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Research article

Robust H_∞ output feedback finite-time control for interval type-2 fuzzy systems with actuator saturation

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Abstract: The finite-time H_∞ performance of the interval type-2 Takagi-Sugeno fuzzy system (IT2 T-S) in presence of immeasurable states and input saturation is investigated. At first, an observer associated with IT2 T-S states is considered to address the problem of immeasurable states. After that, the input saturation is described based on the polyhedron model, and accordingly, a robust H_∞ observer-based finite-time controller is proposed via non-PDC algorithm. Then, on the basis of the Lyapunov function method and LMIs theory, the sufficient conditions for the finite time stability of fuzzy systems are derived. At last, the feasibility of the designed algorithm is verified by two examples of the nonlinear mass-spring-damping system and tunnel diode circuit system, respectively.

Keywords: interval type-2 Takagi-Sugeno fuzzy system; H_∞ observer-based finite-time control; input saturation; stability conditions
Mathematics Subject Classification: 93C42

1. Introduction

The Takagi-Sugeno (T-S) fuzzy system has a strong ability to approximate the nonlinear systems [1–3]. Compared with the T-S fuzzy system based on type-1 theory, the superiority of IT2 T-S is that it can solve the parametric uncertainties existing in the systems. Due to this fact, many significant works based on the IT2 T-S scheme were explored in [4–19]. Among them, the authors in [4] and [5] investigated reasonable initialization enhanced Karnik-Mendel algorithms and weighted-based non-iterative algorithms for interval type-2 fuzzy logic systems. The authors in [6–10] studied the state feedback control design for the IT2 T-S fuzzy systems. Alternatively, [11–13] introduced the output feedback fuzzy control schemes to address this issue. The authors in [14–18] proposed stable fuzzy state feedback controllers for IT2 T-S with the discrete-time form. For IT2 T-S fuzzy interconnected systems, the fuzzy decentralized output feedback controller has been proposed in [19]. However, it is obvious that although the control methods mentioned in the above results can make the T-S systems based on IT2 fuzzy theory reach the asymptotically stable performance, they ignored the finite convergence time problem which is eagerly desired in practical situations.

Note that the control target of the finite time stability is that the trajectory of system state can approximate the periodic orbits or equilibrium values within a finite time. Besides, the systems converge rapidly and have better robustness subject to uncertainties. Hence, the concept of finite-time is introduced in [20–23] and it has received increasing attention and achieved great progress over past decade [24–31]. Among them, [24, 25] respectively studied the H_∞ finite-time control design problem subject to the measurable uncertain continuous-time Markovian jump systems and discrete-time Markovian jump systems, respectively. Alternatively, the authors in [26] and [27] studied the output feedback H_∞ finite-time control design problems. For IT2 T-S fuzzy systems, the finite-time control design problems subject to time delays and actuator faults are studied in [28] and [29]. Besides, the designed approaches in [30] and [31] have extended the fuzzy decentralized control schemes for IT2 T-S fuzzy interconnected systems while the states of each subsystem is available. However, the aforementioned works are based on the assumption that the system states are measurable, thus they are not suitable for the condition in which the states are immeasurable. Besides, they also ignore the phenomenon of the actuator saturation. In practice, the impact of input saturation will extremely damage the control performance of the systems. Hence, the robust H_∞ output feedback finite-time control for IT2 T-S fuzzy systems with input saturation and external disturbances is worthy studying. By far, there are no results on investigating the robust H_∞ output feedback finite-time control for IT2 T-S fuzzy systems in presence of external disturbances and input saturation.

This paper develops the robust observer-based H_∞ output feedback finite-time controller for the IT2 T-S fuzzy systems in presence of external disturbances and input saturation. In the process of control design, an observer associated with IT2 T-S states is firstly designed, and then a robust fuzzy finite-time controller is developed by using the estimating states. After that, a polyhedron model to represent input saturation is introduced. Based on the Lyapunov function scheme and LMIs theory, the sufficient conditions of finite-time stabilization for system with input saturation and external disturbances are gained. The primary contributions of this paper can be roughly concluded as follows

(i) This paper first studied the fuzzy output feedback finite-time H_∞ control design problem for the IT2 T-S fuzzy systems. By developing a state observer, the proposed the fuzzy output feedback finite-time control method can not only guarantee the finite-time stabilization of the closed-loop system, but also solve the state unmeasured problem. Note that although the reference [11–13, 19] also developed the fuzzy output feedback control methods for a class of IT2 T-S fuzzy systems, they do not consider the finite-time stability, that is, it investigate output feedback asymptotic control, not the output feedback finite-time control like this paper.

(ii) Note that [28–31] investigated the finite-time control of IT2 T-S fuzzy system, however, the limitation is that the system state must be completely measurable which are not suitable for systems with immeasurable states like this study. Besides, the impact of input saturation is also investigated in this paper.

The rest of this paper is comprised such that: Section 2 gives the descriptions of the IT2 T-S fuzzy systems, state observer design, actuator saturation functions and necessary definitions and lemmas. The robust finite-time H_∞ controller design, the finite-time stabilization and the finite-time boundedness conditions of the considered system are derived in Section 3. In order to verify the feasibility of the developed scheme, two practical examples are given in Section 4. Section 5 derives the conclusion.

2. Preliminaries

In this section, the descriptions of the IT2 T-S fuzzy system and input saturation functions will be shown.

2.1. IT2 T-S fuzzy system

Consider the IT2 T-S fuzzy system with continue-time form as follows.

Plant rule p: If $\theta_1(x(t))$ is F_{p1} and $\cdots$ and $\theta_i(x(t))$ is F_{pi}, then

$$\begin{cases} \dot{x}(t) = A_p x(t) + B_p sat(u(t)) + D_p w(t) \\ z(t) = E_p x(t) + F_p sat(u(t)) \\ y(t) = C_p x(t) \end{cases} \tag{2.1}$$

where $\theta_s(x(t)) = [\theta_1(x(t)), \theta_2(x(t)), \ldots, \theta_i(x(t))]^T$ represents the premise vector and F_{ps} represents the IT2 fuzzy set. $x(t) \in \Re^n$, $sat(u(t)) \in \Re^m$ and $w(t) \in \Re^h$ are system state vector, saturated control input vector and bounding external disturbances satisfying $L_2(0, \infty]$,respectively. $z(t) \in \Re^g$ is the controlled output and $y(t) \in \Re^l$ is the measured output. A_p, B_p, D_p, E_p, F_p and C_p are known constant matrices with appropriate dimensions.

Based on the IT2 fuzzy logic systems theory, the firing strength of the pth fuzzy rule can be defined within the following interval sets

$$\psi_p(x(t)) \in [\vartheta_p^L(x(t)), \vartheta_p^U(x(t))], \quad p = 1, 2 \ldots, r \tag{2.2}$$

Through utilizing the method of singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier, the IT2 T-S fuzzy system (2.1) is inferred as follows

$$\begin{cases} \dot{x}(t) = \sum_{p=1}^{r} \vartheta_p(x(t))[A_p x(t) + B_p sat(u(t)) + D_p w(t)] \\ z(t) = \sum_{p=1}^{r} \vartheta_p(x(t))[E_p x(t) + F_p sat(u(t))] \\ y(t) = \sum_{p=1}^{r} \vartheta_p(x(t)) C_p x(t) \end{cases} \tag{2.3}$$

in which

$$\vartheta_p(x(t)) = \frac{\vartheta_p^L(x(t))\upsilon_p^L(x(t)) + \vartheta_p^U(x(t))\upsilon_p^U(x(t))}{\sum_{p=1}^{r} [\vartheta_p^L(x(t))\upsilon_p^L(x(t)) + \vartheta_p^U(x(t))\upsilon_p^U(x(t))]}, \quad \sum\nolimits_{p=1}^{r} \vartheta_p(x(t)) = 1,$$

$$\vartheta_p^L(x(t)) = \prod_{s=1}^{i} \underline{\mu}_{F_{ps}}(x(t)) \geq 0, \quad \vartheta_p^U(x(t)) = \prod_{s=1}^{i} \bar{\mu}_{F_{ps}}(x(t)) \geq 0,$$

$$\upsilon_p^L(x(t)) \in [0, 1], \quad \upsilon_p^U(x(t)) \in [0, 1], \quad \upsilon_p^L(x(t)) + \upsilon_p^U(x(t)) = 1.$$

the above $\vartheta_P^L(\theta(x(t)))$,$\vartheta_P^U(\theta(x(t)))$ are the lower and upper grades of memberships, respectively. $\underline{\mu}_{F_{ps}}(x(t))$ and $\bar{\mu}_{F_{ps}}(x(t))$ are the lower and upper membership functions of IT2 fuzzy sets F_{ps}. $\upsilon_P^L(\theta(x(t)))$,$\upsilon_P^U(\theta(x(t)))$ are known weighting coefficient nonlinear functions.

2.2. *Saturation functions*

The aforementioned $u(t)$ is the control signal given by the ideal output feedback fuzzy controller and $sat(u(t)) = [sat(u_1), sat(u_2), \ldots, sat(u_m)]^T$ is the practical control signal, which is used to address saturation problem.

