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Research article

Global existence and asymptotic behavior for a viscoelastic Kirchhoff equation with a logarithmic nonlinearity, distributed delay and Balakrishnan-Taylor damping terms

Abdelbaki Choucha[1,2], **Salah Boulaaras**[3,*] **and Asma Alharbi**[3]

[1] Laboratory of Operator Theory and PDEs: Foundations and Applications, Department of Mathematics, Faculty of Exact Sciences, University of El Oued, Algeria
[2] Department of Mathematics, Faculty of Sciences, Amar Teledji Laghouat University, Algeria
[3] Department of Mathematics, College of Sciences and Arts, ArRass, Qassim University, Saudi Arabia

* **Correspondence:** Email: S.Boularas@qu.edu.sa.

Abstract: A nonlinear viscoelastic Kirchhoff-type equation with a logarithmic nonlinearity, Balakrishnan-Taylor damping, dispersion and distributed delay terms is studied. We establish the global existence of the solutions of the problem and by the energy method we prove an explicit and general decay rate result under suitable hypothesis.

Keywords: Kirchhoff equation; exponential decay; distributed delay term; viscoelastic term; logarithmic nonlinearity
Mathematics Subject Classification: 35B40, 35L20, 35L70, 93D20

1. Introduction and preliminaries

Let $\mathcal{H} = \Omega \times (\tau_1, \tau_2) \times (0, \infty)$, in the present work, we consider the following Kirchhoff equation

$$\begin{cases} |u_t|^p u_{tt} - \Big(\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)}\Big)\Delta u(t) - \Delta u_{tt}(t) \\ \quad + \displaystyle\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho + \beta_1 |u_t(t)|^{m-2} u_t(t) \\ \quad + \displaystyle\int_{\tau_1}^{\tau_2} |\beta_2(s)| |u_t(t-s)|^{m-2} u_t(t-s) ds = u \ln |u|^k. \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad in \ \ \Omega \\ u_t(x,-t) = f_0(x,t), \quad in \ \ \Omega \times (0,\tau_2) \\ u(x,t) = 0, \quad in \ \ \partial\Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\zeta_0, \zeta_1, \sigma, \beta_1, k$ are positive constants, β_2 is a real number. $p \geq 0$ for $N = 1, 2$, and $0 \leq p \leq \frac{4}{N-2}$ for $N \geq 3$, and $m \geq 1$ for $N = 1, 2$, and $1 < m \leq \frac{N+2}{N-2}$ for $N \geq 3$, $\tau_1 < \tau_2$ are non-negative constants and $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function , h is a positive function.

Physically, the relationship between the stress and strain history in the beam inspired by Boltzmann theory called viscoelastic damping term, where the kernel of the term of memory is the function h (See [8, 13, 15–22, 25]. In [3], Balakrishnan and Taylor they proposed a new model of damping called it the Balakrishnan-Taylor damping , as it relates to the span problem and the plate equation. For more depth, here are some papers that focused on the study of this damping [3, 6, 10, 16, 30].

The effect of the delay often appear in many applications and piratical problems and turns a lot of systems into different problems worth studying. Recently, the stability and the asymptotic behavior of evolution systems with time delay especially the distributed delay effect has been studied by many authors [1, 9, 12–14, 24, 25, 27–29, 31, 32, 34]. The great importance of the logarithmic nonlinearity in physics is that they appear in several issues and theories, including symmetry, cosmology, quantum mechanics, as well as nuclear physics. It is also used in many applications such as optical, nuclear and even subterranean physics. Many researchers also touched on this type of problem in several different issues, where the global existence of solutions, stability and blow-up of solutions were studied. For more information, the reader is referred to [4, 5, 7].

Based on all of the above, the combination of these terms of damping (Memory term, Balakrishnan-Taylor damping, logarithmic nonlinearity and the distributed delay terms) in one particular problem with the addition of the distributed delay term ($\int_{\tau_1}^{\tau_2} |\beta_2(s)||u_t(t-s)|^{m-2}u_t(t-s)ds$) we believe that it constitutes a new problem worthy of study and research, different from the above that we will try to shed light on.

Our paper is divided into several sections: in the next section we lay down the hypotheses, concepts and lemmas we need. In the section 3, we state the global existence and in the section 4, we prove the general decay of solutions. Finally, we put a general conclusion.

For studying our problem, in this section we will need some materials.
Firstly, introducing the following hypothesis for k, β_2 and h:
(A1) $h : \mathbb{R}_+ \to \mathbb{R}_+$ are non-increasing C^1 functions satisfying

$$h(t) > 0, \ \ \zeta_0 - \int_0^\infty h(\varrho)d\varrho = l > 0. \tag{1.2}$$

(A2)$\exists \vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing C^1 function, and a constant $1 \leq \theta < \frac{3}{2}$ satisfying

$$\vartheta(t)h^\theta(t) + h'(t) \leq 0, \ \ \forall t \geq 0. \tag{1.3}$$

(A3) $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\beta_2(s)|ds < \beta_1. \tag{1.4}$$

(A4) The constant k in (1.1) is satisfying

$$0 < k < k_0 := 2l\pi e^3. \tag{1.5}$$

Let us introduce

$$(h \circ \psi)(t) := \int_\Omega \int_0^t h(t-\varrho)|\psi(t)-\psi(\varrho)|^2 d\varrho dx.$$

and

$$M(t) := \Big(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\Big).$$

Lemma 1. *(Sobolev-Poincare inequality [2]). Let* $2 \le q < \infty (n = 1, 2)$ *or* $2 \le q < \frac{2n}{n-2} (n \ge 3)$*. Then,* $\exists c_* = c(\Omega, q) > 0$ *such that*

$$\|u\|_q \le c_*\|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

As in [33], taking the following new variables

$$y(x,\rho,s,t) = u_t(x, t - s\rho),$$

which satisfy

$$\begin{cases} sy_t(x,\rho,s,t) + y_\rho(x,\rho,s,t) = 0, \\ y(x,0,s,t) = u_t(x,t). \end{cases} \tag{1.6}$$

So, problem (1.1) can be written as

$$\begin{cases} |u_t|^p u_{tt} - \Big(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)}\Big)\Delta u(t) \\ \quad + \displaystyle\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho - \Delta u_{tt}(t) + \beta_1|u_t(t)|^{m-2}u_t(t) \\ \quad + \displaystyle\int_{\tau_1}^{\tau_2} |\beta_2(s)||y(x,1,s,t)|^{m-2}y(x,1,s,t)ds = u\ln|u|^k. \\ sy_t(x,\rho,s,t) + y_\rho(x,\rho,s,t) = 0. \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \ in \ \Omega \\ y(x,\rho,s,0) = f_0(x,\rho s), \ \ in \ \Omega\times(0,1)\times(0,\tau_2) \\ u(x,t) = 0, \ \ in \ \partial\Omega\times(0,\infty), \end{cases} \tag{1.7}$$

where

$$(x,\rho,s,t) \in \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Now, we give the energy functional.

Lemma 2. *The energy functional E, defined by*

$$\begin{aligned} E(t) \ = \ & \frac{1}{p+2}\|u_t\|_{p+2}^{p+2} + \frac{1}{2}\Big(\zeta_0 - \int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2 \\ & + \frac{1}{2}\|\nabla u_t(t)\|_2^2 + \frac{\zeta_1}{4}\|\nabla u(t)\|_2^4 + \frac{1}{2}(h\circ\nabla u)(t) - \frac{1}{2}\int_\Omega u^2\ln|u|^k dx \\ & + \frac{k}{4}\|u(t)\|_2^2 + \frac{m-1}{m}\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|\|y(x,\rho,s,t)\|_m^m ds d\rho, \end{aligned} \tag{1.8}$$

satisfies

$$E'(t) \ \le \ -\eta_0\|u_t(t)\|_m^m + \frac{1}{2}(h'\circ\nabla u)(t)$$

$$-\frac{1}{2}h(t)\|\nabla u(t)\|_2^2-\frac{\sigma}{4}\Big(\frac{d}{dt}\Big\{\|\nabla u(t)\|_2^2\Big\}\Big)^2\le 0, \tag{1.9}$$

where $\eta_0=\beta_1-\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds>0.$

