



Research article

Soliton solutions and periodic solutions for two models arises in mathematical physics

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Abstract: We aimed in this paper to acquire the periodic wave solutions and soliton solutions and other solutions such as kink-wave solutions for the cubic nonlinear Schrödinger equation with repulsive delta potential (δ -NLSE) and complex coupled Higgs field equation via two mathematical methods Jacobi elliptic function method and generalized Kudryashov method. Some of these solutions are degenerated to solitary wave solutions and periodic wave solutions in the limit case. We also gave the meaning of these solutions physically and the numerical simulation by some figures.

Keywords: cubic nonlinear Schrödinger's equation with repulsive delta potential equation; complex coupled Higgs field equation; Jacobi elliptic function method; generalized Kudryashov method; periodic solutions; soliton solutions

Mathematics Subject Classification: 35Q51, 37K40, 35G20, 35Q60

1. Introduction

Soliton solutions for nonlinear evolution equations (NEEs) play a significant role in soliton theory, physical sciences and engineering. So the finding of exact solutions for NEEs is very important for mathematicians and physicists. Many powerful methods have been suggested to obtain the soliton solutions, such as homogeneous balance technique [1–6], extended tanh method [7, 8], soliton-like, multisoliton-like [9], truncated expansion technique [10, 11], sech-function method [12], Jacobi elliptic function method [13, 14] and Kudryashov's method [15, 16], generalized(G'/G)-expansion method [17, 18], rational generalized(G'/G)-expansion method [19], $\exp(\Phi(\eta))$ -expansion method [20, 21], the fractional sub-equation method [22] and optical solutions [23–26].

We implemented the Jacobi elliptic function method [27–29] and generalized Kudryashov's method to obtain the Jacobi elliptic function solutions, periodic solutions and soliton solutions for the (δ -NLSE)

and complex coupled Higgs field.

Some exact solutions are based on the generalized Hirota bilinear [30] method.

We introduce a brief summary for the two methods which have been applied Jacobi elliptic function method and generalized Kudryashov's method in Section 2. Jacobi elliptic function, periodic solutions and soliton solutions for the cubic nonlinear Schrödinger's equation with repulsive delta potential (δ -NLSE) and complex coupled Higgs field equation using two methods were derived analytically and numerically in Sections 3 and 4. We explained the results in a view of physical perspective in Section 5. Finally, some results were deduced in Section 6.

2. Materials and methods

Suppose that we have a given nonlinear partial differential equations (NPDEs) as follows

$$H_\gamma(x, u, u_1, u_2, \dots, u_n) = 0, \quad \gamma = 1, 2, \dots, m. \quad (2.1)$$

For $m = 2$ such that $u^1 = u, u^2 = v$, and $n = 2$ such as $x^1 = t, x^2 = x$,

Step 1. By using a wave transformation $u(x, t) = u(lx - nt + f), v(x, t) = v(lx - nt + f)$, n is the wave speed and the function.

Transformed Eq (2.1) into an ordinary differential equation (ODE)

$$\begin{aligned} H_1(u, v, u_\zeta, v_\zeta, u_{\zeta\zeta}, v_{\zeta\zeta}, \dots) &= 0, \\ H_2(u, v, u_\zeta, v_\zeta, u_{\zeta\zeta}, v_{\zeta\zeta}, \dots) &= 0, \end{aligned} \quad (2.2)$$

where $\zeta = lx - nt + d$.

Step 2. Suppose the solutions is as follows

$$\begin{aligned} u(\zeta) &= \sum_{i=0}^N a_i \varphi(\zeta)^i, \\ v(\zeta) &= \sum_{i=0}^M b_i \varphi(\zeta)^i, \end{aligned} \quad (2.3)$$

where N , and M determined by balancing the highest order derivatives with the nonlinear term, the parameters a_i and b_i to be determined.

2.1. The Jacobi elliptic function method

The variable $\varphi(\zeta)$ is the solutions of the anzata [31]

$$\varphi'(\zeta) = \sqrt{d_0 + d_1\varphi(\zeta) + d_2\varphi^2(\zeta) + d_3\varphi^3(\zeta) + d_4\varphi^4(\zeta)} \quad (2.4)$$

where d_0, d_1, d_2, d_3 and d_4 are real parameters.

We substitute ansatz Eqs (2.4) and (2.3) into Eq (2.2) with the aid of computerized symbolic computation, we obtain a set of algebraic equations for $d_0, d_1, d_2, d_3, d_4, c, k, a_i$ and b_i .

Inserting the solutions of Eq (2.4) and setting $\zeta = lx \pm nt + f$, then we obtain the exact traveling wave solutions of Eq (2.2).

The Jacobi elliptic functions $sn\zeta = sn(\zeta | m)$, $cn\zeta = cn(\zeta | m)$ and $dn\zeta = dn(\zeta | m)$, m ($0 < m < 1$) is the modulus of the elliptic function, are double periodic possess properties of triangular functions, namely

$$\begin{aligned} sn^2(\zeta) + cn^2(\zeta) &= 1, \\ dn^2(\zeta) + m^2 sn^2(\zeta) &= 1, \\ (cn(\zeta))' &= -sn(\zeta)dn(\zeta), \\ (sn(\zeta))' &= cn(\zeta)dn(\zeta), \\ (dn(\zeta))' &= -m^2 sn(\zeta)cn(\zeta). \end{aligned} \quad (2.5)$$

When $m \rightarrow 0$, the Jacobi elliptic function degenerate to

$$\begin{aligned} sn(\zeta) &\rightarrow \sin(\zeta), & cn(\zeta) &\rightarrow \cos(\zeta), & dn(\zeta) &\rightarrow 1, \\ cs(\zeta) &\rightarrow \cot(\zeta), & ds(\zeta) &\rightarrow csc(\zeta). \end{aligned} \quad (2.6)$$

While $m \rightarrow 1$, the Jacobi elliptic function degenerate to

$$\begin{aligned} sn(\zeta) &\rightarrow \tanh(\zeta), & cn(\zeta) &\rightarrow \operatorname{sech}(\zeta), & dn(\zeta) &\rightarrow \operatorname{sech}(\zeta), \\ cs(\zeta) &\rightarrow \operatorname{csch}(\zeta), & ds(\zeta) &\rightarrow \operatorname{csch}(\zeta), \end{aligned} \quad (2.7)$$

where $\zeta = lx - nt + f$.

2.2. The generalized Kudryashov's method

For $m = 1$, we suppose the solutions of Eq (2.1) can be expressed in the following rational form [32]

$$u(\zeta) = \frac{\sum_{i=0}^N a_i \varphi(\zeta)^i}{\sum_{i=0}^M b_i \varphi(\zeta)^i} \quad (2.8)$$

where N , and M determined by balancing the highest order derivatives with the nonlinear term, the parameters a_i and b_i to be determined.

