AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 4735–4766.
DOI: 10.3934/math.2022263
Received: 15 August 2021
Revised: 17 November 2021
Accepted: 25 November 2021
Published: 24 December 2021

Research article

Pythagorean Cubic fuzzy Hamacher aggregation operators and their application in green supply selection problem

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Abstract: The green chain supplier selection process plays a major role in the environmental decision for the efficient and effective supply chain management. Therefore, the aim of this paper is to develop a mechanism for decision making on green chain supplier problem. First, we define the Hamacher operational law for Pythagorean cubic fuzzy numbers (PCFNs) and study their fundamental properties. Based on the Hamacher operation law of PCFNs, we defined Pythagorean cubic fuzzy aggregation operators by using Hamacher t-norm and t-conorm. Further, we develop a series of Pythagorean cubic fuzzy Hamacher weighted averaging (PCFHWA), Pythagorean cubic fuzzy Hamacher order weighted averaging (PCFHOWA) Pythagorean Cubic fuzzy Hamacher hybrid averaging (PCFHHA), Pythagorean Cubic fuzzy Hamacher weighted Geometric (PCFHWG), Pythagorean Cubic fuzzy Hamacher order weighted Geometric (PCFHOWG), and Pythagorean Cubic fuzzy Hamacher hybrid geometric (PCFHHA) operators. Furthermore, we apply these aggregation operators of Pythagorean Cubic fuzzy numbers to the decision making problem for green supplier selection. We construct an algorithm for the group decision making by using aggregation operators and score function. The proposed decision making method applies to green chain supplier selection problem and find the best green supplier for green supply chain management. The proposed method compared with other group decision techniques under Pythagorean cubic fuzzy information. From the comparison and sensitivity analysis, we concluded that our proposed method is more generalized and effective method.

Keywords: Pythagorean cubic fuzzy sets; Hamacher aggregation operators; Pythagorean Cubic fuzzy Hamacher average operators; Pythagorean Cubic fuzzy Hamacher geometric operators
Mathematics Subject Classification: 03E72, 47S40

1. Introduction

1.1. Green supply chain

The new green activities have been divided into two categories: Single firm activities and supply chain management activities. Single company systems and supply chain systems have been identified as emerging information systems. Emerging information systems has been identified as single business systems and supply chain systems. Green supply chain management (GSCM) has helped with the deployment of eco-efficiency [4], renewable energy sources [20], and sustainable actions [62] in the supply chain (SC), innovation clusters [44], and symbiotic industrial networks [45] in this context. Starting with product design and progressing through raw material selection, manufacturing processes, transportation and delivery, and the final consumer arriving at the final destination. Zhu and Sarkis [61] define GSCM as the integration of environmental thinking with operations management in the SC. According to Large and Thomsen [29], the design process, raw material selection, green procurement, the greener manufacturing process, green distribution, and reverse logistics are all part of GSCM. Centobelli et al. [7] defined the goals of pursuing supply chain sustainable development through the adoption of green practices and enabling technologies using cross-country analysis of LSPs. The value of sustainable development is increasingly known worldwide in Private businesses [12]. This is because the global economy is evolving rapidly for consumption [14, 32]. Here with growing acknowledgment, organizations have begun to accept several forms of susceptibility exercise include Green Supply Chain Management (GSCM) in their reinforcement activities [30, 48, 49]. The GSCM concerns the successful consideration of the production of susceptibility throughout distribution chain in a company [40]. It involves the improvement of company processes in a Organization to attract the supply chain consumer when addressing susceptibility concerns [17, 42]. GSCM helps to reduce waste through diverse processes like product creation, manufacturing resources, product distribution and end of life management of goods [10] in the supply chain activities of an enterprise. This coordinates the tasks Across the supply chain to develop the corporation's susceptibility growth [3, 51]. Successful GSCM assists companies in their productivity by: (a) reducing pollution a and waste; (b) partners with environmentally friendly provider; (c) green product creation and facilities; and (d) decreasing pollution from the conveyance and procurement of goods and facilities [11]. It is difficult to assess the efficiency of GSCM practices. Various decision-makers are sometimes suspected. Often there are numerous and contradictory requirements for the assessment. Hidden uncertainty is still present in the estimation process due to the use of subjective judgments. A lot of work was performed, and different models were built to solve the problem to evaluate the efficiency of GSCM activities from different opinions [2, 22, 41]. These models can usually be categorized from two viewpoints: multi-criteria and multi-objective assessment upgrading [50, 53]. For example, Kannan et al. [22] combined structural interpretive modeling and analytically hierarchy process [9] to estimate the GSCM practices. The research of interpretive structural modeling is interaction between parameters, which contributes to the weights of the assessment criteria. The theoretical hierarchy method is then implemented in order to determine the right GSCM procedure in organizations. A flimsy model for assessing GSCM activities in the organization is developed by Awasthi et al [2]. The TOPSIS model for the evaluation of GSCM activities in the Taiwan electronics industry extended by Shen et al. [41]. For example, For organizations to manage green supply chains, Roghanian et al. [38] suggested a multi-objective optimization model. A multi-objective model for the architecture of multi-echelon supply chains is suggested by Torabi and Hassini [47]. Liu and Papageorgiou [34, 31] have built a linear programming model to analyses the supply GSCM.

1.2. Literature review

The multi-attribute decision making (MADM) approach is the methodology used for decisions making finding the best alternative on the base of some criteria. MADM is the most effective decision-making method and MADM has been commonly applied in human activities [23]. To interact with the MADM process in an indefinite environment, researchers prefer a fuzzy set (FS) rather than a crisp set. In 1965, Zadeh present FS theory [59]. In 1986, Atanassov [1] expressed theory of intuitionistic fuzzy set (IFS), which is the extension of the FS theory of Zadeh. The TOPSIS approach to IFS information was introduced by Kumar and Garg [28]. Garg and Kumar [18] suggested the similarity measures of intuitionistic fuzzy sets based on the principle of set pair analysis and explored their use in decision-making issues, and Garg [19] suggested some robust improved geometric aggregation operators under intuitionistic fuzzy aggregation operators. In 1989, Atanassov and Gargov [46] introduced the concept of Interval-value intuitionistic fuzzy sets (IVIFSs), where the degrees of membership and non-membership are denoted by intervals instead of single numbers. So, IVIFSs is the generalization of IFS. Yager the Pythagorean fuzzy set (PFS) in 2014 [57, 56] and also introduced the aggregation operators like Pythagorean fuzzy weighted averaging (PFWA), Pythagorean fuzzy ordered weighted averaging (PFOWA), Pythagorean fuzzy weighted geometric (PFWG) and Pythagorean fuzzy ordered weighted geometric (PFOWG) operators. Based on PFS the values of membership and non-membership degrees ranging from 0 to 1. So, that the squares of membership and non-membership degrees are less than 1. Peng et al. [36] proposed Pythagorean fuzzy linguistic sets (PFLSs) and their operating laws and score function of PFNs. Jun et al. [21] presented a cubic set obtained by means of an interval valued fuzzy set and a fuzzy set demonstrating the meaning of an interval valued fuzzy set [60] to achieve the most suitable results by means of a fuzzy set [59]. Kaur and Garg [27, 25] suggested the unique definition of a generalized cubic intuitionistic fuzzy set and developed aggregation operators using t-norm operations and explored their applications for group decision-making processes.

The intuitionistic fuzzy set, on the other hand, does not address the ambiguity issues. We were able to solve difficulties involving uncertainty because to this theory. Cubic set theory also describes the satisfied, unsatisfied, and uncertain information that fuzzy sets theory and intuitionistic fuzzy set theory [13, 33, 39]. Cubic set provides more appealing details than FS and IFS [24]. It is one of the more basic types of fuzzy set, similar to IFS, in that each element of a cubic fuzzy set is defined as a pair structure with a positive and negative membership function.

1.3. Motivation and objective

To sum up, these models showed their merits in evaluating the GSCM activities from different perspectives. Though, because: (a) the need to properly consider the needs of multiple DMs; (b) the existence of ambiguity; (c) the requirement on DM's in the assessment process, they are not entirely satisfactory in addressing this problem effectively.

This paper provides a model for a community DM for the assessment of GSCM activities by an association. PCFNs are used to accurately model the ambiguity in decision-making to reflect the decision maker's assessment. To accurately quantify the subjective evaluation, a susceptibility criteria algorithm is defined to determine the cumulative success of the GSCM activities. An example showing the model suggested is a success in addressing related issues in the real world context. The problem of evaluating GSCM activities starts as a decision-making problem for MCDM. Then, an algorithm is developed to solve this problem effectively and followed by an example.

The structure is as follows: Some basic concepts of FS, IFS, PFS and operational laws of Pythagorean fuzzy sets offers in Section 2. In Section 3, we defined the concept of Pythagorean cubic fuzzy Hamacher averaging aggregation operators. In Section 4, we defined the concept of Pythagorean cubic fuzzy Hamacher geometric aggregation operators. In Section 5, addressed the decision frames for PCFSs. In Section 6, a numerical example given to demonstrate the application of the proposed method by using the proposed algorithms. In Section 7, discussed the comparison between the existing methods and proposed method. In Section 8, this work is eventually outlined.

2. Preliminaries

We have familiarized with unique concepts and their relevant properties in this section.

Definition 2.1([59]). Let us assume that X be a fixed set. Then, a FS defined as follows;

$$F = \{\langle x, \hat{u}_F(x)\rangle \,|x \in X\}, \tag{2.1}$$

where $\hat{u}_F$ represent the membership degree of $x \in X$ and mapping from $X \to [0,1]$.

Definition 2.2([1]). Let us assume that X be a fixed set and I is defined on X. Then, IFS is stated as;

$$I = \{\langle x, \hat{u}_I(x), \gamma_I^*(x)\rangle \,|x \in X\}, \tag{2.2}$$

where $\hat{u}_I(x)$ and $\gamma_I^*(x)$ are the function from $X \to [0,1]$, also hold that $0 \le \hat{u}_I(x) \le 1$, $0 \le \gamma_I^*(x) \le 1, \forall x \in X$ and denote the membership and non-membership grade of x of X to set I, respectively.

Definition 2.3([21]). Let us suppose that X is a fixed set. So, a CS stated as;

$$C = \{\langle x, \hat{u}_C(x), \gamma_C^*(x)\rangle \,|x \in X\}, \tag{2.3}$$

where $\hat{u}_C$ is an IVFS in X and γ_C^* is a FS in X.

Definition 2.4([56, 57]). Let us suppose that X be a fix set. Then, a PFS is expressed as;

$$P = \{\langle x, (\hat{u}_P(x), \gamma_P^*(x))\rangle \,|x \in X\}, \tag{2.4}$$

where the function $\hat{u}_P : X \to [0,1]$ represent the membership degree and the function $\gamma_P^* : X \to [0,1]$ represent the non-membership degree of the element of $x \in X$ to P consequently. So, for $x \in X$, it holds that $(\hat{u}_P(x))^2 + (\gamma_P^*(x))^2 \le 1$. And the indeterminacy degree of the Pythagorean fuzzy set as; $\sqrt{1 - (\hat{u}_P(x))^2 + (\gamma_P^*(x))^2}$.

Definition 2.5([26]). Let us suppose that X be a fixed set. Then, a PCFS is expressed as;

$$P = \{\langle x, (\Lambda_P(x), \Gamma_P(x))\rangle \,|x \in X\}, \tag{2.5}$$

where the function $\Lambda_P = \left(\left\langle [\check{c}_{\check{s}}^-, \check{c}_{\check{s}}^+]; \tau_{1\check{s}}^*\right\rangle\right)$ and $\Gamma_P = \left\langle [\check{e}_{\check{s}}^-, \check{e}_{\check{s}}^+]; \tau_{2\check{s}}^*\right\rangle$ represent the membership degree and non-membership degree of the element of $x \in X$ to P consequently. So, for $x \in X$, it holds that $(\Lambda_P(x))^2 + (\Gamma_P(x))^2 \le 1$. And the indeterminacy degree of Pythagorean fuzzy set as; $\sqrt{1 - (\Lambda_P(x))^2 + (\Gamma_P(x))^2}$.

