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Research article

Ostrowski type inequalities for exponentially s-convex functions on time scale

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Abstract: In this paper we establish some new inequalities of Ostrowski type for exponentially s-convex functions and s-convex functions on time scale. We also make comparison of our new results with already existing results by imposing some conditions.

Keywords: Ostrowski type inequalities; s-convex functions; exponentially s-convex functions; time scale
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1. Introduction

Ostrowski formulate a formula in 1938 to calculate the deviation of differentiable functions from its integral mean which is discussed in [1] and known as Ostrowski inequality given by

$$\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi(u)du\right| \leq \sup_{l_1 \leq v \leq l_2} |\xi'(v)|(l_2 - l_1)\left[\frac{(v - \frac{(l_1+l_2)}{2})^2}{(l_2 - l_1)^2} + \frac{1}{4}\right] \tag{1.1}$$

can be proved by Montgomery identity as shown in [2] but this identity on time scale was discussed by M. Bohner and T. Matthews in [3] which is given as

Lemma 1.1. *Let $l_1, l_2, u, v \in T, l_1 < l_2$ and $\xi : [l_1, l_2] \to R$ be differentiable. Then*

$$\xi(v) = \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u + \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \chi(v, u)\xi^{\Delta}(u)\Delta u \tag{1.2}$$

Where

$$\chi(v, u) = \begin{cases} u - l_1, & l_1 \leq u < v \\ u - l_2, & v \leq u \leq l_2 \end{cases} \tag{1.3}$$

Definition 1.1. *([4]) Let $s \in (0,1]$. A function $\xi : I \subseteq R_0 \to R_0$, where $R_0 = [0,\infty)$, is said to be s-convex function in second sense if*

$$\xi(kl_1 + (1-k)l_2) \leq k^s\xi(l_1) + (1-k)^s\xi(l_2)$$

for all $l_1, l_2 \in I$.

Definition 1.2. *([5]) Let $s \in (0,1]$. A function $\xi : I \subseteq R \to R$ is said to be exponentially convex if*

$$\xi(kl_1 + (1-k)l_2) \leq k^s\frac{\xi(l_1)}{e^{\alpha l_1}} + (1-k)^s\frac{\xi(l_2)}{e^{\alpha l_2}}$$

for $l_1, l_2 \in I$ with $l_1 < l_2$, $k \in [0,1]$ and $\alpha \in R$.

Our aim of this paper is to discuss Hermite Hadamard inequality and Ostrowski type inequalities on time scale for exponentially s-convex, s-convex functions.

2. Preliminaries

Definition 2.1. *Time scale is defined as a non-empty close subset of real numbers.*

The most important examples are R (set of real numbers) and Z (set of integers). For $u, v \in T$ where T is a time scale, forward and backward jumped operators σ and ρ respectively are defined as $\sigma(v) = inf\{k \in T : k > v\} \in T$, $\rho(v) = sup\{k \in T : k < v\} \in T$. Supplemented by $inf\phi = supT$ and $sup\phi = infT$.

A point v is said to be right scattered and left scattered if $\sigma(v) > v$ and $\rho(v) < v$ respectively. If a point v is both right and left scattered then it is isolated. If $\sigma(v) = v$ then v is called right dense and it is said to be left dense if $\rho(v) = v$. If the point v is left and right dense both then it is called dense.

Suppose $\zeta_1 \in T$ is right scattered minimum, then $T_k = T - \{\zeta_1\}$ otherwise $T_k = T$. Suppose $\zeta_2 \in T$ is left scattered maximum, then $T^k = T - \{\zeta_2\}$, otherwise $T^k = T$. Moreover $T_k^k = T_k \cap T^k$.

Definition 2.2. *Delta derivative of function $\xi : T \to R$ at $v \in T^k$ is defined to be the number $\xi^\Delta(v)$ (if it exists) satisfying the property that, for any $\epsilon > 0$ there is a neighbourhood U of v such that*

$$|[\xi(\sigma(v)) - \xi(u)] - \xi^\Delta(v)[\sigma(v) - u]| < \epsilon|\sigma(v) - u| \tag{2.1}$$

for all $u \in U$.

