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Research article

A diffusive predator-prey model with generalist predator and time delay

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Abstract: Time delay in the resource limitation of the prey is incorporated into a diffusive predator-prey model with generalist predator. By analyzing the eigenvalue spectrum, time delay inducing instability and Hopf bifurcation are investigated. Some conditions for determining the bifurcation direction and the stability of the bifurcating periodic solution are obtained by utilizing the normal form method and center manifold reduction for partial functional differential equation. The results suggest that time delay can destabilize the stability of coexisting equilibrium and induce bifurcating periodic solution when it increases through a certain threshold.

Keywords: delay; generalist predator; Hopf bifurcation; stability
Mathematics Subject Classification: 34K18, 35B32

1. Introduction

Predator-prey model mainly describes the interaction between two populations with predation relationship. Since predator-prey relationship exists widely in nature, many scholars have studied the predator-prey models [1–4]. Considering the influence of different factors on the population, a variety of predator-prey models have been established [5, 6]. Among these predator-prey models, Leslie-Gower predator-prey model is one of the classical model [7], with the following form

$$\begin{aligned} \dot{u}(t) &= r_1 u(1 - \frac{u}{K_1}) - \varphi(u, v)v, \\ \dot{v}(t) &= r_2 v(1 - \frac{\beta v}{u}). \end{aligned} \tag{1.1}$$

$u(t)$ and $v(t)$ stand for prey and predator's densities. r_1 and K_1 stand for the growth rate and the carrying capacity of the prey. $\varphi(u, v)$ is the functional response. The predator also follows the logistic growth law, where r_2 and u/β stand for the growth rate of predator and the carrying capacity of the predator.

Another classical predator-prey model is Gauss predator-prey model [8], with the form

$$\begin{aligned} \dot{u}(t) &= r_1 u(1 - \frac{u}{K_1}) - \varphi(u, v)v, \\ \dot{v}(t) &= c\varphi(u, v)v - dv. \end{aligned} \tag{1.2}$$

c and d are the conversion rate and death rate.

In predator-prey model, predators are mainly divided into specialist predators and generalist predators. Specialist predators feed almost exclusively on one specie of prey and require more specific environmental conditions. But, the generalist predators feed on many types of species, and can change its diet to another species when its a focal prey population begin to run short [9–11]. In [10], the authors studied a diffusive predator-prey model with generalist predator. They aimed to formalize the conditions in which spatial biological control can be achieved by generalists [10]. In [11], the authors studied the spatiotemporal dynamics and bifurcations of a diffusive predator-prey model with generalist predator and the combined the effect of linear prey harvesting and constant proportion of prey refuge. According to [10], the predator-prey model with generalist predators is of the following form

$$\begin{aligned} \dot{u}(t) &= r_1 u(1 - \frac{u}{K_1}) - \varphi(u, v)v, \\ \dot{v}(t) &= r_2 v(1 - \frac{v}{K_2}) + c\varphi(u, v)v. \end{aligned} \tag{1.3}$$

K_2 stands for the carrying capacity of the predator in absence of focal prey.

Predator-prey models with different functional responses can show different dynamic behaviors. In [10], the authors used the Type II functional response to reflect the effect of predator to the prey. Holling Type II functional response is a kind of prey-dependent functional response. Predator-dependent response function is also important. Such as Beddington-DeAngelis type [12], with the following form

$$\varphi(u, v) = \frac{Bu}{C + A_1 u + A_2 v},$$

where B, C, A_1 and A_2 stand for the maximum predator attack rate, the half-saturation constant, the effect of handling time and the magnitude of interference among predators. Some works all suggest that Beddington-DeAngelis functional response can enrich the dynamics of predator-prey model [13–15].

Moreover, time delay exists widely in the population model. The delayed predator-prey model has attracted wide attention from scholars [16–18]. These results show that time delay can enrich the dynamics of predator-prey model. Considering the generalist predator and discrete time delay τ in the resource limitation of the prey, we propose the following predator-prey model.

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1 \Delta u + r_1 u \left(1 - \dfrac{u(t-\tau)}{K_1}\right) - \dfrac{Buv}{C + A_1 u + A_2 v}, \\ \dfrac{\partial v(x,t)}{\partial t} = d_2 \Delta v + r_2 v \left(1 - \dfrac{v}{K_2}\right) + \dfrac{EBuv}{C + A_1 u + A_2 v}, \quad x \in \Omega, \ t > 0 \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,t) = u_1(x,t) \geq 0, v(x,t) = v_1(x,t) \geq 0, \quad x \in \Omega, t \in [-\tau, 0]. \end{cases} \tag{1.4}$$

All the parameters in the model are positive. $u(x,t)$ and $v(x,t)$ stand for the densities of prey and predator at the location x and time t, respectively. E is conversion rate of prey. The boundary condition is Neumann boundary condition. The aim of this article is to study the dynamics of model (1.4) from the point of view of stability and bifurcation. Whether time delay can induce some new dynamic phenomena?

The organization of this paper is as follows. In Sec. 2, the existence of coexisting equilibrium of the model is given. In Sec. 3, stability of equilibria and the existence of Hopf bifurcation is considered. In Sec. 4, the property of Hopf bifurcation is analyzed. In Sec. 5, some numerical simulations are carried. In Sec. 6, a brief conclusion is given.

2. Equilibrium analysis

Denote $\tilde{u} = u/K_1$, $\tilde{v} = u/K_2$ and $\tilde{t} = tr_2$, system (1.4) is changed to (after dropping tildes):

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1\Delta u + ru\left[1 - u(t-\tau) - \dfrac{av}{1+bu+cv}\right], \\ \dfrac{\partial v(x,t)}{\partial t} = d_2\Delta v + v(1-v) + \dfrac{euv}{1+bu+cv}, \quad x\in\Omega,\ t>0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, \quad x\in\partial\Omega,\ t>0, \\ u(x,t) = u_1(x,t)\geq 0, v(x,t) = v_1(x,t)\geq 0, \quad x\in\Omega, t\in[-\tau,0], \end{cases} \tag{2.1}$$

where

$$d_1 = \frac{D_1}{r_2},\ d_2 = \frac{D_2}{r_2},\ r = \frac{r_1}{b},\ a = \frac{BK_2}{C},\ b = \frac{A_1K_1}{C},\ c = \frac{A_2K_2}{C},\ e = \frac{EBK_1}{Cr_2}. \tag{2.2}$$

We assume $\Omega = (0, l\pi)$, where $l > 0$.