2.1. Operational laws of Pythagorean cubic fuzzy set

Definition 2.6. Let us assume that $P_1^* = \left(\left\langle[\check{c}_1^-, \check{c}_1^+]; \tau_{11}^*\right\rangle, \left\langle[\check{e}_1^-, \check{e}_1^+]; \tau_{21}^*\right\rangle\right)$ and $P_2^* = \left(\left\langle[\check{c}_2^-, \check{c}_2^+]; \tau_{12}^*\right\rangle, \left\langle[\check{e}_2^-, \check{e}_2^+]; \tau_{22}^*\right\rangle\right)$ are two PCFNs, $\gamma^* > 0$. Then, the Hamacher operation for PCFNs are described as given below;

$$1.\ P_1^* \oplus P_2^* = \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{(\check{c}_1^-)^2+(\check{c}_2^-)^2-(\check{c}_1^-)^2(\check{c}_2^-)^2-(1-\gamma^*)(\check{c}_1^-)^2(\check{c}_2^-)^2}{1-(1-\gamma^*)(\check{c}_1^-)^2(\check{c}_2^-)^2}}, \\ \sqrt{\frac{(\check{c}_1^+)^2+(\check{c}_2^+)^2-(\check{c}_1^+)^2(\check{c}_2^+)^2-(1-\gamma^*)(\check{c}_1^+)^2(\check{c}_2^+)^2}{1-(1-\gamma^*)(\check{e}_1^+)^2(\check{e}_2^+)^2}} \end{array} \right]; \\ \sqrt{\frac{(\tau_{11}^*)^2+(\tau_{12}^*)^2-(\tau_{11}^*)^2(\tau_{12}^*)^2-(1-\gamma^*)(\tau_{11}^*)^2(\tau_{12}^*)^2}{1-(1-\gamma^*)(\tau_{11}^*)^2(\tau_{12}^*)^2}} \end{array} \right) \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\check{e}_1^-\check{e}_2^-}{\sqrt{\gamma^*+(1-\gamma^*)((\check{e}_1^-)^2+(\check{e}_2^-)^2-(\check{e}_1^-)^2(\check{e}_2^-)^2)}}, \\ \frac{\check{e}_1^+\check{e}_2^+}{\sqrt{\gamma^*+(1-\gamma^*)((\check{e}_1^+)^2+(\check{e}_2^+)^2-(\check{e}_1^+)^2(\check{e}_2^+)^2)}} \end{array} \right]; \\ \frac{\tau_{21}^*\tau_{22}^*}{\sqrt{\gamma^*+(1-\gamma^*)((\tau_{21}^*)^2+(\tau_{22}^*)^2-(\tau_{21}^*)^2(\tau_{22}^*)^2)}} \end{array} \right) \end{array} \right\};$$

$$2.\ P_1^* \otimes P_2^* = \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \frac{\check{c}_1^-\check{c}_2^-}{\sqrt{\gamma^*+(1-\gamma^*)((\check{c}_1^-)^2+(\check{c}_2^-)^2-(\check{c}_1^-)^2(\check{c}_2^-)^2)}}, \\ \frac{\check{c}_1^+\check{c}_2^+}{\sqrt{\gamma^*+(1-\gamma^*)((\check{c}_1^+)^2+(\check{c}_2^+)^2-(\check{c}_1^+)^2(\check{c}_2^+)^2)}} \end{array} \right]; \\ \frac{\tau_{11}^*\tau_{12}^*}{\sqrt{\gamma^*+(1-\gamma^*)((\tau_{11}^*)^2+(\tau_{12}^*)^2-(\tau_{11}^*)^2(\tau_{12}^*)^2)}} \end{array} \right) \\ \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{(\check{e}_1^-)^2+(\check{e}_2^-)^2-(\check{e}_1^-)^2(\check{e}_2^-)^2-(1-\gamma^*)(\check{e}_1^-)^2(\check{e}_2^-)^2}{1-(1-\gamma^*)(\check{e}_1^-)^2(\check{e}_2^-)^2}}, \\ \sqrt{\frac{(\check{e}_1^+)^2+(\check{e}_2^+)^2-(\check{e}_1^+)^2(\check{e}_2^+)^2-(1-\gamma^*)(\check{e}_1^+)^2(\check{e}_2^+)^2}{1-(1-\gamma^*)(\check{e}_1^+)^2(\check{e}_2^+)^2}} \end{array} \right]; \\ \sqrt{\frac{(\tau_{21}^*)^2+(\tau_{22}^*)^2-(\tau_{21}^*)^2(\tau_{22}^*)^2-(1-\gamma^*)(\tau_{21}^*)^2(\tau_{22}^*)^2}{1-(1-\gamma^*)(\tau_{21}^*)^2(\tau_{22}^*)^2}} \end{array} \right) \end{array} \right\};$$

$$3.\ \lambda P_1^* = \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{(1+(\gamma^*-1)(\check{c}_1^-)^2)^\lambda-(1-(\check{c}_1^-)^2)^\lambda}{(1+(\gamma^*-1)(\check{c}_1^-)^2)^\lambda+(\gamma^*-1)(1-(\check{c}_1^-)^2)^\lambda}}, \\ \sqrt{\frac{(1+(\gamma^*-1)(\check{c}_1^+)^2)^\lambda-(1-(\check{c}_1^+)^2)^\lambda}{(1+(\gamma^*-1)(\check{c}_1^+)^2)^\lambda+(\gamma^*-1)(1-(\check{c}_1^+)^2)^\lambda}} \end{array} \right]; \\ \sqrt{\frac{(1+(\gamma^*-1)(\tau_{11}^*)^2)^\lambda-(1-(\tau_{11}^*)^2)^\lambda}{(1+(\gamma^*-1)(\tau_{11}^*)^2)^\lambda+(\gamma^*-1)(1-(\tau_{11}^*)^2)^\lambda}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{\gamma^*}(\check{e}_1^-)^\lambda}{\sqrt{1+(\gamma^*-1)(1-(\check{e}_1^-)^2)^\lambda+(\gamma^*-1)(\check{e}_1^-)^2)^\lambda}}, \\ \frac{\sqrt{\gamma^*}(\check{e}_1^+)^\lambda}{\sqrt{1+(\gamma^*-1)(1-(\check{e}_1^+)^2)^\lambda+(\gamma^*-1)(\check{e}_1^+)^2)^\lambda}} \end{array} \right]; \\ \sqrt{\frac{(1+(\gamma^*-1)(\tau_{12}^*)^2)^\lambda-(1-(\tau_{12}^*)^2)^\lambda}{(1+(\gamma^*-1)(\tau_{12}^*)^2)^\lambda+(\gamma^*-1)(1-(\tau_{12}^*)^2)^\lambda}} \end{array} \right) \end{array} \right\};$$

4. $$\left(P_1^*\right)^\lambda = \left\{ \begin{array}{c} \left(\left[\begin{array}{c} \frac{\sqrt{\gamma^*}(\check{c}_1^-)^\lambda}{\sqrt{1+(\gamma^*-1)(1-(\check{c}_1^-)^2)^\lambda+(\gamma^*-1)(\check{c}_1^-)^2)^\lambda}}, \\ \frac{\sqrt{\gamma^*}(\check{c}_1^+)^\lambda}{\sqrt{1+(\gamma^*-1)(1-(\check{c}_1^+)^2)^\lambda+(\gamma^*-1)(\check{c}_1^+)^2)^\lambda}} \end{array} \right]; \\ \sqrt{\frac{(1+(\gamma^*-1)(\tau_{11}^*)^2)^\lambda-(1-(\tau_{11}^*)^2)^\lambda}{(1+(\gamma^*-1)(\tau_{11}^*)^2)^\lambda+(\gamma^*-1)(1-(\tau_{11}^*)^2)^\lambda}} \right) \\ \left(\left[\begin{array}{c} \sqrt{\frac{(1+(\gamma^*-1)(\check{e}_1^-)^2)^\lambda-(1-(\check{e}_1^-)^2)^\lambda}{(1+(\gamma^*-1)(\check{e}_1^-)^2)^\lambda+(\gamma^*-1)(1-(\check{e}_1^-)^2)^\lambda}}, \\ \sqrt{\frac{(1+(\gamma^*-1)(\check{e}_1^+)^2)^\lambda-(1-(\check{e}_1^+)^2)^\lambda}{(1+(\gamma^*-1)(\check{e}_1^+)^2)^\lambda+(\gamma^*-1)(1-(\check{e}_1^+)^2)^\lambda}} \end{array} \right]; \\ \sqrt{\frac{(1+(\gamma^*-1)(\tau_{12}^*)^2)^\lambda-(1-(\tau_{12}^*)^2)^\lambda}{(1+(\gamma^*-1)(\tau_{12}^*)^2)^\lambda+(\gamma^*-1)(1-(\tau_{12}^*)^2)^\lambda}} \right) \end{array} \right\}.$$

3. Pythagorean cubic fuzzy Hamacher averaging aggregation operators

In this section, we discuss the Pythagorean cubic fuzzy Hamacher averaging aggregation operators, their basic properties and related theorems.

Definition 3.1. Let $P_{\check{s}}^* = \left(\left\langle[\check{c}_{\check{s}}^-, \check{c}_{\check{s}}^+]; \tau_{1\check{s}}^*\right\rangle, \left\langle[\check{e}_{\check{s}}^-, \check{e}_{\check{s}}^+]; \tau_{2\check{s}}^*\right\rangle\right) (\check{s} \in N)$ be a number of PCFNs. Then, we define the PCFHWA operator given below;

$$PCFHWA_{\hat{w}}(P_1^*, P_2^*, ..., P_n^*) = \bigoplus_{\check{s}=1}^{n} (\hat{w}_{\check{s}} P_{\check{s}}^*) \tag{3.1}$$

where $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$ be the weight vector of $P_{\check{s}}^*(\check{s} \in N)$, and $\hat{w}_{\check{s}} > 0$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1$.

Centered on Hamacher $\oplus$ of the mentioned PCFN operations, we can operate the Theorem 1.

Theorem 3.1. Suppose $P_{\check{s}}^* = \left(\left\langle[\hat{c}_{\check{s}}^-, \hat{c}_{\check{s}}^+]; \tau_{1\check{s}}^*\right\rangle, \left\langle[\check{e}_{\check{s}}^-, \check{e}_{\check{s}}^+]; \tau_{2\check{s}}^*\right\rangle\right) (\check{s} \in N)$ be a group of PCFNs. Then, their accumulated value is also a PCFN, using PCFHWA operator, and

$$PCFHWA_{\hat{w}_{\check{s}}}(P_1^*, P_2^*, ..., P_n^*) = \bigoplus_{\check{s}=1}^{n} (\hat{w}_{\check{s}} P_{\check{s}}^*) \tag{3.2}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-(\hat{c}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-(\hat{c}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-(\hat{c}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-(\hat{c}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}(\check{e}_{\check{s}}^-)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-(\check{e}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left((\check{e}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\check{e}_{\check{s}}^+\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-(\check{e}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left((\check{e}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}(\tau_{2\check{s}}^*)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\tau_{2\check{s}}^*\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left((\tau_{2\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right) \end{array} \right\}$$

where $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$ be the weight vector of $P_{\check{s}}^*(\check{s} = 1, 2, ..., n)$, and $\hat{w}_{\check{s}} > 0, \sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1, \gamma^* > 0$.