In order to model better saturation effect, the polytopic model introduced in [32, 33] will be utilized. Referring from [32, 33], we can obtain such that

$$sat(u(t)) = sat(u(t), \bar{u}) = [sat(u_1), sat(u_2), \ldots, sat(u_m)]^T \tag{2.4}$$

in which $sat\,(u_a) = sign\,(u_a)\,min\,\{\overline{u}_a, |u_a|\}\,, \overline{u} \in \Re^m$ is the saturation level and $u\,(t) \in \Re^m$ represents the control input, $\overline{u}_a$ and u_a are the ath element of the saturation level and the control input, respectively.

Let V be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. For example, when $m = 2$, then

$$V = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

There are 2^m elements in V. Suppose that each that element of V is labeled as H_l, $(l = 1, 2, \ldots, 2^m)$, and denote $H_l^- = I - H_l$. Clearly, H_l^- is also an element of V when $H_l \in V$.

For development, the following lemma associated with input saturation is required.

Lemma 1 [32,33]. Assume $|v_a| \leq \bar{u}_a$, in which v_a and u_a denote the ath element of $v \in \Re^m$ and $u \in \Re^m$, respectively. If $x(t)$ is inside $\bigcap_{j=1}^{r} \left\{ x \in \Re^n \,\middle|\, \left| h_p^j x \right| \leq \bar{u}_p \right\}$, then

$$sat(u(t), \bar{u}) = \sum\nolimits_{l=1}^{2^m} \delta_l (H_l u + H_l^- v) \tag{2.5}$$

in which $0 \leq \delta_l \leq 1$ $(l = 1, \ldots, 2^m)$ are scalars and $\sum_{l=1}^{2^m} \delta_l = 1$.

2.3. *Observer-based controller design*

In this subsection, a state observer will be designed to estimate immeasurable states. Subsequently, the fuzzy controller will be proposed.

The identical membership functions, which rely on estimated states rather than system states, are used to design the fuzzy observer.

Observer Rule q:If $\theta_1(\hat{x}(t))$ is F_{q1} and $\cdots$ and $\theta_q(\hat{x}(t))$ is F_{qi}, then

$$\begin{cases} \dot{\hat{x}}\,(t) = A_q \hat{x}\,(t) + B_q sat\,(u\,(t)) + L_q\,(y\,(t) - \hat{y}\,(t)) \\ \hat{z}\,(t) = E_q \hat{x}\,(t) + F_q sat\,(u\,(t)) \\ \hat{y}\,(t) = C_q \hat{x}\,(t) \end{cases} \tag{2.6}$$

where $\hat{x}\,(t) \in \Re^n$ denotes the estimated state vector, $\hat{z}\,(t) \in \Re^l$ is the estimation output and $\hat{y}\,(t) \in \Re^l$ is the observer output, L_p is observer gain matrix to be solved.

After that, the final state observer is described such that

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{q=1}^{r} \vartheta_q(\hat{x}(t))[A_q \hat{x}(t) + B_q sat(u(t)) + L_q(y(t) - \hat{y}(t))] \\ \hat{z}(t) = \sum_{q=1}^{r} \vartheta_q(\hat{x}(t))[E_q \hat{x}(t) + F_q sat(u(t))] \\ \hat{y}(t) = \sum_{q=1}^{r} \vartheta_q(\hat{x}(t)) C_q \hat{x}(t) \end{cases} \tag{2.7}$$

Due to that the membership functions of fuzzy observer rely on the estimated state variable rather than $x(t)$, thereby, $\hat{x}(t)$ is selected as the membership function representations.

For the purpose of improving the flexibly of fuzzy controller design, by utilizing non-PDC design algorithm, the controller possesses its exclusive membership functions which are different from fuzzy system (2.3) and fuzzy observer (2.7). The details are shown such that:

Controller Rule j:If $g_1(\hat{x}(t))$ is M_{j1} and $\cdots$ and $g_j(\hat{x}(t))$ is $M_{j\tau}$, then

$$u(t) = K_j\hat{x}(t) \tag{2.8}$$

where $K_j(j = 1, 2, \ldots, r)$ are the controller gains to be solved later. Similarly, the jth firing strength can be expressed as

$$\omega_j(\hat{x}(t)) = \left[\eta_j^L(\hat{x}(t)), \eta_j^U(\hat{x}(t))\right],\ j = 1, 2, \ldots, r \tag{2.9}$$

in which

$$\eta_j^L(\hat{x}(t)) = \prod_{\hbar}^{\tau} \underline{\mu}_{M_{j\hbar}}(g_\hbar(\hat{x}(t))) \geq 0,\ \ \eta_j^U(\hat{x}(t)) = \prod_{\hbar}^{\tau} \bar{\mu}_{M_{j\hbar}}(g_\hbar(\hat{x}(t))) \geq 0$$

Then, the fuzzy controller based on IT2 fuzzy theory can be obtained

$$u(t) = \sum\nolimits_{j=1}^{r} \eta_j(\hat{x}(t))K_j\hat{x}(t) \tag{2.10}$$

where

$$\eta_j(\hat{x}(t)) = \frac{\eta_j^L(\hat{x}(t))b_j^L(\hat{x}(t)) + \eta_j^U(\hat{x}(t))b_j^U(\hat{x}(t))}{\sum_{j=1}^{r}[\eta_j^L(\hat{x}(t))b_j^L(\hat{x}(t)) + \eta_j^U(\hat{x}(t))b_j^U(\hat{x}(t))]},\ \ \sum\nolimits_{j=1}^{r} \eta_j(\hat{x}(t)) = 1,$$

$$b_j^L(\hat{x}(t)) \in [0, 1],\ \ b_j^U(\hat{x}(t)) \in [0, 1],\ \ b_j^L(\hat{x}(t)) + b_j^U(\hat{x}(t)) = 1,$$

the above $\underline{\mu}_{M_{j\hbar}}(g_\hbar(\hat{x}(t)))$, $\bar{\mu}_{M_{j\hbar}}(g_\hbar(\hat{x}(t)))$, $\eta_j^L(\hat{x}(t))$, $\eta_j^U(\hat{x}(t))$, $b_j^L(\hat{x}(t))$ and $b_j^U(\hat{x}(t))$ respectively denote the lower and upper membership functions, the lower and upper grades of membership, the tradeoff coefficient nonlinear functions.

Then, from Lemma 1, the observer-based controller in presence of input saturation is given as

$$\begin{aligned} sat(u(t)) &= sat(u(t), \bar{u}) \\ &= \sum\nolimits_{l=1}^{2^m} \delta_l(H_l \sum\nolimits_{j=1}^{r} \eta_j(\hat{x}(t))K_j + H_l^- \sum\nolimits_{j=1}^{r} \eta_j(\hat{x}(t))Z_j)\hat{x}(t) \\ &= \sum\nolimits_{l=1}^{2^m} \sum\nolimits_{j=1}^{r} \delta_l\eta_j(\hat{x}(t))(H_lK_j + H_l^- Z_j)\hat{x}(t) \end{aligned} \tag{2.11}$$

where Z_j is $m \times n$ matrix and h_p^j represents the pth row of Z_j.

Define the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$, controlled output error $\tilde{z}(t) = z(t) - \hat{z}(t)$ and considering equations (2.3), (2.7) and (2.11), one can obtain

$$\begin{cases} \dot{\xi}(t) = \sum_{l=1}^{2^m} \sum_{p=1}^{r} \sum_{q=1}^{r} \sum_{j=1}^{r} \vartheta_p(x(t))\vartheta_q(\hat{x}(t))\eta_j(\hat{x}(t))[A_{pqj}\xi(t) + D_{pqj}w(t)] \\ \tilde{z}(t) = \sum_{l=1}^{2^m} \sum_{p=1}^{r} \sum_{q=1}^{r} \vartheta_p(\theta(x(t)))\vartheta_q(\theta(\hat{x}(t)))[C_{pqj}\xi(t)] \end{cases} \tag{2.12}$$

where

$\xi(t) = \begin{bmatrix} x(t) & \tilde{x}(t) \end{bmatrix}^T$, $A_{pqj} = \begin{bmatrix} A_p + \delta_l B_p H_l K_j + \delta_l B_p H_l^- Z_j & -\delta_l B_p H_l K_j - \delta_l B_p H_l^- Z_j \\ A_p - A_q - L_q C_p - L_q C_q & A_q - L_q C_q \end{bmatrix}$, $D_{pqj} = \begin{bmatrix} D_p \\ D_p \end{bmatrix}$,

$C_{pqj} = \begin{bmatrix} E_p - E_q & E_q \end{bmatrix}$.

Control objective: This study will design a robust H_∞ finite-time output feedback controller for the IT2 T-S fuzzy systems (2.3). The developed controller can make the T-S systems based on IT2 fuzzy theory reach the finite-time stable, in which the input saturation and external disturbances are considered.

Remark 1: Note that the authors in [19] studied the output feedback control design problem for IT2 T-S fuzzy system, the developed controller can make the controlled system achieve asymptotic stability. However, the authors in [19] do not consider the finite-time stability problem. Unlike the above work, in this paper, the robust output feedback finite-time control design problem for T-S systems with input saturation based on IT2 theory is studied. Therefore, the control design and stability analysis are more difficult and challenging.