Proof. Taking the inner product of $(1.7)_1$ with u_t, then integrating over Ω, we find

$$\begin{aligned}
&(|u_t|^p u_{tt}(t),u_t(t))_{L^2(\Omega)}-(M(t)\Delta u(t),u_t(t))_{L^2(\Omega)}-(\Delta u_{tt}(t),u_t(t))_{L^2(\Omega)}\\
&+(\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho,u_t(t))_{L^2(\Omega)}+\beta_1(|u_t|^{m-2}u_t,u_t)_{L^2(\Omega)}\\
&+\int_{\tau_1}^{\tau_2}|\beta_2(s)|(|y(x,1,s,t)|^{m-2}y(x,1,s,t),u_t(t))_{L^2(\Omega)}ds\\
&-(ku\ln|u|,u_t(t))_{L^2(\Omega)}=0.
\end{aligned} \tag{1.10}$$

A calculation direct, gives

$$(|u_t|^p u_{tt}(t),u_t(t))_{L^2(\Omega)} = \frac{1}{p+2}\frac{d}{dt}\Big(\|u_t(t)\|_{p+2}^{p+2}\Big), \tag{1.11}$$

$$-(\Delta u_{tt}(t),u_t(t))_{L^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\Big(\|\nabla u_t(t)\|_2^2\Big), \tag{1.12}$$

by integration by parts, we find

$$\begin{aligned}
&-(M(t)\Delta u(t),u_t(t))_{L^2(\Omega)}\\
=&-(\Big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\Big)\Delta u(t),u_t(t))_{L^2(\Omega)}\\
=&\Big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\Big)\int_\Omega \nabla u(t).\nabla u_t(t)dx\\
=&\Big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u(t),\nabla u_t(t))_{L^2(\Omega)}\Big)\frac{d}{dt}\Big\{\int_\Omega|\nabla u(t)|^2dx\Big\}\\
=&\frac{d}{dt}\Big\{\frac{1}{2}\Big(\zeta_0+\frac{\zeta_1}{2}\|\nabla u\|_2^2\Big)\|\nabla u(t)\|_2^2\Big\}+\frac{\sigma}{4}\frac{d}{dt}\Big\{\|\nabla u(t)\|_2^2\Big\}^2,
\end{aligned} \tag{1.13}$$

and we have

$$\begin{aligned}
&(\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho,u_t(t))_{L^2(\Omega)}\\
=&\int_0^t h(t-\varrho)(\Delta u(\varrho),u_t(t))_{L^2(\Omega)}d\varrho\\
=&-\int_0^t h(t-\varrho)\Big[\int_\Omega \nabla u(x,\varrho)\nabla u(x,t)dx\Big]d\varrho,
\end{aligned} \tag{1.14}$$

and

$$-\nabla u(x,\varrho).\nabla u(x,t)=\frac{1}{2}\frac{d}{dt}\Big\{|\nabla u(x,\varrho)-\nabla u(x,t)(t)|^2\Big\}-\frac{1}{2}\frac{d}{dt}\Big\{|\nabla u(x,t)|^2\Big\}, \tag{1.15}$$

then

$$
\begin{aligned}
& -\int_0^t h(t-\varrho)(\nabla u(\varrho), \nabla u_t(t))_{L^2(\Omega)} d\varrho \\
= & -\int_0^t h(t-\varrho) \int_\Omega \Big[\frac{1}{2}\frac{d}{dt}\Big\{|\nabla u(x,\varrho) - \nabla u(x,t)|^2\Big\}\Big] dxds. \\
& -\int_0^t h(t-\varrho) \int_\Omega \Big[\frac{1}{2}\frac{d}{dt}\Big\{|\nabla u(x,t)|^2\Big\}\Big] dxd\varrho \\
= & \frac{1}{2}\int_0^t h(t-\varrho)\Big[\frac{d}{dt}\Big\{\int_\Omega |\nabla u(x,t) - \nabla u(x,\varrho)|^2 dx\Big\}\Big] d\varrho \\
& -\frac{1}{2}\int_0^t h(t-\varrho)\Big[\frac{d}{dt}\Big\{\|\nabla u(x,t)\|_2^2\Big\}\Big] dxd\varrho. \qquad (1.16)
\end{aligned}
$$

We use (1.2), we obtain

$$
\begin{aligned}
& \frac{1}{2}\int_0^t h(t-\varrho)\Big[\frac{d}{dt}\Big\{\int_\Omega |\nabla u(x,t) - \nabla u(x,\varrho)|^2 dx\Big\}\Big] d\varrho \\
= & \frac{1}{2}\frac{d}{dt}\Big\{\int_0^t h(t-\varrho)\Big[\int_\Omega |\nabla u(x,t) - \nabla u(x,\varrho)|^2 dx\Big]\Big\} d\varrho \\
& -\frac{1}{2}\int_0^t h'(t-\varrho)\Big[\int_\Omega |\nabla u(x,t) - \nabla u(x,\varrho)|^2 dx\Big] d\varrho \\
= & \frac{1}{2}\frac{d}{dt}(h \circ \nabla u)(t) - \frac{1}{2}(h' \circ \nabla u)(t), \qquad (1.17)
\end{aligned}
$$

and

$$
\begin{aligned}
& -\frac{1}{2}\int_0^t h(t-\varrho)\Big[\frac{d}{dt}\Big\{\|\nabla u(t)\|_2^2\Big\}\Big] dxd\varrho \\
= & -\frac{1}{2}\Big(\int_0^t h(t-\varrho)d\varrho\Big)\Big(\frac{d}{dt}\Big\{\|\nabla u(t)\|_2^2\Big\}\Big) dx \\
= & -\frac{1}{2}\Big(\int_0^t h(\varrho)d\varrho\Big)\Big(\frac{d}{dt}\Big\{\|\nabla u(t)\|_2^2\Big\}\Big) dx \qquad (1.18) \\
= & -\frac{1}{2}\frac{d}{dt}\Big\{\Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2\Big\} + \frac{1}{2}h(t)\|\nabla u(t)\|_2^2.
\end{aligned}
$$

By substitying (1.17) and (1.18) into (1.16), gives

$$
\begin{aligned}
& \Big(\int_0^t h(t-\varrho)\Delta u(\varrho)d\varrho, u_t(t)\Big)_{L^2(\Omega)} \\
= & \frac{d}{dt}\Big\{\frac{1}{2}(h \circ \nabla u)(t) - \frac{1}{2}\Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2\Big\} \\
& -\frac{1}{2}(h' \circ \nabla u)(t) + \frac{1}{2}h(t)\|\nabla u(t)\|_2^2, \qquad (1.19)
\end{aligned}
$$

and we have

$$
-(ku\ln|u|, u_t(t))_{L^2(\Omega)} = \frac{d}{dt}\Big\{\frac{k}{4}\|u(t)\|_2^2 - \frac{1}{2}\int_\Omega u^2 \ln|u|^k dx\Big\}. \qquad (1.20)
$$

Now, multiplying the Eq $(1.7)_2$ by $-y|\beta_2(s)|$, and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, and using $(1.6)_2$, we get

$$\begin{aligned}
&\frac{d}{dt}\quad \frac{m-1}{m}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.|y(x,\rho,s,t)|^m dsd\rho dx\\
=&\quad -(m-1)\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\beta_2(s)|.|y|^{m-1}y_\rho dsd\rho dx\\
=&\quad -\frac{m-1}{m}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\beta_2(\varrho)|\frac{d}{d\rho}|y(x,\rho,s,t)|^m dsd\rho dx\\
=&\quad \frac{m-1}{m}\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big(|y(x,0,s,t)|^m-|y(x,1,s,t)|^m\Big)dsdx\\
=&\quad \frac{m-1}{m}\Big(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\Big)\int_\Omega|u_t(t)|^m dx\\
&\quad -\frac{m-1}{m}\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)|.|y\,(x,1,s,t)\,|^m dsdx\\
=&\quad \frac{m-1}{m}\Big(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\Big)\|u_t(t)\|_m^m\\
&\quad -\frac{m-1}{m}\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y\,(x,1,s,t)\,\|_m^m ds, \qquad (1.21)
\end{aligned}$$

and by Young's inequality, we have

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big(|y(x,1,s,t)|^{m-2}y(x,1,s,t),u_t(t)\Big)_{L^2(\Omega)}ds \qquad (1.22)\\
\le&\ \frac{1}{m}\Big(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\Big)\|u_t(t)\|_m^m+\frac{m-1}{m}\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y\,(x,1,s,t)\,\|_m^m ds.
\end{aligned}$$

By replacement (1.11)–(1.13) and (1.19)–(1.22) into (1.10), we find (1.8) and (1.9). Hence, by (1.4), we get the function E is a non-increasing. This completes of the proof. □