Proof. By using the mathematical induction, we have;

(i) If $n = 2$, then by using operation laws of PCFNs, we have

$$PCFHWA_{\hat{w}}(P_1^*, P_2^*) = (P_1^* \oplus P_2^*)$$

$$\left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{\left(1+(\gamma^*-1)(\hat{c}_1^-)^2\right)^{\hat{w}_1} - \left(1-(\hat{c}_1^-)^2\right)^{\hat{w}_1}}{\left(1+(\gamma^*-1)(\hat{c}_1^-)^2\right)^{\hat{w}_1} + (\gamma^*-1)\left(1-(\hat{c}_1^-)^2\right)^{\hat{w}_1}}}, \\ \sqrt{\frac{\left(1+(\gamma^*-1)(\hat{c}_1^+)^2\right)^{\hat{w}_1} - \left(1-(\hat{c}_1^+)^2\right)^{\hat{w}_1}}{\left(1+(\gamma^*-1)(\hat{c}_1^+)^2\right)^{\hat{w}_1} + (\gamma^*-1)\left(1-(\hat{c}_1^+)^2\right)^{\hat{w}_1}}} \end{array} \right]; \\ \sqrt{\frac{\left(1+(\gamma^*-1)(\tau_{11}^*)^2\right)^{\hat{w}_1} - \left(1-(\tau_{11}^*)^2\right)^{\hat{w}_1}}{\left(1+(\gamma^*-1)(\tau_{11}^*)^2\right)^{\hat{w}_1} + \left(\gamma^*-1)(1-(\tau_{11}^*)^2\right)^{\hat{w}_1}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{\gamma^*}(\check{e}_1^-)^{\hat{w}_1}}{\sqrt{1+(\gamma^*-1)\left(1-(\check{e}_1^-)^2\right)^{\hat{w}_1} + (\gamma^*-1)\left((\check{e}_1^-)^2\right)^{\hat{w}_1}}}, \\ \frac{\sqrt{\gamma^*}\left(\check{e}_1^+\right)^{\hat{w}_1}}{\sqrt{1+(\gamma^*-1)\left(1-(\check{e}_1^+)^2\right)^{\hat{w}_1} + (\gamma^*-1)\left((\check{e}_1^+)^2\right)^{\hat{w}_1}}} \end{array} \right]; \\ \frac{\sqrt{\gamma^*}(\tau_{21}^*)^{\hat{w}_1}}{\sqrt{1+(\gamma^*-1)\left(1-(\tau_{21}^*)^2\right)^{\hat{w}_1} + (\gamma^*-1)\left((\tau_{21}^*)^2\right)^{\hat{w}_1}}} \end{array} \right) \end{array} \right\}$$

$$\oplus\left\{\begin{array}{c}\left(\begin{array}{c}\left[\begin{array}{c}\sqrt{\frac{\left(1+(\gamma^*-1)(\hat{c}_2^-)^2\right)^{\hat{w}_2}-\left(1-(\hat{c}_2^-)^2\right)^{\hat{w}_2}}{\left(1+(\gamma^*-1)(\hat{c}_2^-)^2\right)^{\hat{w}_2}+(\gamma^*-1)\left(1-(\hat{c}_2^-)^2\right)^{\hat{w}_2}}},\\ \sqrt{\frac{\left(1+(\gamma^*-1)(\hat{c}_2^+)^2\right)^{\hat{w}_2}-\left(1-(\hat{c}_2^+)^2\right)^{\hat{w}_2}}{\left(1+(\gamma^*-1)(\hat{c}_2^+)^2\right)^{\hat{w}_2}+(\gamma^*-1)\left(1-(\hat{c}_2^+)^2\right)^{\hat{w}_2}}}\end{array}\right];\\ \sqrt{\frac{\left(1+(\gamma^*-1)(\tau_{12}^*)^2\right)^{\hat{w}_2}-\left(1-(\tau_{12}^*)^2\right)^{\hat{w}_2}}{\left(1+(\gamma^*-1)(\tau_{12}^*)^2\right)^{\hat{w}_2}+(\gamma^*-1)\left(1-(\tau_{12}^*)^2\right)^{\hat{w}_2}}}\end{array}\right),\\ \left(\begin{array}{c}\left[\begin{array}{c}\frac{\sqrt{\gamma^*}(\check{e}_2^-)^{\hat{w}_2}}{\sqrt{1+(\gamma^*-1)\left(1-(\check{e}_2^-)^2\right)^{\hat{w}_2}+(\gamma^*-1)\left((\check{e}_2^-)^2\right)^{\hat{w}_2}}},\\ \frac{\sqrt{\gamma^*}\left(\check{e}_2^+\right)^{\hat{w}_2}}{\sqrt{1+(\gamma^*-1)\left(1-(\check{e}_2^+)^2\right)^{\hat{w}_2}+(\gamma^*-1)\left((\check{e}_2^+)^2\right)^{\hat{w}_2}}}\end{array}\right];\\ \frac{\sqrt{\gamma^*}(\tau_{22}^*)^{\hat{w}_2}}{\sqrt{1+(\gamma^*-1)\left(1-(\tau_{22}^*)^2\right)^{\hat{w}_2}+(\gamma^*-1)\left((\tau_{22}^*)^2\right)^{\hat{w}_2}}}\end{array}\right)\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\left[\begin{array}{c}\sqrt{\frac{\prod_{\check{s}=1}^{2}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{2}\left(1-(\hat{c}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{2}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{2}\left(1-(\hat{c}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}}},\\ \sqrt{\frac{\prod_{\check{s}=1}^{2}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{2}\left(1-(\hat{c}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{2}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{2}\left(1-(\hat{c}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}}}\end{array}\right];\\ \sqrt{\frac{\prod_{\check{s}=1}^{2}\left(1+(\gamma^*-1)(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{2}\left(1-(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{2}\left(1+(\gamma^*-1)(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{2}\left(1-(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}}\end{array}\right),\\ \left(\begin{array}{c}\left[\begin{array}{c}\frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{2}(\check{e}_{\check{s}}^-)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{2}1+(\gamma^*-1)\left(1-(\check{e}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{2}\left((\check{e}_{\check{s}}^-)^2\right)^{\hat{w}_{\check{s}}}}},\\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{2}\left(\check{e}_{\check{s}}^+\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{2}1+(\gamma^*-1)\left(1-(\check{e}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{2}\left((\check{e}_{\check{s}}^+)^2\right)^{\hat{w}_{\check{s}}}}}\end{array}\right];\\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{2}(\tau_{2\check{s}}^*)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{2}1+(\gamma^*-1)\left(1-(\tau_{2\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{2}\left((\tau_{2\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}}\end{array}\right)\end{array}\right\}$$

Hence, Eq (3.2) is true for $n = 2$.

Let Eq (3.2) is hold for $n = k$. Then, by the Eq (3.2), we get

$$PCFHWA_{\hat{w}}(P_1^*, ..., P_k^*) = \bigoplus_{\check{s}=1}^{k}(\hat{w}_{\check{s}}P_{\check{s}}^*)$$

$$
= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}-\prod_{\breve{s}=1}^{k}\left(1-(\hat{c}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}}{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left(1-(\hat{c}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}}}, \\ \sqrt{\frac{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}-\prod_{\breve{s}=1}^{k}\left(1-(\hat{c}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}}{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left(1-(\hat{c}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\tau_{1\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}-\prod_{\breve{s}=1}^{k}\left(1-(\tau_{1\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}}{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\tau_{1\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left(1-(\tau_{1\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{\gamma^*}\prod_{\breve{s}=1}^{k}(\breve{e}_{\breve{s}}^-)^{\hat{w}_{\breve{s}}}}{\sqrt{\prod_{\breve{s}=1}^{k}1+(\gamma^*-1)\left(1-(\breve{e}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left((\breve{e}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}}}, \\ \frac{\sqrt{\gamma^*}\prod_{\breve{s}=1}^{k}\left(\breve{e}_{\breve{s}}^+\right)^{\hat{w}_{\breve{s}}}}{\sqrt{\prod_{\breve{s}=1}^{k}1+(\gamma^*-1)\left(1-(\breve{e}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left((\breve{e}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}}} \end{array} \right]; \\ \frac{\sqrt{\gamma^*}\prod_{\breve{s}=1}^{k}(\tau_{2\breve{s}}^*)^{\hat{w}_{\breve{s}}}}{\sqrt{\prod_{\breve{s}=1}^{k}1+(\gamma^*-1)\left(1-(\tau_{2\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left((\tau_{2\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}}} \end{array} \right) \end{array} \right\}
$$

Now, for $n = k + 1$, we get

$$
PCFHWA_{\hat{w}}(P_1^*, ..., P_k^*, P_{k+1}^*) = \bigoplus_{\breve{s}=1}^{k}(\hat{w}_{\breve{s}}P_{\breve{s}}^* \oplus \hat{w}_{\breve{s}+1}P_{\breve{s}+1}^*)
$$

$$
\left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}-\prod_{\breve{s}=1}^{k}\left(1-(\hat{c}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}}{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left(1-(\hat{c}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}}}, \\ \sqrt{\frac{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}-\prod_{\breve{s}=1}^{k}\left(1-(\hat{c}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}}{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left(1-(\hat{c}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\tau_{1\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}-\prod_{\breve{s}=1}^{k}\left(1-(\tau_{1\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}}{\prod_{\breve{s}=1}^{k}\left(1+(\gamma^*-1)(\tau_{1\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left(1-(\tau_{1\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{\gamma^*}\prod_{\breve{s}=1}^{k}(\breve{e}_{\breve{s}}^-)^{\hat{w}_{\breve{s}}}}{\sqrt{\prod_{\breve{s}=1}^{k}1+(\gamma^*-1)\left(1-(\breve{e}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left((\breve{e}_{\breve{s}}^-)^2\right)^{\hat{w}_{\breve{s}}}}}, \\ \frac{\sqrt{\gamma^*}\prod_{\breve{s}=1}^{k}\left(\breve{e}_{\breve{s}}^+\right)^{\hat{w}_{\breve{s}}}}{\sqrt{\prod_{\breve{s}=1}^{k}1+(\gamma^*-1)\left(1-(\breve{e}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left((\breve{e}_{\breve{s}}^+)^2\right)^{\hat{w}_{\breve{s}}}}} \end{array} \right]; \\ \frac{\sqrt{\gamma^*}\prod_{\breve{s}=1}^{k}(\tau_{2\breve{s}}^*)^{\hat{w}_{\breve{s}}}}{\sqrt{\prod_{\breve{s}=1}^{k}1+(\gamma^*-1)\left(1-(\tau_{2\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}+(\gamma^*-1)\prod_{\breve{s}=1}^{k}\left((\tau_{2\breve{s}}^*)^2\right)^{\hat{w}_{\breve{s}}}}} \end{array} \right) \end{array} \right\}
$$

$$\oplus\left\{\begin{array}{l}\left(\begin{array}{c}\left[\begin{array}{c}\sqrt{\frac{(1+(\gamma^*-1)\left(\hat{c}_{k+1}^{-})^2\right)^{\hat{w}_{k+1}}-(1-\left(\hat{c}_{k+1}^{-})^2\right)^{\hat{w}_{k+1}}}{(1+(\gamma^*-1)\left(\hat{c}_{k+1}^{-})^2\right)^{\hat{w}_{k+1}}+(\gamma^*-1)(1-\left(\hat{c}_{k+1}^{-})^2\right)^{\hat{w}_{k+1}}}},\\ \sqrt{\frac{(1+(\gamma^*-1)\left(\hat{c}_{k+1}^{+})^2\right)^{\hat{w}_{k+1}}-\left(1-(\hat{c}_{k+1}^{+})^2\right)^{\hat{w}_{k+1}}}{(1+(\gamma^*-1)\left(\hat{c}_{k+1}^{+})^2\right)^{\hat{w}_{k+1}}+(\gamma^*-1)\left(1-(\hat{c}_{k+1}^{+})^2\right)^{\hat{w}_{k+1}}}}\end{array}\right];\\ \sqrt{\frac{\left(1+(\gamma^*-1)\left(\tau_{1(k+1)}^*\right)^2\right)^{\hat{w}_{k+1}}-\left(1-\left(\tau_{1(k+1)}^*\right)^2\right)^{\hat{w}_{k+1}}}{\left(1+(\gamma^*-1)\left(\tau_{1(k+1)}^*\right)^2\right)^{\hat{w}_{k+1}}+(\gamma^*-1)\left(1-\left(\tau_{1(k+1)}^*\right)^2\right)^{\hat{w}_{k+1}}}}\end{array}\right),\\ \left(\begin{array}{c}\left[\begin{array}{c}\frac{\sqrt{\gamma^*}(\check{e}_{k+1}^{-})^{\hat{w}_{k+1}}}{\sqrt{1+(\gamma^*-1)\left(1-(\check{e}_{k+1}^{-})^2\right)^{\hat{w}_{k+1}}+(\gamma^*-1)\left((\check{e}_{k+1}^{-})^2\right)^{\hat{w}_{k+1}}}},\\ \frac{\sqrt{\gamma^*}\left(\check{e}_{2}^{+}\right)^{\hat{w}_2}}{\sqrt{1+(\gamma^*-1)\left(1-(\check{e}_{k+1}^{+})^2\right)^{\hat{w}_2}+(\gamma^*-1)\left(\left(\check{e}_{k+1}^{+}\right)^2\right)^{\hat{w}_{k+1}}}}\end{array}\right];\\ \frac{\sqrt{\gamma^*}(\tau_{2(k+1)}^*)^{\hat{w}_{k+1}}}{\sqrt{1+(\gamma^*-1)\left(1-\left(\tau_{2(k+1)}^*\right)^2\right)^{\hat{w}_{k+1}}+(\gamma^*-1)\left(\left(\tau_{2(k+1)}^*\right)^2\right)^{\hat{w}_{k+1}}}}\end{array}\right)\end{array}\right\}$$

