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Research article

Ideals on neutrosophic extended triplet groups

Xin Zhou[1] **and Xiao Long Xin**[1,2,*]

[1] School of Science, Xi'an Polytechnic University, 710048 Xi'an, China
[2] School of Mathematics, Northwest University, Xi'an, 710127, China

* **Correspondence:** Email: xlxin@nwu.edu.cn; Tel: +8613571952880.

Abstract: In this paper, we introduce the concept of (prime) ideals on neutrosophic extended triplet groups (NETGs) and investigate some related properties of them. Firstly, we give characterizations of ideals generated by some subsets, which lead to a construction of a NETG by endowing the set consisting of all ideals with a special multiplication. In addition, we show that the set consisting of all ideals is a distributive lattice. Finally, by introducing the topological structure on the set of all prime ideals on NETGs, we obtain the necessary and sufficient conditions for the prime ideal space to become a T_1-space and a Hausdorff space.

Keywords: NETG; ideal; prime ideal space
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1. Introduction

The notion of a neutrosophic extended triplet group (NETG), as a new generalization of the notion of a standard group, is derived from the basic idea of the neutrosophic sets. The concept of neutrosophic sets [8], first introduced by Florentin Smarandache in 1998, is the generalization of classical sets [9], fuzzy sets [13], intuitionistic fuzzy sets [1,9] and so on. Neutrosophic sets are very useful to handle problems consisting uncertainty, imprecision, indeterminacy, incompleteness and falsity. As a result, neutrosophic sets have received wide attention both on practical applications [5–7] and on theory as well [14,15].

Since groups are the most fundamental algebraic structure with respect to some binary operation and play the role of back bone in almost all algebraic structures theory [2,3,10], Smarandache and Ali introduced the notion of a neutrosophic triplet group (NTG) [12] as an application of the basic idea of neutrosophic sets. A semigroup $(N, *)$ is called a neutrosophic triplet group, if every element a in N has its own neutral element (denoted by $neut(a)$) different from the classical identity element of a group, and there exists at least one opposite element (denoted by $anti(a)$) in N relative to $neut(a)$ such that $a * neut(a) = neut(a) * a = a$, and $a * anti(a) = anti(a) * a = neut(a)$. Here, since $neut(a)$ is not allowed to be equal to the classical identity element as a special case, the notion of a neutrosophic extended triplet group (NETG) was introduced in [11] by removing this restriction, and so the classical groups can be regarded as a special case of NETGs. Until now, much research work has been done on NTGs and NETGs [4,16–21] and many meaningful results have been achieved. Similar to the role of subgroups played in group theory, the notion of NT-subgroups is also an important basic concept in NETG theory, which has been proposed in some literatures (see [16,17]). To further study structures of NT-subgroups, in this paper, we shall consider (prime) ideals on NETGs, which are a special kind of NT-subgroups.

This paper is organized in the following way. In Section 2 we will give some necessary definitions and results on NETGs. In Section 3, we shall introduce the concepts of ideals and prime ideals on NETGs. We will give the ideal generation formula and consider the set of all ideals on NETGs. In fact, we will prove that the set of all ideals of a NETG, under inclusion order, is a distributive lattice, and construct a NETG on the set of all ideals by endowing it with a special multiplication. In Section 4, we will introduce the topological space $(Prim(N), \tau)$ induced by all prime ideals of a NETG N and give necessary and sufficient conditions for the topological space to be a T_1-space and a Hausdorff space.

2. Preliminaries

In this section, we will give some concepts and results on NETGs, which will be used in the following sections of this paper.

Definition 1. [11] Let N be a non-empty set together with a binary operation $*$. Then N is called a *neutrosophic extended triplet group* or *NETG* for short, if $(N, *)$ is a semigroup and for any $a \in N$, there exist a neutral of "a" (denoted by $neut(a)$) and an opposite of "a" (denoted by $anti(a)$) such that $neut(a) \in N$, $anti(a) \in N$ and:

$$a * neut(a) = neut(a) * a = a;$$

$$a * anti(a) = anti(a) * a = neut(a).$$

Notice that for every element a of a NETG $(N, *)$, $neut(a)$ is allowed to be equal to the classical identity element of a group, and so all classical groups are special NETGs.

Proposition 1. [16] *Let $(N, *)$ be a NETG. Then for every $a \in N$, the following statements hold*:

(1) *$neut(a)$ is unique*;

(2) $neut(a) * neut(a) = neut(a)$;

(3) $neut(neut(a)) = neut(a)$.

Notice that $anti(a)$ may be not unique for every element a in a NETG $(N, *)$, so we use $\{anti(a)\}$ to denote the set of all opposites of a.