Lemma 3. *Let $\varepsilon_0\in(0,1)$. Then, $\exists d_{\varepsilon_0}>0$ such that*

$$\nu|\ln\nu|\le\nu^2+d_{\varepsilon_0}\nu^{1-\varepsilon_0},\quad \forall\nu>0. \qquad (1.23)$$

Lemma 4. *[11, 23] (Logarithmic Sobolev inequality) Let $u\in H_0^1(\Omega)$ and $a>0$. Then*

$$\int_\Omega u^2\ln|u|dx\le\frac{1}{2}\|u\|_2^2\ln\|u\|_2^2+\frac{a^2}{2\pi}\|\nabla u\|_2^2-(1+\ln a)\|u\|_2^2. \qquad (1.24)$$

Theorem 1. *Suppose that (1.2)–(1.5) are satisfied. Then, for any $u_0,u_1\in H_0^1(\Omega)\cap L^2(\Omega)$, and $f_0\in L^2(\Omega,(0,1),(\tau_1,\tau_2))$, there exists a weak solution u of problem (1.7) such that*

$$\begin{aligned}
&u\in C(]0,T[,H_0^1(\Omega))\cap C^1(]0,T[,L^2(\Omega)),\\
&u_t\in C(]0,T[,H_0^1(\Omega))\cap L^2(]0,T[,L^2(\Omega,(0,1),(\tau_1,\tau_2))).
\end{aligned}$$

2. Global existence

In this section, under smallness condition the global existence result is proved. Introducing the following functionals

$$
\begin{aligned}
J(u) \;=\; & \frac{1}{2}\Big(\zeta_0 - \int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\nabla u_t(t)\|_2^2 - \frac{1}{2}\int_\Omega u^2 \ln |u|^k dx \\
& + \frac{1}{2}(h \circ \nabla u)(t) + \frac{\zeta_1}{4}\|\nabla u(t)\|_2^4 + \frac{k}{4}\|u(t)\|_2^2 \\
& + \frac{m-1}{m}\int_0^1 \int_{\tau_1}^{\tau_2} s|\beta_2(s)|\|y(x,\rho,s,t)\|_m^m ds d\rho,
\end{aligned} \tag{2.1}
$$

and

$$
\begin{aligned}
I(u) \;=\; & \Big(\zeta_0 - \int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 \\
& -3\int_\Omega u^2 \ln |u|^k dx + (h \circ \nabla u)(t).
\end{aligned} \tag{2.2}
$$

Hence

$$
E(t) \;=\; \frac{1}{p+2}\|u_t\|_{p+2}^{p+2} + J(u), \tag{2.3}
$$

and

$$
\begin{aligned}
J(u) \;=\; & \frac{1}{6}\Big\{\Big(\zeta_0 - \int_0^t h(\varrho)d\varrho\Big)\|\nabla u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 + (h \circ \nabla u)(t)\Big\} \\
& + \frac{\zeta_1}{4}\|\nabla u(t)\|_2^4 + \frac{m-1}{m}\int_0^1 \int_{\tau_1}^{\tau_2} s|\beta_2(s)|\|y(x,\rho,s,t)\|_m^m ds d\rho \\
& + \frac{k}{4}\|u(t)\|_2^2 + \frac{1}{3}I(u).
\end{aligned} \tag{2.4}
$$

First, suppose that

$$
e^{-\frac{3}{2}} < a < \sqrt{\frac{2l\pi}{k}}, \tag{2.5}
$$

and we define

$$
C_1 := k(\frac{3}{2} + \ln a), \quad \omega_* := e^{\frac{2C_1 - k}{k}}, \tag{2.6}
$$

the condition (2.5) makes $C_1 > 0$.

Lemma 5. *The following inequalities hold*

$$
k\int_\Omega u^2 \ln |u| dx \le k c_p^3 \|\nabla u\|_2^3, \quad \forall u \in H_0^1(\Omega), \tag{2.7}
$$

and

$$
\Big(\int_\Omega |u|^3 dx\Big)^{1/3} \le c_p \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega), \tag{2.8}
$$

where c_p is the smallest embedding constant of $H_0^1(\Omega)$ in $L^\infty(\Omega)$.

Proof. Let

$$\Omega_1 = \{u \in \Omega : |u| > 1\}, \ \ \Omega_2 = \{u \in \Omega : |u| \leq 1\}.$$

So, by (2.8) and (1.23) , gives

$$\begin{aligned} k\int_{\Omega} u^2 \ln|u|dx &= k\int_{\Omega_1} u^2 \ln|u|dx + k\int_{\Omega_2} u^2 \ln|u|dx \\ &\leq k\int_{\Omega_1} u^2 \ln|u|dx \leq k\int_{\Omega_1} |u|^3 dx \leq k\int_{\Omega} |u|^3 dx \\ &\leq kc_p^3 \|\nabla u\|_2^3. \end{aligned}$$

□

Lemma 6. *Suppose that (1.2), (2.5) and* $u_0, u_1 \in H_0^1(\Omega) \cap L^2(\Omega)$, *and* $f_0 \in L^2(\Omega, (0,1))$ *hold,* $\|u\|_2 < \omega_*$ *and*

$$0 < E(0) < \min\Big\{E_1, \frac{l^2(2\pi l - ka^2)}{36k^2\pi c_p^6}, \frac{\pi l e^2}{2}\Big\}. \tag{2.9}$$

Then,

$$I(u) \geq 0, \ \ \forall t \in [0, T). \tag{2.10}$$

Proof. By (1.2), (2.3) and (1.24), we have

$$\begin{aligned} E(t) &\geq J(u(t)) \\ &\geq \frac{l}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\nabla u_t(t)\|_2^2 - \frac{1}{2}\int_{\Omega} u^2 \ln|u|^k dx \\ &\quad + \frac{1}{2}(h \circ \nabla u)(t) + \frac{\zeta_1}{4}\|\nabla u(t)\|_2^4 + \frac{k}{4}\|u(t)\|_2^2 \\ &\quad + \frac{m-1}{m}\int_0^1 \int_{\tau_1}^{\tau_2} s|\beta_2(s)|\|y(x,\rho,s,t)\|_m^m ds d\rho \\ &\geq \frac{1}{2}\Big(l - \frac{ka^2}{2\pi}\Big)\|\nabla u(t)\|_2^2 + \frac{k}{2}\Big(\frac{3}{2} + \ln a - \frac{1}{2}\ln\|u\|_2^2\Big)\|u\|_2^2. \end{aligned} \tag{2.11}$$

Then, by (2.5) and (2.6), gives

$$E(t) \geq \mathcal{F}(\omega) := \frac{1}{2}C_1\omega^2 - \frac{k}{4}\omega^2 \ln \omega^2, \tag{2.12}$$

where $\omega = \|u\|_2$. After studying the function $\mathcal{F}$, we conclude that exist $\omega_* > 0$ in which $\mathcal{F}$ is increasing on $(0, \omega_*)$, and deceasing on (ω_*, ∞). Furthermore, we have $\lim_{\omega\to+\infty} = -\infty$. and from him

$$\max_{0<\omega<+\infty} \mathcal{F}(\omega) = \frac{1}{2}C_1\omega_*^2 - \frac{k}{4}\omega_*^2 \ln \omega_*^2 := E_1. \tag{2.13}$$

Suppose $\|u\|_2 < \omega_*$ is not true in $[0, T)$. Hence, by continuity of $u(t)$, it follows that there exists $0 < t_0 < T$ satisfying $\|u(x, t_0)\|_2 = \omega_*$. From (2.12) give
$E(t_0) \geq \mathcal{F}(\omega_*) = E_1$. But this is impossible because $E(t) \leq E(0) < E_1, \ \ \forall t \geq 0$.
Now, from (2.11), we get

$$E(t) \geq J(u(t)) \geq \frac{1}{2}\Big(l - \frac{ka^2}{2\pi}\Big)\|\nabla u(t)\|_2^2 > 0,$$

which implies

$$\|\nabla u(t)\|_2^2 \le \Big(\frac{4\pi}{2\pi l - ka^2}\Big)E(t) \le \Big(\frac{4\pi}{2\pi l - ka^2}\Big)E(0). \tag{2.14}$$

Hence, by (2.2), (2.7) and (2.14), we get

$$\begin{aligned} I(t) &\ge l\|\nabla u(t)\|_2^2 - 3\int_\Omega u^2 \ln|u|^k dx \\ &\ge \Big\{l - 3kc_p^3\Big(\frac{4\pi}{2\pi l - ka^2}E(0)\Big)^{1/2}\Big\}\|\nabla u(t)\|_2^2. \end{aligned} \tag{2.15}$$