$$=\left\{\begin{array}{l}\left(\begin{array}{c}\left[\begin{array}{c}\sqrt{\frac{\prod_{\check{s}=1}^{k+1}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{k+1}\left(1-(\hat{c}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{k+1}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{k+1}\left(1-(\hat{c}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}}},\\ \sqrt{\frac{\prod_{\check{s}=1}^{k+!}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{k+1}\left(1-(\hat{c}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{k}\left(1+(\gamma^*-1)(\hat{c}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{k+1}\left(1-(\hat{c}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}}}\end{array}\right];\\ \sqrt{\frac{\prod_{\check{s}=1}^{k+1}\left(1+(\gamma^*-1)(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{k+1}\left(1-(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{k+1}\left(1+(\gamma^*-1)(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{k+1}\left(1-(\tau_{1\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}}\end{array}\right),\\ \left(\begin{array}{c}\left[\begin{array}{c}\frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{k+1}(\check{e}_{\check{s}}^{-})^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{k+1}1+(\gamma^*-1)\left(1-(\check{e}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{k+1}\left((\check{e}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}}},\\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{k+1}\left(\check{e}_{\check{s}}^{+}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{k+1}1+(\gamma^*-1)\left(1-(\check{e}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{k+1}\left((\check{e}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}}}\end{array}\right];\\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{k+1}(\tau_{2\check{s}}^*)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{k+1}1+(\gamma^*-1)\left(1-(\tau_{2\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}+(\gamma^*-1)\prod_{\check{s}=1}^{k+1}\left((\tau_{2\check{s}}^*)^2\right)^{\hat{w}_{\check{s}}}}}\end{array}\right)\end{array}\right\}$$

Thus, Equation (3.2) is true for all $n = k + 1$. Which is required.

It can be easily shown that the following properties exist for the PCFHWA operator.

Theorem 3.2. (Idempotency). If all $P^*_{\check{s}}(\check{s} \in N)$ are equal, i.e., $P^*_{\check{s}} = P^*, \forall\ \check{s}$. Hence,

$$PCFHWA_{\hat{w}_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) = P^*. \tag{3.3}$$

Theorem 3.3. (Boundedness). Suppose $P^*_{\check{s}}(\check{s} \in N)$ is a group of PCFNs, and $P^{*-}_{\check{s}} = \min_{\check{s}} P^*_{\check{s}}, P^{*+}_{\check{s}} = \max_{\check{s}} P^*_{\check{s}}$. Then,

$$P^{*-}_{\check{s}} \leq PCFHWA_{\hat{w}_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) \leq P^{*+}_{\check{s}}. \tag{3.4}$$

Theorem 3.4. (Monotonicity). Suppose $P^{*}_{\check{s}}(\check{s} \in N)$ and $P^{*'}_{\check{s}}(\check{s} \in N)$ be two sets of PCFNs, if $P^{*}_{\check{s}} \leq P^{*'}_{\check{s}}$ for all $\check{s}$. Then,

$$PCFHWA_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) \leq PCFHWA_{\hat{w}_{\check{s}}}(P^{*'}_{1}, P^{*'}_{2}, ..., P^{*'}_{n}). \tag{3.5}$$

Now, about the parameter γ^{*}, we will explore several distinct cases of the PCFHWA operator.

When $\gamma^{*} = 1$, PCFHWA operator decreases to the PCFWA operators as follows:

$$PCFHWA_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) = \bigoplus_{\check{s}=1}^{n} (\hat{w}_{\check{s}} P^{*}_{\check{s}}) \tag{3.6}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - (\hat{c}^{-}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}, \sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - (\hat{c}^{+}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}} \right]; \\ \sqrt{1 - \prod_{\check{s}=1}^{n} (1 - (\tau^{*}_{1\check{s}})^{2})^{\hat{w}_{\check{s}}}} \end{array} \right), \\ \left(\left[\prod_{\check{s}=1}^{n} (\check{e}^{-}_{\check{s}})^{\hat{w}_{\check{s}}}, \prod_{\check{s}=1}^{n} (\check{e}^{+}_{\check{s}})^{\hat{w}_{\check{s}}} \right]; \prod_{\check{s}=1}^{n} (\tau^{*}_{2\check{s}})^{\hat{w}_{\check{s}}} \right) \end{array} \right\}$$

When $\gamma^{*} = 2$, PCFHWA operator reduces to the PCFEWA operator as follows:

$$PCFEWA_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) \tag{3.7}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1+(\hat{c}^{-}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\hat{c}^{-}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\hat{c}^{-}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1-(\hat{c}^{-}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1+(\hat{c}^{+}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\hat{c}^{+}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\hat{c}^{+}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1-(\hat{c}^{+}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1+(\tau^{*}_{1\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\tau^{*}_{1\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\tau^{*}_{1\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1-(\tau^{*}_{1\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{2} \prod_{\check{s}=1}^{n} (\check{e}^{-}_{\check{s}})^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2-(\check{e}^{-}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left((\check{e}^{-}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{2} \prod_{\check{s}=1}^{n} (\check{e}^{+}_{\check{s}})^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2-(\check{e}^{+}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left((\check{e}^{+}_{\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \frac{\sqrt{2} \prod_{\check{s}=1}^{n} (\tau^{*}_{2\check{s}})^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2-(\tau^{*}_{2\check{s}})^{2}\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left((\tau^{*}_{2\check{s}})^{2}\right)^{\hat{w}_{\check{s}}}}} \end{array} \right) \end{array} \right\}$$

Definition 3.2. Let $P^{*}_{\check{s}} = \left(\left\langle [\hat{c}^{-}_{\check{s}}, \hat{c}^{+}_{\check{s}}]; \tau^{*}_{1\check{s}} \right\rangle, \left\langle [\check{e}^{-}_{\check{s}}, \check{e}^{+}_{\check{s}}]; \tau^{*}_{2\check{s}} \right\rangle \right) (\check{s} \in N)$ be a number of PCFNs. Then, we define the PCFHOWA operator as follows:

$$PCFHWA_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) = \bigoplus_{\check{s}=1}^{n} \left(\hat{w}_{\check{s}} P^{*}_{\mho_{\circ}(\check{s})} \right), \tag{3.8}$$

where $(\mho_{\circ}(1), \mho_{\circ}(2), ..., \mho_{\circ}(n))$ is a mapping of $(1, 2, ..., n)$, such that $P^{*}_{\mho_{\circ}(\check{s}-1)} \geq P^{*}_{\mho_{\circ}(\check{s})}$ $\forall$ $\check{s} = 2, 3, ..., n$, and $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$ is the aggregation-related weight vector such that $\hat{w}_{\check{s}} \in [0, 1]$ and $\hat{w}_{\check{s}} > 0$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1, \gamma^* > 0$.

We can drive the Theorem 5 based on Hamacher $\oplus$ operations of the mentioned PCFNs.

Theorem 3.5. Suppose $P^{*}_{\check{s}} = \left(\left\langle [\hat{c}^{-}_{\check{s}}, \hat{c}^{+}_{\check{s}}]; \tau^{*}_{1\check{s}} \right\rangle, \left\langle [\check{e}^{-}_{\check{s}}, \check{e}^{+}_{\check{s}}]; \tau^{*}_{2\check{s}} \right\rangle\right) (\check{s} \in N)$ is a group of PCFNs. So, we define PCFHOWA operator as follows;

$$PCFHOWA_{\hat{w}_{\check{s}}}(P^{*}_1, P^{*}_2, ..., P^{*}_n) = \bigoplus_{\check{s}=1}^{n} \left(\hat{w}_{\check{s}} P^{*}_{\mho_{\circ}(\check{s})}\right) \tag{3.9}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\hat{c}^{-}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{-}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\hat{c}^{-}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{-}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\hat{c}^{+}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{+}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\hat{c}^{+}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{+}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\tau^{*}_{1\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\tau^{*}_{1\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\tau^{*}_{1\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\tau^{*}_{1\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\check{e}^{-}_{\mho_{\circ}(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\check{e}^{-}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\check{e}^{-}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\check{e}^{+}_{\mho_{\circ}(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\check{e}^{+}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\check{e}^{+}_{\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\tau^{*}_{2\mho_{\circ}(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\tau^{*}_{2\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\tau^{*}_{2\mho_{\circ}(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right) \end{array} \right\}$$

where $(\mho_{\circ}(1), \mho_{\circ}(2), ..., \mho_{\circ}(n))$ is a mapping of $(1, 2, ..., n)$, such that $P^{*}_{\mho_{\circ}(\check{s}-1)} \geq P^{*}_{\mho_{\circ}(\check{s})}$ $\forall$ $\check{s} = 2, 3, ..., n$, and $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$ is the aggregation-related weight vector: $\hat{w}_{\check{s}} \in [0, 1]$ and $\hat{w}_{\check{s}} > 0$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1, \gamma^* > 0$.

Can be easily displayed that the following properties exist for the PCFHOWA operator.

Theorem 3.6.(Idempotency). If all $P^{*}_{\check{s}} (\check{s} \in N)$ are equal, i.e., $P^{*}_{\check{s}} = P^{*}$ for all $\check{s}$. Then,

$$PCFHOWA_{\hat{w}_{\check{s}}}(P^{*}_1, P^{*}_2, ..., P^{*}_n) = P^{*}. \tag{3.10}$$

Theorem 3.7(Boundedness). Suppose $P^{*}_{\check{s}} (\check{s} \in N)$ be a group of PCFNs, and $P^{*-}_{\check{s}} = \min_{\check{s}} P^{*}_{\check{s}}, P^{*+}_{\check{s}} = \max_{\check{s}} P^{*}_{\check{s}}$.

Then,

$$P^{*-}_{\check{s}} \leq PCFHOWA_{\hat{w}}(P^{*}_1, P^{*}_2, ..., P^{*}_n) \leq P^{*+}_{\check{s}}. \tag{3.11}$$

Theorem 3.8. (Monotonicity). Suppose $P^{*}_{\check{s}} (\check{s} \in N)$ and $P^{*'}_{\check{s}} (\check{s} \in N)$ be two set of $PCFNs$, if $P^{*}_{\check{s}} \leq P^{*'}_{\check{s}}, \forall\ \check{s}$. So,

$$PCFHOWA_{\hat{w}_{\check{s}}}(P^{*}_1, P^{*}_2, ..., P^{*}_n) \leq PCFHOWA_{\hat{w}}(P^{*'}_1, P^{*'}_2, ..., P^{*'}_n). \tag{3.12}$$

When $\gamma^* = 1$, PCFHOWA operator decrease to PCFOWA operator as follows:

$$PCFHOWA_{\hat{w}_{\check{s}}}(P_1^*, P_2^*, ..., P_n^*) = \bigoplus_{\check{s}=1}^{n} \left(\hat{w}_{\check{s}} P^*_{\mho_\circ(\check{s})}\right) \tag{3.13}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - \left(\hat{c}^-_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}, \sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - \left(\hat{c}^+_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}} \right]; \\ \sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - \left(\tau^*_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}} \end{array} \right), \\ \left(\left[\prod_{\check{s}=1}^{n} \left(\check{e}^-_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}, \prod_{\check{s}=1}^{n} \left(\check{e}^+_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}} \right]; \prod_{\check{s}=1}^{n} \left(\tau^*_{2\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}} \right) \end{array} \right\}$$

When $\gamma^* = 2$, PCFHOWA operator reduces to the PCFEOWA operator as follows:

$$PCFEOWA_{\hat{w}_{\check{s}}}(P_1^*, P_2^*, ..., P_n^*) \tag{3.14}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\hat{c}^-_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}} - \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^-_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\hat{c}^-_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}} + \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^-_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\hat{c}^+_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^+_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\hat{c}^+_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^+_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\tau^*_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\tau^*_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\tau^*_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(1-\left(\tau^*_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\check{e}^-_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\check{e}^-_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(\left(\check{e}^-_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\check{e}^+_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\check{e}^+_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(\left(\check{e}^+_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\tau_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\tau^*_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}} + \prod_{\check{s}=1}^{n}\left(\left(\tau^*_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right) \end{array} \right\}$$

In Definitions (3.1) and (3.2) we discover that the PCFHWA operator tests the PCF statement itself, While the operator of PCFHOWA considers the ordered positions of the PCF arguments rather than the arguments themselves. Hence, weights in both the PCFHWA and PCFHOWA operators show different attributes. However, both the operators find only one of them. To solve this disadvantage, a PCFHHA operator will be suggested in the following.

