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Research article

New characterizations of the generalized Moore-Penrose inverse of matrices

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Abstract: Some new characterizations of the generalized Moore-Penrose inverse are proposed using range, null space, several matrix equations and projectors. Several representations of the generalized Moore-Penrose inverse are given. The relationships between the generalized Moore-Penrose inverse and other generalized inverses are discussed using core-EP decomposition. The generalized Moore-Penrose matrices are introduced and characterized. One relation between the generalized Moore-Penrose inverse and corresponding nonsingular border matrix is presented. In addition, applications of the generalized Moore-Penrose inverse in solving restricted matrix equations are studied.

Keywords: generalized Moore-Penrose inverse; range space; null space; core-EP decomposition
Mathematics Subject Classification: 15A09

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ complex matrices. For $B \in \mathbb{C}^{m\times n}$, let B^*, $r(B)$, $\mathcal{R}(B)$ and $\mathcal{N}(B)$ stand for the conjugate transpose, the rank, the range and the null space, respectively, of B. For $B \in \mathbb{C}^{n\times n}$, the determinant of B is denoted by $detB$. The index of B, denoted by $\mathrm{Ind}(B)$, is the smallest nonnegative integer k such that $r(B^k) = r(B^{k+1})$. $\mathbb{C}^{n\times n}_k$ stands for the set of all $n \times n$ complex matrices with index k. We denote the identity matrix and the zero matrix in $\mathbb{C}^{n\times n}$ by I_n and O. If $\mathbb{C}^m$ is a direct sum of subspaces $\mathcal{L}$ and $\mathcal{M}$, $P_{\mathcal{L},\mathcal{M}}$ is a projector onto $\mathcal{L}$ along $\mathcal{M}$. Also, the orthogonal projector onto $\mathcal{L}$ will be denoted by $P_{\mathcal{L}}$. Given $B \in \mathbb{C}^{m\times n}$ and $x \in \mathbb{C}^m$, if the i-th column of B is replaced by x, then the resulted matrix is written as $B(i \to x)$.

Next we recall the definitions of some generalized inverses. For $B \in \mathbb{C}^{m\times n}$, there exists the MP inverse of B as the unique matrix $B^\dagger \in \mathbb{C}^{n\times m}$ [13] such that $BB^\dagger B = B$, $B^\dagger BB^\dagger = B^\dagger$, $(BB^\dagger)^* = BB^\dagger$, $(B^\dagger B)^* = B^\dagger B$. Moreover, $P_B = BB^\dagger$ and $P_{B^*} = B^\dagger B$ represent the orthogonal projectors on $\mathcal{R}(B)$ and $\mathcal{R}(B^*)$, respectively. If a matrix X fulfills $BXB = B$, X is called $\{1\}$-inverse of B and if $XBX = X$ is satisfied, X is $\{2\}$-inverse of B. Also, $B\{1\}$ stands for the set of all $\{1\}$-inverses of B. Given $B \in \mathbb{C}^{m\times n}$, suppose a matrix X fulfills $XBX = X$, $\mathcal{R}(X) = \mathcal{T}$ and $\mathcal{N}(X) = \mathcal{S}$, then X is denoted as $B^{(2)}_{\mathcal{T},\mathcal{S}}$ [2], where

$\mathcal{T}$, $\mathcal{S}$ are subspaces of $\mathbb{C}^n$, $\mathbb{C}^m$, respectively. When $B^{(2)}_{\mathcal{T},\mathcal{S}}$ exists, it is unique.

The Drazin inverse of $B \in \mathbb{C}^{n\times n}_k$ is denoted by B^D introduced in [5]. If $\mathrm{Ind}(B) = 1$, B^D becomes $B^\#$, which is the group inverse of B. In [1], the core inverse of $B \in \mathbb{C}^{n\times n}_1$ was proposed, written as $B^{\textcircled{\#}}$. $B^{D,\dagger}$ is the DMP inverse of $B \in \mathbb{C}^{n\times n}_k$ introduced in [11]. Moreover, $B^{D,\dagger} = B^D BB^\dagger$, $B^{\dagger,D} = B^\dagger BB^D$, where $B^{\dagger,D}$ [11] is the dual DMP inverse of $B \in \mathbb{C}^{n\times n}_k$. Recently, several characterizations of DMP inverse were investigated in [9, 22]. The core-EP inverse $B^{\textcircled{\dagger}}$ of $B \in \mathbb{C}^{n\times n}_k$ is unique matrix, which satisfies $B^{\textcircled{\dagger}}BB^{\textcircled{\dagger}} = B^{\textcircled{\dagger}}$ and $\mathcal{R}(B^{\textcircled{\dagger}}) = \mathcal{R}((B^{\textcircled{\dagger}})^*) = \mathcal{R}(B^k)$. More research about core-EP inverse can be found in [7, 10, 14, 16, 21]. The weak group inverse of $B \in \mathbb{C}^{n\times n}_k$ is the uniquely determined matrix $B^{\textcircled{w}}$ if $B(B^{\textcircled{w}})^2 = B^{\textcircled{w}}$, $BB^{\textcircled{w}} = B^{\textcircled{\dagger}}B$ [12, 17, 19].

In 2020, the generalized Moore-Penrose inverse (in short, gMP inverse) was introduced by Stojanović and Mosić [15]. More precisely, the gMP inverse of $B \in \mathbb{C}^{n\times n}_k$, defined as $B^{\otimes} = (B^{\textcircled{\dagger}}B)^\dagger B^{\textcircled{\dagger}}$, is the unique solution to the matrix system

$$XBX = X, \quad BX = B(B^{\textcircled{\dagger}}B)^\dagger B^{\textcircled{\dagger}}, \quad XB = (B^{\textcircled{\dagger}}B)^\dagger B^{\textcircled{\dagger}}B.$$

Especially, if $\mathrm{Ind}(B) = 1$, $B^{\otimes}$ becomes $B^\dagger$. For different properties of generalized inverses please see [3, 4].

Inspired by recent investigations about core-EP inverse and weak group inverse, continuing previous work about the gMP inverse, our goal is to give certain new characterizations, representations and properties of the gMP inverse and consider its applications in the restricted matrix equations.

This paper is organized as follows. Section 2 involves several lemmas. In Section 3, we use range space, null space, matrix equations and projectors to characterize the gMP inverse. In Section 4, limit representations of the gMP inverse are presented, as well as maximal classes of matrices such that the general formula of the gMP inverse is satisfied. Section 5 contains several properties of the gMP inverse. In Section 6, we consider the relationship between the gMP inverse and corresponding nonsingular bordered matrix. Also, we apply the gMP inverse to solve restricted matrix equations.

2. Preliminaries

We begin with several lemmas which will be used in later.

Lemma 2.1. [7] *Let $B \in \mathbb{C}^{n\times n}_k$. We have*

(a) $B^{\textcircled{\dagger}} = B^{(2)}_{\mathcal{R}(B^k),\mathcal{N}((B^k)^*)}$,

(b) $BB^{\textcircled{\dagger}} = P_{\mathcal{R}(B^k)}$,

(c) $B^{\textcircled{\dagger}}B = P_{\mathcal{R}(B^k),\mathcal{N}((B^k)^*B)}$.

Lemma 2.2. *Suppose $B \in \mathbb{C}^{n\times n}_k$. Then*

(a) $B^{\otimes} = B^{(2)}_{\mathcal{R}((B^{\textcircled{\dagger}}B)^*),\mathcal{N}(B^{\textcircled{\dagger}})} = B^{(2)}_{\mathcal{R}(B^*B^k),\mathcal{N}((B^k)^*)}$,

(b) $BB^{\otimes} = P_{\mathcal{R}(B(B^{\textcircled{\dagger}}B)^*),\mathcal{N}(B^{\textcircled{\dagger}})} = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$,

(c) $B^{\otimes}B = P_{\mathcal{R}((B^{\textcircled{\dagger}}B)^*)} = P_{\mathcal{R}(B^*B^k)}$.