Solving the following equations,

$$\begin{cases} ru\left[1 - u - \dfrac{av}{1+bu+cv}\right] = 0, \\ v(1-v) + \dfrac{euv}{1+bu+cv} = 0. \end{cases} \tag{2.3}$$

We can obtained that $(0,0)$, $(1,0)$ and $(0,1)$ are three boundary equilibria. And the coexisting equilibrium (u_*, v_*) satisfying $v_* = \frac{(1-u_*)(1+bu_*)}{a-c+cu_*}$. Obviously, $v_* > 0$ implies $\max\{0, \frac{c-a}{c}\} < u_* < 1$. In addition, form (2.3), we can easily obtained that

$$eu_* - eu_*^2 + av_* - av_*^2 = 0.$$

Submitting $v_* = \frac{(1-u_*)(1+bu_*)}{a-c+cu_*}$ into it, yields $h(u_*) = 0$, where

$$\begin{aligned} h(u) &= \left(ab^2 + c^2e\right)u^3 + \beta_2u^2 + \beta_1u - a(1-a+c), \\ \beta_1 &= a - 2ab + a^2b + ac - abc + a^2e - 2ace + c^2e, \\ \beta_2 &= 2ab - ab^2 + abc + 2ace - 2c^2e. \end{aligned} \tag{2.4}$$

Theorem 2.1. *If $c > a - 1$, system (2.1) has at least one coexisting equilibrium (u_*, v_*), where u_* is the root of $h(u_*) = 0$ in interval $(max\{0, \frac{c-a}{c}\}, 1)$ and $v_* = \frac{(1-u_*)(1+bu_*)}{a-c+cu_*}$.*

Proof. By direct calculation, we have $h(1) = a^2(1+b+e) > 0$, $h(\frac{c-a}{c}) = -\frac{a^2(ab-bc-c)^2}{c^3} \leq 0$ and $h(0) = -a(1-a+c)$. If $c > a$, then $\frac{c-a}{c} = \max\{0, \frac{c-a}{c}\}$ and $h(\frac{c-a}{c}) < 0$. Then $h(u_*) = 0$ has at least one root in interval $(\frac{c-a}{c}, 1)$. If $a-1 < c \leq a$, then $0 = \max\{0, \frac{c-a}{c}\}$ and $h(0) < 0$. Then $h(u_*) = 0$ has at least one root in interval $(0,1)$. This completes the proof. □

3. Stability analysis

Linearize system (2.1) at (u_*, v_*)

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = d\Delta \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_1 \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_2 \begin{pmatrix} u(t-\tau) \\ v(t-\tau) \end{pmatrix}, \tag{3.1}$$

where

$$L_1 = \begin{pmatrix} ra_1 & -ra_2 \\ b_1 & -b_2 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} -ru_* & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} a_1 &= \frac{abu_*v_*}{(1+bu_*+cv_*)^2} > 0, \quad a_2 = \frac{u_*a(1+bu_*)}{(1+bu_*+cv_*)^2} > 0, \\ b_1 &= \frac{ev_*(1+cv_*)}{(1+bu_*+cv_*)^2} > 0, \quad b_2 = v_*\left(1 + \frac{ceu_*}{(1+bu_*+cv_*)^2}\right) > 0. \end{aligned} \tag{3.2}$$

The characteristic equation of (3.1) is

$$det(\lambda I - M_n - L_1 - L_2 e^{-\lambda\tau}) = 0, \tag{3.3}$$

where $I = \mathrm{diag}\{1,1\}$ and $M_n = -n^2/l^2 \mathrm{diag}\{d_1, d_2\}$, $n \in \mathbb{N}_0$. Then, we have

$$\lambda^2 + \lambda A_n + B_n + (C_n + \lambda r u_*)e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0, \tag{3.4}$$

where

$$\begin{aligned} A_n &= (d_1+d_2)\frac{n^2}{l^2} - ra_1 + b_2, \\ B_n &= d_1 d_2 \frac{n^4}{l^4} - (a_1 d_2 r - d_1 b_2)\frac{n^2}{l^2} + r(a_2 b_1 - a_1 b_2), \\ C_n &= d_2 r u_* \frac{n^2}{l^2} + b_2 r u_*. \end{aligned}$$

3.1. The non-delay model

When $\tau = 0$, the characteristic Eq (2.1) is

$$\lambda^2 - tr_n\lambda + \Delta_n(r) = 0, \;\; n \in \mathbb{N}_0, \tag{3.5}$$

where

$$\begin{cases} tr_n = r(a_1 - u_*) - b_2 - \frac{n^2}{l^2}(d_1+d_2), \\ \Delta_n = r\left[a_2 b_1 - b_2(a_1 - u_*)\right] - \frac{n^2}{l^2}\left[d_2 r(a_1 - u_*) - b_2 d_1\right] + d_1 d_2 \frac{n^4}{l^4}, \end{cases} \tag{3.6}$$

and the eigenvalues are given by

$$\lambda_{1,2}^{(n)}(r) = \frac{tr_n \pm \sqrt{tr_n^2 - 4\Delta_n}}{2}, \quad n \in \mathbb{N}_0. \tag{3.7}$$

We make the following hypothesis

$$\begin{aligned} &(\mathbf{H_1}) \qquad c > a, \\ &(\mathbf{H_2}) \qquad a - 1 < c \le a, \text{ and } c > a(1 - 1/b). \end{aligned} \tag{3.8}$$

Proposition 3.1. *If hypothesis* ($\mathbf{H_1}$) *(or* ($\mathbf{H_2}$)*) holds, then* $a_1 - u_* < 0$.

Proof. It is easy to obtain that $a_1 - u_* = -\frac{u_*\phi(u_*)}{a(1+bu_*)}$, where $\phi(u) = bcu^2 + 2b(a - c)u + a - ab + bc$. By direct calculation, we have $\phi(1) = a(1 + b) > 0$, $\phi(\frac{c-a}{c}) = a\left(1 + b - \frac{ab}{c}\right)$, $\phi(0) = a - ab + bc$ and $\phi'(\frac{c-a}{c}) = 0$. From the proof of Theorem 2.1, we can obtain that $u_* \in (\frac{c-a}{c}, 1)$ under hypothesis ($\mathbf{H_1}$). And $\phi(\frac{c-a}{c}) > 0$, implying that $\phi(u_*) > 0$. Hence $a_1 - u_* < 0$ under hypothesis ($\mathbf{H_1}$). Similarly, we can verify that $a_1 - u_* < 0$ under hypothesis ($\mathbf{H_2}$). □

Theorem 3.1. *Suppose* ($\mathbf{H_1}$) *(or* ($\mathbf{H_2}$)*) holds. Then the equilibrium* $E_*(u_*, v_*)$ *is locally asymptotically stable.*

Proof. By the Proposition 3.1, we know that $a_1 - u_* < 0$ under hypothesis ($\mathbf{H_1}$) (or ($\mathbf{H_1}$)). Then we have $tr_n < 0$ and $\Delta_n > 0$ for $n \in \mathbb{N}_0$. This implies that all eigenvalues of (3.5) have negative real parts. Then the equilibrium $E_*(u_*, v_*)$ is locally asymptotically stable. □

3.2. The delay model

Now, we study the stability of $E_*(u_*, v_*)$ when $\tau > 0$. Let $i\omega$ ($\omega > 0$) be a solution of Eq (3.4), we have

$$-\omega^2 + i\omega A_n + B_n + (C_n + i\omega r u_*)(cos\omega\tau - isin\omega\tau) = 0.$$

