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Research article

Some new dynamic Steffensen-type inequalities on a general time scale measure space

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Abstract: Our work is based on the multiple inequalities illustrated by Josip Pečarić in 2013, 1982 and Srivastava in 2017. With the help of a positive σ-finite measure, we generalize a number of those inequalities to a general time scale measure space. Besides that, in order to obtain some new inequalities as special cases, we also extend our inequalities to discrete and continuous calculus.

Keywords: Steffensen's inequality; dynamic inequality; dynamic integral; time scale
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1. Introduction

The renowned integral Steffensen's inequality [1] is written as: Let f and g be integrable functions on $[a, b]$ such that f is nonincreasing and $0 \leq g(\Im) \leq 1$ on $[a, b]$. Then

$$\int_{b-\lambda}^{b} f(\Im)d\Im \leq \int_{a}^{b} f(\Im)g(\Im)d\Im \leq \int_{a}^{a+\lambda} f(\Im)d\Im, \tag{1.1}$$

where $\lambda = \int_a^b g(\Im)d\Im$.

It is simple to notice that inequalities (1.1) are reversed if f is nondecreasing.

The discrete version of the Steffensen inequality [2] states:

Theorem A. Assume that $\{f(k)\}_{k=1}^{n}$ is a nonincreasing nonnegative real sequence and $\{g(k)\}_{k=1}^{n}$ is a real sequence such that $0 \leq g(k) \leq 1$ for every k. Furthermore, let that λ_1, $\lambda_2 \in \{1, \dots, n\}$ be such that $\lambda_2 \leq \sum_{k=1}^{n} g(k) \leq \lambda_1$. Then

$$\sum_{k=n-\lambda_2+1}^{n} f(k) \leq \sum_{k=1}^{n} f(k)g(k) \leq \sum_{k=1}^{\lambda_1} f(k). \tag{1.2}$$

Jakšetić et al. [3] established the following interesting results among many other similar results for a positive finite measure μ. States:

Theorem B. Let $\hat{\mu}$ be a positive finite measure on $\mathfrak{B}([a, b])$, and let f, $g : [a, b \to \mathbb{R}$ be measurable functions on $[a, b]$ such that f is nonincreasing and $0 \leq g(\mathfrak{I}) \leq 1$ for all $t \in [a, b]$. Further, let $\hat{\mu}([c, d]) = \int_{[a,b]} g(\mathfrak{I}) d\hat{\mu}(\mathfrak{I})$, where $[c, d] \subseteq [a, b]$. Then

$$\int_{[a,b]} f(\mathfrak{I})g(\mathfrak{I})d\hat{\mu}(\mathfrak{I}) \leq \int_{[c,d]} f(\mathfrak{I})g(\mathfrak{I})d\hat{\mu}(\mathfrak{I}) + \int_{[a,c]} (f(\mathfrak{I}) - f(d))g(\mathfrak{I})d\hat{\mu}(\mathfrak{I}).$$

Also, the authors proved that:

Theorem C. Let f, $g : [a, b] \to \mathbb{R}$ be measurable functions on $[a, b]$ such that f is nonincreasing and $0 \leq g(\mathfrak{I}) \leq 1$ for all $t \in [a, b]$. Further, let $\hat{\mu}([c, d]) = \int_{[a,b]} g(\mathfrak{I}) d\hat{\mu}(\mathfrak{I})$, where $[c, d] \subseteq [a, b]$. If $\hat{\mu}$ is a positive finite measure on $\mathfrak{B}([a, b])$, then

$$\int_{[c,d]} f(\mathfrak{I})d\hat{\mu}(\mathfrak{I}) - \int_{[d,b]} (f(c) - f(\mathfrak{I}))g(\mathfrak{I})d\hat{\mu}(\mathfrak{I}) \leq \int_{[a,b]} f(\mathfrak{I})g(\mathfrak{I})d\hat{\mu}(\mathfrak{I}).$$

In 1982, Pečarić [4] gave speculation of the Steffensen inequality as the following two hypotheses.

Theorem 1.1. Let $\hat{f}$, $\hat{g}$, $\hat{h} : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$ such that $\hat{f}/\hat{h}$ is nonincreasing and $\hat{h}$ is nonnegative. Further, let $0 \leq \hat{g}(\mathfrak{I}) \leq 1$ $\forall \mathfrak{I} \in [a, b]$. Then

$$\int_a^b \hat{f}(\mathfrak{I})\hat{g}(\mathfrak{I})d\mathfrak{I} \leq \int_a^{a+\hat{\wp}} \hat{f} d\mathfrak{I}, \tag{1.3}$$

where $\hat{\wp}$ is the solution of the equation

$$\int_a^{a+\hat{\wp}} \hat{h}(\mathfrak{I})d\mathfrak{I} = \int_a^b \hat{h}(\mathfrak{I})\hat{g}(\mathfrak{I})d\mathfrak{I}.$$

We get the reverse of (1.3), if $f(\hat{\mathfrak{I}})/h(\hat{\mathfrak{I}})$ is nondecreasing.