Definition 2.3. *A function $\xi : T \to R$ is continuous at right dense points of T and its left-sided limit exist at left dense points of T, then ξ is known to be rd-continuous. Denoted by $\xi \in C_{rd}$.*

Theorem 2.1. *Let $\xi : T \to R$ be an rd-continuous function. Then f has an anti-derivative Ξ satisfying $\Xi^\Delta = \xi$.*

Proof. See [6, Theorem 1.74]. □

Definition 2.4. *If $\xi : T \to R$ is an rd-continuous function and $l_1 \in T$, then we define the integral $\Xi(v) = \int_{l_1}^{v} \xi(k)\Delta k$ for $v \in T$.*

Therefore for $\xi \in C_{rd}$ we have $\Xi(l_2) - \Xi(l_1) = \int_{l_1}^{l_2} \xi(k)\Delta k$. Where $\Xi^\Delta = \xi$.

Theorem 2.2. *If $l_1, l_2, l_3 \in T, \beta \in R$ and $\xi_1, \xi_2 \in C_{rd}$, then*

(i) $\int_{l_1}^{l_2}(\xi_1(v) + \xi_2(v))\Delta v = \int_{l_1}^{l_2} \xi_1(v)\Delta v + \int_{l_1}^{l_2} \xi_2(v)\Delta v$,

(ii) $\int_{l_1}^{l_2} \beta\xi_1(v)\Delta v = \beta \int_{l_1}^{l_2} \xi_1(v)\Delta v$,

(iii) $\int_{l_1}^{l_2} \xi_1(v)\Delta v = -\int_{l_2}^{l_1} \xi_1(v)\Delta v$,

(iv) $\int_{l_1}^{l_2} \xi_1(v)\Delta v = \int_{l_1}^{l_3} \xi_1(v)\Delta v + \int_{l_3}^{l_2} \xi_1(v)\Delta v$,

(v) $\int_{l_1}^{l_1} \xi_1(v)\Delta v = 0$,

(vi) $\int_{l_1}^{l_2} \xi_1(v)\xi_2^{\Delta}(v)\Delta v = (\xi_1\xi_2)(l_2) - (\xi_1\xi_2)(l_1) - \int_{l_1}^{l_2} \xi_1^{\Delta}(v)\xi_2(\sigma(v))\Delta v$,

(vii) $\int_{l_1}^{l_2} \xi_1(v)\xi_2^{\Delta}(v) = (\xi_1\xi_2)(l_2) - (\xi_1\xi_2)(l_1) - \int_{l_1}^{l_2} \xi_1^{\Delta}(v)\xi_2(\sigma(v))\Delta v$.

Proof. See [6, Theorem 1.77]. □

Theorem 2.3. *Let $l_1, l_2 \in T$ and $\xi_1, \xi_2 : T \to R$ be rd-continuous. Then*

$$\int_{l_1}^{l_2} |\xi_1(v)\xi_2(v)|\Delta v \leq \left(\int_{l_1}^{l_2} |\xi_1(v)|^p \Delta v\right)^{\frac{1}{p}} \left(\int_{l_1}^{l_2} |\xi_2(v)|^q \Delta v\right)^{\frac{1}{q}}. \tag{2.2}$$

where p, q > 1 and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. See [6. Theorem 6.13]. □

3. Main results

Keeping in mind the integral inequalities and inequalities on time scale [7–15] first we prove the Hermite Hadamard inequality for exponentially s-convex functions on time scale. Throughout this section $K = [l_1, l_2] \subseteq T$.

Theorem 3.1. *Let T be a time scale and $K = [l_1, l_2]$. Let $\xi : K \to R$ is exponentially s-convex function in the second sense on K^0 and Δ-integrable as well. Then for $l_1, l_2 \in K$ with $l_1 < l_2$ and $\alpha \in R$, we have*

$$2^{s-1}\xi\left(\frac{l_1 + l_2}{2}\right) \leq \frac{1}{l_2 - l_1} \int_{l_1}^{l_2} \frac{\xi(w)}{e^{\alpha w}}\Delta w$$
$$\leq \frac{\xi(l_1)}{e^{\alpha l_2}} \int_0^1 \frac{k^s}{e^{\alpha(kl_1+(1-k)l_2)}}\Delta k + \frac{\xi(l_2)}{e^{\alpha l_2}} \int_0^1 \frac{(1-k)^s}{e^{\alpha(kl_1+(1-k)l_2)}}\Delta k. \tag{3.1}$$