2.4. Main definitions and lemmas

For development, the following definitions and lemmas are needed to achieve the control objective.

Definition 1 [20,21]. If for given constants $a_2 > a_1 \geq 0$ and a positive matrix $R > 0$ satisfying

$$\xi^T(0)R\xi(0) \leq a_1 \Rightarrow \xi^T(t)R\xi(t) < a_2, \quad \forall t \in [0, T] \tag{2.13}$$

Then the system (2.3) can be robust finite-time stable within (a_1, a_2, R, T).

Definition 2 [34]. System (2.3) is of robust H_∞ finite-time boundedness performance if there exist positive constants $\bar{\rho} > 0$ and $h > 0$, and the closedloop system based on IT2 T-S fuzzy theory satisfies (2.13) and following H_∞ performance index

$$\int_0^T \tilde{z}^T(t)z(t)dt \leq \bar{\rho}^2 \int_0^T w^T(t)w(t)dt$$

where $\|w(t)\|^2 < h$, and T is a given positive scalar.

Lemma 2 [29]. For full rank matrix $rank(C) = l, F \in \Re^{l\times n}$ with $l < n$,there exist a singular value decomposition (SVD) for $C = C_p$ can be described as $C = O\begin{bmatrix} S & 0 \end{bmatrix} U^T$, in which $V \in \Re^{l\times l}, O \in \Re^{l\times n}, S \in \Re^{n\times n}, U \cdot U^T = I$ and $O \cdot O^T = I$. Let matrices $X > 0$, $X_{11} \in \Re^{l\times l}$, $X_{22} \in \Re^{(n-l)\times(n-l)}$. Then, there exists $\bar{X}$ satisfies $CX = \bar{X}C$ if and only if the following Eigen-value decomposition (EVD) condition holds

$$X = U \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix} U^T$$

3. Stability analysis

The robust finite-time stability analysis and the robust finite-time H_∞ boundedness of the fuzzy system (2.3) will be respectively proved in this section. Based on the Lyapunov function method and LMIs theory, several sufficient conditions will be derived.

Theorem 1: For given positive scalars $T > 0$, $a_1 > 0$, $h > 0$, $\beta > 0$, $\delta_l > 0$, $\bar{\rho} > 0$ and a positive symmetric $R > 0$, system (2.3) can achieve robust finite-time stability within $(a_1, a_2, R, T, \bar{\rho}, h)$, if there exist matrix $P = P^T > 0$ satisfying the following matrix inequalities

$$R < P < R/\lambda_2 \tag{3.1}$$

$$\Pi_{pqj} - \Omega < 0 \tag{3.2}$$

$$\gamma_j \Pi_{pqj} + \gamma_q \Pi_{pjq} - \gamma_j \Omega - \gamma_q \Omega + 2\Omega < 0, \ \ q \le j \tag{3.3}$$

$$\frac{e^{\beta T}(\lambda_1 a_1 + \frac{\bar{\rho}^2 e^{-\beta T} h}{\beta}(1 - e^{-\beta T}))}{\lambda_2} < a_2 \tag{3.4}$$

$$\begin{bmatrix} \bar{u}_p^2 & h_p^j P_1^{-1} \\ P_1^{-1}(h_p^j)^T & P_1^{-1} \end{bmatrix} \ge 0 \tag{3.5}$$

in which

$$\lambda_1 = \lambda_{\max}(\tilde{P}), \ \ \lambda_2 = \lambda_{\min}(\tilde{P}), \ \ \tilde{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}},$$

$$\Pi_{pqj} = \begin{bmatrix} \Pi_{pqj}^{11} & \Pi_{pqj}^{12} & 0 \\ * & \Pi_{pqj}^{22} & 0 \\ * & * & \Pi_{pqj}^{33} \end{bmatrix} < 0, \ \ \Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ * & \Omega_{22} & \Omega_{23} \\ * & * & \Omega_{33} \end{bmatrix},$$

$$\begin{array}{rl} \Pi_{pqj}^{11} = & P_1 A_p + A_p^T P_1 + \delta_l P_1 B_p H_l K_j + \delta_l K_j^T H_l^T B_p^T P_1 + \delta_l P_1 B_p H_l^- Z_j + \delta_l Z_j^T (H_l^-)^T B_p^T P_1 \\ & -\beta P_1 + E_p^T E_p - E_p^T E_q - E_q^T E_p - E_q^T E_q - \beta P_1 + P_1 P_1 \end{array},$$

$$\Pi_{pqj}^{12} = -\delta_l P_1 B_p H_l K_j - \delta_l P_1 B_p H_l^- Z_j + A_p^T P_1 - A_q^T P_1 - C_p^T L_q^T P_1 - C_q^T L_q^T P_1 - E_q^T E_q + E_p^T E_q^T,$$

$$\Pi_{pqj}^{22} = P_1 A_q - P_1 L_q C_q + A_q^T P_1 - C_q^T L_q^T P_1 - \beta P_1 + E_q^T E_q + P_1 P_1, \ \ \Pi_{pqj}^{33} = D_p^T D_p - \bar{\rho}^2 e^{-\beta T} I.$$

Proof: Construct the following Lyapunov function

$$V(\xi(t)) = \xi^T(t) P \xi(t) \tag{3.6}$$

For convenience, one define $P = \text{diag}\{P_1, P_1\}$ and then calculate the time derivative of (3.6), one has

$$\begin{array}{l} \dot{V}(\xi(t)) - \beta V(\xi(t)) + \tilde{z}^T(t)\tilde{z}(t) - \bar{\rho}^2 e^{-\beta T} w^T(t) w(t) \\ = \sum_{l=1}^{2^m} \sum_{p=1}^{r} \sum_{q=1}^{r} \sum_{j=1}^{r} \vartheta_p(x(t)) \vartheta_q(\hat{x}(t)) \eta_j(\hat{x}(t)) \\ \quad \times [\xi^T(t)(PA_{pqj} + A_{pqj}^T P - \beta P + C_{pqj}^T C_{pqj})\xi(t) \\ \quad + \xi^T(t)(PD_{pqj} + D_{pqj}^T P) w(t)] - \bar{\rho}^2 e^{-\beta T} w^T(t) w(t) \end{array} \tag{3.7}$$

Define $\chi(t) = \begin{bmatrix} \xi(t) & w(t) \end{bmatrix}^T$, and then substituting the definition of A_{pqj}, C_{pqj}, D_{pqj} into (3.7), (3.7) can be rewritten as

$$\begin{array}{l} \dot{V}(\xi(t)) - \beta V(\xi(t)) + \tilde{z}^T(t)\tilde{z}(t) - \bar{\rho}^2 e^{-\beta T} w^T(t) w(t) \\ = \sum_{l=1}^{2^m} \sum_{p=1}^{r} \sum_{q=1}^{r} \sum_{j=1}^{r} \vartheta_p(x(t)) \vartheta_q(\hat{x}(t)) \eta_j(\hat{x}(t)) \{ x^T(t)[(P_1 A_p + \delta_l P_1 B_p H_l K_j \\ \quad + \delta_l P_1 B_p H_l^- Z_j) + (P_1 A_p + \delta_l P_1 B_p H_l K_j + \delta_l P_1 B_p H_l^- Z_j)^T - \beta P_1 + P_1 P_1 + E_p^T E_p \\ \quad - E_p^T E_q - E_q^T E_p - E_q^T E_q] x(t) + x^T(t)[(-\delta_l P_1 B_p H_l K_j - \delta_l P_1 B_p H_l^- Z_j) + (P_1 A_p \\ \quad - P_1 A_q - P_1 L_q C_p - P_1 L_q C_q)^T - E_q^T E_q] \tilde{x}(t) + \tilde{x}^T(t)[(P_1 A_p - P_1 A_q - P_1 L_q C_p \\ \quad - P_1 L_q C_q) + (-\delta_l P_1 B_p H_l K_j - \delta_l P_1 B_p H_l^- Z_j)^T + E_q^T E_p - E_q^T E_q] x(t) + \tilde{x}^T(t)[(P_1 A_q \\ \quad - P_1 L_q C_q) + (P_1 A_q - P_1 L_q C_q)^T - \beta P_1 + E_q^T E_q + P_1 P_1] \tilde{x}(t) + w^T(t)(D_p^T D_p - \bar{\rho}^2 e^{-\beta T} I) w(t) \\ \le \sum_{l=1}^{2^m} \sum_{p=1}^{r} \sum_{q=1}^{r} \sum_{j=1}^{r} \vartheta_p(x(t)) \vartheta_q(\hat{x}(t)) \eta_j(\hat{x}(t)) [\chi^T(t) \Pi_{pqj} \chi(t)] \end{array} \tag{3.8}$$

According to [28], the slack matrix is introduced to avoid conservative results as shown such that

$$\sum_{p=1}^{r}\sum_{q=1}^{r}\sum_{j=1}^{r}\vartheta_p(x(t))\vartheta_q(\hat{x}(t))[\vartheta_j(\hat{x}(t))-\eta_j(\hat{x}(t))]\Omega=0 \tag{3.9}$$