The variable $\varphi(\zeta)$ is the solutions which satisfy the first order Bernoulli differential equation

$$\varphi'(\zeta) = \varphi(\zeta)^2 - \varphi(\zeta), \quad (2.9)$$

which has the solution

$$\varphi(\zeta) = \frac{1}{1 + c_1 e^{lx-nt+f}}. \quad (2.10)$$

Inserting the solutions of Eq (2.9) into Eqs (2.8) and (2.1), then we obtain the exact traveling wave solutions such as soliton solutions of Eq (2.1).

3. δ -NLSE equation

3.1. Jacobi elliptic function method

Consider the δ -NLSE equation [33]

$$iu_{tt} + \frac{1}{2} u_{xx} - \alpha \delta u(x, t) - \nu |u|^2 u = 0, \quad i = \sqrt{-1}. \quad (3.1)$$

Where $\alpha, \nu \neq 0$ are real constants, δ is the Dirac measure [34]. The δ potential is called repulsive for $\alpha > 0$ and attractive for $\alpha < 0$. The δ -NLSE equation is a simpler model which describes the resonant nonlinear propagation of light through optical waves [34].

To find the Jacobi elliptic solutions for δ -NLSE equation via Jacobi elliptic function method, we start by assuming a solution of the form [35]

$$u(x, t) = U(\zeta) e^{i(kx + \omega t)}. \quad (3.2)$$

Where $\zeta = lx - nt + f$, l, n represent the wave number and wave frequency of the light pulse, respectively, and $U(\zeta)$ is the cnoidal wave or periodic wave or solitary wave solutions.

We transform the (3.1) into ODE wave transformation using (3.2)

$$\nu U^3 + \alpha \delta U + \frac{n^2}{2l^2} U + U\omega - \frac{1}{2} U'' l^2 = 0. \quad (3.3)$$

Balancing U'' with U^3 , we get $N = 1$. Then, the Jacobi elliptic function solutions for Eq (3.1) as follows:

$$U(\zeta) = a_0 + a_1 \varphi(\zeta) \quad (3.4)$$

where $a_i, i = 0, 1$ are constants which are to be determined later and φ satisfies (2.4) which has the solutions in Table 1.2 in [36].

Inserting (3.4) along (2.4) into (3.3) and collecting the terms with the same order of $\varphi(\zeta)$, and setting it to zero we obtain the Jacobi elliptic function solutions for δ -NLSE equation for $d_0 = 1 - m^2, d_1 = d_3 = 0, d_2 = 2m^2 - 1$ and $d_4 = -m^2$.

$$u(x, t) = \pm \gamma m cn \left(\frac{-2m^2 nt + 2fm^2 + \sqrt{(2m^2 - 1)\sigma x + nt - f}}{2m^2 - 1}, m \right) e^{i(kx + \omega t)}, \quad (3.5)$$

where

$$\sigma = \left(\alpha \delta + \sqrt{\alpha^2 \delta^2 + 2m^2 n^2 + 2\alpha \delta \omega - n^2 + \omega^2} + \omega \right),$$

$$\gamma = \sqrt{-\frac{\sigma}{\nu(2m^2 - 1)}}.$$

When $m \rightarrow 1$, we obtain the soliton solutions for δ -NLSE equation

$$u(x, t) = \pm \gamma m \operatorname{sech} \left(\frac{-2m^2 nt + 2fm^2 + \sqrt{(2m^2 - 1)\sigma x + nt - f}}{2m^2 - 1} \right) e^{i(kx + \omega t)}, \quad (3.6)$$

For $d_0 = 1, d_1 = d_3 = 0, d_2 = -(m^2 + 1)$ and $d_4 = m^2$.

$$\begin{aligned} u(x, t) &= a_1 sn(lx - nt + f, m) e^{i(kx + \omega t)}, \\ u(x, t) &= a_1 cd(lx - nt + f, m) e^{i(kx + \omega t)}, \end{aligned} \quad (3.7)$$

where $k = \frac{n}{l}$. When $m \rightarrow 0$, we obtain the periodic solutions for δ -NLSE equation

$$\begin{aligned} u(x, t) &= a_1 \sin(lx - nt + f) e^{i(kx + \omega t)}, \\ u(x, t) &= a_1 \cos(lx - nt + f) e^{i(kx + \omega t)}. \end{aligned} \quad (3.8)$$

When $m \rightarrow 1$, we obtain the soliton solutions for δ -NLSE equation

$$u(x, t) = a_1 \tanh(lx - nt + f) e^{i(kx + \omega t)}. \quad (3.9)$$

And for $d_0 = m^2, d_1 = d_3 = 0, d_2 = -(m^2 + 1)$ and $d_4 = 1$.

$$\begin{aligned} u(x, t) &= a_1 ns(lx - nt + f, m) e^{i(kx + \omega t)}, \\ u(x, t) &= a_1 dc(lx - nt + f, m) e^{i(kx + \omega t)}. \end{aligned} \quad (3.10)$$

When $m \rightarrow 0$, we obtain the periodic solutions for δ -NLSE equation

$$\begin{aligned} u(x, t) &= a_1 csc(lx - nt + f) e^{i(kx + \omega t)}, \\ u(x, t) &= a_1 sec(lx - nt + f) e^{i(kx + \omega t)}. \end{aligned} \quad (3.11)$$

When $m \rightarrow 1$, we obtain the soliton solutions for δ -NLSE equation

$$u(x, t) = a_1 \coth(lx - nt + f) e^{i(kx + \omega t)}. \quad (3.12)$$

3.2. Generalized Kudryashov's method

In this section we will find the soliton solutions for δ -NLSE equation via generalized Kudryashov's method.

Suppose the solution for Eq (3.1) in the rational form (2.8). Balancing U''' with U^3 , in (3.3) we get $N = M + 1$, we choose $N = 2, M = 1$ Then, we can obtain the soliton solutions for Eq (3.1) according to (2.8) as follows:

$$U(\zeta) = \frac{a_0 + a_1 \varphi(\zeta) + a_2 \varphi(\zeta)^2}{b_0 + b_1 \varphi(\zeta)} \quad (3.13)$$

where $a_i, b_j, i = 0, 1, 2, j = 0, 1$ are constants to be determined and φ satisfies (2.9) which has the solutions (2.10).