Proof. Using the definition of exponential s-convexity of ξ we have

$$2^s\xi\left(\frac{x + y}{2}\right) \leq \frac{\xi(x)}{e^{\alpha x}} + \frac{\xi(y)}{e^{\alpha y}}.$$

Making use of change of variable $x = kl_1 + (1-k)l_2$ and $y = (1-k)l_1 + kl_2$ and taking Δ-integral with respect to $k \in [0,1]$ we get

$$2^s\xi\left(\frac{l_1+l_2}{2}\right) \leq \frac{2}{l_2-l_1}\int_{l_1}^{l_2}\frac{\xi(w)}{e^{\alpha w}}\Delta w$$

and

$$2^{s-1}\xi(\frac{l_1+l_2}{2}) \leq \frac{1}{l_2-l_1}\int_{l_1}^{l_2}\frac{\xi(w)}{e^{\alpha w}}\Delta w. \tag{3.2}$$

Now, we prove second inequality

$$\frac{\xi(kl_1+(1-k)l_2)}{e^{\alpha}(kl_1+(1-k)l_2)} \leq \frac{k^s\frac{\xi(l_1)}{e^{\alpha l_1}} + (1-k)^s\frac{\xi(l_2)}{e^{\alpha l_2}}}{e^{\alpha(kl_1+(1-k)l_2)}}.$$

Taking Δ-integral w.r.t $k \in [0,1]$ we get

$$\frac{1}{l_2-l_1}\int_0^1\frac{\xi(w)}{e^{\alpha w}}\Delta w \leq \frac{\xi(l_1)}{e^{\alpha l_1}}\int_0^1\frac{r^s}{e^{\alpha(kl_1+(1-k)l_2)}}\Delta k + \frac{\xi(l_2)}{e^{\alpha l_2}}\int_0^1\frac{(1-k)^s}{e^{\alpha(kl_1+(1-k)l_2)}}\Delta k. \tag{3.3}$$

Combining (7) and (8) we get (6). □

Corollary 3.1.1. *For $T = R$ we get the Hermite Hadamard inequality for exponentially s-convex functions [5, Theorem 3.2].*

Now, we will discuss Ostrowski inequality for exponentially s-convex function on time scale.

Theorem 3.2. *Let T be a time scale and $K \subseteq T$. Let $\xi : K \to R$ be a differentiable function on K^0 such that $\xi^\Delta \in K$ for $l_1, l_2 \in K$ where $l_1 < l_2$. If ξ^Δ is exponentially s-convex in second sense on $[l_1, l_2]$ for $s \in (0,1]$ and $\sup_{l_1\leq v\leq l_2}|\xi^\Delta(v)| = M$, $v \in [l_1, l_2]$. Then following inequality holds:*

$$\left|\xi(v) - \frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^\sigma(u)\Delta u\right| \leq \frac{M(v-l_1)^2}{l_2-l_1}\int_0^1\left(\frac{k^{s+1}}{e^{\alpha v}} + \frac{k(1-k)^s}{e^{\alpha l_1}}\right)\Delta k + \frac{M(v-l_2)^2}{l_2-l_1}\int_0^1\left(\frac{k^{s+1}}{e^{\alpha v}} + \frac{k(1-k)^s}{e^{\alpha l_2}}\right)\Delta k. \tag{3.4}$$

Proof. Using Montgomery identity

$$\left|\xi(v) - \frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^\sigma(u)\Delta u\right| = \left|\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\chi(v,u)\xi^\Delta(u)\Delta u\right| \leq \frac{1}{l_2-l_1}\left(\int_{l_1}^{t}(u-l_1)|\xi^\Delta(u)|\Delta u + \int_{v}^{l_2}(u-l_2)|\xi^\Delta(u)|\Delta u\right).$$

Making use of change of variables we get

$$\left|\xi(v) - \frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^\sigma(u)\Delta u\right| \leq \frac{1}{l_2-l_1}\int_0^1(v-l_1)^2k|\xi^\Delta(kv+(1-k)l_1)|\Delta k + \frac{1}{l_2-l_1}\int_0^1(v-l_2)^2k|\xi^\Delta(kv+(1-k)l_2)|\Delta k.$$