Then, from (3.8) and (3.9), one has

$$\begin{aligned}
&\sum_{l=1}^{2^m}\sum_{p=1}^{r}\sum_{q=1}^{r}\sum_{j=1}^{r}\vartheta_p(x(t))\vartheta_q(\hat{x}(t))\eta_j(\hat{x}(t))\Pi_{pqj}\\
&=\sum_{l=1}^{2^m}\sum_{p=1}^{r}\sum_{q=1}^{r}\sum_{j=1}^{r}\vartheta_p(x(t))\vartheta_q(\hat{x}(t))[\vartheta_j(\hat{x}(t))(\gamma_j\Pi_{pqj}-\gamma_j\Omega+\Omega)\\
&\quad+(\eta_j(\hat{x}(t))-\gamma_j\vartheta_j(\hat{x}(t)))(\Pi_{pqj}-\Omega)]\\
&=\tfrac{1}{2}\sum_{l=1}^{2^m}\sum_{p=1}^{r}\sum_{q=1}^{r}\sum_{j=1}^{r}\vartheta_p(x(t))\vartheta_q(\hat{x}(t))[\eta_j(\hat{x}(t))(\gamma_j\Pi_{pqj}+\gamma_q\Pi_{pjq}-\gamma_j\Omega-\gamma_q\Omega+2\Omega)]\\
&\quad+\sum_{l=1}^{2^m}\sum_{p=1}^{r}\sum_{q=1}^{r}\sum_{j=1}^{r}\vartheta_p(x(t))\vartheta_q(\hat{x}(t))(\eta_j(\hat{x}(t))-\gamma_j\vartheta_j(\hat{x}(t)))(\Pi_{pqj}-\Omega)
\end{aligned} \tag{3.10}$$

Then, based on Theorem 1, one gets

$$\sum_{l=1}^{2^m}\sum_{p=1}^{r}\sum_{q=1}^{r}\sum_{j=1}^{r}\vartheta_p(x(t))\vartheta_q(\hat{x}(t))\eta_j(\hat{x}(t))[\chi^T(t)\Pi_{pqj}\chi(t)]<0 \tag{3.11}$$

Multiplying (3.11) by $e^{-\beta t}$ and integrating both sides of (24) from 0 to t, then since $\tilde{z}^T(t)\tilde{z}(t)>0$ and $\Pi_{pqj}<0$, one can obtain

$$e^{-\beta t}V(\xi(t))-V(\xi(0))<\bar{\rho}^2e^{-\beta T}\int_0^t e^{-\beta t}w^T(t)w(t)dt \tag{3.12}$$

Then, multiplying (3.12) by $e^{\beta t}$, (3.12) can be represented as

$$V(\xi(t))<e^{\beta t}V(\xi(0))+\bar{\rho}^2e^{-\beta T}e^{\beta t}\int_0^t e^{-\beta t}w^T(t)w(t)dt \tag{3.13}$$

Recalling to $V(\xi(t))=\xi^T(t)P\xi(t)$, $\tilde{P}=R^{-\frac{1}{2}}PR^{-\frac{1}{2}}$ and $\|w(t)\|^2<h$, (3.13) leads to

$$\begin{aligned}
\xi^T(t)P\xi(t)&<e^{\beta t}(\xi^T(0)P\xi(0)+\bar{\rho}^2e^{-\beta T}h\textstyle\int_0^t e^{-\beta t}dt)\\
&<e^{\beta t}(\lambda_1a_1+\frac{\bar{\rho}^2e^{-\beta T}h}{\beta}(1-e^{-\beta t}))\\
&\le e^{\beta T}(\lambda_1a_1+\frac{\bar{\rho}^2e^{-\beta T}h}{\beta}(1-e^{-\beta T}))
\end{aligned} \tag{3.14}$$

in which $\lambda_1=\lambda_{\max}(\tilde{P})$.

Define $\lambda_2=\lambda_{\min}(\tilde{P})$, since $V(\xi(t))=\xi^T(t)P\xi(t)\ge\lambda_2\xi^T(t)R\xi(t)$, then form (3.14), one has

$$\xi^T(t)R\xi(t)<\frac{e^{\beta T}(\lambda_1a_1+\frac{\bar{\rho}^2e^{-\beta T}h}{\beta}(1-e^{-\beta T}))}{\lambda_2},\quad\forall t\in[0,T] \tag{3.15}$$

Suppose $\xi^T(t)R\xi(t)<\frac{e^{\beta T}(\lambda_1a_1+\frac{\bar{\rho}^2e^{-\beta T}h}{\beta}(1-e^{-\beta T}))}{\lambda_2}$, $\forall t\in[0,T]$, then from Definition 1, finite-time stability of fuzzy system (2.3) is ensured within $(a_1,a_2,R,T,\bar{\rho},h)$.

In what follows, we will give the proof of the robust H_∞ output feedback finite-time boundedness performance of the fuzzy system (2.3).

From the inequality (3.7), on can easily find that

$$\dot{V}(\xi(t))-\beta V(\xi(t))<-\tilde{z}^T(t)\tilde{z}(t)+\bar{\rho}^2e^{-\beta T}w^T(t)w(t) \tag{3.16}$$

Similarly, multiplying (3.16) by $e^{-\beta t}$, and then integrating it from 0 to T, one has

$$e^{-\beta T}V(\xi(T)) - V(\xi(0)) < -\int_0^T e^{-\beta t}\tilde{z}^T(t)\tilde{z}(t)dt + \bar{\rho}^2 e^{-\beta T}\int_0^T e^{-\beta t}w^T(t)w(t)dt \tag{3.17}$$

Under the condition of $V(\xi(0)) = 0$ and $V(\xi(T)) > 0$, (3.17) can be rewritten such that

$$\int_0^T e^{-\beta t}\tilde{z}^T(t)\tilde{z}(t)dt < \bar{\rho}^2 e^{-\beta T}\int_0^T e^{-\beta t}w^T(t)w(t)dt \tag{3.18}$$

Since $e^{-\beta T}\int_0^T e^{-\beta t}w^T(t)w(t)dt < \int_0^T e^{-\beta t}\tilde{z}^T(t)\tilde{z}(t)dt$, $\bar{\rho}^2 e^{-\beta T}\int_0^T e^{-\beta t}w^T(t)w(t)dt < \bar{\rho}^2 e^{-\beta T}\int_0^T w^T(t)w(t)dt$, then from (3.18), one can get

$$\int_0^T \tilde{z}^T(t)\tilde{z}(t)dt < \bar{\rho}^2\int_0^T w^T(t)w(t)dt$$

Further, the robust H_∞ finite-time boundedness performance index has been obtained. Hence, from Definition 2, the fuzzy system (2.3) is finite-time and can achieve a robust H_∞ performance.

The solution of positive controller gains K_j, observer gains L_q and positive definite symmetric matrix P are given in following.

Firstly, define $X = P_1^{-1}$, $Y_j = K_jX$, $S_j = Z_jX$ and $T_q = L_q\bar{X}$, by left and right multiplying $\text{diag}\{X, X, I\}$ on both sides of (3.2) and (3.3), thus one can get the following LMIs

$$\bar{\Pi}_{pqj} - \bar{\Omega} < 0 \tag{3.19}$$

$$\gamma_j\bar{\Pi}_{pqj} + \gamma_q\bar{\Pi}_{pjq} - \gamma_j\bar{\Omega} - \gamma_q\bar{\Omega} + 2\bar{\Omega} < 0, \;\; q \le j \tag{3.20}$$

where

$$\Theta_{pqj} = \begin{bmatrix} \Theta_{pqj}^{11} & X & \Theta_{pqj}^{13} & X & 0 \\ * & \Theta_{pqj}^{22} & X & -(E_q^TE_q + E_p^TE_q)^{-1} & 0 \\ * & * & \Theta_{pqj}^{33} & X & 0 \\ * & * & * & -(E_q^TE_q)^{-1}I & 0 \\ * & * & * & * & (-\bar{\rho}^2e^{-\beta T} + D_p^TD_p)I \end{bmatrix},$$

$$\bar{\Omega} = \begin{bmatrix} X\Omega_{11}X & X\Omega_{12}X & X\Omega_{13} \\ * & X\Omega_{22}X & X\Omega_{23} \\ * & * & \Omega_{33} \end{bmatrix},$$

with

$$\Theta_{pqj}^{11} = A_pX + XA_p^T + \delta_lB_pH_lY_j + \delta_lY_j^TH_l^TB_p^T + \delta_lB_pH_l^-S_j + \delta_lS_j^T(H_l^-)^TB_p^T - \beta X + I,$$
$$\Theta_{pqj}^{13} = -\delta_lB_pH_lY_j - \delta_lB_pH_l^-S_j + XA_p^T - XA_q^T - C_p^TT_q^T - C_q^TT_q^T,$$
$$\Theta_{pqj}^{22} = -(E_p^TE_p - E_p^TE_q - E_q^TE_p - E_q^TE_q)^{-1}I, \;\; \Theta_{pqj}^{33} = A_qX + XA_q^T - T_qC_q - C_q^TT_q^T - \beta X + I.$$

Subsequently, by solving LMIs (3.19) and (3.20), one can obtain X, Y_j, S_j and T_q, the it follows P_1, $K_j = Y_jX^{-1}$ and $S_j = H_jX^{-1}$. According to Lemma 2, the SVD of the output matrix $C \in \Re^{l\times n}$, for $C = C_p$ with a full rank $C = O[S, 0]U^T$, and there exists a matrix $\bar{X} \in \Re^{n\times n}$ satisfying $CX = \bar{X}C$, if there exists EVD $X = U\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}U^T$. Then, one can find that $O[S, 0]U^T U\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}U^T = \bar{X}O[S, 0]U^T$. By calculation, one gets $\bar{X} = OSX_{11}S^{-1}O^{-1}$, thus, $\bar{X}^{-1} = OSX_{11}^{-1}S^{-1}O^{-1}$. For bilinear terms $L_qCX = L_q\bar{X}C$, let $T_q = L_q\bar{X}$, then one can obtain $L_q = T_q\bar{X}^{-1} = T_qOSX_{11}^{-1}S^{-1}O^{-1}$.