Proof. Using [15], we get the first equality in (a), (b) and (c), respectively. Using Lemma 2.1, we obtain $\mathcal{R}((B^{\textcircled{\dagger}}B)^*) = \mathcal{N}(B^{\textcircled{\dagger}}B)^\perp = \mathcal{R}(B^*B^k)$, $\mathcal{N}(B^{\textcircled{\dagger}}) = \mathcal{N}((B^k)^*)$. The rest is clear. □

Lemma 2.3. [16] *Let $B \in \mathbb{C}_k^{n\times n}$ and $t = r(B^k)$. Then B is expressed by*

$$B = U\begin{bmatrix} T & S \\ O & N \end{bmatrix} U^*, \tag{2.1}$$

where N is nilpotent with index k, T is $t \times t$ invertible matrix, $U \in \mathbb{C}^{n\times n}$ is unitary. Furthermore, from [6, 16, 17, 20]*, it is known that*

$$B^\dagger = U\begin{bmatrix} T^*\triangle & -T^*\triangle S N^\dagger \\ (I_{n-t} - N^\dagger N)S^*\triangle & N^\dagger - (I_{n-t} - N^\dagger N)S^*\triangle S N^\dagger \end{bmatrix} U^*, \tag{2.2}$$

$$B^D = U\begin{bmatrix} T^{-1} & T^{-k-1}\widetilde{T} \\ O & O \end{bmatrix} U^*, \tag{2.3}$$

$$B^{D,\dagger} = U\begin{bmatrix} T^{-1} & T^{-k-1}\widetilde{T}NN^\dagger \\ O & O \end{bmatrix} U^*, \tag{2.4}$$

$$B^{\dagger,D} = U\begin{bmatrix} T^*\triangle & T^*\triangle T^{-k}\widetilde{T} \\ (I_{n-t} - N^\dagger N)S^*\triangle & (I_{n-t} - N^\dagger N)S^*\triangle T^{-k}\widetilde{T} \end{bmatrix} U^*, \tag{2.5}$$

$$B^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & O \\ O & O \end{bmatrix} U^*, \tag{2.6}$$

$$B^{\textcircled{w}} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ O & O \end{bmatrix} U^*, \tag{2.7}$$

where $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$, $\triangle = [TT^* + S(I_{n-t} - N^\dagger N)S^*]^{-1}$. In addition, $\widetilde{T} = O$ if and only if $S = O$.

The decomposition in (2.1) is known as the core-EP decomposition [16].

Lemma 2.4. *Let $B \in \mathbb{C}_k^{n\times n}$ be given by* (2.1)*. Then*

$$r(B) = r(B^2) \Leftrightarrow N = O.$$

In which case, we have

$$B^\# = U\begin{bmatrix} T^{-1} & T^{-2}S \\ O & O \end{bmatrix} U^*,\ B^{\textcircled{\#}} = U\begin{bmatrix} T^{-1} & O \\ O & O \end{bmatrix} U^*. \tag{2.8}$$

Lemma 2.5. [15] *Let $B \in \mathbb{C}_k^{n\times n}$ be given by* (2.1)*. Then*

$$B^{\otimes} = (B^k(B^k)^\dagger B)^\dagger = U\begin{bmatrix} T^*(TT^* + SS^*)^{-1} & O \\ S^*(TT^* + SS^*)^{-1} & O \end{bmatrix} U^*. \tag{2.9}$$

3. Some characterizations of gMP inverse

Using the results of Lemma 2.2, we have $\mathcal{R}(B^{\otimes}) = \mathcal{R}(B^*B^k)$, $\mathcal{N}(B^{\otimes}) = \mathcal{N}((B^k)^*)$ and $B^{\otimes}BB^{\otimes} = B^{\otimes}$. Now, we will give some necessary and sufficient conditions for a matrix X to be $B^{\otimes}$.

Theorem 3.1. *Let $B \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*
(*a*) $X = B^{\otimes}$;
(*b*) $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$, $BX = B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}$;
(*c*) $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$, $B^{\textcircled{\dagger}}BX = B^{\textcircled{\dagger}}$;
(*d*) $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$, $(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}BX = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}$;
(*e*) $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$, $(B^{\textcircled{\dagger}}B)^*B^{\textcircled{\dagger}}BX = (B^{\textcircled{\dagger}}B)^*B^{\textcircled{\dagger}}$;
(*f*) $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$, $(B^k)^*BX = (B^k)^*$.

Proof. $(a) \Rightarrow (b)$. It is obvious from Lemma 2.2 (a) and the definition of the $B^{\otimes}$.
$(b) \Rightarrow (c)$. It is evident that $B^{\textcircled{\dagger}}BX = B^{\textcircled{\dagger}}B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}BB^{\textcircled{\dagger}} = B^{\textcircled{\dagger}}BB^{\textcircled{\dagger}} = B^{\textcircled{\dagger}}$.
$(c) \Rightarrow (d)$. Obvious.
$(d) \Rightarrow (e)$. Consequently by

$$(B^{\textcircled{\dagger}}B)^*B^{\textcircled{\dagger}}BX = (B^{\textcircled{\dagger}}B)^*B^{\textcircled{\dagger}}B((B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}BX) = (B^{\textcircled{\dagger}}B)^*B^{\textcircled{\dagger}}B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = (B^{\textcircled{\dagger}}B)^*B^{\textcircled{\dagger}}.$$

$(e) \Rightarrow (a)$. By $\mathcal{R}(X) = \mathcal{R}(B^*B^k) = \mathcal{R}((B^{\textcircled{\dagger}}B)^*)$, we get $X = (B^{\textcircled{\dagger}}B)^*L$ for some $L \in \mathbb{C}^{n\times n}$. Applying $((B^{\textcircled{\dagger}}B)^*B^{\textcircled{\dagger}}B)^{\dagger}(B^{\textcircled{\dagger}}B)^* = (B^{\textcircled{\dagger}}B)^{\dagger}$, we verify that

$$X = (B^{\textcircled{\dagger}}B)^*L = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}B(B^{\textcircled{\dagger}}B)^*L = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}BX = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = B^{\otimes}.$$

$(a) \Rightarrow (f)$. By Lemma 2.2, we have $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$, $(B^k)^*BX = (B^k)^*P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)} = (B^k)^*$.
$(f) \Rightarrow (a)$. We have $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$ which gives $X = L(B^k)^*$ for some $L \in \mathbb{C}^{n\times n}$. Pre-multiplying on $(B^k)^*BX = (B^k)^*$ by L, we obtain $XBX = X$. Hence, we have $X = B^{(2)}_{\mathcal{R}(B^*B^k),\mathcal{N}((B^k)^*)} = B^{\otimes}$ by Lemma 2.2 (a). □

Theorem 3.2. *Let $B \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*
(*a*) $X = B^{\otimes}$;
(*b*) $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, $XB = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}B$;
(*c*) $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, $XBB^{\textcircled{\dagger}} = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}$;
(*d*) $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, $XB(B^{\textcircled{\dagger}}B)^* = (B^{\textcircled{\dagger}}B)^*$;
(*e*) $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, $XB(B^{\textcircled{\dagger}}B)^{\dagger} = (B^{\textcircled{\dagger}}B)^{\dagger}$;
(*f*) $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, $XBB^*B^k = B^*B^k$.