Theorem 1.2. Let $\hat{f}$, $\hat{g}$, $\hat{h} : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$ such that $\hat{f}/\hat{h}$ is nonincreasing and $\hat{h}$ is nonnegative. Further, let $0 \leq \hat{g}(\mathfrak{I}) \leq 1$ $\forall \mathfrak{I} \in [a, b]$. Then

$$\int_{b-\hat{\wp}}^b \hat{f}(\mathfrak{I})d\mathfrak{I} \leq \int_a^b \hat{f}(\mathfrak{I})\hat{g}(\mathfrak{I})d\mathfrak{I}, \tag{1.4}$$

where $\hat{\wp}$ gives us the solution of

$$\int_{b-\hat{\wp}}^b \hat{h}(\mathfrak{I})d\mathfrak{I} = \int_a^b \hat{h}(\mathfrak{I})\hat{g}(\mathfrak{I})d\mathfrak{I}. \tag{1.5}$$

We get the reverse of (1.4), if $f(\hat{\mathfrak{I}})/h(\hat{\mathfrak{I}})$ is nondecreasing.

Wu and Srivastava in [5] acquired the accompanying result.

Theorem 1.3. Let $\hat{f}$, $\hat{g}$, $\hat{h} : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$ such that $\hat{f}$ is nonincreasing. Further, let $0 \le \hat{g}(\Im) \le \hat{h}(\Im)\ \forall \Im \in [a, b]$. Then the following integral inequalities hold true:

$$\begin{aligned}\int_{b-\hat{\wp}}^{b} \hat{f}(\Im)\hat{h}(\Im)d\Im &\le \int_{b-\hat{\wp}}^{b} \Big(\hat{f}(\Im)\hat{h}(\Im) - [\hat{f}(\Im) - \hat{f}(b-\hat{\wp})][\hat{h}(\Im) - \hat{g}(\Im)]\Big)d\Im \\ &\le \int_{a}^{b} \hat{f}(\Im)\hat{g}(\Im)d\Im \\ &\le \int_{a}^{a+\hat{\wp}} \Big(\hat{f}(\Im)\hat{h}(\Im) - [\hat{f}(\Im) - \hat{f}(a+\hat{\wp})][\hat{h}(\Im) - \hat{g}(\Im)]\Big)d\Im \\ &\le \int_{a}^{a+\hat{\wp}} \hat{f}(\Im)\hat{h}(\Im)d\Im,\end{aligned}$$

where $\hat{\wp}$ gives us the solution of

$$\int_{a}^{a+\hat{\wp}} \hat{h}(\Im)d\Im = \int_{a}^{b} \hat{g}(\Im)d\Im = \int_{b-\hat{\wp}}^{b} \hat{h}(\Im)d\Im.$$

The following interesting findings were published in [6].

Theorem 1.4. Suppose the integrability of $\hat{g}$, $\hat{h}$, $\hat{f}$, $\psi : [a, b] \to \mathbb{R}$ such that $\hat{f}$ is nonincreasing. Also suppose $0 \le \hat{\psi}(\Im) \le \hat{g}(\Im) \le \hat{h}(\Im) - \hat{\psi}(\Im)$ for all $\Im \in [a, b]$. Then

$$\int_{a}^{b} \hat{f}(\Im)\hat{g}(\Im)d\Im \le \int_{a}^{a+\hat{\wp}} \hat{f}(\Im)\hat{h}(\Im)d\Im - \int_{a}^{b} \Big|(\hat{f}(\Im) - \hat{f}(a+\hat{\wp}))\psi(\Im)\Big|d\Im,$$

where $\hat{\wp}$ is given by

$$\int_{a}^{a+\hat{\wp}} \hat{h}(\Im)d\Im = \int_{a}^{b} \hat{g}(\Im)d\Im.$$

Theorem 1.5. Under the hypotheses of Theorem 1.4. Then

$$\int_{b-\hat{\wp}}^{b} \hat{f}(\Im)\hat{h}(\Im)d\Im + \int_{a}^{b} \Big|(\hat{f}(\Im) - \hat{f}(b-\hat{\wp}))\hat{\psi}(\Im)\Big|d\Im \le \int_{a}^{b} \hat{f}(\Im)\hat{g}(\Im)d\Im,$$

where $\hat{\wp}$ is given by

$$\int_{b-\hat{\wp}}^{b} \hat{h}(\Im)d\Im = \int_{a}^{b} \hat{g}(\Im)d\Im.$$

The calculus of time scales with the intention to unify discrete and continuous analysis (see [7]) was proposed by Hilger [8]. For more details on the time scales calculus we refer to the book by Bohner and Peterson [9].