Definition 3.3. A Pythagorean cubic fuzzy Hamacher hybrid average (PCFHHA) operator is defined as follows:

$$PCFHHWA_{\hat{w}, \omega_{\check{s}}}(P_1^*, P_2^*, ..., P_n^*) = \bigoplus_{\check{s}=1}^{n} \left(\hat{w}_{\check{s}} P^{**}_{\mho_\circ(\check{s})}\right), \tag{3.15}$$

where the corresponding weighting vector is $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$, with $\hat{w}_{\check{s}} \in [0,1]$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1$, $P^{**}_{\mho_\circ(\check{s})}$ be the $\check{s}$th biggest component of the PF argument $P^{**}_{\check{s}}(P^{**}_{\check{s}} = n\omega_{\check{s}} P^{*}_{\check{s}}(\check{s} = 1, 2, ..., n)$, with $\omega_{\check{s}} \in [0,1]$, $\sum_{\check{s}=1}^{n} \omega_{\check{s}} = 1$, and n is the balancing coefficient. Particularly, if $\hat{w}_{\check{s}} = (1/n, 1/n, ..., 1/n)^T$, then PCFHHA is decreases to the PCFHWA operator; if $\hat{w}_{\check{s}} = (1/n, 1/n, ..., 1/n)$ is decreases to the PCFHOWA operator.

Centered on Hamacher sum operations of the demonstrate PCFNs we can develop the Theorem (3.9).

Theorem 3.9. Suppose $P^{*}_{\check{s}} = \left(\left\langle [\hat{c}^{-}_{\check{s}}, \hat{c}^{+}_{\check{s}}]; \tau^{*}_{1\check{s}} \right\rangle, \left\langle [\check{e}^{-}_{\check{s}}, \check{e}^{+}_{\check{s}}]; \tau^{*}_{2\check{s}} \right\rangle\right) (\check{s} \in N)$ be a group of PCFNs. So, their calculated value using PCFHWA operator is also a PCFN,

$$PCFHHA_{\hat{w}_{\check{s}},\omega_{\check{s}}}(P^{*}_1, P^{*}_2, ..., P^{*}_n) = \bigoplus_{\check{s}=1}^{n} \left(\hat{w}_{\check{s}} P^{**}_{\mho_\circ(\check{s})}\right) \tag{3.16}$$

$$= \left\{ \begin{array}{c} \left(\left[\begin{array}{c} \sqrt{\dfrac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\dfrac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\dfrac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \right), \\ \left(\left[\begin{array}{c} \dfrac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\check{e}^{-*}_{\sigma_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \dfrac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \dfrac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \right) \end{array} \right\}$$

Where the corresponding vector weighting is $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)$, with $\hat{w}_{\check{s}} \in [0,1]$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1$, and $P^{**}_{\sigma_\circ(\check{s})}$ is the kth biggest component of the PF argument $P^{**}_{\check{s}}(P^{**}_{\check{s}} = n\omega_{\check{s}} P^{*}_{\check{s}}, (\check{s} \in N)$, $\omega_{\check{s}} = (\omega_1, \omega_2, ..., \omega_n)^T$ is the vector weighting of PF argument $P^{*}_{\check{s}}(\check{s} = 1, 2, ..., n)$, with $\omega_{\check{s}} \in [0,1]$, $\sum_{\check{s}=1}^{n} \omega_{\check{s}} = 1$, and n is the balancing coefficient.

Now, about the parameter γ^*, we will explore several distinct cases of the PCFHHA operator.

When $\gamma^* = 1$, PCFHHA operator decreases to the PCFHA operator as follows:

$$PCFHA_{\hat{w}_{\check{s}},\omega_{\check{s}}}(P^{*}_1, P^{*}_2, ..., P^{*}_n) = \bigoplus_{\check{s}=1}^{n} \left(\hat{w}_{\check{s}} P^{**}_{\mho_\circ(\check{s})}\right) \tag{3.17}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - \left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}, \sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - \left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}} \right]; \\ \sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - \left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}} \end{array} \right), \\ \left(\left[\prod_{\check{s}=1}^{n} \left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^{\hat{w}^*_{\check{s}}}, \prod_{\check{s}=1}^{n} \left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}} \right]; \prod_{\check{s}=1}^{n} \left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}} \right) \end{array} \right\}$$

When $\gamma^* = 2$, PCFHHA operator reduces to the PCFEHA operator as follows:

$$PCFEHA_{\hat{w}_{\check{s}},\omega_1}(P^*_1, P^*_2, ..., P^*_n) \tag{3.18}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(1-\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\tau^{**}_{2\sigma_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n}\left(\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right) \end{array} \right\}$$

4. Pythagorean cubic fuzzy Hamacher geometric aggregation operators

In this section, we discuss the Pythagorean cubic fuzzy Hamacher geometric aggregation operators, their basic properties and related theorems.

Definition 4.1. Suppose $P^*_{\check{s}} = \left(\left\langle[\hat{c}^-_{\check{s}}, \hat{c}^+_{\check{s}}]; \tau^*_{1\check{s}}\right\rangle, \left\langle[\check{e}^-_{\check{s}}, \check{e}^+_{\check{s}}]; \tau^*_{2\check{s}}\right\rangle\right) (\check{s} \in N)$ be a group of PCFNs. So, we define PCFHWG operator as follows:

$$PCFHWG_{\hat{w}}(P^*_1, P^*_2, ..., P^*_n) = \bigotimes_{\check{s}=1}^{n} (P^*_{\check{s}})^{\hat{w}_{\check{s}}}, \tag{4.1}$$

as $P^*_{\check{s}}(\check{s} \in N)$, and $\hat{w}_{\check{s}} > 0$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1, \gamma^* > 0$.

We can drive the Theorem 10 based on Hamacher $\otimes$ operations of the mentioned PCFNs.

Theorem 4.1. Suppose $P^*_{\check{s}} = \left(\left\langle [\hat{c}^-_{\check{s}}, \hat{c}^+_{\check{s}}]; \tau^*_{1\check{s}} \right\rangle, \left\langle [\check{e}^-_{\check{s}}, \check{e}^+_{\check{s}}]; \tau^*_{2\check{s}} \right\rangle\right) (\check{s} \in N)$ be a group of PCFNs. So, their accumulated value is also a PCFN, using PCFHWG operator, and

$$PCFHWG_{\hat{w}_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) = \bigotimes_{\check{s}=1}^{n} (P^*_{\check{s}})^{\hat{w}_{\check{s}}} \tag{4.2}$$

$$= \left\{ \begin{array}{l} \left(\begin{array}{c} \left[\begin{array}{c} \dfrac{\sqrt{\gamma^*} \prod_{\check{s}=1}^{n} (\hat{c}^-_{\check{s}})^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-(\hat{c}^-_{\check{s}})^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left((\hat{c}^-_{\check{s}})^2\right)^{\hat{w}_{\check{s}}}}}, \\ \dfrac{\sqrt{\gamma^*} \prod_{\check{s}=1}^{n} (\hat{c}^+_{\check{s}})^{\omega^*_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-(\hat{c}^+_{\check{s}})^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left((\hat{c}^+_{\check{s}})^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \dfrac{\sqrt{\gamma^*} \prod_{\check{s}=1}^{n} \left(\tau^*_{1\check{s}}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-(\tau^*_{1\check{s}})^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left((\tau^*_{1\check{s}})^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right) \\ \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\dfrac{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)(\check{e}^-_{\check{s}})^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\check{e}^-_{\check{s}})^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)(\check{e}^-_{\check{s}})^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left(1-(\check{e}^-_{\check{s}})^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\dfrac{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)(\check{e}^+_{\check{s}})^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\check{e}^+_{\check{s}})^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)(\check{e}^+_{\check{s}})^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} (\left(1-(\check{e}^+_{\check{s}})^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\dfrac{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)(\tau^*_{2\check{s}})^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\tau^*_{2\check{s}})^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)(\tau^*_{2\check{s}})^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left(1-(\tau^*_{2\check{s}})^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right) \end{array} \right\}$$

Where the corresponding vector weighting is $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$ of $P^*_{\check{s}}(\check{s} \in N)$, and $\hat{w}_{\check{s}} > 0$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1, \gamma^* > 0$.

There are some properties of PCFHWG which can be easily proved as follows:

Theorem 4.2. (Idempotency). If all $P^*_{\check{s}}(\check{s} \in N)$ are same, i.e., $P^*_{\check{s}} = P^*, \forall \check{s}$. So,

$$PCFHWG_{\hat{w}_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) = P^*. \tag{4.3}$$

Theorem 4.3. (Boundedness). Suppose $P^*_{\check{s}}(\check{s} \in N)$ be a group of PCFNs, and $P^{*-}_{\check{s}} = \min_{\check{s}} P^*_{\check{s}}, P^{*+}_{\check{s}} = \max_{\check{s}} P^*_{\check{s}}$. Then,

$$P^{*-}_{\check{s}} \leq PCFHWG_{\hat{w}_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) \leq P^{*+}_{\check{s}}. \tag{4.4}$$

Theorem 4.4. (Monotonicity). Suppose $P^*_{\check{s}}(\check{s} \in N)$ and $P^{*'}_{\check{s}}(\check{s} \in N)$ be two set of PCFNs, if $P^*_{\check{s}} \leq P^{*'}_{\check{s}}$ for all $\check{s}$. Then,

$$PCFHWG_{\hat{w}_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) \leq PCFHWG_{\hat{w}_{\check{s}}}(P^{*'}_1, P^{*'}_2, ..., P^{*'}_n). \tag{4.5}$$

Now, about the parameter γ^*, we will explore several distinct cases of the PCFHWG operator.