Example 1. Consider $Z_6 = \{0, 1, 2, 3, 4, 5\}$ under multiplication $\cdot$ modulo 6, then $(Z_6, \cdot)$ is a NETG, in which $neut(0) = 0$, $\{anti(0)\} = \{0, 1, 2, 3, 4, 5\}$; $neut(1) = 1$, $\{anti(1)\} = \{1\}$; $neut(2) = 4$, $\{anti(2)\} = \{2, 5\}$; $neut(3) = 3$, $\{anti(3)\} = \{1, 3, 5\}$; $neut(4) = 4$, $\{anti(4)\} = \{1, 4\}$; $neut(5) = 1$, $\{anti(5)\} = \{5\}$.

Proposition 2. [20] *Let $(N, *)$ be a NETG. Then the following properties hold: $\forall\ a \in N$, $\forall\ p$, $q \in \{anti(a)\}$*,

(1) $p * neut(a) \in \{anti(a)\}$;

(2) $p * neut(a) = q * neut(a) = neut(a) * q$.

It is well known that in semigroup theory, a^{-1} is called the inverse element of a and it is unique. Similarly, we can define a unary operation $a \mapsto a^{-1}$ by $a^{-1} = anti(a) * neut(a)$ in a NETG $(N, *)$. Then Proposition 2 indicates that this unary operation is well-defined, and in a NETG $(N, *)$, $a^{-1} \in \{anti(a)\}$ and a^{-1} is determined uniquely for every element a of $(N, *)$. Moreover, Theorem 2 in [20] declares that in a NETG $(N, *)$, this unary operation has the following properties:

$$(a^{-1})^{-1} = a, \ a * a^{-1} * a = a, \ a * a^{-1} = a^{-1} * a,$$

which leads a^{-1} to be called the inverse element of a in [20]. Therefore, in the following, we will regard a^{-1} to be $anti(a) * neut(a)$ for every element a of a NETG $(N, *)$, and it holds obviously that $a^{-1} * a = a * a^{-1} = neut(a)$.

Definition 2. [16] Let $(N, *)$ be a NETG. If $a * neut(b) = neut(b) * a$ for all $a, \ b \in N$, then N is called a *weak commutative neutrosophic extended triplet group* or *WCNETG* for short.

Proposition 3. [16] *Let $(N, *)$ be a WCNETG. Then for all $a, \ b \in N$,*

(1) $neut(a) * neut(b) = neut(b * a)$*;*

(2) $anti(a) * anti(b) \in \{anti(b * a)\}$.

Proposition 4. [21] *Let $(N, *)$ be a NETG, then $\forall \ a \in N$, $[neut(a)]^{-1} = neut(a) = neut(a^{-1})$.*

Proposition 5. [21] *Let $(N, *)$ be a WCNETG, then $\forall \ a, \ b \in N$, $(a * b)^{-1} = b^{-1} * a^{-1}$.*

Definition 3. [16,17] Let $(N, *)$ is a NETG. A non-empty subset $S \subseteq N$ is called a *NT-subgroup* of N if it satisfies the following conditions:

(1) $a * b \in S$ for all $a, \ b \in S$;

(2) $\{anti(a)\} \cap S \neq \emptyset$ for all $a \in S$.

Example 2. Consider $Z_6 = \{0, 1, 2, 3, 4, 5\}$ under multiplication $\cdot$ modulo 6, from Example 1 we know $(Z_6, \cdot)$ is a NETG. Then we can list out some NT-subgroups of Z_6. For example, $S_1 = \{0\}$, $S_2 = \{1\}$, $S_3 = \{3\}$, $S_4 = \{4\}$, $S_5 = \{0, \ 1\}$, $S_6 = \{0, \ 3\}$, $S_7 = \{0, \ 4\}$, $S_8 = \{2, \ 4\}$, $S_9 = \{1, \ 3\}$, $S_{10} = \{1, \ 5\}$, $S_{11} = \{0, \ 2, \ 4\}$, $S_{12} = \{0, \ 2, \ 3, \ 4\}$.

3. Ideals of NETGs

In this section, we are going to propose a special kind of NT-subgroups, called (prime) ideals, of NETGs. Besides presenting the lattice structure of the set of all ideals, we will give the ideal generation formula on NETGs and construct a NETG by endowing the set consisting of all ideals with a special multiplication.

Definition 5. Let $(N, *)$ be a NETG. A non-empty subset $S \subseteq N$ is called an *ideal* of N if for all $s \in S$ and $a \in N$,

(1) $s * a \in S$ *and* $a * s \in S$*;*

(2) $\{anti(s)\} \cap S \neq \emptyset$.

We use $Id(N)$ to denote the set of all ideals on N, then $Id(N)$ is a partially ordered set with the inclusion order $\subseteq$.

Remark 1. (1) From Definition 5 we get that ideals must be NT-subgroups, and every NETG itself is an ideal of its own;

(2) An ideal I is called a *proper ideal* if $I \neq N$;

(3) For every subset $X \subseteq N$, we use $\langle X\rangle$ to denote the smallest ideal containing X. Hence, if $(Id(N), \subseteq)$ has the smallest element denoted by I_0, then $\langle\emptyset\rangle = I_0$, and we call $(N, *)$ *a NETG with the smallest ideal* I_0.