Using exponential s-convexity of ξ^{Δ} we get

$$\begin{aligned}
&\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u\right| \\
&\leq \frac{(v-l_1)^2}{l_2-l_1}\int_0^1 \left(k(k^s\frac{|\xi^{\Delta}(v)|}{e^{\alpha v}}) + k((1-k)^s\frac{|\xi^{\Delta}(l_1)|}{e^{\alpha l_1}})\right)\Delta k \\
&+ \frac{(v-l_2)^2}{l_2-l_1}\int_0^1 \left(k(k^s\frac{|\xi^{\Delta}(v)|}{e^{\alpha v}}) + k((1-k)^s\frac{|\xi^{\Delta}(l_2)|}{e^{\alpha l_2}})\right)\Delta k
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{M(v-l_1)^2}{l_2-l_1}\int_0^1 \left(\frac{k^{s+1}}{e^{\alpha v}} + \frac{k(1-k)^s}{e^{\alpha l_1}}\right)\Delta k \\
&+ \frac{M(v-l_2)^2}{l_2-l_1}\int_0^1 \left(\frac{k^{s+1}}{e^{\alpha v}} + \frac{k(1-k)^s}{e^{\alpha l_2}}\right)\Delta k.
\end{aligned}$$

□

Corollary 3.2.1. *If $T = R$, then we obtain Theorem 2.1 given in [16].*

Theorem 3.3. *Suppose that $\xi : K \to R$ be a differentiable mapping on K^0 such that $\xi^{\Delta} \in K$ for $l_1, l_2 \in K$ with $l_1 < l_2$. If $|\xi^{\Delta}|^q$ is exponentially s-convex in the second sense on $[l_1, l_2]$ for some $s \in (0, 1]$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $\sup_{l_1 \leq v \leq l_2} |\xi^{\Delta}(v)| = M, v \in [l_1, l_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u\right| \\
&\leq \frac{M(v-l_1)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{k^s}{e^{\alpha v}} + \frac{(1-k)^s}{e^{\alpha l_1}}\right)\Delta k\right)^{\frac{1}{q}} \\
&+ \frac{M(v-l_2)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{k^s}{e^{\alpha v}} + \frac{(1-k)^s}{e^{\alpha l_2}}\right)\Delta k\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.5}$$

Proof. By Montgomery identity we have

$$\begin{aligned}
&\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u\right| = \left|\frac{1}{l_2-l_1}\int_{l_1}^{l_2} \chi(v,u)\xi^{\Delta}(u)\Delta u\right| \\
&\leq \frac{1}{l_2-l_1}\left(\int_{l_1}^{t}(u-l_1)|\xi^{\Delta}(u)|\Delta u + \int_{v}^{l_2}(u-l_2)|\xi^{\Delta}(u)|\Delta u\right).
\end{aligned}$$

Making use of change of variables we obtain

$$\begin{aligned}
\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u\right| &\leq \frac{1}{l_2-l_1}\int_0^1 (v-l_1)^2 k|\xi^{\Delta}(kv + (1-k)l_1)|\Delta k \\
&+ \frac{1}{l_2-l_1}\int_0^1 (v-l_2)^2 k|\xi^{\Delta}(kv + (1-k)l_2)|\Delta k.
\end{aligned}$$

Using (5) we get

$$\begin{aligned}
&\left|\xi(v)-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^{\sigma}(u)\Delta u\right|\\
&\leq \frac{(v-l_1)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{\frac{1}{p}}\left(\int_0^1 |\xi^{\Delta}(kv+(1-k)l_1)|^q\Delta k\right)^{\frac{1}{q}}\\
&+\frac{(v-l_2)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{\frac{1}{p}}\left(\int_0^1 |\xi^{\Delta}(kv+(1-k)l_2)|^q\Delta k\right)^{\frac{1}{q}}.
\end{aligned}$$