Inserting (3.13) along (2.9) into (3.3) and collecting the terms with the same order of $\varphi(\zeta)$, we obtain the family of soliton solutions for δ -NLSE equation as follows:

Case 1.

$$u(x, t) = -\frac{a_1 e^{-lx + nt - f}}{c_1 b_1} e^{i(kx + \omega t)}, \quad (3.14)$$

Case 2.

$$u(x, t) = -\frac{a_1 e^{lx-nt+f} c_1}{b_1} e^{i(kx+\omega t)}, \quad (3.15)$$

Case 3.

$$u(x, t) = \frac{a_0 (e^{lx-nt+f} c_1 b_0 - b_0 - b_1)}{(e^{lx-nt+f} c_1 b_0 + b_0 + b_1) b_0} e^{i(kx+\omega t)}, \quad (3.16)$$

Case 4.

$$u(x, t) = -2 \frac{a_1 e^{lx-nt+f} c_1}{b_1 (e^{2lx-2nt+2f} c_1^2 - 1)} e^{i(kx+\omega t)}, \quad (3.17)$$

Case 5.

$$u(x, t) = -\frac{(e^{lx-nt+f} c_1 - 1) a_2}{2b_1 (1 + e^{lx-nt+f} c_1)} e^{i(kx+\omega t)}, \quad (3.18)$$

Case 6.

$$u(x, t) = -\frac{a_2 (e^{2lx-2nt+2f} c_1^2 + 1)}{b_1 (e^{2lx-2nt+2f} c_1^2 - 1)} e^{i(kx+\omega t)}, \quad (3.19)$$

where $k = \frac{n}{l}$.

3.3. Physical discussions

In this section, we give the physical meanings of the solutions obtained for δ -NLSE equation (3.1), for (3.5), (3.7) and (3.10) which are Jacobi elliptic pattern solutions, which are the solutions in terms of the Jacobi elliptic functions or cnoidal solutions.

It is interesting to point out that the delicate balance between the nonlinearity effect of $|u|^2 u$ and the dissipative effect of u_{xx} gives rise to solitons, that after full interaction with others, the solitons come back retaining their identities with the same speed and shape. The δ -NLSE equation (3.1) has solitary wave solutions that have exponentially decaying wings. If two solitons of the δ -NLSE equation (3.1) collide, the solitons just pass through each other and emerge unchanged.

We showed the time growth of these Jacobi elliptic solutions by some figures to describe the dynamical properties in 3D graphs and 2D graphs in Figure 1. The solutions (3.8) and (3.11), which are periodic solutions, displayed in 3D graphs and 2D graphs in Figure 2. The solution (3.9), which is kink-soliton solution which has a permanent profile displayed in 2D graph and 3D graph, solutions in (3.12) and (3.14)–(3.19), which soliton-like solutions which is occurred from the balance between nonlinearity U^3 and dispersion U'' , displayed in Figure 3. All figures in 2D for the space evolution $-10 < x < 10$ and figures in 3D for the time evolution $0 < t < 5$ for Jacobi elliptic function, kink-soliton solutions and soliton-like, space evolution for periodic solutions in the interval $-5 < x < 5$ and $-20 < x < 20$.

We note that the wave speed n acts a vital role in a physical structure for solutions for δ -NLSE equation.

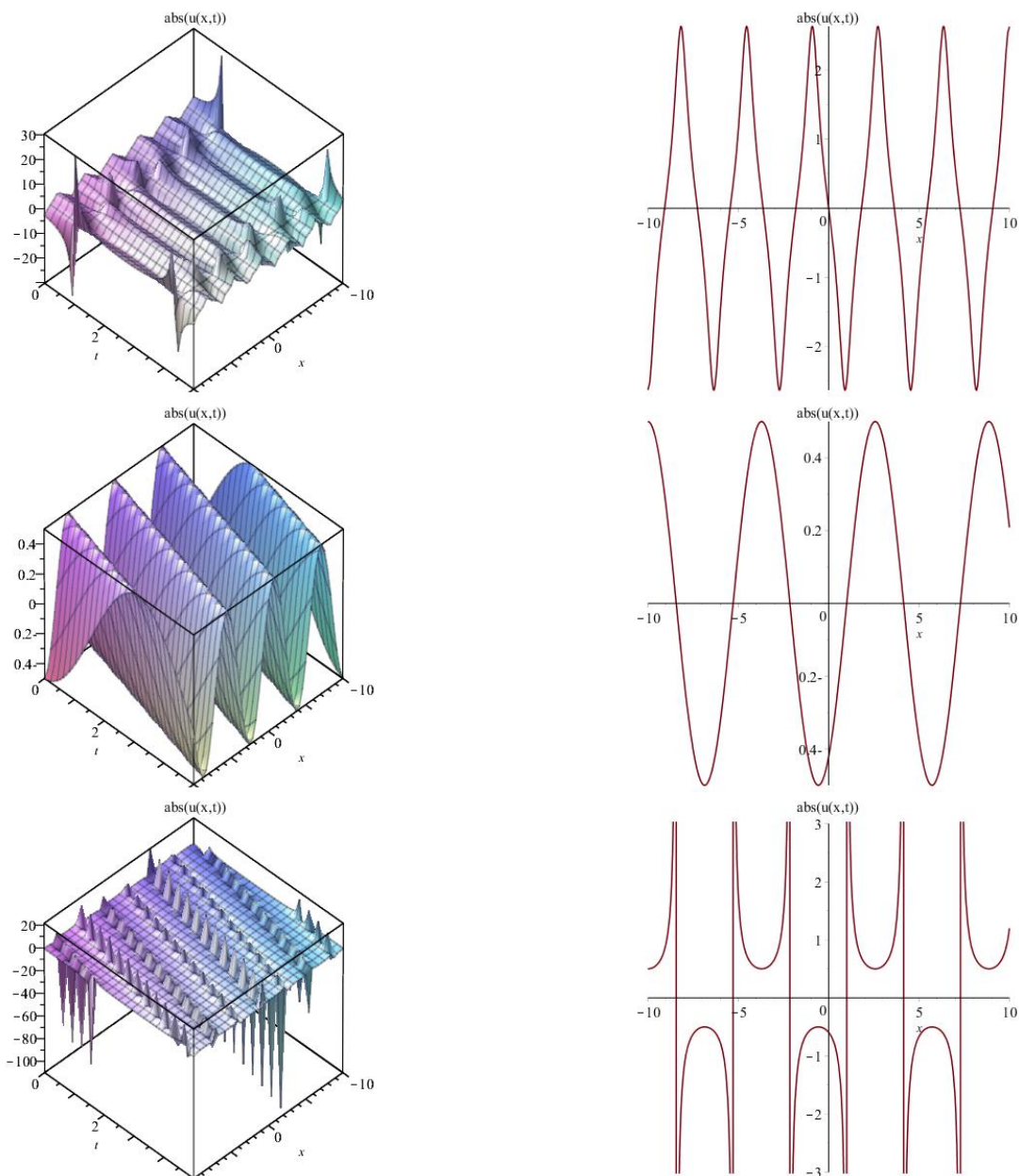


Figure 1. The cnoidal wave solutions of (3.5), (3.7) and (3.10).