When $\gamma^* = 1$, PCFHWG operator decreases to the PCFWG operator as follows:

$$PCFWG_{\hat{w}_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) = \bigotimes_{\check{s}=1}^{n} (P^*_{\check{s}})^{\hat{w}_{\check{s}}} \tag{4.6}$$

$$= \left\{ \begin{array}{c} \left(\left[\prod_{\check{s}=1}^{n} (\hat{c}_{\check{s}}^{-})^{\hat{w}_{\check{s}}}, \prod_{\check{s}=1}^{n} (\hat{c}_{\check{s}}^{+})^{\hat{w}_{\check{s}}} \right]; \prod_{\check{s}=1}^{n} (\tau_{1\check{s}}^{*})^{\hat{w}_{\check{s}}} \right), \\ \left[\sqrt{1 - \prod_{\check{s}=1}^{n} (1 - (\check{e}_{\check{s}}^{-})^2)^{\hat{w}_{\check{s}}}}, \sqrt{1 - \prod_{\check{s}=1}^{n} (1 - (\check{e}_{\check{s}}^{+})^2)^{\hat{w}_{\check{s}}}} \right]; \\ \sqrt{1 - \prod_{\check{s}=1}^{n} (1 - (\tau_{2\check{s}}^{*})^2)^{\hat{w}_{\check{s}}}} \end{array} \right\}$$

When $\gamma^* = 2$, PCFHWG operator reduces to the PCFEWG operator as follows:

$$PCFEWG_{\hat{w}_{\check{s}}}(P_1^*, P_2^*, ..., P_n^*) \tag{4.7}$$

$$= \left\{ \begin{array}{c} \left(\left[\begin{array}{c} \frac{\sqrt{2} \prod_{\check{s}=1}^{n} (\hat{c}_{\check{s}}^{-})^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2-(\hat{c}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left((\hat{c}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{2} \prod_{\check{s}=1}^{n} (\hat{c}_{\check{s}}^{+})^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2-(\hat{c}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left((\hat{c}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \frac{\sqrt{2} \prod_{\check{s}=1}^{n} (\tau_{1\check{s}}^{*})^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2-(\tau_{1\check{s}}^{*})^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left((\tau_{1\check{s}}^{*})^2\right)^{\hat{w}_{\check{s}}}}} \right), \\ \left(\left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1+(\check{e}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\check{e}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\check{e}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1-(\check{e}_{\check{s}}^{-})^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1+(\check{e}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\check{e}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\check{e}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1-(\check{e}_{\check{s}}^{+})^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1+(\tau_{2\check{s}}^{*})^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-(\tau_{2\check{s}}^{*})^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\tau_{2\check{s}}^{*})^2\right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1-(\tau_{2\check{s}}^{*})^2\right)^{\hat{w}_{\check{s}}}}} \right) \end{array} \right\}$$

Definition 4.2. Let $P_{\check{s}}^* = \left(\left\langle [\hat{c}_{\check{s}}^-, \hat{c}_{\check{s}}^+]; \tau_{1\check{s}}^* \right\rangle, \left\langle [\check{e}_{\check{s}}^-, \check{e}_{\check{s}}^+]; \tau_{2\check{s}}^* \right\rangle \right) (\check{s} \in N)$ be a number of PCFNs. So, we define the PCFHOWG operator as follows:

$$PCFHWG_{\hat{w}_{\check{s}}}(P_1^*, P_2^*, ..., P_n^*) = \bigotimes_{\check{s}=1}^{n} \left(P_{\mho_\circ(\check{s})}^* \right)^{\hat{w}_{\check{s}}}, \tag{4.8}$$

where $(\mho_\circ(1), \mho_\circ(2), ..., \mho_\circ(n))$ is a function of $(1, 2, ..., n)$, such that $P_{\mho_\circ(\check{s}-1)}^* \geq P_{\sigma_\circ(\check{s})}^* \ \forall \ \check{s} = 2, 3, ..., n$, and $\hat{w} = (\hat{w}_1, \hat{w}, ..., \hat{w}_n)^T$ is the aggregation-related weight vector such that $\hat{w}_{\check{s}} \in [0, 1]$ and $\hat{w}_{\check{s}} > 0, \sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1, \gamma^* > 0$.

Centered on the Hamacher product operations of the described PCFNs we can develop the Theorem 14.

Theorem 4.6. Suppose $P_{\check{s}}^* = \left(\left\langle [\hat{c}_{\check{s}}^-, \hat{c}_{\check{s}}^+]; \tau_{1\check{s}}^* \right\rangle, \left\langle [\check{e}_{\check{s}}^-, \check{e}_{\check{s}}^+]; \tau_{2\check{s}}^* \right\rangle \right) (\check{s} \in N)$ be a group of PCFNs. Then, their accumulated value is also a PCFN, using PCFHOWG operator, and

$$PCFHOWG_{\hat{w}_{\check{s}}}(P_1^*, P_2^*, ..., P_n^*) = \bigotimes_{\check{s}=1}^{n} (P_{\check{s}}^*)^{\hat{w}_{\check{s}}} \tag{4.9}$$

$$
= \left\{ \begin{array}{l} \left(\left[\begin{array}{l} \dfrac{\sqrt{\gamma^*} \prod_{\check{s}=1}^{n} \left(\hat{c}^{-}_{\mho_\circ(\check{s})} \right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\hat{c}^{-}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left(\left(\hat{c}^{-}_{\mho_\circ(\check{s})}\right)^2 \right)^{\hat{w}_{\check{s}}}}}, \\ \dfrac{\sqrt{\gamma^*} \prod_{\check{s}=1}^{n} \left(\hat{c}^{+}_{\mho_\circ(\check{s})} \right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\hat{c}^{+}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left(\left(\hat{c}^{+}_{\mho_\circ(\check{s})}\right)^2 \right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \dfrac{\sqrt{\gamma^*} \prod_{\check{s}=1}^{n} \left(\tau^{*}_{1\mho_\circ(\check{s})} \right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\tau^{*}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left(\left(\tau^{*}_{1\mho_\circ(\check{s})}\right)^2 \right)^{\hat{w}_{\check{s}}}}} \right) \\ \left(\left[\begin{array}{l} \sqrt{\dfrac{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)\left(\check{e}^{-}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-\left(\check{e}^{-}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)\left(\check{e}^{-}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left(1-\left(\check{e}^{-}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\dfrac{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)\left(\check{e}^{+}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-\left(\check{e}^{+}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)\left(\check{e}^{+}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left(1-\left(\check{e}^{+}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\dfrac{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)\left(\tau^{*}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1-\left(\tau^{*}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1+(\gamma^*-1)\left(\tau^{*}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1) \prod_{\check{s}=1}^{n} \left(1-\left(\tau^{*}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \right) \end{array} \right\}
$$

where $(\mho_\circ(1), \mho_\circ(2), ..., \mho_\circ(n))$ be a mapping of $(1, 2, ..., n)$: $P^{*}_{\mho_\circ(\check{s}-1)} \geq P^{*}_{\mho_\circ(\check{s})}$ $\forall \check{s} = 2, 3, ..., n$, and $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$ is the aggregation-related vector weight: $\hat{w}_{\check{s}} \in [0, 1]$ and $\hat{w}_{\check{s}} > 0$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1, \gamma^* > 0$.

There are some properties of PCFHOWG which can be easily proved as follows:

Theorem 4.7.(Idempotency). If all $P^{*}_{\check{s}} (\check{s} \in N)$ are identical, i.e., $P^{*}_{\check{s}} = P^*$ for all $\check{s}$. Then,

$$PCFHOWG_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) = P^*. \tag{4.10}$$

Theorem 4.8.(Boundedness). Suppose $P^{*}_{\check{s}} (\check{s} = 1, 2, ..., n)$ be a group of PCFNs, and $P^{*-}_{\check{s}} = \min_{\check{s}} P^{*}_{\check{s}}, P^{*+}_{\check{s}} = \max_{\check{s}} P^{*}_{\check{s}}$.

Then,

$$P^{*-}_{\check{s}} \leq PCFHOWG_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) \leq P^{*+}_{\check{s}}. \tag{4.11}$$

Theorem 4.9. (Monotonicity). Suppose $P^{*}_{\check{s}} (\check{s} \in N)$ and $P^{*'}_{\check{s}} (\check{s} \in N))$ be two set of PCFNs, if $P^{*}_{\check{s}} \leq P^{*'}_{\check{s}}$ for all $\check{s}$. Then,

$$PCFHOWG_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) \leq PCFHOWG_{\hat{w}_{\check{s}}}(P^{*'}_{1}, P^{*'}_{2}, ..., P^{*'}_{n}). \tag{4.12}$$

Now, about the parameter γ^*, we will explore several distinct cases of the PCFHOWG operator.

When $\gamma^* = 1$, PCFHOWG operator reduces to the PCFOWG operator as follows:

$$PCFOWG_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) = \bigotimes_{\check{s}=1}^{n} \left(P^{*}_{\mho_\circ(\check{s})} \right)^{\hat{w}_{\check{s}}} \tag{4.13}$$

$$= \left\{ \begin{array}{c} \left(\left[\prod_{\check{s}=1}^{n} \left(\hat{c}^{-}_{\mho_{\circ}(\check{s})} \right)^{\hat{w}_{\check{s}}}, \prod_{\check{s}=1}^{n} \left(\hat{c}^{+}_{\mho_{\circ}(\check{s})} \right)^{\hat{w}_{\check{s}}} \right]; \prod_{\check{s}=1}^{n} \left(\tau^{*}_{1\mho_{\circ}(\check{s})} \right)^{\hat{w}_{\check{s}}} \right), \\ \left[\sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - \left(\check{e}^{-}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}, \sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - (\left(\check{e}^{+}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}} \right]; \\ \sqrt{1 - \prod_{\check{s}=1}^{n} \left(1 - \left(\tau^{*}_{2\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}} \end{array} \right\}$$

When $\gamma^{*} = 2$, PCFHOWG operator reduces to the PCFEOWG operator as follows:

$$PCFEOWG_{\hat{w}_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) \tag{4.14}$$

$$= \left\{ \begin{array}{c} \left(\left[\begin{array}{c} \frac{\sqrt{2} \prod_{\check{s}=1}^{n} \left(\hat{c}^{-}_{\mho_{\circ}(\check{s})} \right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2 - \left(\hat{c}^{-}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(\left(\hat{c}^{-}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{2} \prod_{\check{s}=1}^{n} \left(\hat{c}^{+}_{\mho_{\circ}(\check{s})} \right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2 - \left(\hat{c}^{+}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(\left(\hat{c}^{+}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \frac{\sqrt{2} \prod_{\check{s}=1}^{n} \left(\tau^{*}_{1\mho_{\circ}(\check{s})} \right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} \left(2 - \left(\tau^{*}_{1\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(\left(\tau^{*}_{1\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}} \right), \\ \left(\left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1 + \left(\check{e}^{-}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1 - (\left(\check{e}^{-}_{\mho_{\circ}(\check{s})} \right))^{2} \right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1 + \left(\check{e}^{-}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1 - \left(\check{e}^{-}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1 + \left(\check{e}^{+}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1 - \left(\check{e}^{+}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1 + \left(\check{e}^{+}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1 - \left(\check{e}^{+}_{\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \\ \sqrt{\frac{\prod_{\check{s}=1}^{n} \left(1 + \left(\tau^{*}_{2\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n} \left(1 - \left(\tau^{*}_{2\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n} \left(1 + \left(\tau^{*}_{2\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}} + \prod_{\check{s}=1}^{n} \left(1 - \left(\tau^{*}_{2\mho_{\circ}(\check{s})} \right)^{2} \right)^{\hat{w}_{\check{s}}}}} \right) \end{array} \right\}$$

Definitions (4.1) and (4.2) tell us that the PCFHWG operator weighs the Pythagorean cubic fuzzy argument itself, Whereas the PCFHOWG operator regards the ordered positions of the Pythagorean cubic arguments rather than the arguments themselves. Hence, weights in both the PCFHWG and PCFHOWG operators show different attributes. However, both the operators find only one of them. To solve this disadvantage, a Pythagorean cubic Fuzzy Hamacher Hybrid Average PCFHHG operator will be suggested in the following.

Definition 4.3. A Pythagorean cubic fuzzy Hamacher hybrid geometric PCFHHG operator is defined as follow:

$$PCFHHG_{\hat{w}_{\check{s}}, \omega_{\check{s}}}(P^{*}_{1}, P^{*}_{2}, ..., P^{*}_{n}) = \bigotimes_{\check{s}=1}^{n} \left(P^{**}_{\mho_{\circ}(\check{s})} \right)^{\hat{w}_{\check{s}}}, \tag{4.15}$$

where the corresponding weighting vector is $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$, with $\hat{w}_{\check{s}} \in [0, 1]$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1$, and $P^{**}_{\mho_{\circ}(\check{s})}$ is the $\check{s}$th biggest component of the Pythagorean fuzzy argument $P^{**}_{\check{s}}(P^{**}_{\check{s}} = (P^{*}_{\check{s}})^{n\omega_{\check{s}}}, (\check{s} = 1, 2, ..., n)$, with $\omega^{*}_{\check{s}} \in [0, 1]$, $\sum_{\check{s}=1}^{n} \omega_{\check{s}} = 1$, and n is the balancing coefficient. Particularly, if $\hat{w}_{\check{s}} =$

$(1/n, 1/n, ..., 1/n)^T$, then PCFHHG is decreased to the Pythagorean Cubic fuzzy Hamacher weighted averaging (PCFHG) operator; if $\hat{w} = (1/n, 1/n, ..., 1/n)$ then, PCFHHG is decreased to the Pythagorean Cubic fuzzy Hamacher ordered weighted averaging (PCFHOWG) operator.