Using the definition of exponential s-convexity of $|\xi^{\Delta}|^q$ we have

$$\begin{aligned}
&\left|\xi(v)-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^{\sigma}(u)\Delta u\right|\\
&\leq \frac{(v-l_1)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{\frac{1}{p}}\left(\int_0^1 \left(k^s\frac{|\xi^{\Delta}(v)|^q}{e^{\alpha v}}+(1-k)^s\frac{|\xi^{\Delta}(v)|^q}{e^{\alpha l_1}}\right)\Delta k\right)^{\frac{1}{q}}\\
&+\frac{(v-l_2)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{\frac{1}{p}}\left(\int_0^1 \left(k^s\frac{|\xi^{\Delta}(v)|^q}{e^{\alpha v}}+(1-k)^s\frac{|\xi^{\Delta}(v)|^q}{e^{\alpha l_2}}\right)\Delta k\right)^{\frac{1}{q}}\\
&\leq \frac{M(v-l_1)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{k^s}{e^{\alpha v}}+\frac{(1-k)^s}{e^{\alpha l_1}}\right)\Delta k\right)^{\frac{1}{q}}\\
&+\frac{M(v-l_2)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{k^s}{e^{\alpha v}}+\frac{(1-k)^s}{e^{\alpha l_2}}\right)\Delta k\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 3.3.1. *If $T=R$ then we obtain Theorem 2.2 given in [16].*

Theorem 3.4. *Let us consider a differentiable mapping $\xi: K\to R$ on K^0 such that $\xi^{\Delta}\in K$ for $l_1,l_2\in K$ with $l_1<l_2$. If $|\xi^{\Delta}|^q$ is exponentially s-convex in the second sense on $[l_1,l_2]$ for some $s\in(0,1]$, $q>1$ and $\sup_{l_1\leq v\leq l_2}|\xi^{\Delta}(v)|=M$, $v\in[l_1,l_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\xi(v)-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^{\sigma}(u)\Delta u\right|\\
&\leq \frac{M(v-l_1)^2}{l_2-l_1}\left(\int_0^1 k^p\Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{k^{s+1}}{e^{\alpha v}}+\frac{k(1-k)^s}{e^{\alpha l_1}}\right)\Delta k\right)^{\frac{1}{q}}\\
&+\frac{M(v-l_1)^2}{l_2-l_1}\left(\int_0^1 k^{s+1}\Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{k^{s+1}}{e^{\alpha v}}+\frac{k(1-k)^s}{e^{\alpha l_2}}\right)\Delta k\right)^{\frac{1}{q}}.
\end{aligned}\tag{3.6}$$

Proof. By Montgomery identity we have

$$\left|\xi(v)-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^{\sigma}(u)\Delta u\right|=\left|\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\chi(v,u)\xi^{\Delta}(u)\Delta u\right|$$

$$\leq \frac{1}{l_2-l_1}\left(\int_{l_1}^{t}(u-l_1)|\xi^{\Delta}(u)|\Delta u+\int_{v}^{l_2}(u-l_2)|\xi^{\Delta}(u)|\Delta u\right).$$

Making use of change of variables we obtain

$$\left|\xi(v)-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^{\sigma}(u)\Delta u\right|\leq\left(\frac{1}{l_2-l_1}\right)[\int_0^1(v-l_1)^2k|\xi^{\Delta}(kv+(1-k)l_1)|\Delta k$$
$$+\int_0^1(v-l_2)^2k|\xi^{\Delta}(kv+(1-k)l_2)|\Delta k]$$

It follows that

$$\left|\xi(v)-\frac{1}{l_2-l_1}\int_{l_1}^{l_1}\xi^{\sigma}(u)\Delta u\right|$$
$$\leq\frac{(v-l_1)^2}{l_2-l_1}\left(\int_0^1k\Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1k|\xi^{\Delta}(kv+(1-k)l_1)|^q\Delta k\right)^{\frac{1}{q}}$$
$$+\frac{(v-l_2)^2}{l_2-l_1}\left(\int_0^1k\Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1|\xi^{\Delta}(kv+(1-k)l_2)|^q\Delta k\right)^{\frac{1}{q}}$$

Using the definition of exponential s-convexity of $|\xi^{\Delta}|^q$ we have

$$\left|\xi(v)-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^{\sigma}(u)\Delta u\right|$$
$$\leq\frac{(v-l_1)^2}{l_2-l_1}\left(\int_0^1k\Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1\left(k^{s+1}\frac{|\xi^{\Delta}(v)|^q}{e^{\alpha v}}+k(1-k)^s\frac{|\xi^{\Delta}(v)|^q}{e^{\alpha l_1}}\right)\Delta k\right)^{\frac{1}{q}}$$
$$+\frac{(v-l_2)^2}{l_2-l_1}\left(\int_0^1k\Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1\left(k^{s+1}\frac{|\xi^{\Delta}(t)|^q}{e^{\alpha v}}+k(1-k)^s\frac{|\xi^{\Delta}(v)|^q}{e^{\alpha l_2}}\right)\Delta k\right)^{\frac{1}{q}}$$