Proof. $(a) \Rightarrow (b)$. It is clear by Lemma 2.2 (a) and the definition of $B^{\otimes}$.
$(b) \Rightarrow (c)$. Notice that $XBB^{\textcircled{\dagger}} = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}BB^{\textcircled{\dagger}} = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}$.
$(c) \Rightarrow (a)$. By Lemma 2.1 and $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, we have $X = KB^{\textcircled{\dagger}}$ for some $K \in \mathbb{C}^{n\times n}$. Thus, $X = KB^{\textcircled{\dagger}}BB^{\textcircled{\dagger}} = XBB^{\textcircled{\dagger}} = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = B^{\otimes}$.
$(b) \Rightarrow (d)$. It follows by $(B^{\textcircled{\dagger}}B)^* = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}B(B^{\textcircled{\dagger}}B)^*$.
$(d) \Rightarrow (e)$. We observe that $XB(B^{\textcircled{\dagger}}B)^{\dagger} = XB(B^{\textcircled{\dagger}}B)^*(B^{\textcircled{\dagger}}B(B^{\textcircled{\dagger}}B)^*)^{\dagger} = (B^{\textcircled{\dagger}}B)^{\dagger}$.

$(e) \Rightarrow (a)$. Since $\mathcal{N}(X) = \mathcal{N}((B^k)^*) = \mathcal{N}(BB^{\textcircled{\dagger}})$, we have $X = KBB^{\textcircled{\dagger}}$ for some $K \in \mathbb{C}^{n\times n}$. Hence,

$$X = KBB^{\textcircled{\dagger}} = KBB^{\textcircled{\dagger}}B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = XB(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = B^{\otimes}.$$

$(a) \Rightarrow (f)$. We obtain $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, $XBB^*B^k = P_{\mathcal{R}(B^*B^k)}B^*B^k = B^*B^k$ directly from Lemma 2.2.

$(f) \Rightarrow (a)$. From $XBB^*B^k = B^*B^k$ and $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, we get $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$ implying $X = B^*B^kK$ for some $K \in \mathbb{C}^{n\times n}$. Post-multiplying by K on $XBB^*B^k = B^*B^k$, we get $XBX = X$. Thus, from Lemma 2.2 (a) , $X = B^{\otimes}$. □

Theorem 3.3. *Let $B \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

(a) $X = B^{\otimes}$;

(b) $XBX = X$, $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$, $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$;

(c) $XBX = X$, $XBB^*B^k = B^*B^k$, $BX = B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}$;

(d) $XBX = X$, $XB = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}B$, $(B^k)^*BX = (B^k)^*$.

Proof. $(a) \Rightarrow (b)$. This implication is clear by Lemma 2.2 (a).

$(b) \Rightarrow (c)$. From $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$, we get $\mathcal{R}(BX) = B\mathcal{R}(X) = \mathcal{R}(BB^*B^k) = \mathcal{R}(B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}})$. Since $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$ and $XBX = X$, we get $\mathcal{N}(BX) = \mathcal{N}((B^k)^*) = \mathcal{N}(B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}})$. Further, since BX and $B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}$ are idempotents, we have $BX = B(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}$.

From $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$ and $XBX = X$, we get $\mathcal{R}(XB) = \mathcal{R}(B^*B^k)$ which gives $XBB^*B^k = P_{\mathcal{R}(B^*B^k),\mathcal{N}(XB)}B^*B^k = B^*B^k$.

$(c) \Rightarrow (a)$. By $XBB^*B^k = B^*B^k$, we get $\mathcal{R}(B^*B^k) = \mathcal{R}((B^{\textcircled{\dagger}}B)^{\dagger}) \subseteq \mathcal{R}(XB)$. Then we have

$$X = XB(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = P_{\mathcal{R}(XB),\mathcal{N}(XB)}(B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = B^{\otimes}.$$

$(a) \Rightarrow (d)$. We have the assertion from the definition of $B^{\otimes}$ and Lemma 2.2 (b).

$(d) \Rightarrow (a)$. The equalities $XBX = X$ and $XB = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}B$ give $r(X) = r(XB) = r(B^*B^k) = r((B^k)^*)$. Since $(B^k)^*BX = (B^k)^*$, we get $\mathcal{N}(BX) = \mathcal{N}((B^k)^*)$. By $\mathcal{N}(B^{\textcircled{\dagger}}) = \mathcal{N}((B^k)^*)$, we get

$$X = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}BX = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}}P_{\mathcal{R}(BX),\mathcal{N}((B^k)^*)} = (B^{\textcircled{\dagger}}B)^{\dagger}B^{\textcircled{\dagger}} = B^{\otimes}.$$

□

In [15], $B^{\otimes}$ is characterized by the condition $BB^{\otimes} = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$. Similarly, we characterize the $B^{\otimes}$ by the condition $B^{\otimes}B = P_{\mathcal{R}(B^*B^k)}$.

Theorem 3.4. *Let $B \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. $X = B^{\otimes}$ is the unique solution of equations*

$$XB = P_{\mathcal{R}(B^*B^k)},\ \mathcal{N}(X) \supseteq \mathcal{N}((B^k)^*). \tag{3.1}$$

Proof. Obviously, $X = B^{\otimes}$ satisfies equations (3.1) by Lemma 2.2. It remains to prove the uniqueness.

Assume that X, X_1 satisfy (3.1). By $\mathcal{N}(X) \supseteq \mathcal{N}((B^k)^*)$ and $\mathcal{N}(X_1) \supseteq \mathcal{N}((B^k)^*)$, we get $\mathcal{R}(X^* - X_1^*) \subseteq \mathcal{R}(B^k)$. Since $XB - X_1B = 0$, we get $B^*(X^* - X_1^*) = 0$ which implies $\mathcal{R}(X^* - X_1^*) \subseteq \mathcal{N}(B^*) \subseteq \mathcal{N}((B^k)^*)$. Further, since $\mathrm{Ind}(B) = k$, we get $\mathcal{R}(X^* - X_1^*) \subseteq \mathcal{N}((B^k)^*) \cap \mathcal{R}(B^k) = \{0\}$. Thus, $X^* = X_1^*$ and $X_1 = X = B^{\otimes}$. □

It is interesting to remark that $BX = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$ and $XB = P_{\mathcal{R}(B^*B^k)}$ when $X = B^{\otimes}$ by Lemma 2.2. However, the reverse is invalid which will be shown in the next example.

Example 3.5. *Let*

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^{\otimes} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can easy check that $Ind(B) = 2$*. Also,* X *satisfies* $BX = P_{\mathcal{R}(BB^*B^2),\mathcal{N}((B^2)^*)}$ *and* $XB = P_{\mathcal{R}(B^*B^2)}$*. But,* $X \neq B^{\otimes}$*.*

In the following results, some necessary conditions for the converse implication are presented.

Theorem 3.6. *If* $B \in \mathbb{C}_k^{n\times n}$ *and* $X \in \mathbb{C}^{n\times n}$*, then the following assertions are equivalent:*

(a) $X = B^{\otimes}$;

(b) $BX = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$, $XB = P_{\mathcal{R}(B^*B^k)}$, $XBX = X$;

(c) $BX = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$, $XB = P_{\mathcal{R}(B^*B^k)}$, $r(X) = r(B^k)$;

(d) $BX = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$, $XB = P_{\mathcal{R}(B^*B^k)}$, $XP_{\mathcal{R}(B^k)} = X$;

(e) $BX = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$, $XB = P_{\mathcal{R}(B^*B^k)}$, $P_{\mathcal{R}(B^*B^k)}X = X$.

Proof. $(a) \Rightarrow (b)$. It is obvious by Lemma 2.2.