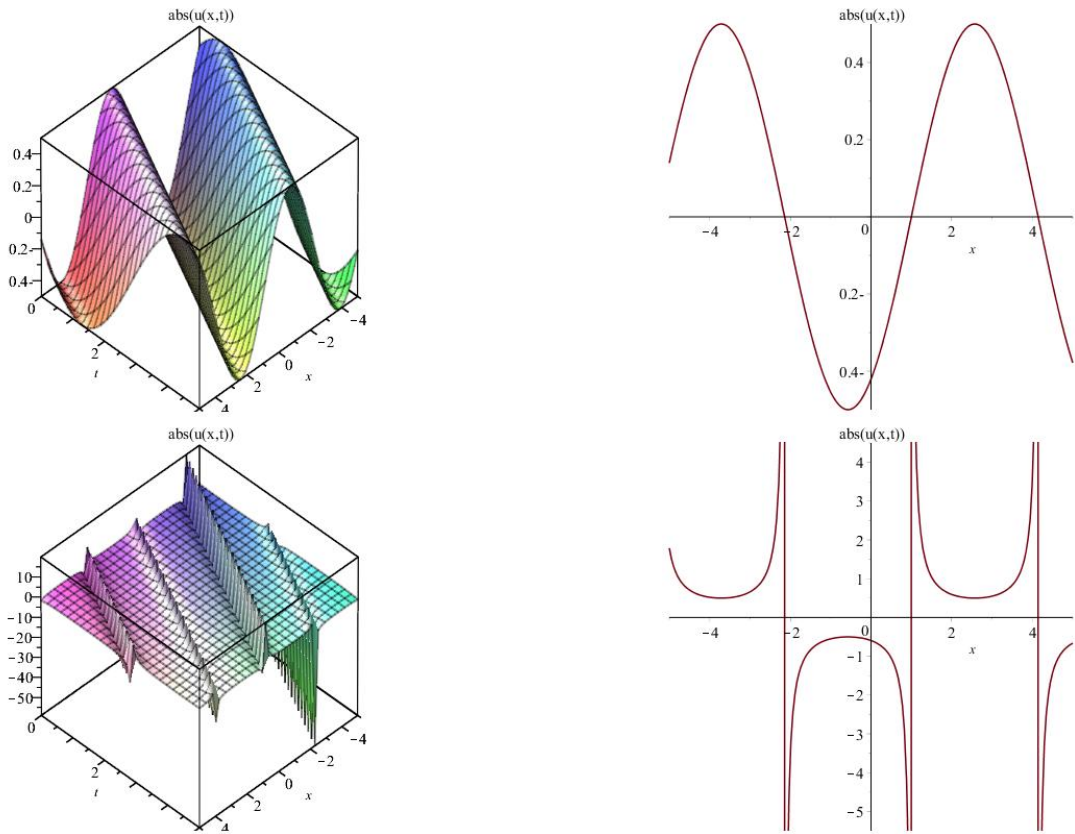


Figure 2. The for periodic wave profiles for various times scale for (3.8) and (3.11).

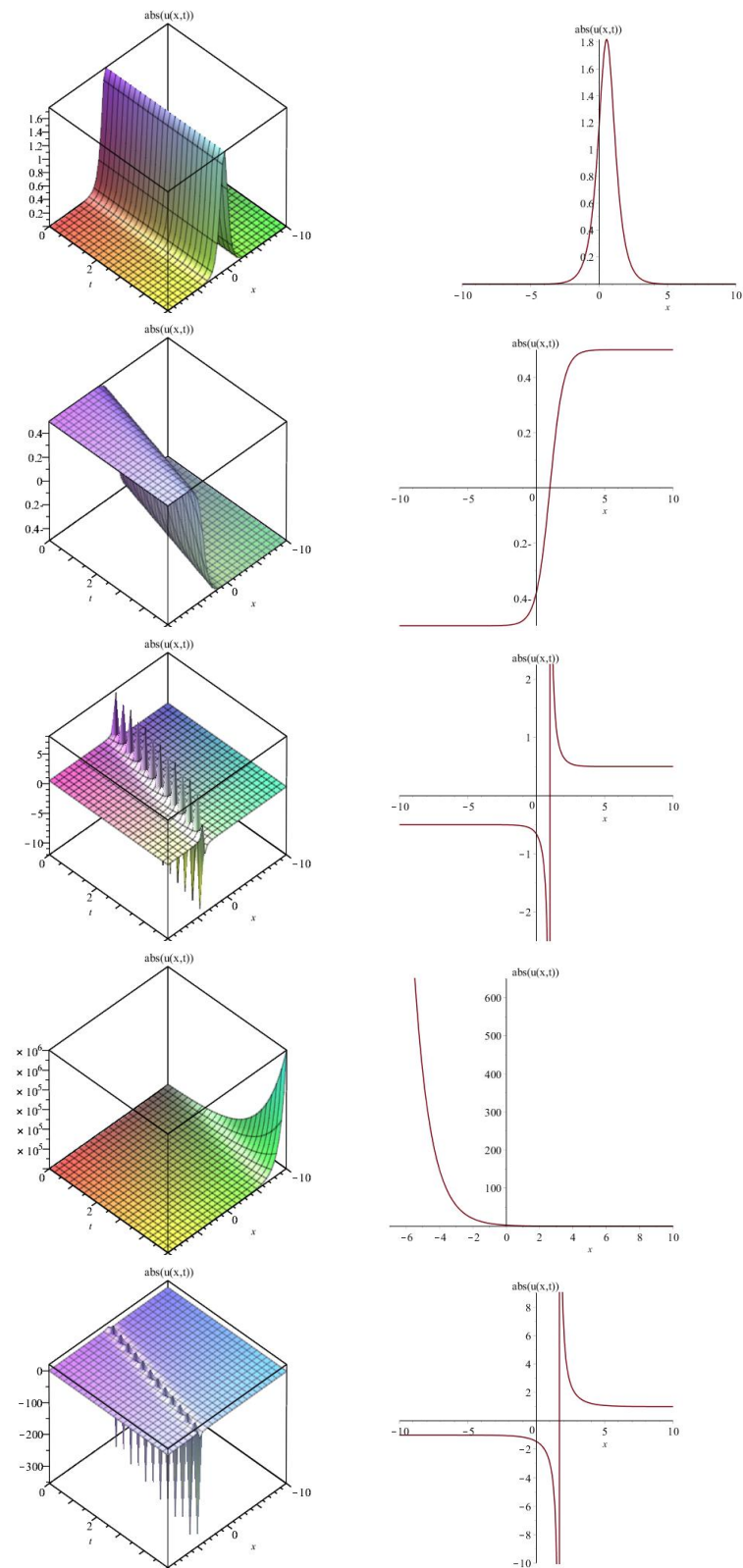


Figure 3. The for kink-soliton solutions and soliton-like wave profiles for different times scale for (3.9), (3.12), (3.14) and (3.16).

Remark: The solutions (3.5), (3.7), (3.10), (3.8), (3.11), (3.9), (3.12), (3.14) and (3.16) of δ -NLSE equation (3.1) are not previously found, that is in [33] the optical solitons obtained using the complex envelope function ansatz method.