From [28, 29], we know that the constraint $\bigcap_{j=1}^{r} \left\{x \in \Re^n \,\middle|\, \left|h_p^j x\right| \leq \bar{u}_p\right\}$ is equivalent to

$$h_p^j P_1^{-1}(h_p^j)^T \leq \bar{u}_p^2 \tag{3.21}$$

By Schur complement, (3.21) is equivalent to the following LMI

$$\begin{bmatrix} \bar{u}_p^2 & h_p^s P_1^{-1} \\ P_1^{-1}(h_p^s)^T & P_1^{-1} \end{bmatrix} \geq 0 \tag{3.22}$$

The design procedures are organized:

Step1: Choose the fuzzy rules, and then give $\bar{u}_p(p = 1, \ldots, r)$, $T > 0$, $a_1 > 0$, $h > 0$, $\bar{\rho} > 0$, $\beta > 0$ and select $0 \leq \delta_l \leq 1$ $(l = 1, \ldots, 2^m)$ which satisfies $\sum_{l=1}^{2^m} \delta_l = 1$, and diagonal matrices H_l, H_l^-, which satisfies $H_l + H_l^- = I_n$.

Step 2: Solve the LMI (3.19), (3.20) to get X, K_j, Z_j and L_q.

Step 3: Substitute the solved gains into (3.1), (3.4) and (3.5) to verify whether they are established.

Step 4: In case of the condition (3.1), (3.4) and (3.5) are not satisfied, increase $\bar{\rho}$, a_2 and $\bar{u}_p(p = 1, \ldots, r)$ and increase β until the conditions are established. Else, decrease $\bar{\rho}$, a_2 and $\bar{u}_p(p = 1, \ldots, r)$ and increase β and go to Steps 1-4 until X cannot be obtained.

Step 5: Represent the fuzzy state observer (2.7) and fuzzy controller (2.11).

4. Simulation results

Example 1 (Mass-spring-damping system) A mass-spring-damping system example as shown in Figure 1 is employed to verify the feasibility of the developed scheme in this section.

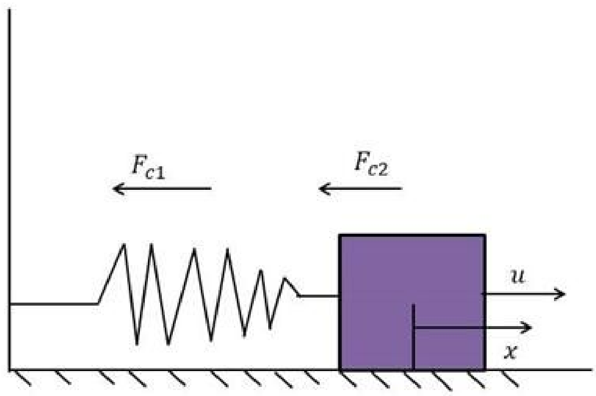

Figure 1. Mass-Spring-Damping system.

The corresponding dynamical equation is shown as follows

$$M\ddot{x} + F_{c2} + F_{c1} = u(t) \tag{4.1}$$

where M is the mass, F_{c1} and $F_{c2} = r\dot{x}$ denote the friction force, F_{c2} is the restoring force of the spring, and $u(t)$ denotes the control input. $F_{c2} = r\dot{x}$ with $r > 0$ and $F_{c1} = d(1 + \varepsilon^2 x^2)x$. Then, (4.1) can be described as

$$M\ddot{x} + r\dot{x} + dx + d\varepsilon^2 x^3 = u(t) \tag{4.2}$$

where x is the displacement from a reference point. Define $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T = \begin{bmatrix} x & \dot{x} \end{bmatrix}^T$ and $\tilde{f}(t) = (-d - d\varepsilon^2 x_1^2(t))/M$. The operating domain are $x_1(t) \in [-2, 2]$ and $d \in [5, 8]N/M$. Let $M = 1$ kg, $r = 2$ N $\cdot$ m/s,and $\varepsilon = 0.3$ m^{-1}. Then, we can calculate that $\tilde{f}_{\max} = -5$ in which $d = 5$ and $x_1(t) = 0$. $\tilde{f}_{\min} = -10.88$, $d = 8$ and $x_1^2(t) = 4$.

System (4.2) is approximated by the following two fuzzy models

Rule 1: If $\theta_1(x(t))$ is F_{11},then:

$$\begin{cases} \dot{x}(t) = A_1 x(t) + B_1 sat(u_1(t)) + D_1 w(t) \\ z(t) = E_1 x(t) + F_1 sat(u_1(t)) \\ y(t) = C_1 x(t) \end{cases} \tag{4.3}$$

Rule 2: If $\theta_1(x(t))$ is F_{21},then:

$$\begin{cases} \dot{x}(t) = A_2 x(t) + B_2 sat(u_2(t)) + D_2 w(t) \\ z(t) = E_2 x(t) + F_2 sat(u_2(t)) \\ y(t) = C_2 x(t) \end{cases} \tag{4.4}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T = \begin{bmatrix} x & \dot{x} \end{bmatrix}^T$, in which $x_1(t)$ represents the premise vector.F_{11} and F_{21} represent the IT2 fuzzy sets of the known function $x_1(t)$.

$$A_1(t) = \begin{bmatrix} 0 & 1 \\ \tilde{f}_{\min} & -\frac{r}{M} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ \tilde{f}_{\max} & -\frac{r}{M} \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}, D_1 = \begin{bmatrix} -0.03 \\ -0.02 \end{bmatrix}, D_2 = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}^T, E_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}^T, F_1 = 0.9, F_2 = 1.1, C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T, C_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T.$$

Finally, the system (4.3) and (4.4) based on IT2 fuzzy theory is represented such that

$$\begin{cases} \dot{x}(t) = \sum_{p=1}^{2} \vartheta_p(\theta(x(t)))[A_p x(t) + B_p sat(u(t)) + D_p w(t)] \\ z(t) = \sum_{p=1}^{2} \vartheta_p(\theta(x(t)))[E_p x(t) + F_p sat(u(t))] \\ y(t) = \sum_{p=1}^{2} \vartheta_p(\theta(x(t)))C_p x(t) \end{cases} \tag{4.5}$$

The fuzzy state observer is designed such that

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{q=1}^{r} \vartheta_q(\hat{x}(t))[A_q \hat{x}(t) + B_q sat(u(t)) + L_q(y(t) - \hat{y}(t))] \\ \hat{z}(t) = \sum_{q=1}^{r} \vartheta_q(\hat{x}(t))[E_q \hat{x}(t) + F_q sat(u(t))] \\ \hat{y}(t) = \sum_{q=1}^{r} \vartheta_q(\hat{x}(t))C_q \hat{x}(t) \end{cases} \tag{4.6}$$

Then the IT2 T-S fuzzy H_∞ output feedback finite-time controller can be obtained as

$$\begin{aligned} sat(u(t)) &= sat(u(t),\bar{u}) \\ &= \sum_{l=1}^{2^m} \delta_l (H_l \sum_{j=1}^{r} \eta_j(\hat{x}(t))K_j + H_l^- \sum_{j=1}^{r} \eta_j(\hat{x}(t))Z_j)\hat{x}(t) \\ &= \sum_{l=1}^{2^m} \sum_{j=1}^{r} \delta_l \eta_j(\hat{x}(t))(H_l K_j + H_l^- Z_j)\hat{x}(t) \end{aligned} \tag{4.7}$$

where $H_1 = 1$, $H_1^- = 0$, $H_2 = 0$, $H_2^- = 1$, $\delta_1 = 0.6$ and $\delta_2 = 0.4$.

The disturbance input is selected as

$$w(t) = \begin{cases} 0.5, & \text{if } 0 < t < 1 \\ 0.1, & \text{if } 2 < t < 5 \\ 0, & \text{else} \end{cases} \tag{4.8}$$

The normalized membership functions of fuzzy system, state observer and controller are selected by Table 1, Table 2 and Table 3, respectively.

Table 1. The membership functions of IT2 fuzzy system.