Then we have

$$\begin{cases} -\omega^2 + B_n + C_n cos\omega\tau + \omega r u_* sin\omega\tau = 0, \\ A_n\omega - C_n sin\omega\tau + \omega r u_* cos\omega\tau = 0. \end{cases}$$

This leads to

$$\omega^4 + (A_n^2 - 2B_n - r^2u_*^2)\omega^2 + B_n^2 - C_n^2 = 0. \tag{3.9}$$

Denote $z = \omega^2$, then (3.9) can be changed into

$$z^2 + (A_n^2 - 2B_n - r^2u_*^2)z + B_n^2 - C_n^2 = 0 = 0, \tag{3.10}$$

and the roots of (3.10) are

$$z^{\pm} = \frac{1}{2}[-(A_n^2 - 2B_n - r^2u_*^2) \pm \sqrt{(A_n^2 - 2B_n - r^2u_*^2)^2 - 4(B_n^2 - C_n^2)}].$$

Under ($\mathbf{H_1}$) (or ($\mathbf{H_2}$)), we have

$$B_n + C_n = \Delta_n > 0.$$

By direct computation,

$$A_n^2 - 2B_n - r^2u_*^2 = \left(d_1^2 + d_2^2\right)\frac{n^4}{l^4} - 2(a_1d_1r - b_2d_2)\frac{n^2}{l^2} + b_2^2 - r\left(2a_2b_1 + r\left(u_*^2 - a_1^2\right)\right),$$
$$B_n - C_n = d_1d_2\frac{n^4}{l^4} - [(d_2r(a_1 + u_*)) - b_2d_1]\frac{n^2}{l^2} + r[a_2b_1 - b_2(a_1 + u_*)].$$

Fix parameters $r,\ a,\ b,\ c,\ e,\ d_1,\ d_2,\ l$, define

$$\mathcal{D} = \{k \in \mathbb{N}_0 \mid \text{Eq (3.10) has positive roots with } n = k.\} \tag{3.11}$$

For $n \in \mathcal{D}$, if $z^+ > 0$, Eq (3.4) has a pair of purely imaginary roots $\pm i\omega_n^+$ at $\tau_n^{j,+}$, $j \in \mathbb{N}_0$; if $z^- > 0$, Eq (3.4) has a pair of purely imaginary roots $\pm i\omega_n^-$ at $\tau_n^{j,-}$, $j \in \mathbb{N}_0$, where

$$\begin{aligned} \omega_n^{\pm} &= \sqrt{z_n^{\pm}}, \quad \tau_n^{j,\pm} = \tau_n^{0,\pm} + \frac{2j\pi}{\omega_n^{\pm}},\ (j = 0, 1, 2, \cdots), \\ \tau_n^{0,\pm} &= \frac{1}{\omega_n^{\pm}}\arccos\frac{(C_n - ru_*A_n)(\omega_n^{\pm})^2 - B_nC_n}{C_n^2 + r^2u_*^2(\omega_n^{\pm})^2}. \end{aligned} \tag{3.12}$$

From (3.12), we have $\tau_n^{0,\pm} < \tau_n^{j,\pm}$ $(j \in \mathbb{N})$. For $k \in \mathcal{D}$, define the smallest τ so that the stability will change, $\tau_* = \min\{\tau_k^{0,\pm} \text{ or } \tau_k^{0,+} \mid k \in \mathcal{D}\}$.

Lemma 3.1. *Suppose* $(\mathbf{H_1})$ *(or* $(\mathbf{H_2})$*) holds. If* $(A_n^2 - 2B_n - r^2u_*^2)^2 - 4(B_n^2 - C_n^2) > 0$*, then* $Re(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}} > 0$, $Re(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}} < 0$ *for* $\tau \in \mathcal{D}$ *and* $j \in \mathbb{N}_0$.

Proof. Differentiating two sides of (3.4) with respect τ, we have

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda + A_n + ru_*e^{-\lambda\tau}}{(C_n + \lambda ru_*)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Then

$$\begin{aligned} [\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}]_{\tau=\tau_n^{j,\pm}} &= \mathrm{Re}[\frac{2\lambda + A_n + ru_*e^{-\lambda\tau}}{(C_n + \lambda ru_*)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}]_{\tau=\tau_n^{j,\pm}} \\ &= [\frac{1}{\Lambda}\omega^2(2\omega^2 + A_n^2 - 2B_n - r^2u_*^2)]_{\tau=\tau_n^{j,\pm}} \\ &= \pm[\frac{1}{\Lambda}\omega^2\sqrt{(A_n^2 - 2B_n - r^2u_*^2)^2 - 4(B_n^2 - C_n^2)}]_{\tau=\tau_n^{j,\pm}}, \end{aligned}$$

where $\Lambda = \omega^4b_2^2 + C_n^2\omega^2 > 0$. Therefore $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}} > 0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}} < 0$. □

Theorem 3.2. *Suppose* $(\mathbf{H_1})$ *(or* $(\mathbf{H_2})$*) holds. For system* (2.1)*, the following statements are true.*

(i) $E_*(u_*, v_*)$ *is locally asymptotically stable for all* $\tau \geq 0$ *when* $\mathbb{D} = \emptyset$.

(ii) $E_*(u_*, v_*)$ *is locally asymptotically stable for* $\tau \in [0, \tau_*)$*, and unstable for* $\tau \in [\tau_*, \tau_* + \epsilon)$ *with some* ϵ *when* $\mathcal{D} \neq \emptyset$.

(iii) System (2.1) *undergoes a Hopf bifurcation at the equilibrium* $E_*(u_*, v_*)$ *when* $\tau = \tau_n^{j,+}$ *(or* $\tau = \tau_n^{j,-}$*),* $j \in \mathbb{N}_0$, $n \in \mathcal{D}$ *when* $\mathcal{D} \neq \emptyset$.

Using the same process, we can obtain the following theorem about the stability of boundary equilibria.

Theorem 3.3. *For system* (2.1)*, the following statements are true.*

(i) $(0,0)$ *is always unstable for all* $\tau \geq 0$*;*

(ii) $(1,0)$ *is always unstable when* $\tau = 0$*;*

(iii) $(0,1)$ *is locally asymptotically stable for* $a > 1 + c$ *and* $\tau \geq 0$*; and unstable for* $a < 1 + c$ *and* $\tau \geq 0$.