4. Complex coupled Higgs field equation

4.1. Jacobi elliptic function method

Consider the complex coupled Higgs field equation [37]

$$\begin{aligned} u_{tt} - u_{xx} - \alpha u + \beta (|u|)^2 u - 2uv &= 0, \\ v_{tt} + v_{xx} - \beta (|u|^2)_{xx} &= 0, \end{aligned} \quad (4.1)$$

describes a system of conserved scalar nucleons interacting with neutral scalar mesons, where v is a complex scalar nucleon field and u is a real scalar meson field [38]. In case of $\alpha < 0, \beta < 0$, the Eq (4.1) transformed into nonlinear Klein-Gordon equation and for $\alpha > 0, \beta > 0$, the Eq (4.1) transformed into coupled Higgs field equation.

To find the Jacobi elliptic solutions for complex coupled Higgs field equation via Jacobi elliptic function method, we start by assuming a solution of the form [35]

$$u(x, t) = U(\zeta)e^{i(kx+\omega t)}, \quad v(x, t) = V(\zeta). \quad (4.2)$$

Where $\zeta = lx - nt + f$, l, n represent the wave number and wave frequency of the light pulse, respectively, and $U(\zeta)$ and $V(\zeta)$ are the periodic wave profile.

We transform the (4.1) into ODE wave transformation using (4.2)

$$\begin{aligned} \beta U^3 + \frac{Un^2\omega^2}{l^2} - U\omega^2 - 2\frac{U^3l^2\beta}{l^2+n^2} - \alpha U + U''(n^2 - l^2) &= 0, \\ V &= \frac{l^2\beta U^2}{l^2+n^2}. \end{aligned} \quad (4.3)$$

Balancing U'' with U^3 , we get $N = 2$. Then, the Jacobi elliptic function solutions for Eq (4.1) as follows:

$$U(\zeta) = a_0 + a_1\varphi(\zeta), \quad (4.4)$$

where $a_i, i = 0, 1$ are constants which are to be determined later and φ satisfies (2.4).

Inserting (4.4) along (2.4) into (4.3) and collecting the terms with the same order of $\varphi(\zeta)$, and setting it to zero we obtain the Jacobi elliptic function solutions for complex coupled Higgs field equation for $d_0 = 1 - m^2, d_1 = d_3 = 0, d_2 = 2m^2 - 1$ and $d_4 = -m^2$.

$$\begin{aligned} u(x, t) &= a_1 cn \left(lx - nt + f, \sqrt{\frac{\beta}{2(l^2+n^2)} a_1} \right) e^{i(kx+\omega t)}, \\ v(x, t) &= \frac{\beta l^2 a_1^2}{l^2+n^2} \left(cn \left(lx - nt + f, -\sqrt{\frac{\beta}{2(l^2+n^2)} a_1} \right) \right)^2, \end{aligned} \quad (4.5)$$

where $k = -\frac{n\omega}{l}$.

When $m \rightarrow 1$, we obtain the soliton solutions for complex coupled Higgs field equation

$$\begin{aligned} u(x, t) &= \operatorname{sech}(x-1) e^{i(kx+\omega t)}, \\ v(x, t) &= 2 \operatorname{sech}^2(x-1). \end{aligned} \quad (4.6)$$

For $d_0 = 1, d_1 = d_3 = 0, d_2 = -(m^2 + 1)$ and $d_4 = m^2$.

$$\begin{aligned} u(x, t) &= a_1 \operatorname{sn}(lx - nt + f, m) e^{i(kx+\omega t)}, \\ v(x, t) &= \frac{l^2 \beta (a_0 + a_1 \operatorname{sn}(lx - nt + f, m))^2}{l^2 + n^2}, \\ u(x, t) &= a_1 \operatorname{cd}(lx - nt + f, m) e^{i(kx+\omega t)}, \\ v(x, t) &= \frac{l^2 \beta (a_0 + a_1 \operatorname{cd}(lx - nt + f, m))^2}{l^2 + n^2}. \end{aligned} \quad (4.7)$$

When $m \rightarrow 1$, we obtain the soliton solutions for complex coupled Higgs field equation

$$\begin{aligned} u(x, t) &= a_1 \tanh(x-1) e^{i(kx+\omega t)}, \\ v(x, t) &= 2 (|\tanh(x-1)|)^2. \end{aligned} \quad (4.8)$$

And for $d_0 = m^2, d_1 = d_3 = 0, d_2 = -(m^2 + 1)$ and $d_4 = 1$.

$$\begin{aligned} u(x, t) &= a_1 \operatorname{ns} \left(lx + I \sqrt{\frac{\beta a_1^2}{2} + l^2 t + f, m} \right) e^{i(kx+\omega t)}, \\ v(x, t) &= -2 l^2 \left(\operatorname{ns} \left(lx + I \sqrt{\frac{\beta a_1^2}{2} + l^2 t + f, m} \right) \right)^2, \\ u(x, t) &= a_1 \operatorname{dc} \left(lx + I \sqrt{\frac{\beta a_1^2}{2} + l^2 t + f, m} \right) e^{i(kx+\omega t)}, \\ v(x, t) &= -2 l^2 \left(\operatorname{dc} \left(lx + I \sqrt{\frac{\beta a_1^2}{2} + l^2 t + f, m} \right) \right)^2, \end{aligned} \quad (4.9)$$

where

$$m = -\frac{1}{l} \sqrt{\frac{\beta l^2 a_1^2 - \beta \omega^2 a_1^2 + 4 l^4 - 4 l^2 \omega^2 - 2 l^2 \alpha}{\beta a_1^2 + 4 l^2}}.$$

When $m \rightarrow 1$, we obtain the soliton solutions for complex coupled Higgs field equation

$$\begin{aligned} u(x, t) &= \operatorname{coth}(-x + i\sqrt{2}t) e^{i(kx+\omega t)}, \\ v(x, t) &= 2 \left(\left| \operatorname{coth}(x - 2i\sqrt{2}t) \right| \right)^2. \end{aligned} \quad (4.10)$$

4.2. Generalized Kudryashov's method

In this section we will find the soliton solutions for complex coupled Higgs field equation using generalized Kudryashov's method.

Let us assume the solution for Eq (4.1) in the rational form (2.8). Balancing U'' with U^3 , in (3.3) we get $N = M + 1$, we choose $N = 2, M = 1$ Then, we can obtain the soliton solutions for Eq (4.1) according to (2.8) as follows:

$$\begin{aligned} U(\zeta) &= \frac{a_0 + a_1\varphi(\zeta) + a_2\varphi(\zeta)^2}{b_0 + b_1\varphi(\zeta)}, \\ V(\zeta) &= \frac{l^2\beta U^2}{l^2 + n^2}, \end{aligned} \quad (4.11)$$

where $a_i, b_j, i = 0, 1, 2, j = 0, 1$ are constants to be determined.