$\vartheta_1^L(x(t)) = \frac{-\tilde{f}(x_1(t))+\tilde{f}_{\max}}{\tilde{f}_{\max}-\tilde{f}_{\min}}$, with $d = 5$	$\vartheta_1^U(x(t)) = \frac{-\tilde{f}(x_1(t))+\tilde{f}_{\max}}{\tilde{f}_{\max}-\tilde{f}_{\min}}$ with $d = 8$
$\vartheta_2^L(x(t)) = \frac{\tilde{f}(x_1(t))-\tilde{f}_{\min}}{\tilde{f}_{\max}-\tilde{f}_{\min}}$, with $d = 8$	$\vartheta_2^U(x(t)) = \frac{\tilde{f}(x_1(t))-\tilde{f}_{\min}}{\tilde{f}_{\max}-\tilde{f}_{\min}}$, with $d = 5$
$\upsilon_1^L(x(t))=\sin^2(x_1(t))$	$\upsilon_1^U(x(t)) = 1 - \sin^2(x_1(t))$
$\upsilon_2^L(x(t)) = 1 - \sin^2(x_1(t))$	$\upsilon_2^U(x(t))=\sin^2(x_1(t))$

Table 2. The membership functions of fuzzy state observer.

$\vartheta_1^L(\hat{x}(t)) = \frac{-\tilde{f}(\hat{x}_1(t))+\tilde{f}_{\max}}{\tilde{f}_{\max}-\tilde{f}_{\min}}$, with $d = 5$	$\vartheta_1^U(\hat{x}(t)) = \frac{-\tilde{f}(\hat{x}_1(t))+\tilde{f}_{\max}}{\tilde{f}_{\max}-\tilde{f}_{\min}}$ with $d = 8$
$\vartheta_2^L(\hat{x}(t)) = \frac{\tilde{f}(\hat{x}_1(t))-\tilde{f}_{\min}}{\tilde{f}_{\max}-\tilde{f}_{\min}}$, with $d = 8$	$\vartheta_2^U(\hat{x}(t)) = \frac{\tilde{f}(\hat{x}_1(t))-\tilde{f}_{\min}}{\tilde{f}_{\max}-\tilde{f}_{\min}}$, with $d = 5$
$\upsilon_1^L(\hat{x}(t))=\sin^2(\hat{x}_1(t))$	$\upsilon_1^U(\hat{x}(t)) = 1 - \sin^2(\hat{x}_1(t))$
$\upsilon_2^L(\hat{x}(t)) = 1 - \sin^2(\hat{x}_1(t))$	$\upsilon_2^U(\hat{x}(t))=\sin^2(\hat{x}_1(t))$

Table 3. The membership functions of fuzzy controller.

$\eta_1^L(\hat{x}(t)) = 1 - \frac{1}{9}\hat{x}_1^2(t)$	$\eta_1^U(\hat{x}(t)) = 1 - \frac{1}{27}\hat{x}_1^2(t)$
$\eta_2^L(\hat{x}(t)) = \frac{1}{27}\hat{x}_1^2(t)$	$\eta_2^U(\hat{x}(t)) = \frac{1}{9}\hat{x}_1^2(t)$
$b_1^L(\hat{x}(t))= 0.6\sin^2(\hat{x}_1(t))$	$b_1^U(\hat{x}(t)) = 1 - 0.6\sin^2(\hat{x}_1(t))$
$b_2^L(\hat{x}(t)) = 1 - 0.6\sin^2(\hat{x}_1(t))$	$b_2^U(\hat{x}(t))= 0.6\sin^2(\hat{x}_1(t))$

For given $a_1 = 5.8426$, $a_2 = 12.8613$, $h = 3, \beta = 0.01, \bar{\rho} = 0.3, T = 10$, and then by solving LMIs in Theorem 1 with the saturation levels $\bar{u} = 0.18$, we are able to get the constants and positive definite matrices as follows:

$$K_1 = \begin{bmatrix} 0.3223 \\ -0.1475 \end{bmatrix}^T, K_2 = \begin{bmatrix} 0.3223 \\ -0.1475 \end{bmatrix}^T, Z_1 = \begin{bmatrix} 0.4835 \\ -0.2212 \end{bmatrix}^T,$$

$$Z_2 = \begin{bmatrix} 0.4835 \\ -0.2212 \end{bmatrix}^T, L_1 = \begin{bmatrix} 0.4834 \\ -0.2212 \end{bmatrix}, L_2 = \begin{bmatrix} 0.4834 \\ -0.2212 \end{bmatrix}.$$

The results are carried out by choosing the initial conditions $x(0) = \begin{bmatrix} 0.7 & -0.6 \end{bmatrix}^T$, $\hat{x}(0) = \begin{bmatrix} 0.6 & -0.5 \end{bmatrix}^T$. Then, Figures 2-6 show the responses of closed-loop system. In addition, by Figures 2-3, from which we can see that all the estimations converge to zero asymptotically. Besides, the precise condition $\xi^T(0)R\xi(0) \leq a_1 = 5.8426$ is verified, and operating domain of the membership functions is established, i.e, $x_1(t) \in [-2, 2]$. Figure 4 is the response of the saturated control input u which satisfies the conditions that $\bar{u} \leq 0.18$. From Figure 5, one can find that $\xi^T(t)R\xi(t)$ stays within the bound of $a_2 = 12.8613$, which implies that the definition of robust H_∞ finite-time stability is satisfied. Figure 6 shows the distance attenuation for the fuzzy system, from which one can know that the robust H_∞ finite-time boundedness is satisfied.

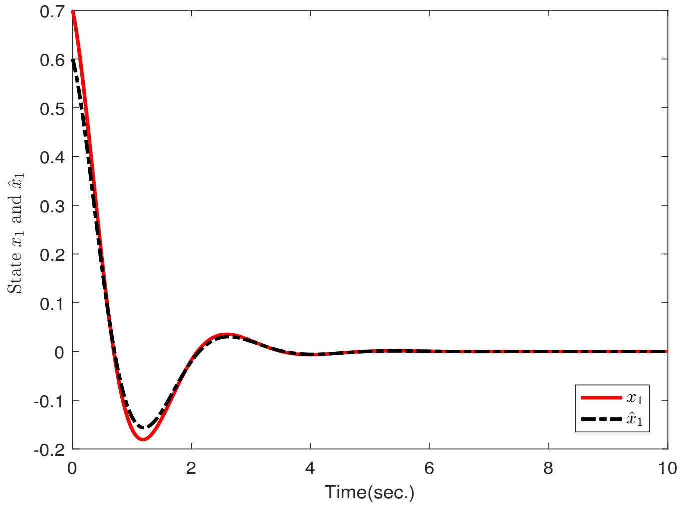

Figure 2. The trajectories of $x_1(t)$ and $\hat{x}_1(t)$.

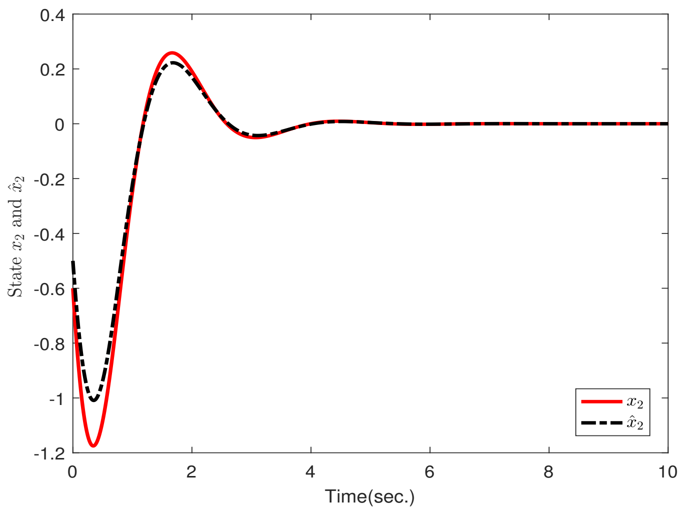

Figure 3. The trajectories of $x_2(t)$ and $\hat{x}_2(t)$.

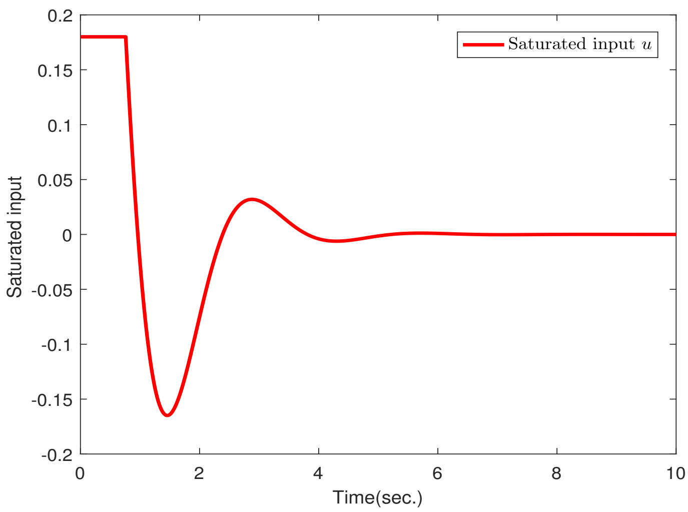

Figure 4. The trajectory of control input.