4. Properties of Hopf bifurcation

Now, we will study the property of Hopf bifurcation by the method in [19,20]. For a critical value $\tau_n^{j,+}$ (or $\tau_n^{j,-}$), we denote it as $\tilde{\tau}$. Let $\tilde{u}(x,t) = u(x,\tau t) - u_*$ and $\tilde{v}(x,t) = v(x,\tau t) - v_*$, then the system (2.1) is (drop the tildes)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tau[d_1 \Delta u + r\,(u+u_*)\left(1 - u(t-1) - u_* - \dfrac{a\,(v+v_*)}{1 + b\,(u+u_*) + c\,(v+v_*)}\right)], \\ \dfrac{\partial v}{\partial t} = \tau[d_2 \Delta v + (v+v_*)\,(1 - v - v_*) + \dfrac{e\,(u+u_*)\,(v+v_*)}{1 + b\,(u+u_*) + c\,(v+v_*)}]. \end{cases} \tag{4.1}$$

Denote $\tau = \tilde{\tau} + \varepsilon$, and $U = (u(x,t), v(x,t))^T$. In the phase space $\mathbb{C}_1 := C([-1,0], X)$, (4.1) can be rewritten as

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t, \varepsilon), \tag{4.2}$$

where $L_\varepsilon(\varphi)$ and $F(\varphi, \varepsilon)$ are

$$L_\varepsilon(\phi) = \varepsilon \begin{pmatrix} ra_1\phi_1(0) - ra_2\phi_2(0) - ru_*\phi_1(-1) \\ b_1\phi_1(0) - b_2\phi_2(0) \end{pmatrix}, \tag{4.3}$$

$$F(\phi, \varepsilon) = \varepsilon D \Delta \phi + L_\varepsilon(\phi) + f(\phi, \varepsilon), \tag{4.4}$$

with

$$\begin{aligned} f(\phi,\varepsilon) &= (\tilde{\tau} + \varepsilon)(F_1(\phi,\varepsilon), F_2(\phi,\varepsilon))^T, \\ F_1(\phi,\varepsilon) &= r\,(\phi_1(0) + u_*)\left(1 - \phi_1(-1) - u_* - \frac{a\,(\phi_2(0) + v_*)}{1 + b\,(\phi_1(0) + u_*) + c\,(\phi_2(0) + v_*)}\right) \\ &\quad - ra_1\phi_1(0) + ra_2\phi_2(0) + ru_*\phi_1(-1), \\ F_2(\phi,\varepsilon) &= (\phi_2(0) + v_*)\,(1 - \phi_2(0) - v_*) + \frac{e\,(\phi_1(0) + u_*)\,(\phi_2(0) + v_*)}{1 + b\,(\phi_1(0) + u_*) + c\,(\phi_2(0) + v_*)} - b_1\phi_1(0) + b_2\phi_2(0), \end{aligned}$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in \mathbb{C}_1$.

Consider the linear equation

$$\frac{dU(t)}{dt} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t). \tag{4.5}$$

We know that $\Lambda_n := \{i\omega_n\tilde{\tau}, -i\omega_n\tilde{\tau}\}$ are characteristic roots of

$$\frac{dz(t)}{dt} = -\tilde{\tau} D \frac{n^2}{l^2} z(t) + L_{\tilde{\tau}}(z_t). \tag{4.6}$$

By Riesz representation theorem, there exists a 2×2 matrix function $\eta^n(\sigma, \tilde{\tau})$, $(-1 \le \sigma \le 0)$, whose elements are of bounded variation functions such that

$$-\tilde{\tau} D \frac{n^2}{l^2} \phi(0) + L_{\tilde{\tau}}(\phi) = \int_{-1}^{0} d\eta^n(\sigma, \tau)\phi(\sigma)$$

for $\phi \in C([-1, 0], \mathbb{R}^2)$.

Choose

$$\eta^n(\sigma, \tau) = \begin{cases} \tau E & \sigma = 0, \\ 0 & \sigma \in (-1, 0), \\ -\tau F & \sigma = -1, \end{cases} \tag{4.7}$$

where

$$E = \begin{pmatrix} ra_1 - d_1 \frac{n^2}{l^2} & -ra_2 \\ b_1 & -b_2 - d_2 \frac{n^2}{l^2} \end{pmatrix}, \qquad F = \begin{pmatrix} -ru_* & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.8}$$

Define the bilinear paring

$$\begin{aligned} (\psi, \varphi) &= \psi(0)\varphi(0) - \int_{-1}^{0} \int_{\xi=0}^{\sigma} \psi(\xi - \sigma) d\eta^n(\sigma, \tilde{\tau}) \varphi(\xi) d\xi \\ &= \psi(0)\varphi(0) + \tilde{\tau} \int_{-1}^{0} \psi(\xi + 1) F \varphi(\xi) d\xi, \end{aligned} \tag{4.9}$$

for $\varphi \in C([-1, 0], \mathbb{R}^2)$, $\psi \in C([0, 1], \mathbb{R}^2)$. $A(\tilde{\tau})$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_n \tilde{\tau}$, and they are also eigenvalues of A^*.

Define $p_1(\theta) = (1, \xi)^T e^{i\omega_n \tilde{\tau}\sigma}$ $(\sigma \in [-1, 0])$, $q_1(r) = (1, \eta) e^{-i\omega_n \tilde{\tau} r}$ $(r \in [0, 1])$, where

$$\xi = \frac{b_1}{b_2 + d_2 n^2/l^2 + i\omega_n}, \quad \eta = \frac{a_2 r}{-b_2 - d_2 n^2/l^2 + i\omega_n}.$$

Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)^T$ with

$$\Phi_1(\sigma) = \frac{p_1(\sigma) + p_2(\sigma)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{i\omega_n \tilde{\tau}\sigma}\right) \\ \mathrm{Re}\left(\xi \mathrm{e}^{i\omega_n \tilde{\tau}\sigma}\right) \end{pmatrix}, \ \Phi_2(\sigma) = \frac{p_1(\sigma) - p_2(\sigma)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{i\omega_n \tilde{\tau}\sigma}\right) \\ \mathrm{Im}\left(\xi \mathrm{e}^{i\omega_n \tilde{\tau}\sigma}\right) \end{pmatrix},$$

for $\theta \in [-1, 0]$, and

$$\Psi_1^*(r) = \frac{q_1(r) + q_2(r)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{-i\omega_n \tilde{\tau} \mathrm{r}}\right) \\ \mathrm{Re}\left(\eta \mathrm{e}^{-i\omega_n \tilde{\tau} \mathrm{r}}\right) \end{pmatrix}, \ \Psi_2^*(r) = \frac{q_1(r) - q_2(r)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{-i\omega_n \tilde{\tau} \mathrm{r}}\right) \\ \mathrm{Im}\left(\eta \mathrm{e}^{-i\omega_n \tilde{\tau} \mathrm{r}}\right) \end{pmatrix},$$

for $r \in [0, 1]$. Then we can compute by (4.9)

$$D_1^* := (\Psi_1^*, \Phi_1), \ D_2^* := (\Psi_1^*, \Phi_2), \ D_3^* := (\Psi_2^*, \Phi_1), \ D_4^* := (\Psi_2^*, \Phi_2).$$

Define $(\Psi^*, \Phi) = (\Psi_j^*, \Phi_k) = \begin{pmatrix} D_1^* & D_2^* \\ D_3^* & D_4^* \end{pmatrix}$ and construct a new basis Ψ for P^* by

$$\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1} \Psi^*.$$

Then $(\Psi, \Phi) = I_2$. In addition, define $f_n := (\beta_n^1, \beta_n^2)$, where

$$\beta_n^1 = \begin{pmatrix} \cos \frac{n}{l} x \\ 0 \end{pmatrix}, \quad \beta_n^2 = \begin{pmatrix} 0 \\ \cos \frac{n}{l} x \end{pmatrix}.$$

We also define

$$c \cdot f_n = c_1 \beta_n^1 + c_2 \beta_n^2, \quad \text{for } \ c = (c_1, c_2)^T \in \mathbb{C}_1,$$

and

$$< u, v > := \frac{1}{l\pi} \int_0^{l\pi} u_1 \overline{v_1} dx + \frac{1}{l\pi} \int_0^{l\pi} u_2 \overline{v_2} dx$$

for $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u, v \in X$ and $< \varphi, f_0 >= (< \varphi, f_0^1 >, < \varphi, f_0^2 >)^T$.