Example 3. Refer to Example 1, we can list out all ideals of Z_6: $I_1 = \{0\}$, $I_2 = \{0, 2, 4\}$, $I_3 = \{0, 3\}$, $I_4 = \{0, 2, 3, 4\}$, $I_5 = Z_6$.

Example 4. Let $N = \{a, b, c\}$, and we define multiplication $*$ on N as shown in Table 1.

$*$	a	b	c
a	a	b	a
b	b	b	b
c	a	b	c

Table 1

It is easy to verify that $(N, *)$ is a WCNETG, in which $neut(a) = a$, $\{anti(a)\} = \{a, c\}$, $a^{-1} = a$; $neut(b) = b$, $\{anti(b)\} = \{a, b, c\}$, $b^{-1} = b$; $neut(c) = c$, $\{anti(c)\} = \{c\}$, $c^{-1} = c$.

There are only three ideals of N that are $I_1 = \{b\}$, $I_2 = \{a, b\}$, $I_3 = \{a, b, c\}$.

Proposition 6. *Let $(N, *)$ be a NETG and I an ideal of N. Then for every $a \in I$, $neut(a) \in I$ and $a^{-1} \in I$.*

Proposition 7. *Let $(N, *)$ be a NETG, then for any subsets $I, J \in Id(N)$, we have $I \cap J \in Id(N)$ and $I \cup J \in Id(N)$.*

Proof. From Proposition 6 it holds obviously. □

Corollary 1. *Let $(N, *)$ be a NETG, then $(Id(N), \cap, \cup)$ is a distributive lattice.*

Theorem 1. *Let $(N, *)$ be a WCNETG, then for every non-empty subset $X \subseteq N$*

$$\langle X\rangle = \{neut(x) * y^{-1} : x \in X, y \in N\}.$$

Proof. Let $A = \{neut(x) * y^{-1} : x \in X, y \in N\}$, then for every $a \in A$, there exist $x \in X$ and $y \in N$ such that $a = neut(x) * y^{-1}$. Thus, by Proposition 5, $a^{-1} = (neut(x) * y^{-1})^{-1} = (y^{-1})^{-1} * [neut(x)]^{-1} = (y^{-1})^{-1} * neut(x) = neut(x) * (y^{-1})^{-1} \in A$. Moreover, for every $b \in N$, $a * b = [neut(x) * y^{-1}] * b = neut(x) * (y^{-1} * b) = neut(x) * (b^{-1} * y)^{-1} \in A$, and $b * a = b * [neut(x) * y^{-1}] = b * [y^{-1} * neut(x)] = (b * y^{-1}) * neut(x) = neut(x) * (y * b^{-1})^{-1} \in A$. Therefore, A is an ideal. Since for every $m \in X$, $m = neut(m)*m = neut(m)*(m^{-1})^{-1} \in A$, we have $X \subseteq A$. Let I be an ideal and $X \subseteq I$. Then for every $n \in A$, there exist $p \in X$ and $q \in N$ such that $n = neut(p) * q^{-1}$. By Proposition 6 and $p \in X \subseteq I$, we have $neut(p) \in I$, and so $n = neut(p)*q^{-1} \in I$. Thus, $A \subseteq I$. Hence, $\langle X\rangle = A = \{neut(x)*y^{-1} : x \in X, y \in N\}$. □

Proposition 8. *Let $(N, *)$ be a WCNETG, then for any $a, b \in N$, $\langle a\rangle \cap \langle b\rangle = \langle a * b\rangle$.*

Proof. From Theorem 1 we know $\langle a\rangle = \{neut(a) * c^{-1} : c \in N\}$, $\langle b\rangle = \{neut(b) * d^{-1} : d \in N\}$ and $\langle a * b\rangle = \{neut(a * b) * k^{-1} : k \in N\}$.

Let $x \in \langle a\rangle \cap \langle b\rangle$, then there exist $c, d \in N$ such that $x = neut(a) * c^{-1}$ and $x = neut(b) * d^{-1}$. By Proposition 1 and Proposition 3, we obtain that $neut(x) = neut(neut(a) * c^{-1}) = neut(c^{-1}) * neut(neut(a)) = neut(c^{-1})*neut(a)$. Then by Proposition 4, $x = x*neut(x) = [neut(b)*d^{-1}]*[neut(c^{-1})*$

$neut(a)] = neut(b) * [d^{-1} * (neut(c^{-1}) * neut(a))] = neut(b) * [(d^{-1} * neut(c^{-1})) * neut(a)] = neut(b) * [neut(a) * (d^{-1} * neut(c^{-1}))] = [neut(b) * neut(a)] * [d^{-1} * neut(c^{-1})] = neut(a * b) * [d^{-1} * (neut(c))^{-1}] = neut(a * b) * [neut(c) * d]^{-1} \in \langle a * b \rangle$. Hence, $\langle a \rangle \cap \langle b \rangle \subseteq \langle a * b \rangle$.

