *Fractal Fract.,* **4** (2020), 13. https://doi.org/10.3390/fractalfract4020013

10. A. Shaikh, K. S. Nisar, V. Jadhav, S. K. Elagan, M. Zakarya, Dynamical behaviour of HIV/AIDS model using fractional derivative with Mittag-Leffler kernel, *Alex. Eng. J.,* **64** (2022), 2601–2610. https://doi.org/10.1016/j.aej.2021.08.030

11. F. Rahman, A. Ali, S. Saifullah, Analysis of time-fractional Φ^4-equation with singular and non-singular kernels, *Int. J. Appl. Comput. Math.,* **7** (2021), 192. https://doi.org/10.1007/s40819-021-01128-w

12. T. Jin, X. Yang, H. Xia, H. Ding, Reliability index and option pricing formulas of the first-hitting time model based on the uncertain fractional-order differential equation with Caputo type, *Fractals,* **29** (2021), 2150012. https://doi.org/10.1142/S0218348X21500122

13. E. K. Akgül, A. Akgül, D. Baleanu, Laplace transform method for economic models with constant proportional Caputo derivative, *Fractal Fract.,* **4** (2020), 30. https://doi.org/10.3390/fractalfract4030030

14. S. Ahmad, A. Ullah, A. Akgül, M. De la Sen, A novel homotopy perturbation method with applications to nonlinear fractional order KdV and Burger equation with exponential-decay kernel, *J. Funct. Space.,* **2021** (2021), 11. https://doi.org/10.1155/2021/8770488

15. K. M. Owolabi, A. Atangana, Analysis and application of new fractional Adams-Bashforth scheme with Caputo-Fabrizio derivative, *Chaos Soliton. Fract.,* **105** (2017), 111–119. https://doi.org/10.1016/j.chaos.2017.10.020

16. N. Heymans, J. C. Bauwens, Fractal rheological models and fractional differential equations for viscoelastic behavior, *Rheol. Acta,* **33** (1994), 210–219. https://doi.org/10.1007/BF00437306

17. F. B. Tatom, The relationship between fractional calculus and fractals, *Fractals*, **3** (1995), 217–229.

18. M. Al-Refai, T. Abdeljawad, Analysis of the fractional diffusion equations with fractional derivative of non-singular kernel, *Adv. Differ. Equ.,* **2017** (2017), 315. https://doi.org/10.1186/s13662-017-1356-2

19. T. Abdeljawad, Fractional operators with exponential kernels and a Lyapunov type inequality, *Adv. Differ. Equ.,* **2017** (2017), 313. https://doi.org/10.1186/s13662-017-1285-0

20. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.,* **1** (2015), 1–13.

21. S. A. Khan, K. Shah, F. Jarad, G. Zaman, Existence theory and numerical solutions to smoking model under Caputo-Fabrizio fractional derivative, *Chaos*, **29** (2019), 013128. https://doi.org/10.1063/1.5079644

22. T. Abdeljawad, D. Baleanu, On fractional derivatives with exponential kernel and their discrete versions, *Rep. Math. Phys.,* **80** (2017), 11–27. https://doi.org/10.1016/S0034-4877(17)30059-9

23. S. Saifullah, A. Ali, M. Irfan, K. Shah, Time-fractional Klein-Gordon equation with solitary/shock waves solutions, *Math. Probl. Eng.,* **2021** (2021), 15. https://doi.org/10.1155/2021/6858592

24. S. Ahmad, A. Ullah, A. Akgül, M. De la Sen, A study of fractional order Ambartsumian equation involving exponential decay kernel, *AIMS Math.,* **6** (2021), 9981–9997. https://doi.org/10.3934/math.2021580

25. E. J. Moore, S. Sirisubtawee, S. Koonprasert, A Caputo-Fabrizio fractional differential equation model for HIV/AIDS with treatment compartment, *Adv. Differ. Equ.*, **2019** (2019), 200. https://doi.org/10.1186/s13662-019-2138-9

26. J. Wang, Y. Zhou, M. Medved, Picard and weakly Picard operators technique for nonlinear differential equations in Banach spaces, *J. Math. Anal. Appl.*, **389** (2012), 261–274. https://doi.org/10.1016/j.jmaa.2011.11.059