Rewrite Eq (4.1) as the following abstract form

$$\frac{dU(t)}{dt} = A_{\tilde{\tau}} U_t + R(U_t, \varepsilon), \tag{4.10}$$

where

$$R(U_t, \varepsilon) = \begin{cases} 0, & \theta \in [-1, 0), \\ F(U_t, \varepsilon), & \theta = 0. \end{cases} \tag{4.11}$$

The solution is

$$U_t = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n + h(x_1, x_2, \varepsilon), \tag{4.12}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\Psi, < U_t, f_n >),$$

and

$$h(x_1, x_2, \varepsilon) \in P_S \mathbb{C}_1, \quad h(0, 0, 0) = 0, \quad Dh(0, 0, 0) = 0.$$

Then

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n + h(x_1, x_2, 0). \tag{4.13}$$

Let $z = x_1 - i x_2$, and notice that $p_1 = \Phi_1 + i \Phi_2$. Then we have

$$\Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{z + \overline{z}}{2} \\ \frac{i(z - \overline{z})}{2} \end{pmatrix} f_n = \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n,$$

and

$$h(x_1, x_2, 0) = h(\frac{z + \overline{z}}{2}, \frac{i(z - \overline{z})}{2}, 0).$$

Equation (4.13) is

$$\begin{aligned} U_t &= \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n + h(\frac{z + \overline{z}}{2}, \frac{i(z - \overline{z})}{2}, 0) \\ &= \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n + W(z, \overline{z}), \end{aligned} \tag{4.14}$$

where

$$W(z,\overline{z}) = h(\frac{z+\overline{z}}{2}, \frac{i(z-\overline{z})}{2}, 0).$$

From [19], z satisfies

$$\dot{z} = i\omega_n \tilde{\tau} z + g(z,\overline{z}), \tag{4.15}$$

where

$$g(z,\overline{z}) = (\Psi_1(0) - i\Psi_2(0)) < F(U_t,0), f_n > . \tag{4.16}$$

Let

$$W(z,\overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{4.17}$$

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{4.18}$$

then

$$\begin{aligned}
u_t(0) &= \frac{1}{2}(z+\overline{z})\cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots,\\
v_t(0) &= \frac{1}{2}(\xi + \overline{\xi}\overline{z})\cos\left(\frac{nx}{l}\right) + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots,\\
u_t(-1) &= \frac{1}{2}(ze^{-i\omega_n\tilde{\tau}} + \overline{z}e^{i\omega_n\tilde{\tau}})\cos(\frac{nx}{l}) + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots,\\
v_t(-1) &= \frac{1}{2}(\xi ze^{-i\omega_n\tilde{\tau}} + \overline{\xi}\overline{z}e^{i\omega_n\tilde{\tau}})cos(\frac{nx}{l}) + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + \cdots,
\end{aligned}$$

and

$$\begin{aligned}
\overline{F}_1(U_t,0) = \frac{1}{\tilde{\tau}}F_1 &= \alpha_1 u_t^2(0) + \alpha_2 u_t(0)v_t(0) + \alpha_3 v_t^2(0) + \alpha_4 u_t^3(0)\\
&+ \alpha_5 u_t^2(0)v_t(0) + \alpha_6 u_t(0)v_t^2(0) + \alpha_7 v_t^3(0) + O(4),
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\overline{F}_2(U_t,0) = \frac{1}{\tilde{\tau}}F_2 &= -v_t^2(0) + \beta_1 u_t^2(-1) + \beta_2 u_t(-1)v_t(-1) + \beta_3 v_t^2(-1)\\
&+ \beta_4 u_t^3(-1) + \beta_5 u_t^2(-1)v_t(-1) + \beta_6 u_t(-1)v_t^2(-1) + \beta_7 v_t^3(-1) + O(4),
\end{aligned} \tag{4.20}$$

with $\alpha_1 = \frac{arv_*(b+bcv_*)}{(1+bu_*+cv_*)^3}$, $\alpha_2 = -\frac{ar(1+cv_*+b(u_*+2cu_*v_*))}{(1+bu_*+cv_*)^3}$, $\alpha_3 = \frac{aru_*(c+bcu_*)}{(1+bu_*+cv_*)^3}$, $\alpha_4 = -\frac{ab^2rv_*(1+cv_*)}{(1+bu_*+cv_*)^4}$, $\alpha_5 = \frac{abr\left(1-c^2v_*^2+b(u_*+2cu_*v_*)\right)}{(1+bu_*+cv_*)^4}$, $\alpha_6 = \frac{acr\left(1-b^2u_*^2+cv_*+2bcu_*v_*\right)}{(1+bu_*+cv_*)^4}$, $\alpha_7 = -\frac{6ac^2ru_*(1+bu_*)}{(1+bu_*+cv_*)^4}$, $\beta_1 = -\frac{ev_*(b+bcv_*)}{(1+bu_*+cv_*)^3}$, $\beta_2 = \frac{e(1+cv_*+b(u_*+2cu_*v_*))}{(1+bu_*+cv_*)^3}$, $\beta_3 = -1 - \frac{eu_*(c+bcu_*)}{(1+bu_*+cv_*)^3}$, $\beta_4 = \frac{b^2ev_*(1+cv_*)}{(1+bu_*+cv_*)^4}$, $\beta_5 = -\frac{be\left(1-c^2v_*^2+b(u_*+2cu_*v_*)\right)}{(1+bu_*+cv_*)^4}$, $\beta_6 = -\frac{ce\left(1-b^2u_*^2+cv_*+2bcu_*v_*\right)}{(1+bu_*+cv_*)^4}, \beta_7 = \frac{c^2eu_*(1+bu_*)}{(1+bu_*+cv_*)^4}$.