Conversely, let $y \in \langle a*b \rangle$, then there exist $k \in N$ such that $y = neut(a*b)*k^{-1} = [neut(b)*neut(a)]*k^{-1} = neut(b)*[neut(a)*k^{-1}] = neut(b)*[(neut(a))^{-1}*k^{-1}] = neut(b)*[k*neut(a)]^{-1} \in \langle b \rangle$. Moreover, $y = neut(a * b) * k^{-1} = [neut(b) * neut(a)] * k^{-1} = [neut(a) * neut(b)] * k^{-1} = neut(a) * [neut(b) * k^{-1}] = neut(a) * [(neut(b))^{-1} * k^{-1}] = neut(a) * [k * neut(b)]^{-1} \in \langle a \rangle$. Therefore, $y \in \langle a \rangle \cap \langle b \rangle$, which implies $\langle a * b \rangle \subseteq \langle a \rangle \cap \langle b \rangle$. □

Let A and B be two non-empty subsets of N, then we shall use the notation $A * B = \{a * b : a \in A,\ b \in B\}$.

Theorem 2. *Let* $(N, *)$ *be a WCNETG. Then for any* $I_1,\ I_2 \in Id(N)$*, we have*

$$\langle I_1 * I_2 \rangle = \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}.$$

Proof. (1) First of all, we will prove that $\{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$ is an ideal of N.

(i) Let $x \in \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$ and $y \in N$. Then there exist $a \in I_1$ and $b \in I_2$ such that $x = a^{-1} * b^{-1}$. Hence, by Proposition 5, $x * y = (a^{-1} * b^{-1}) * y = a^{-1} * (b^{-1} * y) = a^{-1} * [(b^{-1} * y)^{-1}]^{-1} = a^{-1} * [y^{-1} * (b^{-1})^{-1}]^{-1} = a^{-1} * (y^{-1} * b)^{-1}$. Since I_2 is an ideal of N and $b \in I_2$, we have $y^{-1} * b \in I_2$. Thus, $x * y = a^{-1} * (y^{-1} * b)^{-1} \in \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$. Similarly, we also can prove that $y * x \in \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$.

(ii) Let $z \in \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$, then there exist $c \in I_1$ and $d \in I_2$ such that $z = c^{-1} * d^{-1} = (d * c)^{-1}$, so by Proposition 4, $z^{-1} = [(d * c)^{-1}]^{-1} = [(d * c)^{-1}]^{-1} * neut([(d * c)^{-1}]^{-1}) = [(d * c)^{-1}]^{-1} * [neut((d * c)^{-1})]^{-1}$. Since I_1 is an ideal and $c \in I_1$, we have $d * c \in I_1$. Then by Proposition 6, we can get $(d * c)^{-1} \in I_1$. Similarly, since I_2 is an ideal and $d \in I_2$, we can get $(d * c)^{-1} \in I_2$, and then $neut((d * c)^{-1}) \in I_2$. Therefore, $z^{-1} \in \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$, which means $\{anti(z)\} \cap \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\} \neq \emptyset$.

By (i) and (ii), we conclude that $\{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$ is an ideal of N.

(2) Next we are going to prove that $I_1 * I_2 \subseteq \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$. For every $x \in I_1$ and $y \in I_2$, by Proposition 6, $x^{-1} \in I_1$ and $y^{-1} \in I_2$, so $x * y = (x^{-1})^{-1} * (y^{-1})^{-1} \in \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$.

(3) Let I be an ideal of N and $I_1 * I_2 \subseteq I$. For every $m \in I_1$ and $n \in I_2$, by Proposition 6, $m^{-1} \in I_1$ and $n^{-1} \in I_2$, so $m^{-1} * n^{-1} \in I_1 * I_2 \subseteq I$. By arbitrariness of m and n, we have $\{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\} \subseteq I$, which implies $\{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$ is the smallest ideal containing $I_1 * I_2$. Therefore, $\langle I_1 * I_2 \rangle = \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\}$. □

Theorem 3. *Let* $(N, *)$ *be a WCNETG. If we define a multiplicative operation* & *on* $Id(N)$ *by: For all* $I_1,\ I_2 \in Id(N)$*,*

$$I_1 \& I_2 = \{a^{-1} * b^{-1} : a \in I_1,\ b \in I_2\},$$

then $(Id(N), \&)$ *is a NETG.*

Proof. From Theorem 2, we know the multiplicative operation & is well-defined on $Id(N)$.

(1) Firstly, we are going to prove & is associative on $Id(N)$.