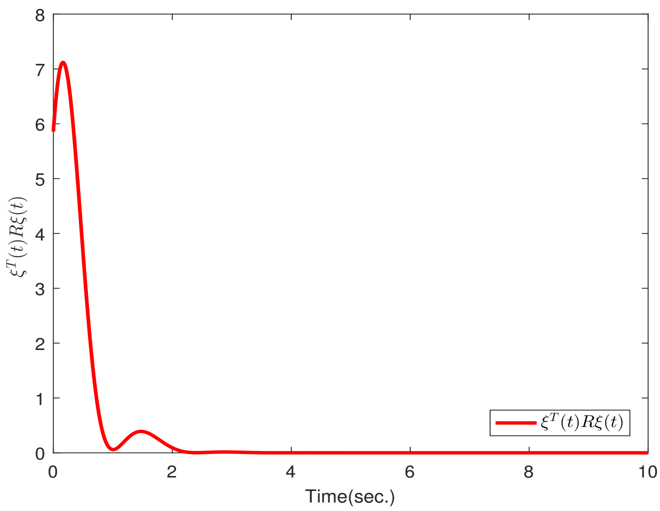

Figure 5. The trajectory of $\xi^T(t)R\xi(t)$.

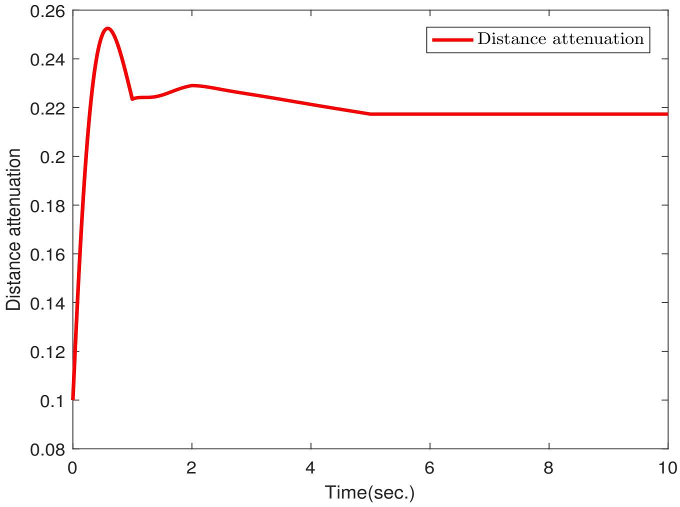

Figure 6. The trajectory of distance attenuation.

Example 2 (Tunnel diode circuit system) A nonlinear tunnel diode circuit system example as shown in Figure 7 is employed to verify the feasibility of the developed scheme in this subsection.

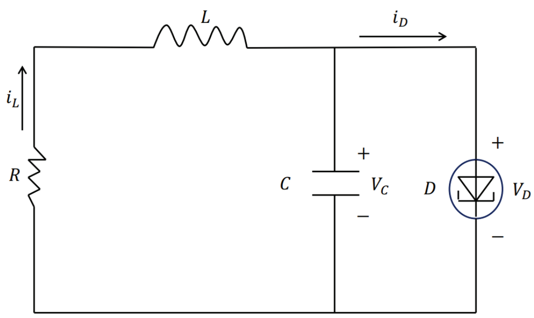

Figure 7. Tunnel diode circuit system.

where $i_L(t)$ is the inductor current, and $i_D(t)$ depicts the nonlinear volt-ampere characteristic of tunnel diode. V_C is the voltage of the capacitor C. V_D is the voltage of the tunnel diode D, R and L are electrical resistance.

The corresponding dynamical equation is shown as follows

$$i_D(t) = 0.002V_D(t) + \tau V_D^3(t) \tag{4.9}$$

in which $\tau \in [0.01, 0.03]$ is an uncertain parameter. Let $x_1(t) = V_D(t)$, $x_2(t) = i_D(t)$ and $g(V_D(t)) = 0.002 + \tau V_D^2(t)$ as nonlinear function composed of the terminal voltage across the tunnel diode. Here, state variable is supposed to be $x_1(t) \in [-3, 3]$. Then, one can find that $g_{\max} = 0.0272$ with $\tau = 0.03$ and $x_1(t) = 3$. $g_{\min} = 0.002$ with $\tau = 0.01$ and $x_1(t) = 0$.

Then, (4.9) can be described as

$$\begin{cases} \dot{x}_1(t) = -50g(V_D(t))x_1(t) + 50x_2(t) + u(t) \\ \dot{x}_2(t) = -x_1(t) - 10x_2(t) + u(t) \end{cases} \tag{4.10}$$

Then system (4.10) is approximated by the following two fuzzy models

Rule 1: If $\theta_1(x(t))$ is F_{11},then:

$$\begin{cases} \dot{x}(t) = A_1x(t) + B_1sat(u_1(t)) + D_1w(t) \\ z(t) = E_1x(t) + F_1sat(u_1(t)) \\ y(t) = C_1x(t) \end{cases} \tag{4.11}$$

Rule 2: If $\theta_1(x(t))$ is F_{21},then:

$$\begin{cases} \dot{x}(t) = A_2x(t) + B_2sat(u_2(t)) + D_2w(t) \\ z(t) = E_2x(t) + F_2sat(u_2(t)) \\ y(t) = C_2x(t) \end{cases} \tag{4.12}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$, in which $x_1(t)$ represents the premise vector.F_{11} and F_{21} represent the IT2 fuzzy sets of the known function $x_1(t)$.

$$A_1 = \begin{bmatrix} -\frac{g_{\min}}{0.02} & 50 \\ -1 & -10 \end{bmatrix}, A_2 = \begin{bmatrix} -\frac{g_{\max}}{0.02} & 50 \\ -1 & -10 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D_1 = \begin{bmatrix} -0.03 \\ -0.02 \end{bmatrix}, D_2 = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}^T, E_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}^T, F_1 = 0.9, F_2 = 1.1, C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T, C_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T.$$

The normalized membership functions of fuzzy system, state observer and controller are selected by Table 4, Table 5 and Table 6, respectively.

Table 4. The membership functions of IT2 fuzzy system.

$\vartheta_1^L(x(t)) = \frac{g_{\max}-g(V_D(t))}{g_{\max}-g_{\min}}$ with $\tau = 0.03$	$\vartheta_1^U(x(t)) = \frac{g_{\max}-g(V_D(t))}{g_{\max}-g_{\min}}$ with $\tau = 0.01$
$\vartheta_2^L(x(t)) = 1 - \vartheta_1^U(x(t))$	$\vartheta_2^U(x(t)) = 1 - \vartheta_1^L(x(t))$
$\upsilon_1^L(x(t))=\sin^2(V_D(t))$	$\upsilon_1^U(x(t)) = 1 - \sin^2(V_D(t))$
$\upsilon_2^L(x(t)) = 1 - \sin^2(V_D(t))$	$\upsilon_2^U(x(t))=\sin^2(V_D(t))$

Table 5. The membership functions of fuzzy state observer.

$\vartheta_1^L(\hat{x}(t)) = \frac{g_{\max}-g(\hat{V}_D(t))}{g_{\max}-g_{\min}}$ with $\tau = 0.03$	$\vartheta_1^U(\hat{x}(t)) = \frac{g_{\max}-g(\hat{V}_D(t))}{g_{\max}-g_{\min}}$ with $\tau = 0.01$
$\vartheta_2^L(\hat{x}(t)) = 1 - \vartheta_1^U(\hat{x}(t))$	$\vartheta_2^U(\hat{x}(t)) = 1 - \vartheta_1^L(\hat{x}(t))$
$\upsilon_1^L(\hat{x}(t))=\sin^2(\hat{V}_D(t))$	$\upsilon_1^U(\hat{x}(t)) = 1 - \sin^2(\hat{V}_D(t))$
$\upsilon_2^L(\hat{x}(t)) = 1 - \sin^2(\hat{V}_D(t))$	$\upsilon_2^U(\hat{x}(t))=\sin^2(\hat{V}_D(t))$

Table 6. The membership functions of fuzzy controller.

$\eta_1^L(\hat{x}(t)) = 1 - \frac{1}{9}\hat{V}_D^2(t)$	$\eta_1^U(\hat{x}(t)) = 1 - \frac{1}{27}\hat{V}_D^2(t)$
$\eta_2^L(\hat{x}(t)) = \frac{1}{27}\hat{V}_D^2(t)$	$\eta_2^U(\hat{x}(t)) = \frac{1}{9}\hat{V}_D^2(t)$
$b_1^L(\hat{x}(t))=\sin^2(\hat{V}_D(t))$	$b_1^U(\hat{x}(t)) = 1 - \sin^2(\hat{V}_D(t))$
$b_2^L(\hat{x}(t)) = 1 - \sin^2(\hat{V}_D(t))$	$b_2^U(\hat{x}(t))=\sin^2(\hat{V}_D(t))$

For given $\delta_1 = 0.7$, $\delta_2 = 0.3$, $a_1 = 2.9403$, $a_2 = 10.4953$, $h = 0.3$, $\beta = 0.2$, $\bar{\rho} = 0.3$, $T = 6$, and then by solving LMIs in Theorem 1 with the saturation levels $\bar{u} = 0.82$, we are able to get the constants and positive definite matrices as follows:

$$K_1 = \begin{bmatrix} -5.7955 \\ -2.4495 \end{bmatrix}^T, K_2 = \begin{bmatrix} -5.7955 \\ -2.4495 \end{bmatrix}^T, Z_1 = \begin{bmatrix} -13.5228 \\ -5.7155 \end{bmatrix}^T,$$

$$Z_2 = \begin{bmatrix} -13.5228 \\ -5.7155 \end{bmatrix}^T, L_1 = \begin{bmatrix} 0.2208 \\ 1.4482 \end{bmatrix}, L_2 = \begin{bmatrix} 0.2295 \\ 1.4482 \end{bmatrix}.$$

The results are carried out by choosing the initial conditions the same as Example 1. Then, Figures 8-12 show the responses of closed-loop system. In this example, $\xi^T(0)R\xi(0) \le a_1 = 2.9403$, saturated control input $\bar{u} \le 0.82$ and $\xi^T(t)R\xi(t) \le a_2 = 10.4953$.