We can drive Theorem (4.10) based on the Hamacher $\otimes$ operations of the previously mentioned PCFNs.

Theorem 4.10. Suppose $P^*_{\check{s}} = \left(\left\langle[\hat{c}^-_{\check{s}}, \hat{c}^+_{\check{s}}]; \tau^*_{1\check{s}}\right\rangle, \left\langle[\check{e}^-_{\check{s}}, \check{e}^+_{\check{s}}]; \tau^*_{2\check{s}}\right\rangle\right)$ $(\check{s} \in N)$ be a group of PCFNs. So their accumulated value is also a PCFN, using PCFHHG operator, and

$$PCFHHG_{\hat{w}_{\check{s}},\omega_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) = \bigotimes_{\check{s}=1}^{n} \left(P^{**}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}} \tag{4.16}$$

$$= \left\{\begin{array}{c} \left(\left[\begin{array}{c} \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array}\right]; \\ \frac{\sqrt{\gamma^*}\prod_{\check{s}=1}^{n}\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n} 1+(\gamma^*-1)\left(1-\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}\right) \\ \left(\left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}\right.}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left((1+(\gamma^*-1)\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}\right.}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array}\right]; \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} - \prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+(\gamma^*-1)\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}} + (\gamma^*-1)\prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}\right) \end{array}\right\}$$

Where the corresponding weighting vector is $\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$, with $\hat{w}_{\check{s}} \in [0, 1]$, $\sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1$, and $P^{**}_{\mho_\circ(\check{s})}$ is the *kth* biggest component of the PF argument $P^{**}_{\check{s}}(P^{**}_{\check{s}} = (P^*_{\check{s}})^{n\omega_{\check{s}}}, (\check{s} = 1, 2, ..., n)$, with $\omega_{\check{s}} \in [0, 1]$, $\sum_{\check{s}=1}^{n} \omega_{\check{s}} = 1$, and n is the balancing coefficient.

When $\gamma^* = 1$, PCFHHG operator decreases to the PCFHG operator as follows:

$$PCFHG_{\hat{w}_{\check{s}},\omega_{\check{s}}}(P^*_1, P^*_2, ..., P^*_n) = \bigotimes_{\check{s}=1}^{n} \left(P^{**}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}} \tag{4.17}$$

$$= \left\{\begin{array}{c} \left(\left[\prod_{\check{s}=1}^{n}\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}, \prod_{\check{s}=1}^{n}\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}\right]; \prod_{\check{s}=1}^{n}\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}\right), \\ \left[\sqrt{1-\prod_{\check{s}=1}^{n}\left(1-\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}, \sqrt{1-\prod_{\check{s}=1}^{n}\left(1-\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}\right]; \\ \sqrt{1-\prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}} \end{array}\right\}$$

When $\gamma^* = 2$, PCFHHG operator decreases to the PCFEHG operator as follows:

$$PCFEHG_{\hat{w}_{\check{s}}}(P_1^*, P_2^*, ..., P_n^*) \tag{4.18}$$

$$= \left\{ \begin{array}{l} \left(\left[\begin{array}{c} \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}+\prod_{\check{s}=1}^{n}\left(\left(\hat{c}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}+\prod_{\check{s}=1}^{n}\left(\left(\hat{c}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \frac{\sqrt{2}\prod_{\check{s}=1}^{n}\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^{\hat{w}_{\check{s}}}}{\sqrt{\prod_{\check{s}=1}^{n}\left(2-\left(\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)\right)^{\hat{w}_{\check{s}}}+\prod_{\check{s}=1}^{n}\left(\left(\tau^{**}_{1\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \right), \\ \left(\left[\begin{array}{c} \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{n}\left(1-\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}+\prod_{\check{s}=1}^{n}\left(1-\left(\check{e}^{-*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}}, \\ \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{n}\left(1-\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}+\prod_{\check{s}=1}^{n}\left(1-\left(\check{e}^{+*}_{\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \end{array} \right]; \sqrt{\frac{\prod_{\check{s}=1}^{n}\left(1+\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}-\prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}{\prod_{\check{s}=1}^{n}\left(1+\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}+\prod_{\check{s}=1}^{n}\left(1-\left(\tau^{**}_{2\mho_\circ(\check{s})}\right)^2\right)^{\hat{w}_{\check{s}}}}} \right) \end{array} \right\}$$

5. Model for MCGDM with Pythagorean cubic Hamacher fuzzy information

MCGDM includes multiple DM in setting and rating goals. Alternatives available regarding multiple, often conflicting criteria [41, 35, 37, 54, 55]. A variety of tasks typically include evaluating and choosing appropriate green supply chain management (GSCM) strategies within an enterprise. First, it is important to identify all the alternatives and the assessment criteria. The feasibility of the alternatives to the GSCM procedure and the weight of the appraisal criterion must be calculated. At last, for the calculation of the overall value of the performance index for each alternative, the alternative score and the weight of the parameters must be considered aggregated. In human decision making uncertainty is always present. In order to model the uncertainty better, Pythagorean cubic fuzzy numbers are used to represent DM's assessments. This is because (a) the suitability of Pythagorean cubic fuzzy numbers to resolve ambiguity in DM, and (b) the flexibility of particular evaluations using membership and non-member status. The proposed model begins by assessing the success ranking for each GSCM practise alternative $A_z(z = 1, 2, ..., n)$ for each $C_s(s = 1, 2, ..., n)$ parameter.

The method set out above for the proposed model can be summarized as given below;

Step 1. Construct Pythagorean cubic fuzzy decision matrix.

$$R = \left(\left\langle[\check{c}^-_{ij}, \check{c}^+_{ij}]; \tau^*_{ij}\right\rangle, \left\langle[\check{e}^-_{ij}, \check{e}^+_{ij}]; \tau^*_{ij}\right\rangle\right)(i = 1, 2, ..., m; j = 1, 2, ..., n).$$

Normalize the aggregated matrix R to $R^/$. The criteria are divided as benefit criteria and cost criteria. If all the criteria are of the same type, then normalization is not required. Where as cost type criteria can be converted to benefit criteria by the below normalization formula, if R has both benefit criteria and cost criteria.

$$R^{/} = \begin{cases} \left(\left\langle [\check{c}_{ij}^{-}, \check{c}_{ij}^{+}]; \tau_{ij}^{*} \right\rangle, \left\langle [\check{e}_{ij}^{-}, \check{e}_{ij}^{+}]; \tau_{ij}^{*} \right\rangle\right), & \text{for benefit type} \\ \left(\left\langle [\check{e}_{ij}^{-}, \check{e}_{ij}^{+}]; \tau_{ij}^{*} \right\rangle, \left\langle [\check{c}_{ij}^{-}, \check{c}_{ij}^{+}]; \tau_{ij}^{*} \right\rangle\right), & \text{for cost type} \end{cases},$$

$R^{/}$ is the complement of R. Thus, we get the normalized PCFSs decision matrix.

Step 2. The parameters weight $\hat{w}$ can be modified as follows to promote the decision-making process as given below;

$$\hat{w}_{\check{s}} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T, \hat{w}_{\check{s}} > 0, \sum_{\check{s}=1}^{n} \hat{w}_{\check{s}} = 1, \gamma^* > 0. \tag{5.1}$$

Step 3. Utilize the proposed aggregation operators to calculate the Pythagorean cubic fuzzy values for the alternatives $A_{1i}(i = 1, 2, ..., n)$. i.e., the proposed operators to stem the collective overall preference values of the alternatives A_i^*, where $\hat{w} = (\hat{w}_1, \hat{w}_2, ..., \hat{w}_n)^T$ is the criteria weight vector.

Step 4. The performance index value for every alternative can be calculated in relation to all parameters using the score function.

$$S(P_{\check{s}}^*) = \frac{1}{2}\left\{\left(1 + (\frac{\hat{c}_{\check{s}}^{-} + \hat{c}_{\check{s}}^{+} - \tau_{1\check{s}}^{*}}{3}\right) - \left(\frac{\check{e}_{\check{s}}^{-} + \check{e}_{\check{s}}^{+} - \tau_{2\check{s}}^{*}}{3}\right)\right\}. \tag{5.2}$$

Step 5. Rank the alternatives in decreased sequence of their index values for performance.

6. Case study

This segment provides an example of how to analyse the existing GSCM practises to ensure that an organization can choose the best GSCM practises. Owing to the extreme competitiveness and the networked nature of the industry, a renowned Taiwanese organization is contemplating a Green Supply Chain plan to improve its action [10]. The company is one of Taiwan's biggest producers of professional printed circuit boards. It devises the next generation of development to improve their competitiveness. Because of the rising market demand, the business has to produce green products to satisfy the growing customer situation. This contributes to finding the implementation of a successful GSCM protocol by the organization.

The assessment of the GSCM activities identified as E_1, E_2, E_3 includes four top managers from four different divisions within the organization. A number of meetings were held. This results in four parameters for determining the four GSCM operations within an organization. Such requirements include the Green Design (C_1), Green Manufacturing (C_2) and Green Manufacturing Investments (C_3), and green marketing (C_4).

Green Design (C_1) is linked to an organization's ability to decrease its negative environmental impact of product designs [30]. This is also calculated by the organization's ability to eliminate the use of toxic chemicals, follow the ideals of susceptibility, improve technological capacities and the energy usage in the organization's operations. Green Manufacturing (C_2) considers enhancing the production processes to minimize toxic matter generation [58]. To manufacture goods with less waste and less emissions, a company needs to. This is also calculated by how many resources are used, how green the energy is, and how hazardous waste is reduced [3, 15]. Green Purchasing (C_3) deals with the process of obtaining and consuming goods and facilities which are less environmentally destructive than their substitutes [49]. In this respect, many other aspects, including the green picture, green proficiency and green process management, are generally taken into account [51, 5]. Green Marketing (C_4) aims to promote the green qualities of safe products and services for the surrounding [16]. This is also assessed by the implementation of advertisement ICTs, and the provision of training and information on the benefits of green services and goods.

Step 1. By choosing the alternatives and criteria given above, the output score for all alternative options can be calculated from A_1, A_2, A_3 and A_4 as shown in Tables 1–3 for each criterion.

Table 1. Performance assessment of GSCM Practices.

	C_1	C_2	C_3	C_4
A_1	$\begin{pmatrix} \langle[.7,.8];.7\rangle, \\ \langle[.4,.5];.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.5\rangle, \\ \langle[.4,.5];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.8];.7\rangle, \\ \langle[.5,.7];.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.4,.5];.7\rangle, \\ \langle[.7,.8];.5\rangle \end{pmatrix}$
A_2	$\begin{pmatrix} \langle[.7,.8];.6\rangle, \\ \langle[.4,.5];.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.7\rangle, \\ \langle[.5,.6];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.5];.7\rangle, \\ \langle[.5,.7];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.5\rangle, \\ \langle[.4,.5];.6\rangle \end{pmatrix}$
A_3	$\begin{pmatrix} \langle[.6,.7];.7\rangle, \\ \langle[.5,.6];.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.6\rangle, \\ \langle[.4,.5];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.4,.5];.5\rangle, \\ \langle[.7,.6];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.5,.6];.8\rangle, \\ \langle[.6,.7];.5\rangle \end{pmatrix}$
A_4	$\begin{pmatrix} \langle[.6,.1];.2\rangle, \\ \langle[.4,.5];.1\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.4,.5];.4\rangle, \\ \langle[.7,.8];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.2];.1\rangle, \\ \langle[.5,.6];.3\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.1\rangle, \\ \langle[.8,.6];.4\rangle \end{pmatrix}$

Table 2. Performance assessment of GSCM Practices.