The family of soliton solutions for complex coupled Higgs field equation as follows

$$\begin{aligned} u(x, t) &= -\frac{c_1^2 a_2 e^\zeta - 4 c_1 a_1 e^{\frac{\zeta}{2}} + 2 c_1 a_2 e^{\frac{\zeta}{2}} - 4 a_1 - 3 a_2}{2 b_1 (c_1 e^{\frac{\zeta}{2}} + 3) (1 + c_1 e^{\zeta/2})} e^{i(kx + \omega t)}, \\ v(x, t) &= \frac{l^2 \beta}{(l^2 + n^2)} \left(\frac{c_1^2 a_2 e^\zeta - 4 c_1 a_1 e^{\frac{\zeta}{2}} + 2 c_1 a_2 e^{\frac{\zeta}{2}} - 4 a_1 - 3 a_2}{2 b_1 (c_1 e^{\frac{\zeta}{2}} + 3) (1 + c_1 e^{\zeta/2})} \right)^2 \end{aligned} \quad (4.12)$$

for

$$n = \pm \frac{\sqrt{-4 l^2 b_1^2 - 2 a_2^2 \beta}}{2 b_1} \text{ and } \omega = \mp \left(\sqrt{\frac{4 l^2 b_1^2 + a_2^2 \beta - 4 b_1^2 \alpha}{2(4 l^2 b_1^2 + a_2^2 \beta)}} \right),$$

where

$$\zeta = \frac{2 l x b_1 + \sqrt{-4 l^2 b_1^2 - 2 a_2^2 \beta} t + 2 f b_1}{b_1}, \quad k = -\frac{n \omega}{l}.$$

4.3. Physical interpretation

In this section, we will display numerical simulations to the problem of this paper that add extra flavor to our analytical solutions of Eq (4.1), the physical meanings of the solutions obtained for complex coupled Higgs field equation (4.1), for (4.5), (4.7) and (4.9) which are Jacobi elliptic pattern solutions, which are cnoidal solutions, we showed the time growth of these Jacobi elliptic solutions by some figures to describe the dynamical properties in 3D graphs and 2D graphs in Figures 4–6. The solutions (4.6), which is soliton-solutions, (4.8) and (4.10), which is kink-soliton solutions which have a permanent profile displayed in 2D graph and 3D graph by taking suitable values of the parameters to visualize the mechanism of the (4.1) in Figures 7–10., solutions in (4.12), which are soliton-like solutions from balance between nonlinearity U^3 and dispersion U'' , displayed in Figure 6. All figures in 2D for the space evolution $-10 < x < 10$ and figures in 3D for the time evolution $0 < t < 5$ for Jacobi elliptic function, kink-soliton solutions and soliton-like, space evolution for periodic solutions in the interval $-5 < x < 5$ and $-20 < x < 20$.

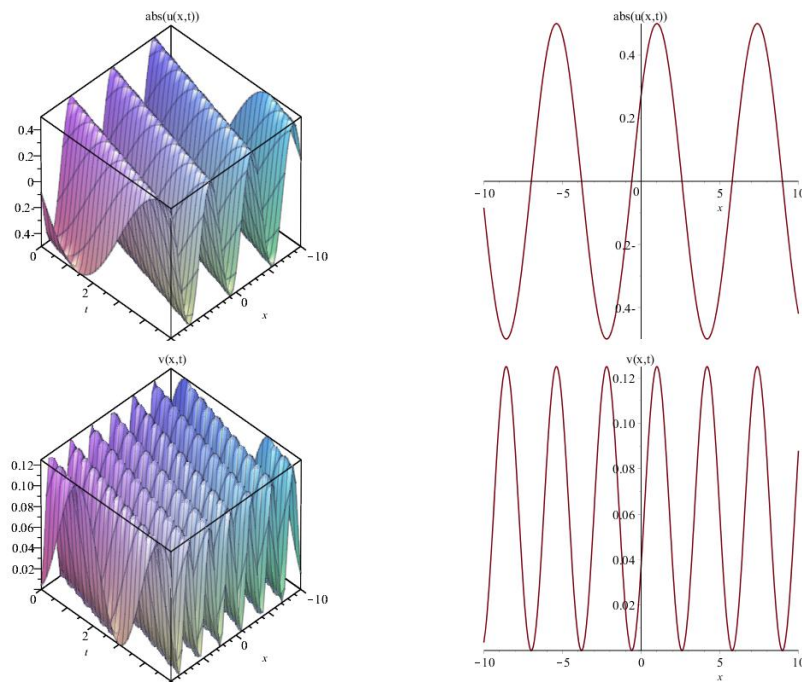


Figure 4. The for periodic wave profiles for various times scale for (4.5) using Jacobi elliptic function method.

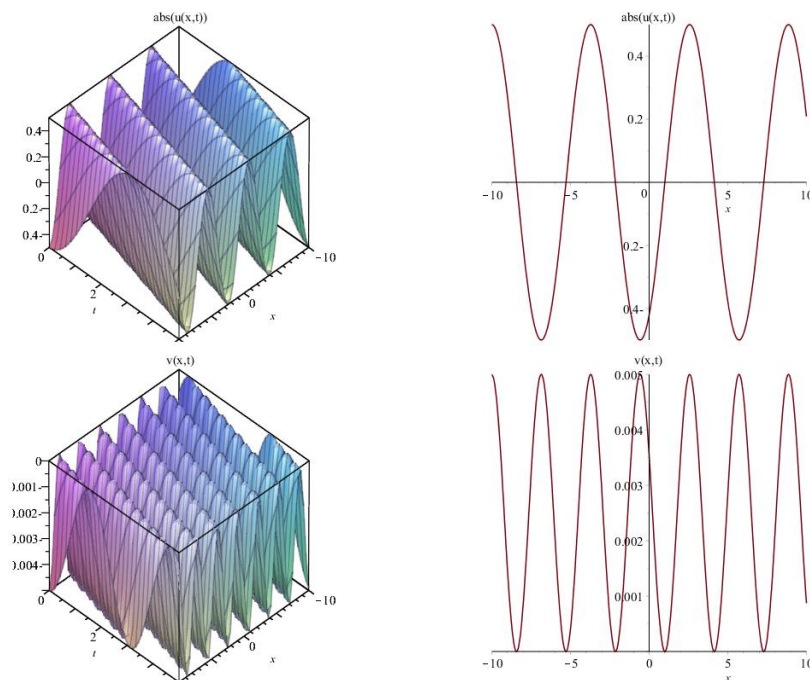


Figure 5. The for periodic wave profiles for various times scale for (4.7) using Jacobi elliptic function method.