$(b) \Rightarrow (c)$. By $XB = P_{\mathcal{R}(B^*B^k)}$ and $XBX = X$, we get $r(X) = r(XB) = r(B^*B^k) = r(B^k)$.

$(c) \Rightarrow (d)$. From $BX = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$ and $r(X) = r(B^k)$, we obtain $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$, which gives $XP_{\mathcal{R}(B^k)} = X$.

$(d) \Rightarrow (e)$. Since $BX = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}$ and $XP_{\mathcal{R}(B^k)} = X$, we get $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$. Since $XB = P_{\mathcal{R}(B^*B^k)}$, we get $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$ which implies $P_{\mathcal{R}(B^*B^k)}X = X$.

$(e) \Rightarrow (a)$. Similar as $(d) \Rightarrow (e)$, we have $\mathcal{R}(X) = \mathcal{R}(B^*B^k)$ and $\mathcal{N}(X) = \mathcal{N}((B^k)^*)$. Then $XBX = XP_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)} = X$. Thus, we obtain $X = B^{\otimes}$ directly from Lemma 2.2 (a). □

The inverse B^{-1} of a invertible matrix B is the unique matrix, which satisfies

$$r\left(\begin{bmatrix} B & I_n \\ I_n & B^{-1} \end{bmatrix}\right) = r(B).$$

The similar characterizations for some generalized inverses can be found in [9, 10, 12]. We have analogous characterization for the gMP inverse.

Theorem 3.7. *Let* $B \in \mathbb{C}_k^{n\times n}$ *with* $r(B^k) = t$*. Then there exists a unique matrix* P *which satisfies*

$$(B^k)^*P = O, \quad PBB^*B^k = O, \quad P^2 = P, \quad r(P) = n - t, \tag{3.2}$$

a unique matrix Q *that satisfies*

$$(B^k)^*BQ = O, \quad QB^*B^k = O, \quad Q^2 = Q, \quad r(Q) = n - t, \tag{3.3}$$

and a unique matrix K *that satisfies*

$$r\left(\begin{bmatrix} B & I_n - P \\ I_n - Q & K \end{bmatrix}\right) = r(B). \tag{3.4}$$

Moreover, the matrix K *is the gMP inverse* $B^{\otimes}$ *of* B *and*

$$P = P_{\mathcal{N}((B^k)^*),\mathcal{R}(BB^*B^k)}, \quad Q = P_{\mathcal{N}((B^k)^*B)}. \tag{3.5}$$

Proof. We can verify that

$$\begin{aligned} \text{the condition (3.2) holds} \quad &\Leftrightarrow \quad (B^k)^*(I_n - P) = (B^k)^*, (I_n - P)BB^*B^k = BB^*B^k, \\ &\qquad (I_n - P)^2 = I_n - P \text{ and } r(I_n - P) = t \\ &\Leftrightarrow \quad I_n - P = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)} \\ &\Leftrightarrow \quad P = P_{\mathcal{N}((B^k)^*),\mathcal{R}(BB^*B^k)}. \end{aligned}$$

Similarly, we verify that $Q = P_{\mathcal{N}((B^k)^*B)}$ is a unique matrix which satisfies (3.3).

Using (3.5), Lemma 2.2 and elementary computations, we get

$$r\left(\begin{bmatrix} B & I_n - P \\ I_n - Q & K \end{bmatrix}\right) = r\left(\begin{bmatrix} B & P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)} \\ P_{\mathcal{R}(B^*B^k)} & K \end{bmatrix}\right) = r(B) + r(K - B^{\otimes}).$$

Now

$$\text{the condition (3.4) holds} \Leftrightarrow r(K - B^{\otimes}) = 0 \Leftrightarrow K = B^{\otimes}.$$

□

Example 3.8. *(see [15, Example 2.1]) Let*

$$B = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $Ind(B) = 2$. *From (2.9), we get*

$$B^{\otimes} = (B^2(B^2)^{\dagger}B)^{\dagger} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \end{bmatrix}.$$

The block matrix

$$\begin{aligned} L &= \begin{bmatrix} B & I_4 - P \\ I_4 - Q & K \end{bmatrix} = \begin{bmatrix} B & P_{\mathcal{R}(BB^*B^2),\mathcal{N}((B^2)^*)} \\ P_{\mathcal{R}(B^*B^2)} & B^{\otimes} \end{bmatrix} \\ &= \left[\begin{array}{cccc|cccc} 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

satisfies $r(L) = r(B) = 3$. *In addition, the matrix*

$$P = P_{\mathcal{N}((B^2)^*),\mathcal{R}(BB^*B^2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

satisfies (3.2). *Further, we can check that*

$$Q = P_{\mathcal{N}((B^2)^*B)} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{5}{6} & -\frac{1}{6} \\ -\frac{1}{3} & 0 & -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

satisfies (3.3). *Therefore, Theorem 3.7 is valid in this example.*

4. Representations of gMP inverse

For a nonsingular matrix B, B^{-1} can be characterized in term of a well-known limit process

$$B^{-1} = \lim_{\lambda \to 0} (\lambda I_n + B)^{-1}, \tag{4.1}$$

when $\lambda \notin \sigma(-B)$. Limit representations for several generalized inverses, such as DMP inverse, core-EP inverse and the weak group inverse were studied in [9, 10, 12, 19, 22]. At the beginning of this section, we present limit expressions of the gMP inverse.

Theorem 4.1. *Let* $B \in \mathbb{C}_k^{n\times n}$. *We have*

$$B^{\otimes} = \lim_{\lambda \to 0} B^*(\lambda I_n + B^k(B^k)^* BB^*)^{-1} B^k(B^k)^* \tag{4.2}$$

$$= \lim_{\lambda \to 0} B^* B^k(B^k)^*(\lambda I_n + BB^*B^k(B^k)^*)^{-1}. \tag{4.3}$$

Proof. We denote $M = B^*(\lambda I_n + B^k(B^k)^*BB^*)^{-1}B^k(B^k)^*$. Let B be given by (2.1), $L = T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*$ and $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$. A straightforward calculation gives that

$$\begin{aligned} M &= B^*(\lambda I_n + B^k(B^k)^*BB^*)^{-1}B^k(B^k)^* \\ &= U \begin{bmatrix} T^* & O \\ S^* & N^* \end{bmatrix} \begin{bmatrix} \lambda I_t + L(TT^* + SS^*) & LSN^*) \\ O & \lambda I_{n-t} \end{bmatrix}^{-1} \begin{bmatrix} L & O \\ O & O \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^* & O \\ S^* & N^* \end{bmatrix} \begin{bmatrix} (\lambda I_t + L(TT^* + SS^*))^{-1} & -\frac{1}{\lambda}(\lambda I_t + L(TT^* + SS^*))^{-1}LSN^*) \\ O & \frac{1}{\lambda} I_{n-t} \end{bmatrix} \begin{bmatrix} L & O \\ O & O \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^*(\lambda I_t + L(TT^* + SS^*))^{-1}L & O \\ S^*(\lambda I_t + L(TT^* + SS^*))^{-1}L & O \end{bmatrix} U^*. \end{aligned}$$

Applying (4.1), we get

$$\lim_{\lambda \to 0} M = \lim_{\lambda \to 0} U \begin{bmatrix} T^*(\lambda I_t + L(TT^* + SS^*))^{-1}L & 0 \\ S^*(\lambda I_t + L(TT^* + SS^*))^{-1}L & 0 \end{bmatrix} U^*$$

$$= U\begin{bmatrix} T^*(TT^*+SS^*)^{-1}L^{-1}L & O \\ S^*(TT^*+SS^*)^{-1}L^{-1}L & O \end{bmatrix}U^*$$
$$= U\begin{bmatrix} T^*(TT^*+SS^*)^{-1} & O \\ S^*(TT^*+SS^*)^{-1} & O \end{bmatrix}U^* = B^{\bigotimes}.$$

Similarly, (4.3) can be verified. □

The purpose of the following example is to illustrate the equation (4.2) of Theorem 4.1. The equation (4.3) can be verified similarly.