Assume that I_1, I_2 and I_3 are three arbitrary ideals of N. Now we will prove $(I_1 \& I_2) \& I_3 = I_1 \& (I_2 \& I_3)$. By definition of &, we know $(I_1 \& I_2) \& I_3 = \{x^{-1} * y^{-1} : x \in I_1 \& I_2,\ y \in I_3\}$, and $I_1 \& (I_2 \& I_3) = \{x^{-1} * y^{-1} : x \in I_1,\ y \in I_2 \& I_3\}$. For every $m \in (I_1 \& I_2) \& I_3$, there exist $x \in I_1 \& I_2$

and $y \in I_3$ such that $m = x^{-1} * y^{-1}$, and there exist $a \in I_1$ and $b \in I_2$ such that $x = a^{-1} * b^{-1}$. Therefore, by Propositions 4 and 5, $m = x^{-1} * y^{-1} = (a^{-1} * b^{-1})^{-1} * y^{-1} = [(b^{-1})^{-1} * (a^{-1})^{-1}] * y^{-1} = (b * a) * y^{-1} = [b * (neut(a) * a)] * y^{-1} = [(b * neut(a)) * a] * y^{-1} = [(neut(a) * b) * a] * y^{-1} = neut(a) * [(b * a) * y^{-1}] = (neut(a))^{-1} * [((b * a)^{-1})^{-1} * y^{-1}]$. From Proposition 6 and Remark 3 we can get $neut(a) \in I_1$. Since I_2 is an ideal and $b \in I_2$, we have $b * a \in I_2$, and by Proposition 6 again, $(b * a)^{-1} \in I_2$. Hence, $m = (neut(a))^{-1} * [((b * a)^{-1})^{-1} * y^{-1}] \in I_1\&(I_2\&I_3)$. By arbitrariness of m, we conclude that $(I_1\&I_2)\&I_3 \subseteq I_1\&(I_2\&I_3)$. We also can prove $I_1\&(I_2\&I_3) \subseteq (I_1\&I_2)\&I_3$ in the same way. Consequently, $(I_1\&I_2)\&I_3 = I_1\&(I_2\&I_3)$. By arbitrariness of I_1, I_2 and I_3, we can conclude that & is associative on $Idl(N)$.

(2) From (1) we know $(Id(N), \&)$ is a semigroup. Now we are going to prove $(Id(N), \&)$ is a NETG.

Assume that I is an arbitrary ideal of N. It is clear that $I\&I = \{a^{-1} * b^{-1} : a \in I,\ b \in I\} \subseteq I$. Conversely, for every $x \in I$, since $neut(x) \in I$ and $x^{-1} \in I$, by Proposition 4, we have $x = neut(x) * x = [neut(x)]^{-1} * (x^{-1})^{-1} \in I\&I$, so by arbitrariness of x, we have $I \subseteq I\&I$. Therefore, $I\&I = I$, which implies $neut(I) = I$ and $I \in \{anti(I)\}$.

By arbitrariness of I, we conclude that $(Id(N), \&)$ is a NETG. □

Definition 6. Let P be a proper ideal of a NETG $(N, *)$. Then P is said to be a *prime ideal*, if for any $x,\ y \in N,\ x * y \in P$ implies $x \in P$ or $y \in P$.

We use $Prim(N)$ to denote the set of all ideals on N. It is easy to see that I is a prime ideal of the NETG $(N, *)$ if and only if for all non-empty subsets $A,\ B \subseteq N,\ A * B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

Example 5. Refer to Example 1 and Example 3, $I_2 = \{0,\ 2,\ 4\}$, $I_3 = \{0,\ 3\}$ and $I_4 = \{0,\ 2,\ 3,\ 4\}$ are all prime ideals of the NETG $(Z_6, \cdot)$. Moreover, in Example 4, $I_2 = \{a,\ b\}$ is the unique prime ideal of the NETG $(N, *)$.

Theorem 4. *Let $(N, *)$ be a WCNETG and I an ideal of N such that $x \notin I$. Then there exists a prime ideal P such that $I \subseteq P$ and $x \notin P$.*

Proof. Assume that I is an ideal and $x \notin I$. Let $\Pi = \{J \in Id(N) | I \subseteq J \text{ and } x \notin J\}$, then $I \in \Pi$. From Proposition 6, it is clear that Π satisfies Zorn's Lemma. Let P be a maximal element of Π, then $x \notin P$. Since $P \neq N$, we can choose $a,\ b \in N$ such that $a \notin P$ and $b \notin P$, then $P \subsetneq P \cup \langle a\rangle \in Id(N)$ and $P \subsetneq P \cup \langle b\rangle \in Id(N)$. By the maximality of P, we have $x \in P \cup \langle a\rangle$ and $x \in P \cup \langle b\rangle$. Thus, by Corollary 1 and Proposition 8, we have $x \in (P \cup \langle a\rangle) \cap (P \cup \langle b\rangle) = P \cup (\langle a\rangle \cap \langle b\rangle) = P \cup \langle a * b\rangle$. If $a * b \in P$, then $P \cup \langle a * b\rangle = P$, and so $x \in P$, which is a contradiction. Thus, $a * b \notin P$, which implies P is a prime ideal. □

4. Prime ideal spaces

In this section, we shall define the topological structure on the collection of all prime ideals of a NETG and study some topological properties of the space. Necessary and sufficient conditions will be proposed for the space becoming a T_1-space and a Hausdorff space.