According (2.5), (2.9) and (2.15), we obtain

$$I(t) \ge 0. \tag{2.16}$$

This completes the proofs. □

3. General decay

In this section, we state and prove the asymptotic behavior of the system (1.7). For this goal, we set

$$\Psi(t) := \frac{1}{p+1}\int_\Omega u(t)|u_t|^p u_t(t)dx + \frac{\sigma}{4}\|\nabla u(t)\|_2^4 + \int_\Omega \nabla u(t)\nabla u_t(t)dx, \tag{3.1}$$

and

$$\Phi(t) := \int_\Omega\Big(\Delta u_t - \frac{1}{p+1}|u_t|^p u_t\Big)\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho dx, \tag{3.2}$$

and

$$\Theta(t) := \int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho. \tag{3.3}$$

Lemma 7. *The functional $\Psi(t)$ defined in (3.1) satisfies, for any $\varepsilon > 0$*

$$\begin{aligned} \Psi'(t) \le\ & \frac{1}{p+1}\|u_t\|_{p+2}^{p+2} - (\frac{l}{2} - \varepsilon(c_1+c_2))\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 \\ &+ c(h\circ\nabla u)(t) + \|\nabla u_t\|_2^2 + k\int_\Omega u^2\ln|u|dx \\ &+ c(\varepsilon)\Big(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2}|\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds\Big). \end{aligned} \tag{3.4}$$

Proof. A differentiation of (3.1) and using $(1.7)_1$, gives

$$\Psi'(t) = \frac{1}{p+1}\|u_t\|_{p+2}^{p+2} + \int_\Omega |u_t|^p u_{tt} u dx + \sigma\|\nabla u\|_2^2\int_\Omega \nabla u_t\nabla u dx$$

$$
\begin{aligned}
&+\int_\Omega \nabla u(t)\nabla u_{tt}(t)dx+\|\nabla u_t\|_2^2\\
=\;& \frac{1}{p+1}\|u_t\|_{p+2}^{p+2}-\zeta_0\|\nabla u\|_2^2-\zeta_1\|\nabla u\|_2^4\underbrace{-\beta_1\int_\Omega |u_t|^{m-2}u_t u dx}_{J_1}\\
&+\underbrace{\int_\Omega \nabla u(t)\int_0^t h(t-\varrho)\nabla u(\varrho)d\varrho dx}_{J_2}+\|\nabla u_t\|_2^2+k\int_\Omega u^2\ln|u|dx\\
&\underbrace{-\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)|^{m-2}y(x,1,s,t).udsdx}_{J_3}. \qquad (3.5)
\end{aligned}
$$

We estimate the last 3 terms of the RHS of (3.5). Applying Hölder's, Sobolev-Poincare and Young's inequalities, (1.2) and (1.8), we find

$$
\begin{aligned}
J_1 &\le \varepsilon\beta_1^m\|u\|_m^m+c(\varepsilon)\|u_t\|_m^m\\
&\le \varepsilon\beta_1^m c_p^m\|\nabla u\|_2^m+c(\varepsilon)\|u_t\|_m^m\\
&\le \varepsilon\beta_1^m c_p^m\Big(\frac{E(0)}{l}\Big)^{(m-2)/2}\|\nabla u\|_2^2+c(\varepsilon)\|u_t\|_m^m\\
&\le \varepsilon c_1\|\nabla u\|_2^2+c(\varepsilon)\|u_t\|_m^m, \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
J_2 &\le (\zeta_0-l)\|\nabla u\|_2^2+\frac{\varepsilon_4}{2}\|\nabla u\|_2^2+\frac{c}{\varepsilon_4}(h\circ\nabla u)(t)\\
&\le (\zeta_0-l+\frac{\varepsilon_4}{2})\|\nabla u\|_2^2+\frac{c}{\varepsilon_4}(h\circ\nabla u)(t),
\end{aligned}
$$

by letting $\varepsilon_4=l$, we get

$$
J_2 \le (\zeta_0-\frac{l}{2})\|\nabla u\|_2^2+c(h\circ\nabla u)(t). \qquad (3.7)
$$

Similarly to J_1, we have

$$
J_3 \le \varepsilon c_2\|\nabla u\|_2^2+c(\varepsilon)\int_{\tau_1}^{\tau_2}|\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds. \qquad (3.8)
$$

Combining (3.6)–(3.8) and (3.5), we get

$$
\begin{aligned}
\Psi'(t) \le\;& \frac{1}{p+1}\|u_t\|_{p+2}^{p+2}-(\frac{l}{2}-\varepsilon(c_1+c_2))\|\nabla u\|_2^2-\zeta_1\|\nabla u\|_2^4\\
&+k\int_\Omega u^2\ln|u|dx+\|\nabla u_t\|_2^2+c(h\circ\nabla u)(t)\\
&+c(\varepsilon)\Big(\|u_t\|_m^m+\int_{\tau_1}^{\tau_2}|\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds\Big).
\end{aligned}
$$

□

Lemma 8. *The functional* $\Phi(t)$ *defined in (3.37) satisfies, for any* $\delta > 0$

$$\begin{aligned}
\Phi'(t) \leq & -\frac{1}{p+1}\Big(\int_0^t h(\varrho)d\varrho\Big)\|u_t\|_{p+2}^{p+2} + \delta\Big(\zeta_0 + 2(\zeta_0 - l)^2 + 1\Big)\|\nabla u\|_2^2 \\
& +\zeta_1\delta\|\nabla u\|_2^4 + \delta\frac{\sigma E(0)}{l}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2 \\
& +\Big(2c(\delta + \frac{1}{\delta})\Big)(h \circ \nabla u)(t) + c(\varepsilon_0, \delta)(h \circ \nabla u)^{1/(1+\varepsilon_0)}(t) \\
& +c(\delta)\Big(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2} |\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds\Big) \\
& +\Big(\delta_1(1 + c(E(0))^p) - \int_0^t h(\varrho)d\varrho\Big)\|\nabla u_t\|_2^2 \\
& -\Big(\frac{h(0)c_p^2}{4\delta_1} + c(\delta_1)\Big)(h' \circ \nabla u)(t). \qquad (3.9)
\end{aligned}$$

Proof. A differentiation of (3.37) and using $(1.7)_1$, gives

$$\begin{aligned}
\Phi'(t) = & \int_\Omega \Big(\Delta u_{tt} - u_{tt}|u_t|^p\Big)\int_0^t h(t-\varrho)(u(t) - u(\varrho))d\varrho dx \\
& + \int_\Omega \Big(\Delta u_t - \frac{1}{p+1}|u_t|^p u_t\Big)\int_0^t h'(t-\varrho)(u(t) - u(\varrho))d\varrho dx \\
& -\frac{1}{p+1}\Big(\int_0^t h(\varrho)d\varrho\Big)\|u_t\|_{p+2}^{p+2} - \Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u_t\|_2^2 \\
= & \underbrace{-(\zeta_0 + \zeta_1\|\nabla u\|_2^2)\int_\Omega \nabla u \int_0^t h(t-\varrho)(\nabla u(t) - \nabla u(\varrho))d\varrho dx}_{J_1} \\
& \underbrace{-\sigma\int_\Omega \nabla u \nabla u_t dx. \int_\Omega \nabla u \int_0^t h(t-\varrho)(\nabla u(t) - \nabla u(\varrho))d\varrho dx}_{J_2} \\
& \underbrace{+\int_\Omega \Big(\int_0^t h(t-\varrho)\nabla u(\varrho)d\varrho\Big).\Big(\int_0^t h(t-\varrho)(\nabla u(t) - \nabla u(\varrho))d\varrho\Big)dx}_{J_3} \\
& \underbrace{+\beta_1\int_\Omega |u_t|^{m-2}u_t\Big(\int_0^t h(t-\varrho)(u(t) - u(\varrho))d\varrho\Big)dx}_{J_4} \\
& \underbrace{+\int_\Omega\int_{\tau_1}^{\tau_2} |\beta_2(s)||y(x,1,s,t)|^{m-2}y(x,1,s,t).\Big(\int_0^t h(t-\varrho)(u(t) - u(\varrho))d\varrho\Big)ds dx}_{J_5} \\
& \underbrace{-\frac{1}{p+1}\int_\Omega |u_t|^p u_t \int_0^t h'(t-\varrho)(u(t) - u(\varrho))d\varrho dx}_{J_6}
\end{aligned}$$

$$\underbrace{-\int_\Omega \nabla u_t \int_0^t h'(t-\varrho)(\nabla u(t)-\nabla u(\varrho))d\varrho dx}_{J_7}$$

$$\underbrace{-k\int_\Omega u\ln|u|.\Big(\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big)dsdx}_{J_8}$$

$$-\frac{1}{p+1}\Big(\int_0^t h(\varrho)d\varrho\Big)\|u_t\|_{p+2}^{p+2}-\Big(\int_0^t h(\varrho)d\varrho\Big)\|\nabla u_t\|_2^2. \tag{3.10}$$