$$\leq\frac{M(v-l_1)^2}{l_2-l_1}\left(\int_0^1k\Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{k^{s+1}}{e^{\alpha v}}+\frac{k(1-k)^s}{e^{\alpha l_1}}\right)\Delta k\right)^{\frac{1}{q}}$$
$$+\frac{M(v-l_2)^2}{l_2-l_1}\left(\int_0^1k\Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{k^{s+1}}{e^{\alpha v}}+\frac{k(1-k)^s}{e^{\alpha l_2}}\right)\Delta k\right)^{\frac{1}{q}}.$$

□

Corollary 3.4.1. *If $T = R$ then we obtain Theorem 2.3 given in [16].*

Theorem 3.5. *Let $\xi : K \to R$ be a differentiable mapping on K^0 such that $\xi^{\Delta} \in K$ for $l_1, l_2 \in K$ with $l_1 < l_2$. If $|\xi^{\Delta}|^q$ is exponentially s-concave on $[l_1, l_2]$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds:*

$$\left|\xi(v)-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi^{\sigma}(u)\Delta u\right|\leq\left(\int_0^1k^p\Delta k\right)^{\frac{1}{p}}\frac{(v-l_1)^2}{l_2-l_1}2^{\frac{s-1}{q}}\left|\xi^{\Delta}\left(\frac{v+l_1}{2}\right)\right|$$
$$+\left(\int_0^1k^p\Delta k\right)^{\frac{1}{p}}\frac{(v-l_2)^2}{l_2-l_1}2^{\frac{s-1}{q}}\left|\xi^{\Delta}\left(\frac{v+l_2}{2}\right)\right|. \tag{3.7}$$

Proof. Using Montgomery identity and making use of variables we get

$$\begin{aligned}
&\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u\right| \\
&\leq \frac{1}{l_2 - l_1}\int_0^1 (v - l_1)^2 k|\xi^k v + (1-k)l_1)|\Delta k \\
&+ \frac{1}{l_2 - l_1}\int_0^1 (v - l_2)^2 k|\xi^{\Delta}(kv + (1-k)l_2)|\Delta k \\
&\leq \frac{(v-l_1)^2}{l_2 - l_1}\left(\int_0^1 k^p \Delta k\right)^{\frac{1}{p}}\left(\int_0^1 |\xi^{\Delta}(kv + (1-k)l_1)|^q \Delta k\right)^{\frac{1}{q}} \\
&+ \frac{(v-l_2)^2}{l_2 - l_1}\left(\int_0^1 k^p \Delta k\right)^{\frac{1}{p}}\left(\int_0^1 |\xi^{\Delta}(kv + (1-k)l_2)|^q \Delta k\right)^{\frac{1}{q}}. \qquad (3.8)
\end{aligned}$$

Since $|\xi^{\Delta}|^q$ is exponentially s-concave, by (6) we have

$$\left(\int_0^1 |\xi^{\Delta}(kv + (1-k)l_1)|^q \Delta k\right)^{\frac{1}{q}} \leq 2^{s-1}\left|\xi^{\Delta}\left(\frac{v + l_1}{2}\right)\right|^q \qquad (3.9)$$

and

$$\left(\int_0^1 |\xi^{\Delta}(kv + (1-k)l_2)|^q \Delta k\right)^{\frac{1}{q}} \leq 2^{s-1}\left|\xi^{\Delta}\left(\frac{v + l_2}{2}\right)\right|^q. \qquad (3.10)$$

Using (14) and (15) in (13) we get the conclusion. □

Corollary 3.5.1. *If $T = R$ then we obtain Theorem 2.4 given in [16].*

Now we discuss some results for s-convex functions.