Let $(N, *)$ be a NETG. For $A \subseteq N$, we define $H(A) = \{P \in Prim(N) | A \nsubseteq P\}$, and for any $a \in N$, $H(a) = H(\{a\})$.

Proposition 9. *Let $(N, *)$ be a NETG. Then for any $A,\ B \subseteq N,\ A \subseteq B$ implies $H(A) \subseteq H(B)$.*

Proof. If $H(A) = \emptyset$, then $H(A) \subseteq H(B)$. If $H(A) \neq \emptyset$, then for every $P \in H(A)$, we have $A \nsubseteq P$, which implies $A \neq \emptyset$ and there exists $a \in A$ such that $a \notin P$. Since $A \subseteq B$, we get $a \in B$, thus, $B \nsubseteq P$, and so $P \in H(B)$. Therefore, $H(A) \subseteq H(B)$. □

Proposition 10. *Let $(N, *)$ be a NETG with the smallest ideal I_0. Then for any $A,\ B \subseteq N$, the following statements hold:*

(1) $H(N) = Prim(N),\ H(\emptyset) = H(I_0) = \emptyset$;

(2) *if* $\{A_i\}_{i\in\Lambda}$ *is any family of subsets of N, then* $H(\bigcup_{i\in\Lambda} A_i) = \bigcup_{i\in\Lambda} H(A_i)$;

(3) $H(A) = H(\langle A\rangle)$;

(4) $H(A) \cap H(B) = H(\langle A\rangle \cap \langle B\rangle)$.

Proof. (1) It holds obviously.

(2) Since for every $i \in \Lambda$, $A_i \subseteq \bigcup_{i\in\Lambda} A_i$, by Proposition 9, we have $H(A_i) \subseteq H(\bigcup_{i\in\Lambda} A_i)$ and so $\bigcup_{i\in\Lambda} H(A_i) \subseteq H(\bigcup_{i\in\Lambda} A_i)$. Conversely, if $H(\bigcup_{i\in\Lambda} A_i) = \emptyset$, then $H(\bigcup_{i\in\Lambda} A_i) \subseteq \bigcup_{i\in\Lambda} H(A_i)$. If $H(\bigcup_{i\in\Lambda} A_i) \neq \emptyset$, then for every $P \in H(\bigcup_{i\in\Lambda} A_i)$, we have $\bigcup_{i\in\Lambda} A_i \nsubseteq P$. Thus, there exists $i_0 \in \Lambda$ such that $A_{i_0} \nsubseteq P$, and so $P \in H(A_{i_0}) \subseteq \bigcup_{i\in\Lambda} H(A_i)$. Hence, $H(\bigcup_{i\in\Lambda} A_i) \subseteq \bigcup_{i\in\Lambda} H(A_i)$. Therefore, $H(\bigcup_{i\in\Lambda} A_i) = \bigcup_{i\in\Lambda} H(A_i)$.

(3) Since $A \subseteq \langle A\rangle$, we have $H(A) \subseteq H(\langle A\rangle)$. Conversely, if $H(\langle A\rangle) = \emptyset$, then $H(\langle A\rangle) \subseteq H(A)$. If $H(\langle A\rangle) \neq \emptyset$, then for every $P \in H(\langle A\rangle)$, we have $\langle A\rangle \nsubseteq P$. Since $A \subseteq P$ implies $\langle A\rangle \subseteq P$, a contradiction arises. Hence, $A \nsubseteq P$, which means $P \in H(A)$. Thus, $H(\langle A\rangle) \subseteq H(A)$. Therefore, $H(A) = H(\langle A\rangle)$.

(4) Applying (3), we only need to prove $H(\langle A\rangle)\cap H(\langle B\rangle) = H(\langle A\rangle\cap\langle B\rangle)$. Since $\langle A\rangle\cap\langle B\rangle \subseteq \langle A\rangle$, $\langle B\rangle$, we have $H(\langle A\rangle \cap \langle B\rangle) \subseteq H(\langle A\rangle) \cap H(\langle B\rangle)$. Conversely, if $H(\langle A\rangle) \cap H(\langle B\rangle) = \emptyset$, then $H(\langle A\rangle) \cap H(\langle B\rangle) \subseteq H(\langle A\rangle \cap \langle B\rangle)$. If $H(\langle A\rangle) \cap H(\langle B\rangle) \neq \emptyset$, then for every $P \in H(\langle A\rangle) \cap H(\langle B\rangle)$, we have $\langle A\rangle \nsubseteq P$ and $\langle B\rangle \nsubseteq P$, which implies there exist $a \in \langle A\rangle$ and $b \in \langle B\rangle$ such that $a \notin P$ and $b \notin P$. Since P is a prime ideal, we conclude $a*b \notin P$. However, $a*b \in \langle A\rangle\cap\langle B\rangle$. Thus, $\langle A\rangle\cap\langle B\rangle \nsubseteq P$, that is, $P \in H(\langle A\rangle\cap\langle B\rangle)$. Hence, $H(\langle A\rangle) \cap H(\langle B\rangle) \subseteq H(\langle A\rangle \cap \langle B\rangle)$. Therefore, $H(\langle A\rangle) \cap H(\langle B\rangle) = H(\langle A\rangle \cap \langle B\rangle)$. □

Proposition 11. *Let $(N, *)$ be a NETG. Then for any $x,\ y \in N$, the following statements hold:*

(1) $H(x) \cap H(y) = H(x * y)$;

(2) $\bigcup_{x\in N} H(x) = Prim(N)$;

(3) $H(x) = H(neut(x)) = H(x^{-1})$.