We estimate the terms of the RHS of (3.10). Applying Hölder's, Sobolev-Poincare and Young's inequalities, (1.2) and (1.8), we find

$$\begin{aligned}|J_1| &\le (\zeta_0+\zeta_1\|\nabla u\|_2^2)\Big(\delta\|\nabla u\|_2^2+\frac{(\zeta_0-l)}{4\delta}(h\circ\nabla u)(t)\Big)\\ &\le \delta\zeta_0\|\nabla u\|_2^2+\delta\zeta_1\|\nabla u\|_2^4+\Big(\frac{\zeta_0(\zeta_0-l)}{4\delta}+\frac{\zeta_1(\zeta_0-l)E(0)}{4l\delta}\Big)(h\circ\nabla u)(t),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}J_2 &\le \delta\sigma\Big(\int_\Omega \nabla u\nabla u_t dx\Big)^2\|\nabla u\|_2^2+\frac{\sigma(\zeta_0-l)}{4\delta}(h\circ\nabla u)(t)\\ &\le \delta\frac{\sigma E(0)}{l}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2+\frac{\sigma(\zeta_0-l)}{4\delta}(h\circ\nabla u)(t),\end{aligned} \tag{3.12}$$

$$\begin{aligned}|J_3| &\le \delta\int_\Omega\Big(\int_0^t h(t-\varrho)(|\nabla u(t)-\nabla u(\varrho)|-\nabla|u(t)|)d\varrho\Big)^2dx\\ &\quad+\frac{1}{4\delta}\int_\Omega\Big(\int_0^t h(t-\varrho)(\nabla u(t)-\nabla u(\varrho))d\varrho\Big)^2dx\\ &\le 2\delta(\zeta_0-l)^2\|\nabla u\|_2^2+c\Big(\delta+\frac{1}{\delta}\Big)(h\circ\nabla u)(t),\end{aligned} \tag{3.13}$$

$$\begin{aligned}|J_4| &\le c(\delta)\|u_t\|_m^m+\delta\beta_1^m\int_\Omega\Big(\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big)^m dx\\ &\le c(\delta)\|u_t\|_m^m+\delta\beta_1^m(\zeta_0-l)^{m-1}c_p^m\int_0^t h(t-\varrho)\|\nabla u(t)-\nabla u(\varrho)\|_2^m d\varrho\\ &\le c(\delta)\|u_t\|_m^m+\delta\Big(\beta_1^m(\zeta_0-l)^{m-1}c_p^m(\frac{E(0)}{l})^{(m-2)/2}\Big)(h\circ\nabla u)(t)\\ &\le c(\delta)\|u_t\|_m^m+\delta c_3(h\circ\nabla u)(t),\end{aligned} \tag{3.14}$$

Similarly, we have

$$|J_5| \le c(\delta)\int_{\tau_1}^{\tau_2}|\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds+\delta c_4(h\circ\nabla u)(t). \tag{3.15}$$

By exploiting the Sobolev embedding, we have

$$\begin{aligned}|J_6| &\le \frac{1}{p+1}\Big(\delta_1\|u_t\|_{2(p+1)}^{2(p+1)} + \frac{c}{\delta_1}\int_\Omega\int_0^t(-h'(t-\varrho))|u(t)-u(\varrho)|^2 d\varrho dx\Big)\\ &\le c\delta_1(E(0))^p\|\nabla u_t\|_2^2 - c(\delta_1)(h'\circ\nabla u)(t),\end{aligned} \tag{3.16}$$

and

$$|J_7| \le \delta_1\|\nabla u_t\|_2^2 - \frac{h(0)}{4\delta_1}(h'\circ\nabla u)(t). \tag{3.17}$$

Applying (1.24) for $\nu = |u|$, using the embedding of $H_0^1(\Omega)$ in $L^\infty(\Omega)$ and performing the same calulactions as before,weget, for any $\varepsilon_5 > 0$ and any $\varepsilon_0 \in (0,1)$,

$$\begin{aligned}|J_8| &\le k\int_\Omega(u^2 + d_{\varepsilon_0}u^{1-\varepsilon_0}).\Big|\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big|dx\\ &\le c\int_\Omega u^2\Big|\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big|dx + \varepsilon_5\int_\Omega u^2 dx\\ &\quad + c(\varepsilon_0,\varepsilon_5)\int_\Omega\Big|\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big|^{\frac{2}{1+\varepsilon_0}}dx\\ &\le c\varepsilon_5\|\nabla u\|_2^2 + \frac{c}{\varepsilon_5}\int_\Omega\Big|\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big|^2 dx\\ &\quad + c(\varepsilon_0,\varepsilon_5)\int_\Omega\Big|\int_0^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\Big|^{\frac{2}{1+\varepsilon_0}}dx,\end{aligned}$$

then, by letting $\varepsilon_5 = \frac{\delta}{c}$ and using Hölder's inequality, we get

$$|J_8| \le \delta\|\nabla u\|_2^2 + \frac{c}{\delta}(h\circ\nabla u)(t) + c(\varepsilon_0,\delta)(h\circ\nabla u)^{1/(1+\varepsilon_0)}(t). \tag{3.18}$$

According (3.11)–(3.18) and (3.10), we get (3.9). □

Lemma 9. *The functional* $\Theta(t)$ *defined in (3.3) satisfies*

$$\begin{aligned}\Theta'(t) &\le -\eta_1\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho\\ &\quad -\eta_1\int_{\tau_1}^{\tau_2}|\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds + \beta_1\|u_t(t)\|_m^m.\end{aligned} \tag{3.19}$$

Proof. By differentiating of $\Theta(t)$, and using $(1.7)_2$, gives

$$\begin{aligned}\Theta'(t) &= -m\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\beta_2(s)|.|y|^{m-1}y_\rho(x,\rho,s,t)\,ds d\rho dx\\ &= -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\beta_2(s)|.|y(x,\rho,s,t)|^m ds d\rho dx\\ &\quad -\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big[e^{-s}|y(x,1,s,t)|^m - |y(x,0,s,t)|^m\Big]ds dx.\end{aligned}$$

Applying $y(x,0,s,t) = u_t(x,t)$, and $e^{-s} \leq e^{-s\rho} \leq 1$, for any $0 < \rho < 1$, and we set $\eta_1 = e^{-\tau_2}$, we obtain

$$\begin{aligned}\Theta'(t) \;\leq\; & -\eta_1 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\beta_2(s)|.|y(x,\rho,s,t)|^m ds d\rho dx \\ & -\eta_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\beta_2(s)||y(x,1,s,t)|^m ds dx + \int_{\tau_1}^{\tau_2} |\beta_2(s)| ds \int_\Omega |u_t|^m(t) dx,\end{aligned}$$

using (1.4), we find (3.19).

□

Now, we introduce the functional

$$\mathcal{G}(t) \;:=\; E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \Phi(t) + \varepsilon_3 \Theta(t), \tag{3.20}$$

for some positive constants $\varepsilon_i, i = 1,2,3$ to be determined.