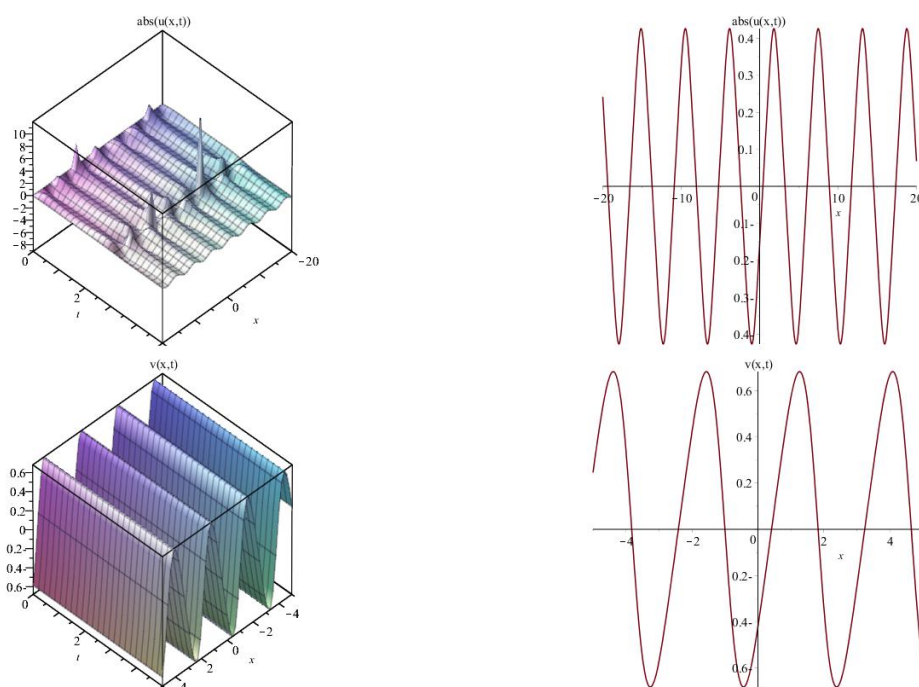


Figure 6. The for periodic wave profiles for various times scale for (4.9) using Jacobi elliptic function method.

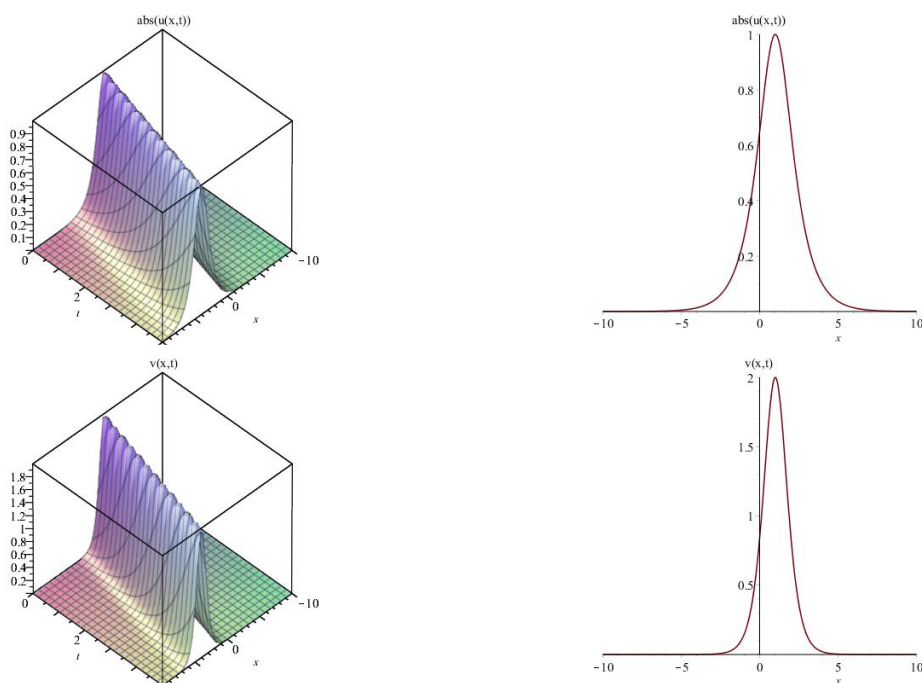


Figure 7. The for soliton wave solutions for various times scale for (4.6) using Jacobi elliptic function method.

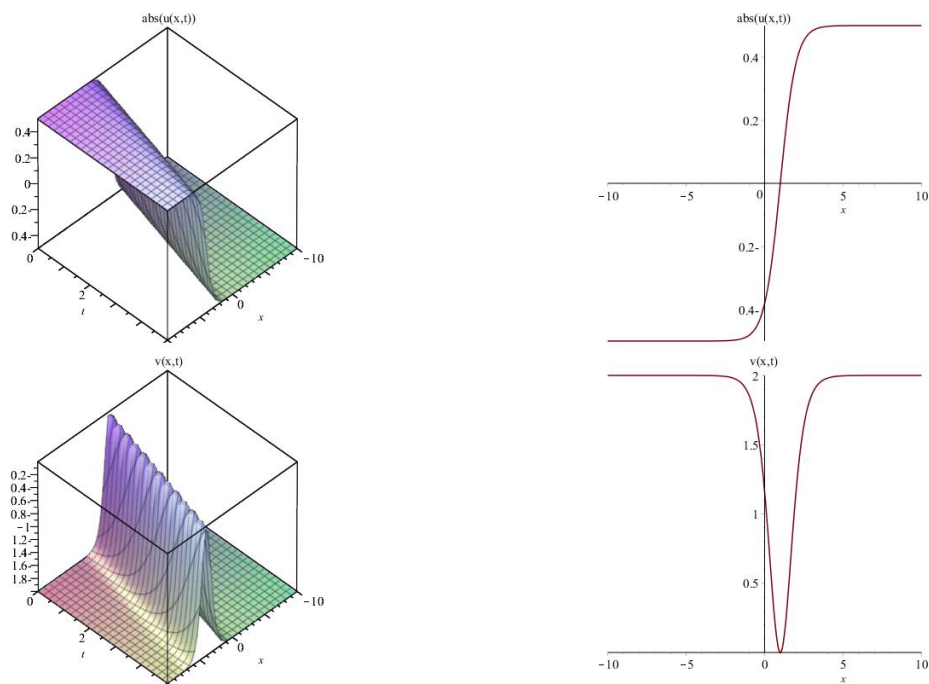


Figure 8. The for kink-soliton solutions for various times scale for (4.8) using Jacobi elliptic function method.