Theorem 3.6. *Let T be a time scale and $K = [l_1, l_2] \subseteq T$ such that $l_1 < l_2 \in T$. Let $\xi : K \to R$ be a delta differentiable on K^0 such that $\xi^{\Delta} \in K$, for $l_1, l_2 \in K$ with $l_1 < l_2$. If $|\xi^{\Delta}|$ is s-convex on K for some fixed $s \in (0, 1]$ and $\sup_{l_1 \leq v \leq l_2} \left|\xi^{\Delta}(v)\right| = M$ for $v \in K$, then following inequality holds:*

$$\begin{aligned}
\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u\right| &\leq \frac{M(v-l_1)^2}{l_2 - l_1}\int_0^1 \left([k^{s+1} + k(1-k)^s]\right)\Delta k \\
+ \frac{M(v-l_2)^2}{l_2 - l_1}\int_0^1 \left([k^{s+1} + k(1-k)^s]\right)\Delta k. \qquad (3.11)
\end{aligned}$$

Proof. The proof is analogous to Theorem 3.2 only difference is to use definition of s-convex function $|\xi^{\Delta}|$ instead of exponentially s-convexity. □

Corollary 3.6.1. *If $T = R$, then we obtain Theorem 2 given in [17].*

Theorem 3.7. *Let T be a time scale and $K = [l_1, l_2] \subseteq T$ such that $l_1 < l_2 \in T$. Let $\xi : K \to R$ be a delta differentiable on K^0 such that $\xi^{\Delta} \in K$, for $l_1, l_2 \in K$ with $l_1 < l_2$. If $|\xi^{\Delta}|^q$ is s-convex on K for* *some fixed* $s \in (0,1]$, p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\sup_{l_1 \le v \le l_2} \left|\xi^{\Delta}(v)\right| = M$ *for* $v \in K$*, then following inequality holds:*

$$\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u\right| \le \frac{M(v-l_1)^2}{l_2 - l_1}\left(\int_0^1 k^p \Delta k\right)^{\frac{1}{p}} \times$$
$$\left(\int_0^1 \left[k^s + (1-k)^s\right]\Delta k\right)^{\frac{1}{q}} + \frac{M(v-l_2)^2}{l_2 - l_1}\left(\int_0^1 k^p \Delta k\right)^{\frac{1}{p}}\left(\int_0^1 \left[k^s + (1-k)^s\right]\Delta k\right)^{\frac{1}{q}}. \tag{3.12}$$

Proof. Proof is analogous to Theorem 3.3 but in place of definition of exponential s-convexity we use s-convexity of $|\xi^{\Delta}|^q$. □

Corollary 3.7.1. *If* $T = R$*, then we obtain Theorem 3 given in [17].*

Theorem 3.8. *Let* T *be a time scale and* $K = [l_1, l_2] \subseteq T$ *such that* $l_1 < l_2 \in T$*. Let* $\xi : K \to R$ *be a delta differentiable on* K^0 *such that* $\xi^{\Delta} \in K$ *for* l_1, $l_2 \in K$ *with* $l_1 < l_2$*. If* $|\xi^{\Delta}|^q$ *is s-convex in second sense on* K *for some fixed* $s \in (0,1]$, $q > 1$ *and* $\sup_{l_1 \le v \le l_2} \left|\xi^{\Delta}(v)\right| = M$ *for* $v \in K$*, then following inequality holds:*

$$\left|\xi(v) - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi^{\sigma}(u)\Delta u\right|$$
$$\le \frac{M(v-l_1)^2}{l_2 - l_1}\left(\int_0^1 k^p \Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1 \left[k^s + (1-k)^s\right]\Delta k\right)^{\frac{1}{q}}$$
$$+ \frac{M(v-l_2)^2}{l_2 - l_1}\left(\int_0^1 k^p \Delta k\right)^{1-\frac{1}{q}}\left(\int_0^1 \left[k^s + (1-k)^s\right]\Delta k\right)^{\frac{1}{q}}. \tag{3.13}$$

Proof. Proof is analogous to Theorem 3.4 but we use definition of s-convexity of $|\xi^{\Delta}|^q$ instead of exponential s-convexity. □

Corollary 3.8.1. *If* $T = R$*, then we obtain Theorem 4 given in [17].*

4. Conclusions

From Theorem 3.1 we obtain the Hermite-Hadamard inequality for exponentially s-convex functions on time scale. From Theorems 3.2–3.5 we obtain Ostrowski type inequalities for exponentially s-convex functions on time scale. From Theorems 3.6–3.8 we obtain Ostrowski type inequalities for s-convex functions on time scale.

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Conflict of interest

The authors declare that there is no interest regarding the publication of this paper.

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