Lemma 10. *There exist $\mu_1, \mu_2 > 0$, such that*

$$\mu_1 E(t) \leq \mathcal{G}(t) \leq \mu_2 E(t). \tag{3.21}$$

Proof. From (3.1), by using Hölder inequality (for $q_1 = \frac{p+2}{p+1}$, $q_2 = p+2$), Young's inequality (for $\kappa > 0$), and embedding $H_0^1 \hookrightarrow L^{2(p+1)}$, $\|u_t\|_{p+2}^p \leq [(p+2)E(0)]^{\frac{p}{(p+2)}}$, we find

$$\begin{aligned}\Psi(t) \;\leq\; & \frac{1}{p+1}\|u_t(t)\|_{p+2}^{p+1}\|u(t)\|_{p+2} + \frac{1}{2}\Big(\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2\Big) \\ \leq\; & \frac{\kappa}{2(p+1)^2}\|u_t(t)\|_{p+2}^{2(p+1)} + \frac{1}{2\kappa}\|u(t)\|_{p+2}^2 \\ & + \frac{1}{2}\Big(\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2\Big) \\ \leq\; & \frac{\kappa}{2(p+1)^2}\|u_t(t)\|_{p+2}^{p}\|u_t(t)\|_{p+2}^{p+2} + \frac{1}{2\kappa}\|u(t)\|_{p+2}^2 \\ & + \frac{1}{2}\Big(\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2\Big) \\ \leq\; & \frac{\kappa[(p+2)E(0)]^{\frac{p}{(p+2)}}}{2(p+1)^2}\|u_t(t)\|_{p+2}^{p+2} + c(\kappa)\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\nabla u_t(t)\|_2^2,\end{aligned} \tag{3.22}$$

where $c(\kappa) = (\frac{C_0}{2\kappa} + \frac{1}{2})$, with C_0 comes from the embedding $H_0^1 \hookrightarrow L^{2(p+1)}$.
According to the relations (3.22), (3.37)–(3.39) and by using Hölder, Young's and poincare inequalities, we get

$$\begin{aligned}|\mathcal{G}(t) - E(t)| \;\leq\; & \varepsilon_1\Big(\frac{\kappa[(p+2)E(0)]^{\frac{p}{(p+2)}}}{2(p+1)^2}\|u_t(t)\|_{p+2}^{p+2} + c(\kappa)\|\nabla u(t)\|_2^2\Big) \\ & + (\varepsilon_1 + \varepsilon_2)\frac{1}{2}\|\nabla u_t(t)\|_2^2 + \varepsilon_1 \frac{\sigma}{4}\|\nabla u(t)\|_2^4\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon_2 \frac{1}{2(p+1)} \|u_t(t)\|_{2(p+1)}^{2(p+1)} + \varepsilon_2 \frac{(\zeta_0 - l)c(p)}{2}(h \circ \nabla u)(t) \\
&+\varepsilon_3 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\rho s} |\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho, \qquad (3.23)
\end{aligned}
$$

where $c(p) = (\frac{c_p}{p+1} + 1)$.
By $e^{-\rho s} < 1$, we find

$$
|\mathcal{G}(t) - E(t)| \leq C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \kappa) E(t). \qquad (3.24)
$$

We pick $\kappa = 1$ and choosing $\varepsilon_1, \varepsilon_2$ and ε_3 sufficiently small, then (3.21) follows from (3.24).

□

Lemma 11. *Suppose that (1.2)–(1.5),(2.5) and (2.9) hold, let $\varepsilon_0 \in (0,1)$. There exist $k_1, k_2, t_0 > 0$ satisfying*

$$
\mathcal{G}'(t) \leq -k_1 E(t) + k_2 (h \circ \nabla u)(t) + c(\varepsilon_0)(h \circ \nabla u)^{1/(1+\varepsilon_0)}(t), \quad t \geq t_0. \qquad (3.25)
$$

Proof. Since the function h is a positive and continuous, for all $t_0 > 0$, we have

$$
\int_0^t h(\varrho) d\varrho \geq \int_0^{t_0} h(\varrho) d\varrho := h_0, \ \ \forall t \geq t_0.
$$

By using the relation (1.9) with the results of Lemmas 7, 8 and 9,
then, for $t \geq t_0$, we get

$$
\begin{aligned}
\mathcal{G}'(t) &:= E'(t) + \varepsilon_1 \Psi'(t) + \varepsilon_2 \Phi'(t) + \varepsilon_3 \Theta'(t) \\
&\leq \Big\{ \frac{1}{p+1}(\varepsilon_1 - \varepsilon_2 h_0) \Big\} \|u_t\|_{p+2}^{p+2} + \Big\{ \varepsilon_2 \zeta_1 \delta - \varepsilon_1 \zeta_1 \Big\} \|\nabla u\|_2^4 \\
&\quad + \Big\{ \varepsilon_2 \delta (\zeta_0 + 2(\zeta_0 - l)^2 + 1) - \varepsilon_1 (\frac{l}{2} - \varepsilon(c_1 + c_2)) \Big\} \|\nabla u\|_2^2 \\
&\quad + \Big\{ \varepsilon_1 + \varepsilon_2 [\delta_1 (1 + c(E(0))^p) - h_0] \Big\} \|\nabla u_t\|_2^2 \\
&\quad + \Big\{ \varepsilon_2 \delta \frac{\sigma E(0)}{l} - \frac{\sigma}{4} \Big\} \Big(\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 \Big)^2 \\
&\quad + \Big\{ c\varepsilon_1 + 2c\varepsilon_2 (\delta + \frac{1}{\delta}) \Big\} (h \circ \nabla u)(t) + c(\varepsilon_0, \delta)(h \circ \nabla u)^{1/(1+\varepsilon_0)}(t) \\
&\quad + \Big\{ \frac{1}{2} - \varepsilon_2 (\frac{h(0) c_p^2}{4\delta_1} + c(\delta_1)) \Big\} (h' \circ \nabla u)(t) \\
&\quad + \Big\{ \varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) + \varepsilon_3 \beta_1 - \eta_0 \Big\} \|u_t\|_m^m + k\varepsilon_1 \int_\Omega u^2 \ln |u| dx \\
&\quad + \Big\{ \varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) - \eta_1 \varepsilon_3 \Big\} \int_{\tau_1}^{\tau_2} |\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds \\
&\quad - \eta_1 \varepsilon_3 \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho. \qquad (3.26)
\end{aligned}
$$

Using (1.8) ,we obtain, for any $\gamma > 0$,

$$
\begin{aligned}
\mathcal{G}'(t) \;\leq\; & -\gamma E(t) + \frac{1}{p+1}\Big\{\varepsilon_1 - \varepsilon_2 h_0 + \frac{\gamma(p+1)}{p+2}\Big\}\|u_t\|_{p+2}^{p+2} \\
& + \Big\{\varepsilon_2\zeta_1\delta - \varepsilon_1\zeta_1 + \frac{\gamma\zeta_1}{4}\Big\}\|\nabla u\|_2^4 \\
& + \Big\{\varepsilon_2\delta(\zeta_0 + 2(\zeta_0 - l)^2 + 1) - \varepsilon_1(\frac{l}{2} - \varepsilon(c_1 + c_2)) + \frac{\gamma}{2}(\zeta_0 - h_0)\Big\}\|\nabla u\|_2^2 \\
& + \Big\{\varepsilon_1 + \varepsilon_2[\delta_1(1 + c(E(0))^p) - h_0] + \frac{\gamma}{2}\Big\}\|\nabla u_t\|_2^2 \\
& + \Big\{\varepsilon_2\delta\frac{\sigma E(0)}{l} - \sigma\Big\}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2 + \frac{k\gamma}{4}\|u\|_2^2 \\
& + \Big\{c\varepsilon_1 + 2c\varepsilon_2(\delta + \frac{1}{\delta}) + \frac{\gamma}{2}\Big\}(h\circ\nabla u)(t) + c(\varepsilon_0,\delta)(h\circ\nabla u)^{1/(1+\varepsilon_0)}(t) \\
& + \Big\{\frac{1}{2} - \varepsilon_2(\frac{h(0)c_p^2}{4\delta_1} + c(\delta_1))\Big\}(h'\circ\nabla u)(t) \\
& + \Big\{\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) + \varepsilon_3\beta_1 - \eta_0\Big\}\|u_t\|_m^m + k(\varepsilon_1 - \frac{\gamma}{2})\int_\Omega u^2 \ln|u|dx \\
& + \Big\{\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) - \eta_1\varepsilon_3\Big\}\int_{\tau_1}^{\tau_2} |\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds \\
& \Big\{-\eta_1\varepsilon_3 + \frac{\gamma(m-1)}{m}\Big\}\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(s)|\|y(x,\rho,s,t)\|_m^m ds d\rho. \qquad (3.27)
\end{aligned}
$$