To further verify the feasibility of the proposed fuzzy output feedback finite-time controller, we apply the fuzzy controller in [19], which is designed based on asymptotic stability theory to control system (4.9). Especially pointed out that, the controlled IT2 T-S interconnected system in [19] has been revised as IT2 T-S fuzzy system, thus can be compared further. In the simulation, we utilize the same IT2 T-S fuzzy system and the same initial conditions of $x(t)$ and $\hat{x}(t)$. The controller and observer gain matrices are as follows:

$$K_1 = \begin{bmatrix} 13.8487 \\ 10.1729 \end{bmatrix}^T, K_2 = \begin{bmatrix} -19.8188 \\ -10.9547 \end{bmatrix}^T, Z_1 = \begin{bmatrix} -69.9688 \\ -22.7800 \end{bmatrix}^T,$$

$$Z_2 = \begin{bmatrix} 8.5887 \\ 24.7800 \end{bmatrix}^T, L_1 = \begin{bmatrix} 2.8107 \\ 8.4807 \end{bmatrix}, L_2 = \begin{bmatrix} 2.8107 \\ 8.4807 \end{bmatrix}.$$

The simulation results are also depicted by Figs. 8-9. From Figs. 8-9, we clearly know that the variables and their estimations of the controlled system in this study have faster convergent rates than those by [19].

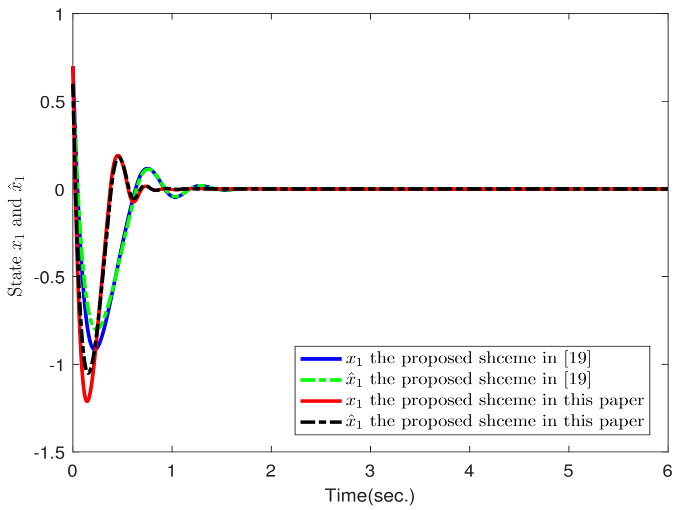

Figure 8. The trajectories of $x_1(t)$ and $\hat{x}_1(t)$.

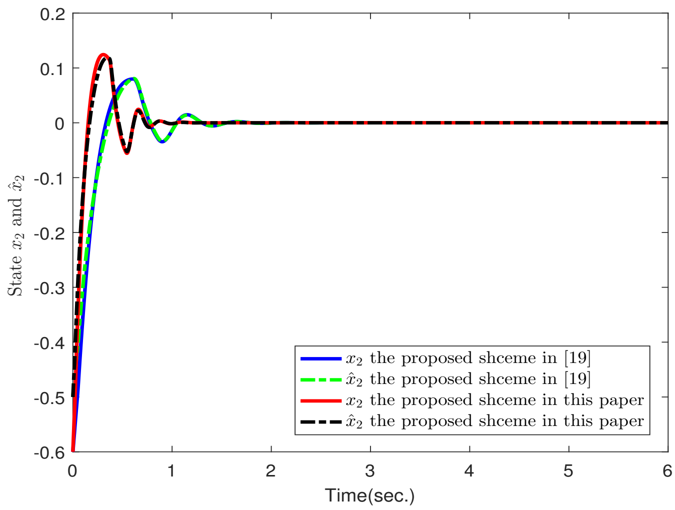

Figure 9. The trajectories of $x_2(t)$ and $\hat{x}_2(t)$.

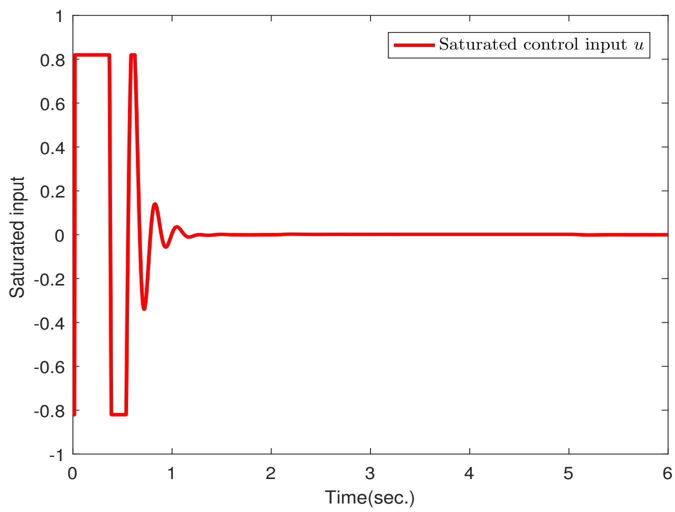

Figure 10. The trajectory of control input.

5. Conclusions

The robust H_∞ observer-based finite-time control problem for IT2 T-S fuzzy system in presence actuator saturation has been studied. In order to obtain the finite-time stability and robust H_∞ finite-time boundedness for the controlled plant, a fuzzy state observer and a fuzzy robust H_∞ finite-time controller with saturation limitation have been proposed. Then, on the basis of the Lyapunov function method and LMIs theory, the finite-time stability sufficient conditions are derived to ensure of IT2 T-S fuzzy systems in presence of input saturation limitation. At last, two practical examples of nonlinear mass-spring-damping and tunnel diode circuit systems is given to verify the feasibility of the developed fuzzy control scheme.

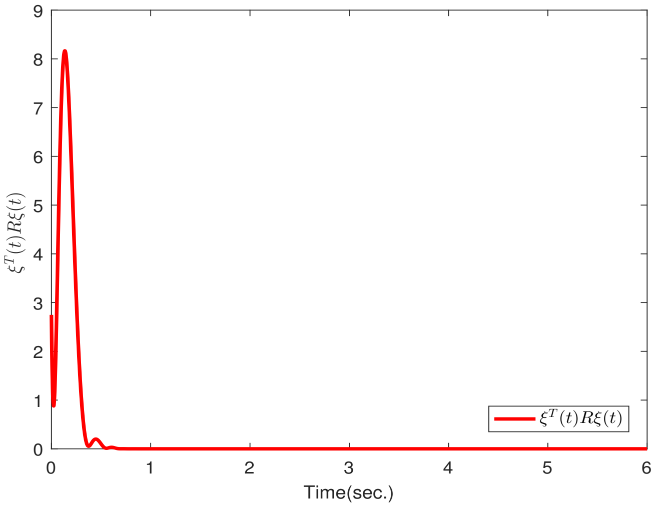

Figure 11. The trajectory of $\xi^T(t)R\xi(t)$.

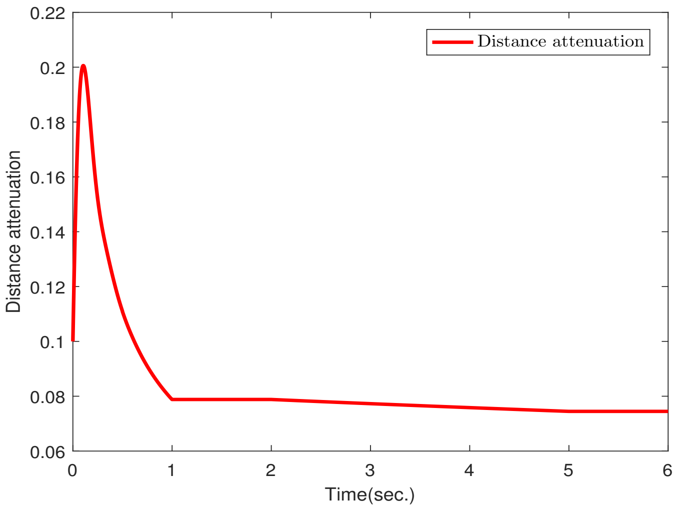

Figure 12. The trajectory of distance attenuation.

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Conflict of interest

The authors declare that there is no conflicts of interest.

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