Hence,

$$\begin{aligned}
\overline{F}_1(U_t,0) &= \cos^2(\frac{nx}{l})(\frac{z^2}{2}\chi_{20} + z\overline{z}\chi_{11} + \frac{\overline{z}^2}{2}\overline{\chi}_{20}) + \frac{z^2\overline{z}}{2}(\chi_1\cos\frac{nx}{l} + \chi_2\cos^3\frac{nx}{l}) + \cdots,\\
\overline{F}_2(U_t,0) &= \cos^2(\frac{nx}{l})(\frac{z^2}{2}\varsigma_{20} + z\overline{z}\varsigma_{11} + \frac{\overline{z}^2}{2}\overline{\varsigma}_{20}) + \frac{z^2\overline{z}}{2}(\varsigma_1\cos\frac{nx}{l} + \varsigma_2\cos^3\frac{nx}{l}) + \cdots,
\end{aligned} \tag{4.21}$$

$$< F(U_t,0), f_n >=\tilde{\tau}(\overline{F}_1(U_t,0)f_n^1+\overline{F}_2(U_t,0)f_n^2)$$
$$=\frac{z^2}{2}\tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}\Gamma+z\bar{z}\tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varsigma_{11}\end{pmatrix}\Gamma+\frac{\bar{z}^2}{2}\tilde{\tau}\begin{pmatrix}\overline{\chi}_{20}\\ \overline{\varsigma}_{20}\end{pmatrix}\Gamma+\frac{z^2\bar{z}}{2}\tilde{\tau}\begin{pmatrix}\kappa_1\\ \kappa_2\end{pmatrix}+\cdots, \tag{4.22}$$

with

$$\Gamma=\frac{1}{l\pi}\int_0^{l\pi}\cos^3(\frac{nx}{l})dx,$$
$$\kappa_1=\frac{\chi_1}{l\pi}\int_0^{l\pi}\cos^2(\frac{nx}{l})dx+\frac{\chi_2}{l\pi}\int_0^{l\pi}\cos^4(\frac{nx}{l})dx,$$
$$\kappa_2=\frac{\varsigma_1}{l\pi}\int_0^{l\pi}\cos^2(\frac{nx}{l})dx+\frac{\varsigma_2}{l\pi}\int_0^{l\pi}\cos^4(\frac{nx}{l})dx$$

and

$\chi_{20}=\frac{1}{2}e^{-i\tau\omega_n}\left(-r+e^{i\tau\omega_n}(\alpha_1+\xi(\alpha_2+\alpha_3\xi))\right),$

$\chi_{11}=-\frac{1}{4}e^{-i\tau\omega_n}\left(\left(1+e^{2i\tau\omega_n}\right)r-e^{i\tau\omega_n}(2\alpha_1+2\alpha_3\bar{\xi}\xi+\alpha_2(\bar{\xi}+\xi))\right),$

$\chi_1=W_{11}^1(0)\left(-e^{-i\tau\omega_n}r+2\alpha_1+\alpha_2\xi\right)+W_{11}^2(0)(\alpha_2+2\alpha_3\xi)+\frac{1}{2}W_{20}^1(0)\left(-e^{i\tau\omega_n}r+2\alpha_1+\alpha_2\bar{\xi}\right)$
$\quad+\frac{1}{2}W_{20}^2(0)(\alpha_2+2\alpha_3\bar{\xi})-rW_{11}^1(-1)-\frac{rW_{20}^1(-1)}{2},$

$\chi_2=\frac{1}{4}(3\alpha_4+\alpha_5(\bar{\xi}+2\xi)+\xi(2\alpha_6\bar{\xi}+\alpha_6\xi+3\alpha_7\bar{\xi}\xi)),$

$\varsigma_{20}=\frac{1}{2}(\beta_1+\xi(\beta_2+\beta_3\xi)),\ \varsigma_{11}=\frac{1}{4}(2\beta_1+2\beta_3\bar{\xi}\xi+\beta_2(\bar{\xi}+\xi)),$

$\varsigma_1=W_{11}^1(0)(2\beta_1+\beta_2\xi)+W_{11}^2(0)(\beta_2+2\beta_3\xi)+W_{20}^1(0)\left(\beta_1+\frac{\beta_2\bar{\xi}}{2}\right)+\frac{1}{2}W_{20}^2(0)(\beta_2+2\beta_3\bar{\xi}),$

$\varsigma_2=\frac{1}{4}(3\beta_4+\beta_5(\bar{\xi}+2\xi)+\xi(2\beta_6\bar{\xi}+\beta_6\xi+3\beta_7\bar{\xi}\xi)).$

Denote

$$\Psi_1(0)-i\Psi_2(0):=(\gamma_1\ \ \gamma_2).$$

Notice that

$$\frac{1}{l\pi}\int_0^{l\pi}\cos^3\frac{nx}{l}dx=0,\ \ n=1,2,3,\cdots,$$

and we have

$$(\Psi_1(0)-i\Psi_2(0)) < F(U_t,0), f_n >=$$
$$\frac{z^2}{2}(\gamma_1\chi_{20}+\gamma_2\varsigma_{20})\Gamma\tilde{\tau}+z\bar{z}(\gamma_1\chi_{11}+\gamma_2\varsigma_{11})\Gamma\tilde{\tau}+\frac{\bar{z}^2}{2}(\gamma_1\overline{\chi}_{20}+\gamma_2\overline{\varsigma}_{20})\Gamma\tilde{\tau}$$
$$+\frac{z^2\bar{z}}{2}\tilde{\tau}[\gamma_1\kappa_1+\gamma_2\kappa_2]+\cdots. \tag{4.23}$$

Then by (4.16), (4.18) and (4.23), we have $g_{20}=g_{11}=g_{02}=0$, for $n=1,2,3,\cdots$. If $n=0$, we have:

$$g_{20}=\gamma_1\tilde{\tau}\chi_{20}+\gamma_2\tilde{\tau}\varsigma_{20},\quad g_{11}=\gamma_1\tilde{\tau}\chi_{11}+\gamma_2\tilde{\tau}\varsigma_{11},\quad g_{02}=\gamma_1\tilde{\tau}\overline{\chi}_{20}+\gamma_2\tilde{\tau}\overline{\varsigma}_{20}.$$

And for $n\in\mathbb{N}_0$, $g_{21}=\tilde{\tau}(\gamma_1\kappa_1+\gamma_2\kappa_2)$.

From [19], we have

$$\dot{W}(z,\overline{z}) = W_{20}z\dot{z} + W_{11}\dot{z}\overline{z} + W_{11}z\dot{\overline{z}} + W_{02}\overline{z}\dot{\overline{z}} + \cdots,$$

$$A_{\tilde{\tau}}W(z,\overline{z}) = A_{\tilde{\tau}}W_{20}\frac{z^2}{2} + A_{\tilde{\tau}}W_{11}z\overline{z} + A_{\tilde{\tau}}W_{02}\frac{\overline{z}^2}{2} + \cdots,$$

and

$$\dot{W}(z,\overline{z}) = A_{\tilde{\tau}}W + H(z,\overline{z}),$$

where

$$\begin{aligned} H(z,\overline{z}) &= H_{20}\frac{z^2}{2} + W_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots \\ &= X_0F(U_t,0) - \Phi(\Psi, < X_0F(U_t,0), f_n > \cdot f_n). \end{aligned} \tag{4.24}$$

Hence, we have

$$(2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{20} = H_{20}, \quad -A_{\tilde{\tau}}W_{11} = H_{11}, \quad (-2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{02} = H_{02}, \tag{4.25}$$

that is

$$W_{20} = (2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{20}, \quad W_{11} = -A_{\tilde{\tau}}^{-1}H_{11}, \quad W_{02} = (-2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{02}. \tag{4.26}$$