Example 4.2. *Let*

$$B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

with $Ind(B) = 2$*. By (2.9), we get*

$$B^{\bigotimes} = (B^2(B^2)^\dagger B)^\dagger = \begin{bmatrix} \frac{3}{41} & -\frac{1}{41} & 0 & 0 \\ -\frac{1}{41} & \frac{14}{41} & 0 & 0 \\ \frac{7}{41} & -\frac{16}{41} & 0 & 0 \\ \frac{8}{41} & \frac{11}{41} & 0 & 0 \end{bmatrix}.$$

On the other hand,

$$B^*(\lambda I_4 + B^2(B^2)^*BB^*)^{-1}B^2(B^2)^* = \begin{bmatrix} \frac{35\lambda+303}{\lambda^2+542\lambda+4141} & \frac{-37\lambda-101}{\lambda^2+542\lambda+4141} & 0 & 0 \\ \frac{-37\lambda-101}{\lambda^2+542\lambda+4141} & \frac{42\lambda+1414}{\lambda^2+542\lambda+4141} & 0 & 0 \\ \frac{107\lambda+707}{\lambda^2+542\lambda+4141} & \frac{-116\lambda-1616}{\lambda^2+542\lambda+4141} & 0 & 0 \\ \frac{68\lambda+808}{\lambda^2+542\lambda+4141} & \frac{-69\lambda+1111}{\lambda^2+542\lambda+4141} & 0 & 0 \end{bmatrix}.$$

After simplification, it follows that $\lim_{\lambda\to 0} B^*(\lambda I_4 + B^2(B^2)^*BB^*)^{-1}B^2(B^2)^* = B^{\bigotimes}$.

In [15], the authors established maximal classes of operators for which the representations of the gMP inverse are still valid. Two operator matrix forms for the gMP inverse were given. From (2.9), we have that $B^{\bigotimes} = (B^k(B^k)^\dagger B)^\dagger$. We study maximal classes of complex matrices such that this form of expression for gMP inverse is still valid.

Theorem 4.3. *Let* $B \in \mathbb{C}_k^{n\times n}$ *be given by* (2.1) *and* $r(B^k) = t$*. The following are equivalent:*

(*a*) $B^{\bigotimes} = (B^kXB)^\dagger$;

(*b*) $B^kXB = P_{B^k}B$;

(*c*) $X = (B^k)^\dagger P_B + Y - P_{(B^k)^*}YP_B$, *where* $Y \in \mathbb{C}^{n\times n}$ *is arbitrary;*

(*d*) *X is given by*

$$X = U\begin{bmatrix} Y_1 + (T^k)^*L^{-1}(I_t - T^kY_1 - \widetilde{T}Y_3) & Y_2 - (T^k)^*L^{-1}(T^kY_2 + \widetilde{T}Y_4)NN^\dagger \\ Y_3 + (\widetilde{T})^*L^{-1}(I_t - T^kY_1 - \widetilde{T}Y_3) & Y_4 - (\widetilde{T})^*L^{-1}(T^kY_2 + \widetilde{T}Y_4)NN^\dagger \end{bmatrix}U^*,$$

where $L = T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*$, $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$, $Y_1 \in \mathbb{C}^{t\times t}$, $Y_2 \in \mathbb{C}^{t\times(n-t)}$, $Y_3 \in \mathbb{C}^{(n-t)\times t}$ *and* $Y_4 \in \mathbb{C}^{(n-t)\times(n-t)}$ *are arbitrary.*

Proof. $(a) \Rightarrow (b)$. We have $B^kXB = P_{B^k}B$ since $B^{\bigotimes} = (B^k(B^k)^{\dagger}B)^{\dagger}$.

$(b) \Rightarrow (c)$. Obviously, $(B^k)^{\dagger}P_B$ satisfies the equation

$$B^kXB = P_{B^k}B. \tag{4.4}$$

Applying [2, Ch.2 Theorem 1] to (4.4), $X = (B^k)^{\dagger}P_B + Y - P_{(B^k)^*}YP_B$ is the general solution of (4.4), where $Y \in \mathbb{C}^{n\times n}$ is arbitrary.

$(c) \Rightarrow (a)$. By computation, we get $B^kXB = B^k(B^k)^{\dagger}B$. Therefore, $(B^kXB)^{\dagger} = (B^k(B^k)^{\dagger}B)^{\dagger} = B^{\bigotimes}$.

$(c) \Leftrightarrow (d)$. Using (2.1), we have

$$B^k = U\begin{bmatrix} T^k & \widetilde{T} \\ O & O \end{bmatrix}U^*. \tag{4.5}$$

Applying [8, Lemma 1] to (4.5), we get

$$(B^k)^{\dagger} = U\begin{bmatrix} (T^k)^*(T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*)^{-1} & O \\ (\widetilde{T})^*(T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*)^{-1} & O \end{bmatrix}U^*. \tag{4.6}$$

Next,

$$Y = U\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^*,$$

where $Y_1 \in \mathbb{C}^{t\times t}$, $Y_2 \in \mathbb{C}^{t\times(n-t)}$, $Y_3 \in \mathbb{C}^{(n-t)\times t}$ and $Y_4 \in \mathbb{C}^{(n-t)\times(n-t)}$ are arbitrary. By direct calculation, we get that $X = (B^k)^{\dagger}P_B + Y - P_{(B^k)^*}YP_B$ is equivalent with

$$X = U\begin{bmatrix} Y_1 + (T^k)^*L^{-1}(I_t - T^kY_1 - \widetilde{T}Y_3) & Y_2 - (T^k)^*L^{-1}(T^kY_2 + \widetilde{T}Y_4)NN^{\dagger} \\ Y_3 + (\widetilde{T})^*L^{-1}(I_t - T^kY_1 - \widetilde{T}Y_3) & Y_4 - (\widetilde{T})^*L^{-1}(T^kY_2 + \widetilde{T}Y_4)NN^{\dagger} \end{bmatrix}U^*,$$

where $L = T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*$, $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$, Y_1, Y_2, Y_3, Y_4 are arbitrary. □

Theorem 4.4. *Let $B \in \mathbb{C}_k^{n\times n}$, $a \neq 0$. Suppose that P and Q^* are full column rank matrices which satisfy $\mathcal{N}((B^k)^*) = \mathcal{R}(P)$ and $\mathcal{R}(B^*B^k) = \mathcal{N}(Q)$. Let $E_P = I_n - PP^{\dagger}$, $F_Q = I_n - Q^{\dagger}Q$. Then,*

$$B^{\bigotimes} = (B^*E_PB + aQ^*Q)^{-1}B^*E_P \tag{4.7}$$

$$= F_QB^*(BF_QB^* + aPP^*)^{-1}. \tag{4.8}$$

Proof. We show that $B^*E_PB + aQ^*Q$ is nonsingular. Assume that $(B^*E_PB + aQ^*Q)x = 0$ for some $x \in \mathbb{C}^n$. Then, we have $aQ^*Qx = -B^*E_PBx$,

$$x \in \mathcal{R}(Q^*Q) \cap \mathcal{R}(B^*E_PB) = \mathcal{R}(Q^*) \cap \mathcal{R}(B^*E_P) = \mathcal{R}(B^*B^k)^{\perp} \cap \mathcal{R}(B^*B^k) = \{0\},$$

which implies $Q^*Qx = 0$ and $B^*E_PBx = 0$. Hence $Qx = 0$, $E_PBx = 0$ yield

$$x \in \mathcal{N}(Q) \cap \mathcal{N}(E_PB) = \mathcal{R}(B^*B^k) \cap \mathcal{R}(B^*B^k)^{\perp} = \{0\}.$$

Thus $x = 0$ and $B^*E_PB + aQ^*Q$ is nonsingular. Hence, since $\mathcal{R}(B^{\bigotimes}) = \mathcal{R}(B^*B^k) = \mathcal{N}(Q)$, we get $QB^{\bigotimes} = O$. By Lemma 2.2 (b) and $\mathcal{N}(E_P) = \mathcal{R}(P) = \mathcal{N}((B^k)^*)$, we obtain $E_PBB^{\bigotimes} = E_P$. Therefore, $B^{\bigotimes} = (B^*E_PB + aQ^*Q)^{-1}B^*E_P$.