Proof. (1) If $H(x * y) \neq \emptyset$, then for every $P \in H(x * y)$, we have $x * y \notin P$. If $x \in P$ or $y \in P$, then $x * y \in P$, which is a contradiction. Hence, $x \notin P$ and $y \notin P$, that is, $P \in H(x) \cap H(y)$. Thus, $H(x * y) \subseteq H(x) \cap H(y)$. Conversely, if $H(x) \cap H(y) \neq \emptyset$, then for every $P \in H(x) \cap H(y)$, we have $x \notin P$ and $y \notin P$. Since P is a prime ideal, we conclude $x * y \notin P$, that is, $P \in H(x * y)$. Hence, $H(x)\cap H(y) \subseteq H(x*y)$. Therefore, $H(x)\cap H(y) = H(x*y)$, and when $H(x*y) = \emptyset$ or $H(x)\cap H(y) = \emptyset$, the equation holds obviously.

(2) It follows from Proposition 10 obviously, because $N = \bigcup_{x\in N}\{x\}$ and $Prim(N) = H(N)$.

(3) It follows from Proposition 6 obviously. □

Let $(N, *)$ be a NETG with the smallest ideal I_0, and let $\tau = \{H(X)|X \subseteq N\}$ be a subset of the power set of $Prim(N)$. Then by Proposition 10, we obtain the following conclusions:

(1) $\emptyset,\ Prim(N) \in \tau$;

(2) *if* $H(A),\ H(B) \in \tau$, *then* $H(A) \cap H(B) \in \tau$;

(3) *if* $\{H(A_i)|i \in \Lambda\} \subseteq \tau$, *then* $\bigcup_{i\in\Lambda} H(A_i) \in \tau$.

Therefore, τ is a topology on $Prim(N)$. We call $(Prim(N), \tau)$ a *prime ideal space* and $H(X)$ an *open set in* τ.

Proposition 12. *Let $(N, *)$ be a NETG with the smallest ideal I_0, then $\{H(a)|a \in N\}$ is a base of topology $(Prim(N), \tau)$.*

Proof. For every non-empty subset $A \subseteq N$, $A = \bigcup_{a\in N}\{a\}$. From Proposition 10 we get that $H(A) =$

$\bigcup_{a\in A} H(a)$. Moreover, $H(\emptyset) = H(I_0)$. □

Proposition 13. *Let* $(N, *)$ *be a NETG. Then for any* $I,\ J \in Id(N)$*, the following hold:*

(1) $H(I \cap J) = H(I) \cap H(J)$;

(2) $H(I \cup J) = H(I) \cup H(J)$.

Proof. (1) Obviously, $H(I \cap J) \subseteq H(I) \cap H(J)$. Conversely, if $H(I) \cap H(J) = \emptyset$, then $H(I) \cap H(J) \subseteq H(I \cap J)$. If $H(I) \cap H(J) \neq \emptyset$, then for every $P \in H(I) \cap H(J)$, $I \nsubseteq P$ and $J \nsubseteq P$, which implies there exist $a \in I$ and $b \in J$ such that $a \notin P$ and $b \notin P$. Since P is a prime ideal, we have $a * b \notin P$. However, $a * b \in I \cap J$, thus, $I \cap J \nsubseteq P$, which means $P \in H(I \cap J)$. Hence, $H(I) \cap H(J) \subseteq H(I \cap J)$. Therefore, $H(I \cap J) = H(I) \cap H(J)$.

(2) Obviously, $H(I) \cup H(J) \subseteq H(I \cup J)$. Conversely, if $H(I \cup J) = \emptyset$, then $H(I \cup J) \subseteq H(I) \cup H(J)$. If $H(I \cup J) \neq \emptyset$, then for every $P \in H(I \cup J)$, we have $I \cup J \nsubseteq P$. Since $I \subseteq P$ and $J \subseteq P$ imply $I \cup J \subseteq P$, a contradiction arises. Hence, $I \nsubseteq P$ or $J \nsubseteq P$, and so $P \in H(I)$ or $P \in H(J)$, which implies $P \in H(I) \cup H(J)$. Thus, $H(I \cup J) \subseteq H(I) \cup H(J)$. Therefore, $H(I \cup J) = H(I) \cup H(J)$. □

Theorem 5. *Let* $(N, *)$ *be a WCNETG with the smallest ideal* I_0*. Then the lattice* $(Id(N), \cap, \cup)$ *is isomorphic with the lattice of all open sets in* $(\tau, \cap, \cup)$*.*