Then

$$\begin{aligned} H(z,\overline{z}) &= -\Phi(0)\Psi(0) < F(U_t,0), f_n > \cdot f_n \\ &= -\left(\frac{p_1(\theta)+p_2(\theta)}{2}, \frac{p_1(\theta)-p_2(\theta)}{2i}\right)\begin{pmatrix} \Phi_1(0) \\ \Phi_2(0) \end{pmatrix} < F(U_t,0), f_n > \cdot f_n \\ &= -\frac{1}{2}[p_1(\theta)(\Phi_1(0) - i\Phi_2(0)) + p_2(\theta)(\Phi_1(0) + i\Phi_2(0))] < F(U_t,0), f_n > \cdot f_n \\ &= -\frac{1}{2}[(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02})\frac{z^2}{2} + (p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11})z\overline{z} + (p_1(\theta)g_{02} + p_2(\theta)\overline{g}_{20})\frac{\overline{z}^2}{2}] + \cdots. \end{aligned}$$

Therefore,

$$H_{20}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}) \cdot f_0 & n = 0, \end{cases}$$

$$H_{11}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}) \cdot f_0 & n = 0, \end{cases}$$

$$H_{02}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{02} + p_2(\theta)\overline{g}_{20}) \cdot f_0 & n = 0, \end{cases}$$

and

$$H(z,\overline{z})(0) = F(U_t,0) - \Phi(\Psi, < F(U_t,0), f_n >) \cdot f_n,$$

where

$$H_{20}(0)=\begin{cases}\tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}\cos^2(\frac{nx}{l}), & n\in\mathbb{N},\\ \tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}-\frac{1}{2}(p_1(0)g_{20}+p_2(0)\overline{g}_{02})\cdot f_0, & n=0.\end{cases}\tag{4.27}$$

$$H_{11}(0)=\begin{cases}\tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varsigma_{11}\end{pmatrix}\cos^2(\frac{nx}{l}), & n\in\mathbb{N},\\ \tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varsigma_{11}\end{pmatrix}-\frac{1}{2}(p_1(0)g_{11}+p_2(0)\overline{g}_{11})\cdot f_0, & n=0.\end{cases}$$

By the definition of $A_{\tilde{\tau}}$ and (4.25), we have

$$\dot{W}_{20}=A_{\tilde{\tau}}W_{20}=2i\omega_n\tilde{\tau}W_{20}+\frac{1}{2}(p_1(\theta)g_{20}+p_2(\theta)\overline{g}_{02})\cdot f_n,\quad -1\le\theta<0.$$

That is

$$W_{20}(\theta)=\frac{i}{2i\omega_n\tilde{\tau}}(g_{20}p_1(\theta)+\frac{\overline{g}_{02}}{3}p_2(\theta))\cdot f_n+E_1e^{2i\omega_n\tilde{\tau}\theta},$$

where

$$E_1=\begin{cases}W_{20}(0) & n=1,2,3,\cdots,\\ W_{20}(0)-\frac{i}{2i\omega_n\tilde{\tau}}(g_{20}p_1(\theta)+\frac{\overline{g}_{02}}{3}p_2(\theta))\cdot f_0 & n=0.\end{cases}$$

By the definition of $A_{\tilde{\tau}}$ and (4.25), we have that for $-1\le\theta<0$

$$\begin{aligned}&-(g_{20}p_1(0)+\frac{\overline{g}_{02}}{3}p_2(0))\cdot f_0+2i\omega_n\tilde{\tau}E_1-A_{\tilde{\tau}}(\frac{i}{2\omega_n\tilde{\tau}}(g_{20}p_1(0)+\frac{\overline{g}_{02}}{3}p_2(0))\cdot f_0)\\&-A_{\tilde{\tau}}E_1-L_{\tilde{\tau}}(\frac{i}{2\omega_n\tilde{\tau}}(g_{20}p_1(0)+\frac{\overline{g}_{02}}{3}p_2(0))\cdot f_n+E_1e^{2i\omega_n\tilde{\tau}\theta})\\&=\tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}-\frac{1}{2}(p_1(0)g_{20}+p_2(0)\overline{g}_{02})\cdot f_0.\end{aligned}$$

As

$$A_{\tilde{\tau}}p_1(0)+L_{\tilde{\tau}}(p_1\cdot f_0)=i\omega_0p_1(0)\cdot f_0,$$

and

$$A_{\tilde{\tau}}p_2(0)+L_{\tilde{\tau}}(p_2\cdot f_0)=-i\omega_0p_2(0)\cdot f_0,$$

we have

$$2i\omega_nE_1-A_{\tilde{\tau}}E_1-L_{\tilde{\tau}}E_1e^{2i\omega_n}=\tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}\cos^2(\frac{nx}{l}),\quad n\in\mathbb{N}_0.$$

That is

$$E_1=\tilde{\tau}E\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}\cos^2(\frac{nx}{l}),$$

where

$$E = \begin{pmatrix} 2i\omega_n\tilde{\tau} + d_1\frac{n^2}{l^2} - ra_1 & ra_2 \\ -b_1e^{-2i\omega_n\tilde{\tau}} & 2i\omega_n\tilde{\tau} + d_2\frac{n^2}{l^2} - \alpha - b_2e^{-2i\omega_n\tilde{\tau}} \end{pmatrix}^{-1}.$$

Similarly, from (4.26), we have

$$-\dot{W}_{11} = \frac{i}{2\omega_n\tilde{\tau}}(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}) \cdot f_n, \quad -1 \leq \theta < 0.$$

That is

$$W_{11}(\theta) = \frac{i}{2i\omega_n\tilde{\tau}}(p_1(\theta)\overline{g}_{11} - p_1(\theta)g_{11}) + E_2.$$

Similarly, we have

$$E_2 = \tilde{\tau}E^* \begin{pmatrix} \chi_{11} \\ \varsigma_{11} \end{pmatrix} \cos^2(\frac{nx}{l}),$$

where

$$E^* = \begin{pmatrix} d_1\frac{n^2}{l^2} - ra_1 & ra_2 \\ -b_1 & d_2\frac{n^2}{l^2} - b_2 - \alpha \end{pmatrix}^{-1}.$$

Thus, we have

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_n\tilde{\tau}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{1}{2}g_{21}, \quad \mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau_n^j))}, \\ T_2 &= -\frac{1}{\omega_n\tilde{\tau}}[Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_n^j))], \quad \beta_2 = 2Re(c_1(0)). \end{aligned} \tag{4.28}$$

Theorem 4.1. *When $\mu_2 > 0$ (or $\mu_2 < 0$), the bifurcating periodic solutions exists for $\tau > \tau_n^{j,\pm}$ (or $\tau < \tau_n^{j,\pm}$), and are orbitally asymptotically stable (or unstable) when $\beta_2 < 0$ (or $\beta_2 > 0$).*

5. Numerical simulations

According to the reference [21], we choose $r_1 = 19.3$, $K_1 = 400$, $r_2 = 8.8$, $K_2 = 5$, $B = 10.76$, $C = 60.6$, $A_1 = 0.00728$, $A_2 = 1$. By (2.2), we can obtain a, b, c, e and r. Fix $d_1 = 0.1$, $d_2 = 0.2$, $l = 2$. We give the coexisting equilibrium (Figure 1), and bifurcation diagram (Figure 2) of system (2.1) with parameter E. It shows that the density of prey (predator) in the coexisting equilibrium decreases (increases) and the stable region increases with the increase of parameter E. This indicates that increasing parameter E is benefit to prey and predator to reach the steady state (coexisting equilibrium).