Using the Logarithmic Sobolev inequality (1.24), we get

$$
\begin{aligned}
\mathcal{G}'(t) \;\leq\; & -\gamma E(t) + \frac{1}{p+1}\Big\{\varepsilon_1 - \varepsilon_2 h_0 + \frac{\gamma(p+1)}{p+2}\Big\}\|u_t\|_{p+2}^{p+2} \\
& + \Big\{\varepsilon_2\zeta_1\delta - \varepsilon_1\zeta_1 + \frac{\gamma\zeta_1}{4}\Big\}\|\nabla u\|_2^4 \\
& + \Big\{\varepsilon_2\delta(\zeta_0 + 2(\zeta_0 - l)^2 + 1) - \varepsilon_1(\frac{l}{2} - \varepsilon(c_1 + c_2)) \\
& \qquad + \frac{\gamma}{2}(\zeta_0 - h_0) + (\varepsilon_1 - \frac{\gamma}{2})\frac{kc_p a^2}{2\pi}\Big\}\|\nabla u\|_2^2 \\
& + \Big\{\varepsilon_1 + \varepsilon_2[\delta_1(1 + c(E(0))^p) - h_0] + \frac{\gamma}{2}\Big\}\|\nabla u_t\|_2^2 \\
& + \Big\{\varepsilon_2\delta\frac{\sigma E(0)}{l} - \sigma\Big\}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2 \\
& + \Big\{c\varepsilon_1 + 2c\varepsilon_2(\delta + \frac{1}{\delta}) + \frac{\gamma}{2}\Big\}(h\circ\nabla u)(t) + c(\varepsilon_0,\delta)(h\circ\nabla u)^{1/(1+\varepsilon_0)}(t) \\
& + \Big\{\frac{1}{2} - \varepsilon_2(\frac{h(0)c_p^2}{4\delta_1} + c(\delta_1))\Big\}(h'\circ\nabla u)(t) \\
& + \Big\{\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) + \varepsilon_3\beta_1 - \eta_0\Big\}\|u_t\|_m^m \\
& + \Big\{\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) - \eta_1\varepsilon_3\Big\}\int_{\tau_1}^{\tau_2} |\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds
\end{aligned}
$$

$$\Big\{-\eta_1\varepsilon_3+\frac{\gamma(m-1)}{m}\Big\}\int_0^1\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x,\rho,s,t)\|_m^m ds d\rho$$
$$-\frac{k}{2}\Big\{(\varepsilon_1-\frac{\gamma}{2})\Big(2(1+\ln a)-\ln\|u\|_2^2\Big)-\frac{\gamma}{2}\Big\}\|u\|_2^2. \tag{3.28}$$

Using (1.9), (2.1), (2.4), and (2.10), we find

$$\ln\|u\|_2^2\le\ln\Big(\frac{4}{k}J(t)\Big)\le\ln\Big(\frac{4}{k}E(t)\Big)\le\ln\Big(\frac{4}{k}E(0)\Big). \tag{3.29}$$

According (2.9) and (3.29), we have

$$2(1+\ln a)-\ln\|u\|_2^2>0.$$

Next, we carefully choose our constants.
Letting $\delta_1=\frac{h_0}{2(1+c(E(0))^p)}$, and we choose ε small enough such that

$$\frac{l}{2}-\varepsilon(c_1+c_2)>0.$$

Then, we pick δ small enough such that

$$\delta<\min\Big\{\frac{h_0(\frac{l}{2}-\varepsilon(c_1+c_2))}{2(\zeta_0+2(\zeta_0-l)^2+1)};\frac{h_0}{2}\Big\}.$$

For any fixed δ_1,δ and ε, we select $\varepsilon_1,\varepsilon_2$ and ε_3 so small satisfying

$$\frac{h_0}{2}\varepsilon_2<\varepsilon_1<h_0\varepsilon_2.$$

$$\mu_3:=-\varepsilon_2\delta(\zeta_0+2(\zeta_0-l)^2+1)+\varepsilon_1(\frac{l}{2}-\varepsilon(c_1+c_2))>0,$$

and

$$\begin{aligned}&\frac{1}{2}-\varepsilon_2(\frac{h(0)c_p^2}{4\delta_1}+c(\delta_1))>0,\\&\varepsilon_1c(\varepsilon)+\varepsilon_2c(\delta)+\varepsilon_3\beta_1-\eta_0<0,\\&\varepsilon_1c(\varepsilon)+\varepsilon_2c(\delta)-\eta_1\varepsilon_3<0,\\&\varepsilon_2\delta\frac{\sigma E(0)}{l}-\sigma<0.\end{aligned}$$

Finally, we choose γ,k small enough such that

$$\begin{aligned}&\varepsilon_1-\varepsilon_2h_0+\frac{\gamma(p+1)}{p+2}<0,\quad \varepsilon_1-\frac{\gamma}{2}>0,\\&\varepsilon_2\delta-\varepsilon_1+\frac{\gamma}{4}<0,\quad -\eta_1\varepsilon_3+\frac{\gamma(m-1)}{m}<0,\\&-\mu_3+\frac{\gamma}{2}(\zeta_0-h_0)+(\varepsilon_1-\frac{\gamma}{2})\frac{kc_pa^2}{2\pi}<0,\end{aligned}$$

$$\varepsilon_1 - \varepsilon_2[h_0 - \delta_1(1 + c(E(0))^p)] + \frac{\gamma}{2} < 0,$$
$$(\varepsilon_1 - \frac{\gamma}{2})\Big(2(1 + \ln a) - \ln \|u\|_2^2\Big) - \frac{\gamma}{2} > 0.$$

Therefore, (3.28) becomes, for positive constants $k_i, i = 1, 2$

$$\mathcal{G}'(t) \leq -k_1 E(t) + k_2(h \circ \nabla u)(t) + c(\varepsilon_0)(h \circ \nabla u)^{1/(1+\varepsilon_0)}(t), \quad \forall t \geq t_0.$$

□

Remark 1. *By* (1.2)*,* (2.3)*,* (2.4) *and* (2.10)*, we have*

$$E(t) \geq J(t) \geq \frac{l}{6}\|\nabla u(t)\|_2^2, \tag{3.30}$$

then, by (1.9)

$$\|\nabla u\|_2^2 \leq \frac{6}{l}E(0). \tag{3.31}$$

Hence, using (1.9) *and Young's inequality, gives*

$$\begin{aligned}
|E'(t)| &\leq -\eta_0\|u_t(t)\|_m^m + \frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2}h(t)\|\nabla u(t)\|_2^2 \\
&\quad -\frac{\sigma}{4}\Big(\frac{d}{dt}\Big\{\|\nabla u(t)\|_2^2\Big\}\Big)^2 \\
&\leq \frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2}h(t)\|\nabla u(t)\|_2^2 \\
&\leq \int_\Omega \int_0^t h'(t-\varrho)(\|\nabla u(t)\|_2^2 + \|\nabla u(\varrho)\|_2^2)d\varrho dx - \frac{1}{2}h(t)\|\nabla u(t)\|_2^2 \\
&\leq \frac{6}{l}\Big(2h(0) - \frac{3}{2}h(t)\Big)E(0) \\
&\leq cE(0).
\end{aligned} \tag{3.32}$$

Corollary 1. *Suppose that (1.2)–(1.5) hold, let u is a solution of (1.7). Then*

$$\vartheta(t)(h \circ \nabla u)(t) \leq c\Big(-E'(t)\Big)^{1/(2\theta-1)}, \tag{3.33}$$

and, for all $\varepsilon_0 \in (0, 1)$

$$\vartheta(t)(h \circ \nabla u)^{1/(1+\varepsilon_0)}(t) \leq c(\varepsilon_0)\Big(-E'(t)\Big)^{1/(2\theta-1)(1+\varepsilon_0)}. \tag{3.34}$$

Theorem 2. *Suppose that (1.2)–(1.5) are satisfied, let* (u_0, u_1, f_0) *satisfy (2.9),* $\varsigma \in (0, 2\theta - 1)$*. Then, for k small enough,* $\exists \Gamma > 0$ *such that the solution of (1.7) satisfies*

$$E(t) \leq \Gamma\Big(1 + \int_{t_0}^t \vartheta^{2\theta-1+\varsigma}(\varrho)d\varrho\Big)^{-1/(2\theta-2+\varsigma)}, \quad \forall t \geq t_0. \tag{3.35}$$

Hence, if there exist $\varsigma_1 \in (0, 2\theta - 1)$ *and* $t_0 > 0$ *such that*

$$\int_{t_0}^{\infty} \left(1 + \int_{t_0}^{t} \vartheta^{2\theta-1+\varsigma_1}(\varrho)d\varrho\right)^{-1/(2\theta-2+\varsigma_1)} dt < \infty. \tag{3.36}$$

Then, for all $r \in (0, \theta)$ *and* $t_0 > 0$, $\exists \Gamma > 0$ *such that the solution of (1.7) satisfies*

$$E(t) \leq \Gamma\left(1 + \int_{t_0}^{t} \vartheta^{\theta+r}(\varrho)d\varrho\right)^{-1/(\theta-1+r)}, \quad \forall t \geq t_0. \tag{3.37}$$