Similarly, (4.8) can be verified. □

5. Some properties of gMP inverse

In [15], the authors discussed equivalent conditions for $B^{\otimes} \in B\{1\}$, $B^{\otimes} = B^{\textcircled{\dagger}}$. We now consider the relationships between the gMP inverse and other generalized inverses using core-EP decomposition.

For convenience, we introduce several matrix classes. Symbols $\mathbb{C}_n^{\mathrm{CM}}$, $\mathbb{C}_n^{\mathrm{EP}}$ and $\mathbb{C}_n^{\mathrm{OP}}$ stand for the sets of all core matrices, EP-matrices and orthogonal projectors, respectively, i.e.,

$$\begin{aligned}
\mathbb{C}_n^{\mathrm{CM}} &= \{B \mid B \in \mathbb{C}^{n\times n}, r(B) = r(B^2)\},\\
\mathbb{C}_n^{\mathrm{EP}} &= \{B \mid B \in \mathbb{C}^{n\times n}, \mathcal{R}(B) = \mathcal{R}(B^*)\},\\
\mathbb{C}_n^{\mathrm{OP}} &= \{B \mid B \in \mathbb{C}^{n\times n}, B^2 = B = B^*\}.
\end{aligned}$$

Theorem 5.1. *Let $B \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) $B^{\otimes} \in B\{1\}$;
(b) $B \in \mathbb{C}_n^{\mathrm{CM}}$;
(c) $B^{\otimes} = B^{\dagger}$;
(d) $BB^{\otimes} = P_B$;
(e) $B^{\otimes}B = P_{B^*}$.

Proof. Assume that B is given by (2.1) and $\triangle = [TT^* + S(I_{n-t} - N^{\dagger}N)S^*]^{-1}$.

$(a) \Leftrightarrow (b)$. Using (2.1) and (2.9), we get

$$\begin{aligned}
BB^{\otimes}B = B \Leftrightarrow & \begin{bmatrix} T & S \\ NS^*(TT^* + SS^*)^{-1}T & NS^*(TT^* + SS^*)^{-1}S \end{bmatrix} = \begin{bmatrix} T & S \\ O & N \end{bmatrix}\\
\Leftrightarrow & \; NS^*(TT^* + SS^*)^{-1}T = O,\; NS^*(TT^* + SS^*)^{-1}S = N\\
\Leftrightarrow & \; NS^* = O,\; N = O\\
\Leftrightarrow & \; N = O\\
\Leftrightarrow & \; B \in \mathbb{C}_n^{\mathrm{CM}}.
\end{aligned}$$

$(b) \Leftrightarrow (c)$. Using (2.2) and (2.9), we obtain

$$\begin{aligned}
B^{\otimes} = B^{\dagger} \quad \Leftrightarrow \quad & \begin{bmatrix} T^*(TT^* + SS^*)^{-1} & O \\ S^*(TT^* + SS^*)^{-1} & O \end{bmatrix} = \begin{bmatrix} T^*\triangle & -T^*\triangle SN^{\dagger} \\ (I_{n-t} - N^{\dagger}N)S^*\triangle & N^{\dagger} - (I_{n-t} - N^{\dagger}N)S^*\triangle SN^{\dagger} \end{bmatrix}\\
\Leftrightarrow \quad & T^*(TT^* + SS^*)^{-1} = T^*\triangle,\; SN^{\dagger} = O, N^{\dagger} = O,\; S^*(TT^* + SS^*)^{-1} = (I_{n-t} - N^{\dagger}N)S^*\triangle\\
\Leftrightarrow \quad & N = O\\
\Leftrightarrow \quad & B \in \mathbb{C}_n^{\mathrm{CM}}.
\end{aligned}$$

$(c) \Rightarrow (d)$. It is obvious.

$(d) \Rightarrow (a)$. It follows by multiplying $BB^{\otimes} = P_B$ from the right side by B.

$(c) \Rightarrow (e) \Rightarrow (a)$. It is similar to $(c) \Rightarrow (d) \Rightarrow (a)$. □

Theorem 5.2. *Let $B \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

(a) $B \in \mathbb{C}_n^{\mathrm{EP}}$;
(b) $B^{\otimes} = B^{\textcircled{\#}}$;
(c) $B^{\otimes} = B^{\#}$;
(d) $B^{\otimes} B = P_B$;
(e) $BB^{\otimes} = P_{B^*}$.

Proof. Assume that B is given by (2.1) and $\triangle = [TT^* + S(I_{n-t} - N^{\dagger}N)S^*]^{-1}$.

$(a) \Leftrightarrow (b)$. Using (2.8) and (2.9), we get

$$\begin{aligned} B^{\otimes} = B^{\textcircled{\#}} &\Leftrightarrow \begin{bmatrix} T^*(TT^* + SS^*)^{-1} & O \\ S^*(TT^* + SS^*)^{-1} & O \end{bmatrix} = \begin{bmatrix} T^{-1} & O \\ O & O \end{bmatrix} \text{and } N = O \\ &\Leftrightarrow N = O,\ S = O \\ &\Leftrightarrow B \in \mathbb{C}_n^{\mathrm{EP}}. \end{aligned}$$

$(a) \Leftrightarrow (c)$. We can verify that $B^{\otimes} = B^{\#}$ is equivalent with $N = O$, $S = O$, that is $B \in \mathbb{C}_n^{\mathrm{EP}}$.

By (2.1) and (2.2), we have

$$BB^{\dagger} = U \begin{bmatrix} I_t & O \\ O & NN^{\dagger} \end{bmatrix} U^*, \tag{5.1}$$

$$B^{\dagger}B = U \begin{bmatrix} T^*\triangle T & T^*\triangle S(I_{n-t} - N^{\dagger}N) \\ (I_{n-t} - N^{\dagger}N)S^*\triangle T & NN^{\dagger} + (I_{n-t} - N^{\dagger}N)S^*\triangle S(I_{n-t} - N^{\dagger}N) \end{bmatrix} U^*. \tag{5.2}$$

Using (2.1) and (2.9), we get

$$BB^{\otimes} = U \begin{bmatrix} I_t & O \\ NS^*(TT^* + SS^*)^{-1} & O \end{bmatrix} U^*, \tag{5.3}$$

$$B^{\otimes}B = U \begin{bmatrix} T^*(TT^* + SS^*)^{-1}T & T^*(TT^* + SS^*)^{-1}S \\ S^*(TT^* + SS^*)^{-1}T & S^*(TT^* + SS^*)^{-1}S \end{bmatrix} U^*. \tag{5.4}$$

$(a) \Leftrightarrow (d)$. Compared (5.1) with (5.4), we get that $B^{\otimes}B = P_B$ is equivalent with $S = O$ and $N = O$, that is $B \in \mathbb{C}_n^{\mathrm{EP}}$.