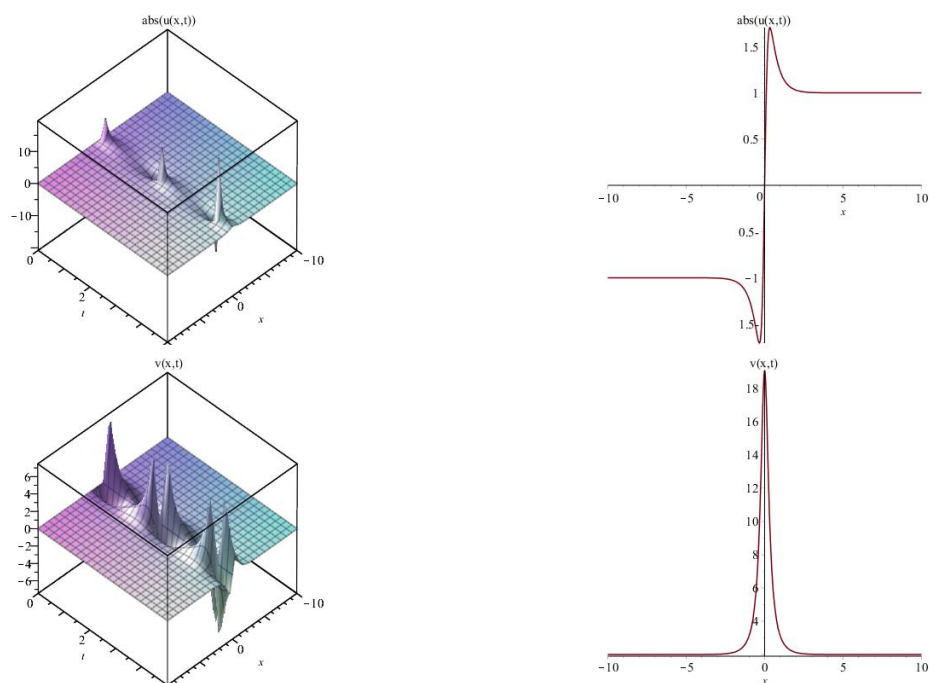


Figure 9. The for kink-soliton solutions for various times scale for (4.10) using Jacobi elliptic function method.

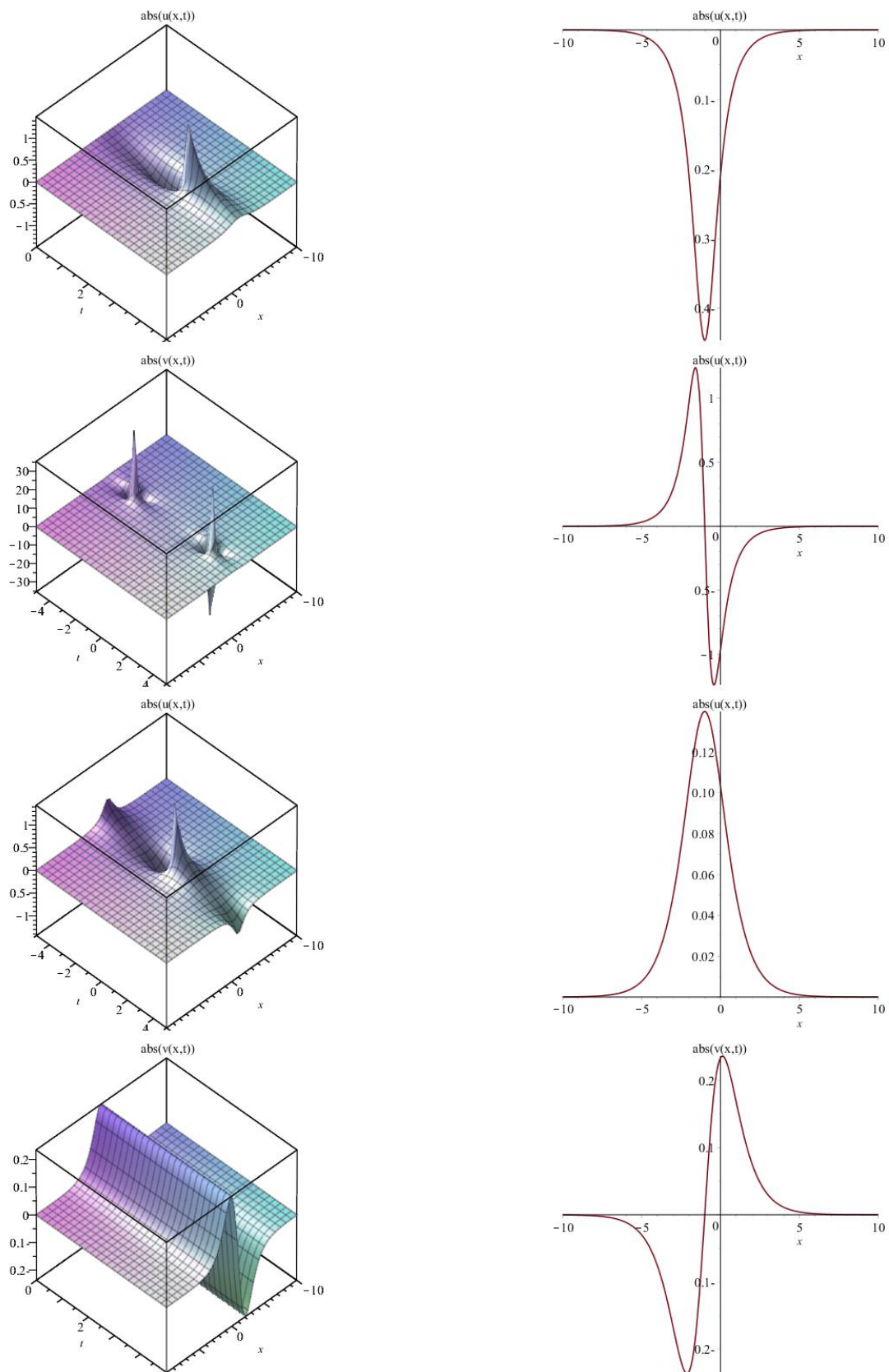


Figure 10. The for soliton like solutions for various times scale for (4.12) using generalized Kudryashov's method.

Remark: The solutions (4.5)–(4.10) and (4.12) for complex coupled Higgs field equation (4.1) are not previously found, that is some of them were explored before and in this paper we gave new solutions, in [39], which is used an algebraic method is applied to construct solitary wave solutions and Jacobi periodic wave solutions, the authors in [40] constructed the traveling wave solutions using (G'/G) -expansion method and the authors in [37] constructed the soliton solutions and triangular solutions using tanh method.

5. Conclusions

In this paper we successfully constructed the Jacobi elliptic function solutions, periodic wave solutions and soliton solutions, which obtained as a limit of the periodic wave solutions for δ -NLSE equation (3.1) and complex coupled Higgs field equation (4.1) using the Jacobi elliptic function method and generalized Kudryashov method utilized in treating the equations with the aid of symbolic computation using Mathematica. Other solutions are constructed such as kink-soliton solutions. We investigate these results numerically by some figures. As such, we compared the obtained results with previously works to prove that we explored some results and new results were derived and evaluated computationally owing to the less constant parameters. These solutions with free parameters have rich structures and may be important to explain some physical phenomena. It is worthy to note that the two methods are straightforward, concise and can be widely used to other nonlinear partial differential equations which arise in the theory of solitons in mathematical physics.

Acknowledgments

The authors would like to express their sincere thanks to the referees for suggestions.

Conflict of interest

There is no conflict of interests regarding the publication of this paper.

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