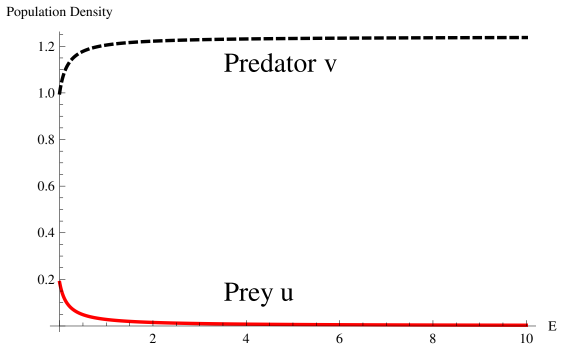

Figure 1. The coexisting equilibrium of system (2.1) with parameter E.

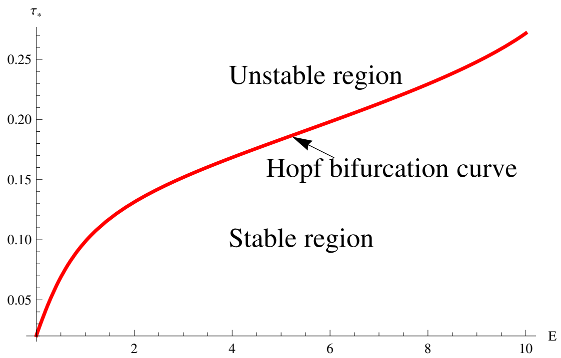

Figure 2. Bifurcation diagram of system (2.1) with parameter E.

When $E = 8.8$, we can obtain $(u_*, v_*) \approx (0.0037,\ 1.2370)$ is a unique coexisting equilibrium. Hypothesis $(\mathbf{H_1})$ is satisfied. We have $\tau_* = \tau_0^0 \approx 0.2440$. Then $E_*(u_*, v_*)$ is local stable when $\tau \in [0, \tau_*)$ (shown in Figure 3). When $\tau = \tau_*$, Hopf bifurcation occurs. We can obtain

$$\mu_2 \approx 81.0352 > 0, \ \ \beta_2 \approx -556.5305 < 0, \ \ \text{and} \ \ T_2 \approx 2035.9058 > 0.$$

Then, when $\tau > 3.4595$, the local stable bifurcating periodic solutions exists (shown in Figure 4).

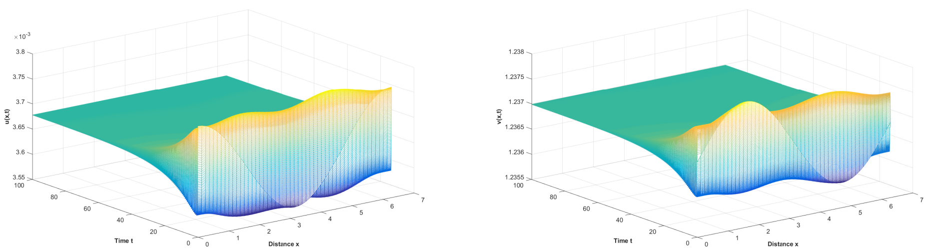

Figure 3. Numerical simulations of system (2.1) for $\tau = 0.2$, and initial condition is $(0.00367 + 0.0001\cos x,\ 1.23699 + 0.001\sin x)$.

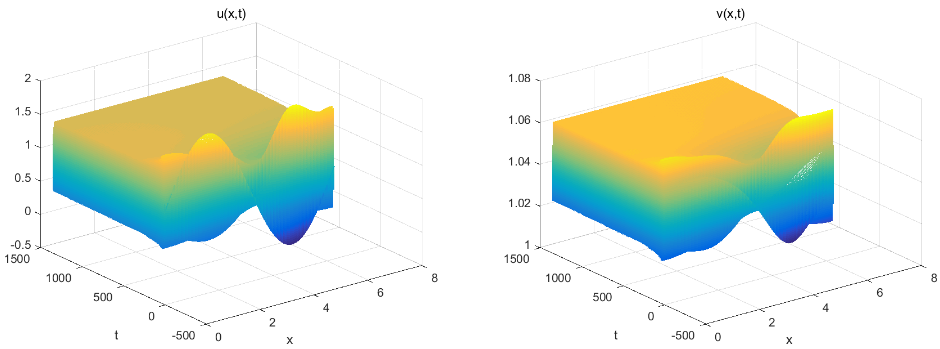

Figure 4. Numerical simulations of system (2.1) for $\tau = 0.3$, and initial condition is $(0.00367 + 0.0001\cos x,\ 1.23699 + 0.001\sin x)$.

6. Conclusions

In this paper, we propose a diffusive predator-prey system with generalist predator and time delay in the resource limitation of the prey. We obtained that system (2.1) has three boundary equilibria: $(0, 0)$ (predator and prey extinction equilibrium), $(1, 0)$ (predator extinction equilibrium) and $(0, 1)$ (prey extinction equilibrium). We mainly analyze the stability and Hopf bifurcation of coexisting equilibrium. By the theory of normal form and center manifold method, we give some parameters that determining the property of Hopf bifurcation: Bifurcation direction and the stability of the bifurcating periodic solution.

Since the predators are generalist type and have other food resource, they will not be extinct. This is in agreement with the Theorem 3.3. When the predator attack rate is large enough $a > 1 + c$, all the prey will be caught by the predator. This will lead to the extinction of the prey, and the predator will reach a balanced state. It is also in agreement with the Theorem 3.3 that $(0, 1)$ is local asymptotically stable for $a > 1 + c$. When the predator attack rate is not large enough $a < 1 + c$, then the prey and predator will coexist.

The conversion rate E can affect the the density of prey (predator) in the coexisting equilibrium. With the increase of conversion rate, the density of prey (predator) will decrease (increase) and the stable region will increase. In addition, time delay will also affect the stability of the equilibrium point when the parameters satisfying the condition $\mathcal{D} \neq \emptyset$. Specifically, when the time delay is small than the critical value τ_*, the prey and the predator will coexist and tend to the coexisting equilibrium. But when time delay is larger than the critical value τ_*, the prey and predator will exhibit oscillatory behavior. In addition, the spatial inhomogeneous periodic solutions may exist, but they are generally unstable.

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Conflict of interest

The authors declare that they have no competing interests.

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