Proof. Define a mapping $\Phi : Id(N) \to \tau$ by $\Phi(I) = H(I)$ for every $I \in Id(N)$. Let $I,\ J \in Id(N)$, by Proposition 7 and Proposition 13, we have $\Phi(I \cup J) = H(I \cup J) = H(I) \cup H(J) = \Phi(I) \cup \Phi(J)$. Similarly, we get $\Phi(I \cap J) = \Phi(I) \cap \Phi(J)$. Hence, $\Phi : Id(N) \to \tau$ is a lattice homomorphism. For any $H(X) \in \tau$, by Proposition 10, $H(X) = H(\langle X \rangle) = \Phi(\langle X \rangle)$. Thus, Φ is surjective. On the other hand, let $A,\ B \in Id(N)$ and $\Phi(A) = \Phi(B)$, then $H(A) = H(B)$. If $A \neq B$, then there exists $a \in A$ such that $a \notin B$. By Theorem 4, there exists a prime ideal P such that $B \subseteq P$ and $a \notin P$, which implies $A \nsubseteq P$. Thus, $P \in H(A) = H(B)$. Hence, $B \nsubseteq P$, which is a contradiction. Therefore, $A = B$, and so Φ is injective. □

Theorem 6. *Let* $(N, *)$ *be a NETG with the smallest ideal* I_0*. Then the following statements are equivalent:*

(1) $(Prim(N), \tau)$ *is a* T_1*-space;*

(2) *every prime ideal is maximal in* $(Prim(N), \subseteq)$;

(3) *every prime ideal is minimal in* $(Prim(N), \subseteq)$.

Proof. (1)⇒(2) Assume that $(Prim(N), \tau)$ is a T_1-space. Let $P \in Prim(N)$. If there exists a prime ideal Q such that $P \subset Q$, then there exist subsets $A,\ B \subseteq X$ such that $P \in H(A)$ but $Q \notin H(A)$, and $Q \in H(B)$ but $P \notin H(B)$. Hence, $B \subseteq P \subset Q$, and so $Q \notin H(B)$, which is a contradiction. Therefore, P is maximal in $(Prim(N), \subseteq)$.

(2)⇒(3) It holds obviously.

(3)⇒(1) Assume that every prime ideal is minimal in $(Prim(N), \subseteq)$. Let $P,\ Q$ be two distinct elements of $Prim(N)$. Since P and Q are minimal in $(Prim(N), \subseteq)$, we have $P \nsubseteq Q$ and $Q \nsubseteq P$, and so there exist $a \in P \setminus Q$ and $b \in Q \setminus P$. Hence, $P \in H(b) \setminus H(a)$ and $Q \in H(a) \setminus H(b)$. Therefore, $(Prim(N), \tau)$ is a T_1-space. □

Lemma 1. *Let* $(N, *)$ *be a NETG. Then for every* $I,\ J \in Prim(N)$*, we have* $I \cup J \in Prim(N)$*.*

Theorem 7. *Let* $(N, *)$ *be a NETG with the smallest ideal* I_0*. Then* $(Prim(N), \tau)$ *is a* T_1*-space if and only if* $(Prim(N), \tau)$ *is a Hausdorff-space.*

Proof. We only prove necessity. Suppose that $(Prim(N), \tau)$ is a T_1-space. Let P and Q be two distinct elements of $Prim(N)$. From Theorem 6 we know P and Q are both maximal in $(Prim(N), \subseteq)$, which implies $P \nsubseteq Q$ and $Q \nsubseteq P$, and so there exist $a \in P \setminus Q$ and $b \in Q \setminus P$. Hence, $P \in H(b) \setminus H(a)$ and $Q \in H(a) \setminus H(b)$. If $H(a) \cap H(b) \neq \emptyset$, then there exists a prime ideal $K \in H(a) \cap H(b)$, which means $a \notin K$ and $b \notin K$, so $P \neq K$. Since P is maximal in $(Prim(N), \subseteq)$ and from Lemma 1 we get $P \subseteq P \cup K \in Prim(N)$, we can conclude that $P \cup K = P$ must holds. Thus, $K \subset P$, which is in contradiction with the fact that K is maximal in $(Prim(N), \subseteq)$. Therefore, $H(a) \cap H(b) = \emptyset$. Hence, $(Prim(N), \tau)$ is a Hausdorff-space. □

5. Conclusions

In this paper, inspired by the research work in properties of NT-subgroups in NETGs, we proposed and investigated ideals of NETGs, which are a special kind of NT-subgroups. After studying the lattice structure of $(Id(N), \subseteq)$, we presented the characterization of the smallest ideal generated by a subset. Moreover, we defined a special multiplication on the set of all ideals of a NETG, which constructed a new NETG. At last, we investigated some topological properties of the prime ideal space. Future research will consider applying fuzzy set theory and rough set theory to the research of algebraic structure of NETGs.

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Conflicts of interest

The authors declare no conflicts of interest.

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