	C_1	C_2	C_3	C_4
A_1	$\begin{pmatrix} \langle[.7,.8];.4\rangle, \\ \langle[.4,.5],.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.5\rangle, \\ \langle[.5,.6];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.8\rangle, \\ \langle[.5,.6];.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.7\rangle, \\ \langle[.4,.6];.4\rangle \end{pmatrix}$
A_2	$\begin{pmatrix} \langle[.6,.7];.7\rangle, \\ \langle[.7,.8];.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.7\rangle, \\ \langle[.7,.8];.9\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.5,.6];.6\rangle, \\ \langle[.4,.5];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.4,.5];.5\rangle, \\ \langle[.4,.5];.3\rangle \end{pmatrix}$
A_3	$\begin{pmatrix} \langle[.6,.7];.7\rangle, \\ \langle[.5,.6];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.5\rangle, \\ \langle[.4,.5];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.7\rangle, \\ \langle[.4,.5];.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.9\rangle, \\ \langle[.4,.5];.3\rangle \end{pmatrix}$
A_4	$\begin{pmatrix} \langle[.5,.6];.6\rangle, \\ \langle[.6,.7];.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.5\rangle, \\ \langle[.5,.6];.8\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.3];.4\rangle, \\ \langle[.4,.5];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.5\rangle, \\ \langle[.4,.6];.7\rangle \end{pmatrix}$

Table 3. Performance assessment of GSCM Practices.

	C_1	C_2	C_3	C_4
A_1	$\begin{pmatrix} \langle[.7,.8];.6\rangle, \\ \langle[.4,.6];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.8];.6\rangle, \\ \langle[.5,.6];.8\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.8];.6\rangle, \\ \langle[.5,.6];.8\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.4];.6\rangle, \\ \langle[.8,.7];.9\rangle \end{pmatrix}$
A_2	$\begin{pmatrix} \langle[.5,.7];.5\rangle, \\ \langle[.4,.5];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.4,.5];.6\rangle, \\ \langle[.8,.4];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.8];.6\rangle, \\ \langle[.5,.6];.8\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.3,.4];.8\rangle, \\ \langle[.8,.9];.6\rangle \end{pmatrix}$
A_3	$\begin{pmatrix} \langle[.6,.8];.2\rangle, \\ \langle[.5,.6];.9\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.6\rangle, \\ \langle[.5,.6];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.8,.5];.6\rangle, \\ \langle[.5,.3];.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.5,.6];.8\rangle, \\ \langle[.4,.7];.6\rangle \end{pmatrix}$
A_4	$\begin{pmatrix} \langle[.6,.8];.6\rangle, \\ \langle[.5,.6];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.5,.7];.7\rangle, \\ \langle[.4,.6];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.1];.2\rangle, \\ \langle[.5,.6];.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.4,.5];.8\rangle, \\ \langle[.7,.8];.6\rangle \end{pmatrix}$

Step 2. Centered on this data the parameters of the weight vector can be determined using Equation (5.1). The corresponding parameters was calculated as $\hat{w} = (0.25, 0.35, 0.4)^T$.

Step 3, 4. Using the proposed aggregation operators and Eq (5.2), each alternative's output index value can be calculated in regard of all criteria and their respective ranking. It indicates that in Table 4, with the highest performance index of 0.8566, alternative A_4 has the highest performance value.

Table 4. The ranking of GCSM practices Alternatives.

Alternative	Performance index value	Ranking
A_1	−0.02774	4
A_2	0.79762	2
A_3	0.76802	3
A_4	0.8566	1

7. Comparative analysis

7.1. Validity test

In this subsection, first we take a worse alternative of each expert, as given in Table 5.

Table 5. Rating values of the worse alternative.

	C_1	C_2	C_3	C_4
E_1	$\begin{pmatrix} \langle[.6,.6];.7\rangle, \\ \langle[.7,.8];.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.5\rangle, \\ \langle[.4,.5];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.8\rangle, \\ \langle[.5,.6];.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.7\rangle, \\ \langle[.4,.5];.5\rangle \end{pmatrix}$
E_2	$\begin{pmatrix} \langle[.7,.8];.4\rangle, \\ \langle[.4,.5];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.7\rangle, \\ \langle[.4,.6];.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.5,.6];.6\rangle, \\ \langle[.4,.5];.7\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.9\rangle, \\ \langle[.4,.5];.3\rangle \end{pmatrix}$
E_3	$\begin{pmatrix} \langle[.4,.7];.5\rangle, \\ \langle[.5,.6];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.7\rangle, \\ \langle[.5,.6];.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.6,.7];.5\rangle, \\ \langle[.4,.5];.3\rangle \end{pmatrix}$	$\begin{pmatrix} \langle[.7,.8];.7\rangle, \\ \langle[.4,.5];.9\rangle \end{pmatrix}$

Since, different MCGDM approaches produce different results (ranking) when applied to the same DM problem, the results are uncertain. In order to analyze the reliability and validity of the MCGDM methods, Wang & Triantaphyllou [52] provided the following test conditions:

Test criteria 1: The MCGDM method is successful if the best alternative remains unchanged and the nonoptimal alternative is replaced with a worse alternative without increasing the relative importance of each decision-criteria.

Test criteria 2: An efficient MCGDM approach should obey transitive properties.

Test criteria 3: When decomposing the MCGDM problem into subproblems and applying the proposed MCGDM approach to these subproblems for ranking alternatives, the MCGDM approach is successful. The overall ranking of the alternatives is identical to the overall ranking of the problem.

The following criteria are used to determine the validity of the proposed solution.

7.1.1. Validity check with Criteria 1

In order to assess the validity of the existing method with Criteria 1, the non-optimal alternative A_4 is replaced by the worse alternative $A_4^{/}$ in the original decision matrix for each expert, and the rating values are given in Table 6.

Now, using the proposed aggregation operators, we get the computed the scores of the alternatives are $S(A_1) = 0.716$, $S(A_2) = 0.837$, $S(A_3) = 0.742$, and $S(A_4) = 0.873$. As a result, the final ranking of the alternatives indicates that A_4 remains the best option and that the approach established meets the test criteria 1.

7.1.2. Validity check with criteria 2 and 3

We decomposed the original decision-making problem into sub-DM problems, which included these alternatives, (A_1, A_2, A_3), (A_2, A_3, A_4) and (A_1, A_3, A_4) in order to evaluate the specified MCGDM method with Criteria 2 and 3. When we use the MCGDM method to solve these subproblems, we get the following ranking of alternatives: $A_2 > A_3 > A_1$, $A_4 > A_2 > A_3$ and $A_4 > A_3 > A_1$,. We get the final ranking order as $A_4 > A_2 > A_3 > A_1$ by adding the ranking of alternatives to these small problems. The resulting ranking is identical to that of a non-decomposed problem, revealing a transitive property. As a result, the specified MCGDM method is compatible with Criteria 2 and 3.

7.2. Comparison with existing methods

A comparative study is conducted to demonstrate the new model's validity. Within this portion, we compare the output of the defined MCGDM method with some of the current methods of Pythagorean fuzzy set; like as [6, 43, 26]. On the basis of this environment, we applied the current methods, and their results are given in Table 6.

Table 6. A Descriptive research review

Method	Ranking
IFGA [6]	$A_4 > A_3 > A_1 > A_2$
PFHWA [43]	$A_4 > A_2 > A_3 > A_1$
PFHOWA [43]	$A_4 > A_3 > A_2 > A_1$
PCFWA [26]	$A_4 > A_3 > A_2 > A_1$
PCFOWA [26]	$A_4 > A_3 > A_2 > A_1$
PCFHWA (proposed)	$A_4 > A_2 > A_3 > A_1$
PCFHOWA (proposed)	$A_4 > A_2 > A_3 > A_1$

From the analysis of Table 6, we see that the existing approaches have the deficiency of data and these approaches are not suitable for solving and ranking the developed numerical example. Therefore, the proposed methods are more effective and capable than the existing techniques.

7.3. Verification

The TOPSIS method is used to verify the results provided by the proposed aggregation operators in this section.

TOPSIS method

We use the TOPSIS method to check the numerical problem from Section 6 in this section.

To solve the problem in Section 6, we use the TOPSIS method, which includes the following steps:

Step 1. Normalize Tables 1–3 decision matrix. Since, all of the measure values are of the same type, i.e., benefit type, there is no need for normalization.

Step 2. Identifying the PIS $\Xi^+$and NIS Ξ^-, which are defined as,

$$\Xi^+ = (\zeta_1^+, ..., \zeta_4^+), \Xi^- = (\zeta_1^-, ..., \zeta_4^-),$$

where

$$\zeta_j^+ = \max\{\zeta_{ij}/1 \leq i \leq 4\} \text{ and } \zeta_j^- = \min\{\zeta_{ij}/1 \leq i \leq 4\},$$

which are calculated by using the score function;

$$S(P^*_{\check{s}}) = \frac{1}{2}\left\{\left(1 + (\frac{\hat{c}^-_{\check{s}} + \hat{c}^+_{\check{s}} - \tau^*_{1\check{s}}}{3}\right) - \left(\frac{\check{e}^-_{\check{s}} + \check{e}^+_{\check{s}} - \tau^*_{2\check{s}}}{3}\right)\right\}$$

Step 3. Calculate the distance for each alternative to Ξ^+ and Ξ^- using the proposed distance measures with criteria weight vector $\hat{w} = (0.25, 0.35, 0.4)^T$. i.e.,

$$d_i^+ = \sqrt{\Sigma_{j=1}^m w_j(\zeta_j^+ - \zeta_{ij})^2}, \text{ and } d_i^- = \sqrt{\Sigma_{j=1}^m w_j(\zeta_j^- - \zeta_{ij})^2}.$$

Step 4. Calculate the closeness coefficients to the ideal solution by each alternative by applying the Equation,

$$\Theta_i = d_i^-/(d_i^- + d_i^+)(i = 1, ..., 4),$$

the overall closeness coefficients are obtained.

Step 5. Give ranking to the alternatives based on score value and select the best one. We obtain the ranking result as;

$$A_4 > A_3 > A_1 > A_2$$

All the calculation results and the alternative ranking are given in Table 7. According to the calculations of overall coefficients the best one with largest closeness coefficient is A_4.

Hence, by the TOPSIS method it is again verified that A_4 is the best alternative, as given in Table 7.

Table 7. Obtained values and their ranking.

Alternatives	Distance between alternative to PIS (d_i^+)	Distance between alternative to NIS (d_i^-)	Closeness coefficients of the alternative (Θ_i)	Ranking
A_1	0.103	0.217	0.245	3
A_2	0.171	0.193	0.284	4
A_3	0.183	0.221	0.212	2
A_4	0.265	0.223	0.162	1

8. Conclusions

It's not an easy task to evaluate the efficiency of existing GSCM practices in society. This is because such an appraisal requires multiple decision-makers and various parameters with complexity in community DM processes. In this paper, we introduced a multi-criteria community DM model for assessing effectively the performance of GSCM activities within an organization. The complexity of the evaluation process is appropriately modeled by the use of PCFNs. Also, we have developed some Hamacher averaging and geometric aggregation operators to aggregate Pythagorean cubic fuzzy information. The main features of these proposed operators are evaluated. Then, we used these operators to develop several methods to solve the PCF multiple attribute decision-making issues. Finally, a practical example is established to confirm the method established and to prove its practicality and effectiveness. Furthermore, the comparison of the proposed and current aggregation was given and discussed how our proposed method is more powerful than other current aggregation operators. The example described displays that this model established can effectively and efficiently solve the multi-criteria DM problem. This will help organizations understand more about the value of GSCM activities to boost their susceptibility management growth performance. One relates to the importance of the decision result on the decision-maker's inputs. One more is the need to evaluate direct and indirect benefits and costs together when determining the existing GSCM practices. Further analysis should be executed to help address these two problems through the correct use of corporate evidence and knowledge in management and improvement designing current optimization frameworks to overcome the performance appraisal problem.

In the future, the application of our proposed model can be applied in DM using Dombi t-norm and t-conorm operation, Bonferroni mean operator, Maclaurin symmetric mean operator, q-rung orthopair fuzzy set, Spherical fuzzy set, and T-Spherical fuzzy set.

Acknowledgments

The authors would like to thank the Deanship of Scientfic Research at Umm Al-Qura University for supporting this work by grant number 19-SCI-1-01-0041.

Conflict of interest

The authors declare that they have no conflicts of interest.

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