$(a) \Leftrightarrow (e)$. Using (5.2) and (5.3), it can be verified that $BB^{\otimes} = P_{B^*}$ if and only if $B \in \mathbb{C}_n^{\mathrm{EP}}$. □

Theorem 5.3. *Let $B \in \mathbb{C}^{n\times n}$. The following are equivalent:*

(a) $B \in \mathbb{C}_n^{\mathrm{OP}}$;
(b) $B^{\otimes} = P_B$;
(c) $B^{\otimes} = P_{B^*}$;
(d) $BB^{\textcircled{\#}} = B^{\otimes}$;
(e) $BB^{\#} = B^{\otimes}$;
(f) $B^{\textcircled{\#}}B = B^{\otimes}$.

Proof. Assume that B is given by (2.1).

$(a) \Leftrightarrow (b)$. By (2.9) and (5.1),

$$\begin{aligned} B^{\otimes} = P_B &\Leftrightarrow \begin{bmatrix} T^*(TT^* + SS^*)^{-1} & O \\ S^*(TT^* + SS^*)^{-1} & O \end{bmatrix} = \begin{bmatrix} I_t & O \\ O & NN^{\dagger} \end{bmatrix} \\ &\Leftrightarrow N = O,\ S = O,\ T = I_t \\ &\Leftrightarrow B \in \mathbb{C}_n^{\mathrm{OP}}. \end{aligned}$$

$(a) \Leftrightarrow (c)$. Using (2.9) and (5.2), we have that $B^{\otimes} = P_{B^*}$ is equivalent with $S = O$, $N = O$ and $T = I_t$, that is $B \in \mathbb{C}_n^{\mathrm{OP}}$.

$(a) \Leftrightarrow (d)$. Using (2.1), (2.8) and (2.9), we get

$$\begin{aligned} B^{\otimes} = BB^{\textcircled{\#}} &\Leftrightarrow \begin{bmatrix} T^*(TT^*+SS^*)^{-1} & O \\ S^*(TT^*+SS^*)^{-1} & O \end{bmatrix} = \begin{bmatrix} I_t & O \\ O & O \end{bmatrix} \text{and } N = O \\ &\Leftrightarrow N = O,\ S = O,\ T = I_t \\ &\Leftrightarrow B \in \mathbb{C}_n^{\mathrm{OP}}. \end{aligned}$$

$(a) \Leftrightarrow (e)$. By (2.1), (2.8) and (2.9), we get

$$\begin{aligned} B^{\otimes} = BB^{\#} &\Leftrightarrow \begin{bmatrix} T^*(TT^*+SS^*)^{-1} & O \\ S^*(TT^*+SS^*)^{-1} & O \end{bmatrix} = \begin{bmatrix} I_t & T^{-1}S \\ O & O \end{bmatrix} \text{and } N = O \\ &\Leftrightarrow S = O,\ N = O,\ T = I_t \\ &\Leftrightarrow B \in \mathbb{C}_n^{\mathrm{OP}}. \end{aligned}$$

$(e) \Leftrightarrow (f)$. We observe that $B^{\textcircled{\#}}B = B^{\#}BB^{\dagger}B = B^{\#}B = BB^{\#}$. □

In [18, Definition 4.1], Wang and Liu defined the weak group matrices using the commutability: $BB^{\textcircled{w}} = B^{\textcircled{w}}B$. Inspired by that, we introduce the generalized Moore-Penrose matrices using gMP inverse.

Definition 1. A matrix $B \in \mathbb{C}^{n\times n}$ is called a generalized Moore-Penrose matrix (in short, gMP matrix) if $BB^{\otimes} = B^{\otimes}B$.

The set of all $n \times n$ gMP matrices is denoted by $\mathbb{C}_n^{\otimes}$, that is $\mathbb{C}_n^{\otimes} = \{B \mid B \in \mathbb{C}^{n\times n}, BB^{\otimes} = B^{\otimes}B\}$.

It is widely known that $B \in \mathbb{C}_n^{\mathrm{EP}}$ if and only if $B^{\#} = B^{\dagger}$. In order to get similar characterizations of the gMP matrices, we state the next lemma.

Lemma 5.4. *Let $B \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *The following conditions are equivalent:*

(a) $B \in \mathbb{C}_n^{\otimes}$;
(b) $S = O$;
(c) $\mathcal{R}(B^k) = \mathcal{R}(B^*B^k)$;
(d) $B^{\otimes} \in \mathbb{C}_n^{\mathrm{EP}}$.

Proof. $(a) \Leftrightarrow (b)$. It can be directly verified using (5.3) and (5.4).

$(a) \Leftrightarrow (c)$. By Lemma 2.2, we obtain

$$\begin{aligned} B \in \mathbb{C}_n^{\otimes} &\Leftrightarrow \mathcal{R}(BB^*B^k) = \mathcal{R}(B^*B^k) \text{ and } \mathcal{N}((B^k)^*) = \mathcal{N}((B^k)^*B) \\ &\Leftrightarrow \mathcal{R}(B^k) = \mathcal{R}(B^*B^k). \end{aligned}$$

$(d) \Leftrightarrow (c)$. Notice that $B^{\otimes} \in \mathbb{C}_n^{\mathrm{EP}}$ is equivalent with $\mathcal{R}(B^{\otimes}) = \mathcal{R}((B^{\otimes})^*)$. Using Lemma 2.2 (a), we can verify that $\mathcal{R}(B^{\otimes}) = \mathcal{R}((B^{\otimes})^*)$ if and only if $\mathcal{R}(B^k) = \mathcal{R}(B^*B^k)$. □

Remark 5.5. *Let $B \in \mathbb{C}_k^{n\times n}$ be given by* (2.1). *Using [6, Theorem 4.4], we know that B is a k-core EP matrix (that is, $B^kB^{\textcircled{\dagger}} = B^{\textcircled{\dagger}}B^k$) if and only if $S = O$. In [18, Theorem 2.4], the authors presented that B is i-EP matrix (that is, $BB^{\textcircled{\dagger}} = B^{\textcircled{\dagger}}B$) if and only if $S = O$. Thus, generalized Moore-Penrose matrices are the same as the k-core EP matrices and i-EP matrices.*

Theorem 5.6. *Let $B \in \mathbb{C}^{n\times n}$. The following conditions are equivalent:*

(a) $B \in \mathbb{C}_n^{\otimes}$;
(b) $B^{\otimes} = B^{D}$;
(c) $B^{\otimes} = B^{D,\dagger}$;
(d) $B^{\otimes} = B^{\textcircled{\dagger}}$;
(e) $B^{\otimes} = B^{\dagger,D}$;
(f) $B^{\otimes} = B^{\textcircled{w}}$.

Proof. Let $B \in \mathbb{C}^{n\times n}$ be given by (2.1).

If $B \in \mathbb{C}_n^{\otimes}$, by Lemma 5.4 (b), we have that $(b)-(f)$ hold.

On the contrary, we only need to prove that each of the conditions $(b)-(f)$ is equivalent with $S = O$.

$(b) \Rightarrow (a)$. If $B^{\otimes} = B^{D}$, by (2.3) and (2.9), we obtain $S = O$.

$(c) \Rightarrow (a)$. If $B^{\otimes} = B^{D,\dagger}$, it follows from (2.4) and (2.9) that $S = O$.

$(d) \Rightarrow (a)$. If $B^{\otimes} = B^{\textcircled{\dagger}}$, from (2.6) and (2.9), it is obvious that $S = O$.

$(e) \Rightarrow (a)$. If $B^{\otimes} = B^{\dagger,D}$, using (2.5) and (2.9), we obtain $\widetilde{T} = O$, which leads to $S = O$.