Proof. Multiplying (3.25) by $\vartheta(t)$, using Corollary 1 and (3.32), we find

$$\begin{aligned}
\vartheta(t)\mathcal{G}'(t) &\leq -k_1\vartheta(t)E(t) + c(-E'(t))^{1/(2\theta-1)} + c(-E'(t))^{1/(2\theta-1)(1+\varepsilon_0)} \\
&\leq -k_1\vartheta(t)E(t) + c(-E'(t))^{\varepsilon_0/(2\theta-1)(1+\varepsilon_0)}(-E'(t))^{1/(2\theta-1)(1+\varepsilon_0)} \\
&\quad +c(-E'(t))^{1/(2\theta-1)(1+\varepsilon_0)} \\
&\leq -k_1\vartheta(t)E(t) + c(-E'(t))^{1/(2\theta-1)(1+\varepsilon_0)}, \quad \forall t \geq t_0.
\end{aligned} \tag{3.38}$$

Multiply (3.38) by $\vartheta^{\eta}(t)E^{\eta}(t)$, with $\eta = (2\theta-1)(1+\varepsilon_0) - 1$, and using the fact that $\vartheta' \leq 0$ to get

$$\vartheta^{\eta+1}(t)E^{\eta}(t)\mathcal{G}'(t) \leq -k_1\vartheta^{\eta+1}(t)E^{\eta+1}(t) + c(\vartheta E)^{\eta}(t)(-E'(t))^{1/(\eta+1)}.$$

By using Young's inequality, with $q = \eta + 1$ and $q^* = (\eta+1)/\eta$, gives, for all $\varepsilon' > 0$,

$$\begin{aligned}
\vartheta^{\eta+1}(t)E^{\eta}(t)\mathcal{G}'(t) &\leq -k_1\vartheta^{\eta+1}(t)E^{\eta+1}(t) + c(\varepsilon'\vartheta^{\eta+1}(t)E^{\eta+1}(t) - c(\varepsilon')E'(t)) \\
&= -(k_1 - c\varepsilon')\vartheta^{\eta+1}(t)E^{\eta+1}(t) - c(\varepsilon')E'(t), \quad \forall t \geq t_0.
\end{aligned}$$

We select $0 < \varepsilon' < \frac{k_1}{c}$ and recalling $\vartheta' \leq 0$ and $E' \leq 0$, to find, for $k_3 = k_1 - \varepsilon' c$

$$(\vartheta^{\eta+1}E^{\eta}\mathcal{G})'(t) \leq \vartheta^{\eta+1}(t)E^{\eta}(t)\mathcal{G}'(t) \leq -k_3\vartheta^{\eta+1}(t)E^{\eta+1}(t) - cE'(t), \quad \forall t \geq t_0,$$

which implies

$$(\vartheta^{\eta+1}E^{\eta}\mathcal{G} + cE)'(t) \leq -k_3\vartheta^{\eta+1}(t)E^{\eta+1}(t), \quad \forall t \geq t_0.$$

Let

$$\mathcal{Y}(t) := (\vartheta^{\eta+1}E^{\eta}\mathcal{G} + cE)(t) \sim E(t), \tag{3.39}$$

we obtain

$$\mathcal{Y}'(t) \leq -c\vartheta^{\eta+1}(t)\mathcal{Y}^{\eta+1}(t) = -c\vartheta^{1/(2\theta-1)(1+\varepsilon_0)}(t)\mathcal{Y}^{1/(2\theta-1)(1+\varepsilon_0)}(t), \quad \forall t \geq t_0. \tag{3.40}$$

Integrating of (3.40) over (t_0, t) and using (3.39), we get (3.35) with $\varsigma = (2\theta-1)\varepsilon_0$.

Remark 2. *Using (3.35) and (3.36), we can easily show that*

$$\int_0^{\infty} E(t)dt < \infty. \tag{3.41}$$

At this point, to prove (3.37), let the functional

$$\varphi(t) := \int_0^t (\|\nabla u(t) - \nabla u(t-\varrho)\|_2^2) d\varrho, \tag{3.42}$$

by using (3.31), (3.35), (3.36) and (3.41), we find

$$\begin{aligned} \varphi(t) &\leq 2\int_0^t (\|\nabla u(t)\| + \|\nabla u(t-\varrho)\|_2^2) d\varrho \\ &\leq \frac{12}{l}\int_0^t (E(t) + E(t-\varrho)) d\varrho \\ &\leq \frac{24}{l}\int_0^t E(\varrho) d\varrho \leq \frac{24}{l}\int_0^\infty E(\varrho) d\varrho < \infty. \end{aligned} \tag{3.43}$$

Hence

$$\sup_{t>0} \varphi^{1-(1/\theta)}(t) < \infty. \tag{3.44}$$

Suppose that $\varphi(t) > 0$. Then, since ϑ is non-increasing, we get

$$\vartheta(t)(h \circ \nabla u)(t) \leq \frac{\varphi(t)}{\varphi(t)} \int_0^t (\vartheta^\theta(\varrho) h^\theta(\varrho))^{1/\theta} (\|\nabla u(t) - \nabla u(t-\varrho)\|_2^2) d\varrho,$$

by Jensen's inequality to obtain

$$\vartheta(t)(h \circ \nabla u)(t) \leq \varphi(t)\Big(\frac{1}{\varphi(t)} \int_0^t \vartheta^\theta(\varrho) h^\theta(\varrho) (\|\nabla u(t) - \nabla u(t-\varrho)\|_2^2) d\varrho\Big)^{1/\theta}.$$

Hence, by (1.3) and (3.44) we find

$$\begin{aligned} \vartheta(t)(h \circ \nabla u)(t) &\leq \varphi^{1-(1/\theta)}(t)\Big(\vartheta^{\theta-1}(0) \int_0^t \vartheta(\varrho) h^\theta(\varrho) (\|\nabla u(t) - \nabla u(t-\varrho)\|_2^2) d\varrho\Big)^{1/\theta} \\ &\leq c(-h' \circ \nabla u)^{1/\theta}(t). \end{aligned}$$

From (1.9), we have

$$\vartheta(t)(h \circ \nabla u)(t) \leq c(-E'(t))^{1/\theta}(t). \tag{3.45}$$

Since ϑ is non-increasing function, we get

$$\begin{aligned} \vartheta(t)(h \circ \nabla u)^{1/(1+\varepsilon_0)}(t) &\leq \Big(\vartheta^{\varepsilon_0}(t)\vartheta(t)(h \circ \nabla u)(t)\Big)^{1/(1+\varepsilon_0)} \\ &\leq \Big(\vartheta^{\varepsilon_0}(0)\vartheta(t)(h \circ \nabla u)(t)\Big)^{1/(1+\varepsilon_0)} \\ &\leq c(\vartheta(t)(h \circ \nabla u)(t))^{1/(1+\varepsilon_0)} \\ &\leq c(-E'(t))^{1/(\theta(1+\varepsilon_0))}(t). \end{aligned} \tag{3.46}$$

If $\varphi(t) = 0$, then $\varrho \to \nabla u(\varrho)$ is a constant function on $[0, t]$. Therefore

$$(h \circ \nabla u)(t) = 0,$$

and hence (3.45) and (3.46) hold.
At this point, multiplying (3.25) by $\vartheta(t)$ and we use (3.32), (3.45) and (3.46) to obtain, for any $t \geq t_0$ (as for (3.38))

$$\vartheta(t)\mathcal{G}'(t) \leq -k_1\vartheta(t)E(t) + c(-E'(t))^{1/(2\theta-1)(1+\varepsilon_0)}, \quad \forall t \geq t_0. \tag{3.47}$$

Inequality (3.32) with $2\theta-1$ replaced by θ is exactely (3.47). Then, the proof of (3.37) can be completed as for the one of (3.35) (by taking $\eta = \theta(1+\varepsilon_0) - 1$ and $\varsigma = \theta\varepsilon_0$). The proof is complete. □

4. Conclusions

The purpose of this work was to study the global existence of the solutions for a nonlinear viscoelastic Kirchhoff-type equation with a logarithmic nonlinearity, Balakrishnan-Taylor damping, dispersion and distributed delay terms, and by the energy method we prove an explicit and general decay rate result under suitable hypothesis. This type of problem is frequently found in some mathematical models in applied sciences.

In the next work, we will try to using the same method with same problem. But in added of other damping terms.

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Conflict of interest

All authors declare no conflict of interest.

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