$(f) \Rightarrow (a)$. If $B^{\otimes} = B^{\textcircled{w}}$, it is obtained by (2.7) and (2.9) that $S = O$. □

6. Applications of gMP inverse

In present section, we consider the relationship between the gMP inverse and nonsingular bordered matrix, which is applied to the Cramer's rule of the restricted matrix equation.

Theorem 6.1. *Let $B \in \mathbb{C}_k^{n\times n}$ with $r(B^k) = t$. Assume that $P \in \mathbb{C}^{n\times(n-t)}$ and $Q \in \mathbb{C}^{(n-t)\times n}$ satisfy $r(P) = r(Q) = n - t$, $\mathcal{N}((B^k)^*) = \mathcal{R}(P)$ and $\mathcal{R}(B^*B^k) = \mathcal{N}(Q)$. Then the bordered matrix*

$$B_1 = \begin{bmatrix} B & P \\ Q & O \end{bmatrix}$$

is invertible with

$$B_1^{-1} = \begin{bmatrix} B^{\otimes} & (I_n - B^{\otimes}B)Q^\dagger \\ P^\dagger(I_n - BB^{\otimes}) & P^\dagger(BB^{\otimes}B - B)Q^\dagger \end{bmatrix}.$$

Proof. Let $Z = \begin{bmatrix} B^{\otimes} & (I_n - B^{\otimes}B)Q^\dagger \\ P^\dagger(I_n - BB^{\otimes}) & P^\dagger(BB^{\otimes}B - B)Q^\dagger \end{bmatrix}$. We only need to verify that $B_1Z = I_{2n-t}$.

Since $\mathcal{R}(B^{\otimes}) = \mathcal{R}(B^*B^k) = \mathcal{N}(Q)$, we get $QB^{\otimes} = O$. Since Q is full row rank matrix, we get $QQ^\dagger = I_{n-t}$. Using

$$\mathcal{R}(I_n - BB^{\otimes}) = \mathcal{N}(BB^{\otimes}) = \mathcal{N}((B^k)^*) = \mathcal{R}(P) = \mathcal{R}(PP^\dagger),$$

we get $PP^\dagger(I_n - BB^{\otimes}) = I_n - BB^{\otimes}$. Then, we have

$$\begin{aligned} B_1Z &= \begin{bmatrix} BB^{\otimes} + PP^\dagger(I_n - BB^{\otimes}) & B(I_n - B^{\otimes}B)Q^\dagger + PP^\dagger(BB^{\otimes}B - B)Q^\dagger \\ QB^{\otimes} & Q(I_n - B^{\otimes}B)Q^\dagger \end{bmatrix} \\ &= \begin{bmatrix} BB^{\otimes} + I_n - BB^{\otimes} & B(I_n - B^{\otimes}B)Q^\dagger - (I_n - BB^{\otimes})BQ^\dagger \\ O & QQ^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I_n & O \\ O & I_{n-t} \end{bmatrix}. \end{aligned}$$

Thus, $Z = B_1^{-1}$. The proof is completed. □

Using the gMP inverse, we will solve the restricted matrix equation.

Theorem 6.2. *Let $B \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times m}$, $D \in \mathbb{C}^{n\times m}$. If $\mathcal{R}(D) \subseteq \mathcal{R}(BB^*B^k)$, then the restricted matrix equation*

$$BX = D, \quad \mathcal{R}(X) \subseteq \mathcal{R}(B^*B^k) \tag{6.1}$$

has unique solution $X = B^{\otimes}D$.

Proof. If $\mathcal{R}(D) \subseteq \mathcal{R}(BB^*B^k)$, then $BB^{\otimes}D = P_{\mathcal{R}(BB^*B^k),\mathcal{N}((B^k)^*)}D = D$. Clearly, $X = B^{\otimes}D$ is a solution of (6.1). $X = B^{\otimes}D$ also satisfies the restricted condition because $\mathcal{R}(X) \subseteq \mathcal{R}(B^{\otimes}) = \mathcal{R}(B^*B^k)$. Finally, we show the uniqueness. If X_1 also satisfies (6.1), then

$$X = B^{\otimes}D = B^{\otimes}BX_1 = P_{\mathcal{R}(B^*B^k)}X_1 = X_1,$$

since $\mathcal{R}(X_1) \subseteq \mathcal{R}(B^*B^k)$. □

Next, we show a Cramer's rule for solving the restricted matrix equation (6.1).

Theorem 6.3. *Let $B \in \mathbb{C}_k^{n\times n}$, $X, D \in \mathbb{C}^{n\times m}$. Suppose that P and Q^* are full column rank matrices which satisfy $\mathcal{N}((B^k)^*) = \mathcal{R}(P)$ and $\mathcal{R}(B^*B^k) = \mathcal{N}(Q)$. Then the elements of the unique solution $X = [x_{ij}]$ of the restricted matrix equation* (6.1) *are given by*

$$x_{ij} = \frac{det\begin{bmatrix} B(i \to d_j) & P \\ Q(i \to 0) & O \end{bmatrix}}{det\begin{bmatrix} B & P \\ Q & O \end{bmatrix}}, \ i = 1, 2, ..., n, \ j = 1, 2, ..., m, \tag{6.2}$$

where d_j denotes the j-th column of D.

Proof. Since X is the solution of the restricted matrix equation (6.1), we have $\mathcal{R}(X) \subseteq \mathcal{R}(B^*B^k) = \mathcal{N}(Q)$. It follows that $QX = O$ and

$$\begin{bmatrix} B & P \\ Q & O \end{bmatrix}\begin{bmatrix} X & O \\ O & O \end{bmatrix} = \begin{bmatrix} BX & O \\ O & O \end{bmatrix} = \begin{bmatrix} D & O \\ O & O \end{bmatrix}.$$

By Theorem 6.1, we have $X = B^{\otimes}D$. Now, (6.2) follows by the Cramer's rule. □

Example 6.4. *Let B and $B^{\otimes}$ as in Example* 3.8. *Let*

$$D = \begin{bmatrix} 18 & 12 & -36 \\ 0 & 16 & 32 \\ 9 & 6 & -18 \\ 0 & 0 & 0 \end{bmatrix}, \ P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

We get

$$B^*B^2 = \begin{bmatrix} 8 & 0 & 4 & 10 \\ 0 & 8 & 0 & 0 \\ 4 & 0 & 2 & 5 \\ 4 & 0 & 2 & 5 \end{bmatrix}, \ BB^*B^2 = \begin{bmatrix} 24 & 0 & 12 & 30 \\ 0 & 16 & 0 & 0 \\ 12 & 0 & 6 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check $\mathcal{R}(D) \subseteq \mathcal{R}(BB^*B^2)$. *Also, the matrix*

$$X = B^{\otimes} D = \begin{bmatrix} 6 & 4 & -12 \\ 0 & 8 & 16 \\ 3 & 2 & -6 \\ 3 & 2 & -6 \end{bmatrix}$$

satisfies matrix equation $BX = D$ *and restricted condition* $\mathcal{R}(X) \subseteq \mathcal{R}(B^*B^2)$. *After simple calculations, we observe that results of* (6.2) *are the same as the matrix* X.

7. Conclusions

In this paper, new characteristics of the gMP inverse are derived by using rang space, null space, matrix equations and projectors. Some representations for the gMP inverse of matrices are obtained. Several properties of the gMP inverse are discussed. Additionally, applications of the gMP inverse in solving restricted matrix equation are presented.

The future perspectives for research are proposed:

1). Iterative algorithms and splitting methods to compute the gMP inverse.

2). Perturbations and continuity of the gMP inverse could be studied.

3). Considering the gMP inverse of tensors.

4). Studying relation of the gMP inverse and some partial order.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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