Lately, several dynamic inequalities on time scales has been investigated by using exclusive authors who have been inspired with the aid of a few applications (see [10–18]). Some authors created different results regarding fractional calculus on time scales to provide associated dynamic inequalities (see [19–27]).

In this article, we explore new generalizations of the integral Steffensen inequality given in [4–6] via general time scale measure space with a positive σ-finite measure. We also retrieve some of the integral inequalities known in the literature as special cases of our tests.

2. Main results

In what follows $\mathfrak{B}([a,b]_{\mathbb{T}})$ is Borel σ-algebra $[a,b]$. Next, we enroll the accompanying suppositions for the verifications of our primary outcomes:

(S_1) $([a,b]_{\mathbb{T}}, \mathfrak{B}([a,b]_{\mathbb{T}}), \hat{\mu})$ is time scale measure space with a positive σ-finite measure on $\mathfrak{B}([a,b]_{\mathbb{T}})$.

(S_2) $\eta, \Upsilon, \Xi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ are $\Delta\hat{\mu}$-integrable functions on $[a,b]_{\mathbb{T}}$.

(S_3) η/Ξ is nonincreasing and Ξ is nonnegative.

(S_4) $0 \leq \Upsilon(\Im) \leq 1$ for all $\Im \in [a,b]_{\mathbb{T}}$.

(S_5) $\hat{\wp} \in [0,\infty)$.

(S_6) η is nonincreasing.

(S_7) $1 \leq \Upsilon(\Im) \leq \Xi(\Im)$ for all $\Im \in [a,b]_{\mathbb{T}}$.

(S_8) $0 \leq \psi(\Im) \leq \Upsilon(\Im) \leq \Xi(\Im) - \psi(\Im)$ for all $\Im \in [a,b]_{\mathbb{T}}$.

(S_9) $0 \leq M \leq \Upsilon(\Im) \leq 1 - M$ for all $\Im \in [a,b]_{\mathbb{T}}$.

(S_{10}) $0 \leq \psi(\Im) \leq \Upsilon(\Im) \leq 1 - \psi(\Im)$ for all $\Im \in [a,b]_{\mathbb{T}}$.

$\hat{\wp}$ is the solution of the equations listed below:

(S_{11}) $\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} = \int_{[a,b]_{\mathbb{T}}} \Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}$.

(S_{12}) $\int_{[b-\hat{\wp},b]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} = \int_{[a,b]_{\mathbb{T}}} \Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}$.

(S_{13}) $\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} = \int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu} = \int_{[b-\hat{\wp},b]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu}$.

(S_{14}) $\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} = \int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu}$.

(S_{15}) $\int_{[b-\hat{\wp},b]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} = \int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu}$.

Presently, we are prepared to state and explain the principle results that make bigger numerous effects inside the literature.

Theorem 2.1. Let S_1, S_2, S_3, S_4, S_5 and S_{11} be satisfied. Then

$$\int_{[a,b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} \leq \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \eta(\Im)\Delta\hat{\mu}. \tag{2.1}$$

We get the reverse of (2.1), if η/Ξ is nondecreasing.

Proof. From our hypotheses, we observe that,

$$\begin{aligned}&\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \eta(\Im)\Delta\hat{\mu} - \int_{[a,b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}\\ &= \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Xi(\Im)[1-\Upsilon(\Im)]\frac{\eta(\Im)}{\Xi(\Im)}\Delta\hat{\mu} - \int_{[a+\hat{\wp},b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}\end{aligned}$$

$$\geq \frac{\eta(a+\hat{\wp})}{\Xi(a+\hat{\wp})}\int_{[a,a+\hat{\wp}]_{\mathbb{T}}}\Xi(\Im)[1-\Upsilon(\Im)]\Delta\hat{\mu}-\int_{[a+\hat{\wp},b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\frac{\eta(a+\hat{\wp})}{\Xi(a+\hat{\wp})}\Big[\int_{[a,a+\hat{\wp}]_{\mathbb{T}}}\Xi(\Im)\Delta\hat{\mu}-\int_{[a,a+\hat{\wp}]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}\Big]-\int_{[a+\hat{\wp},b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\frac{\eta(a+\hat{\wp})}{\Xi(a+\hat{\wp})}\Big[\int_{[a,b]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}-\int_{[a,a+\hat{\wp}]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}\Big]-\int_{[a+\hat{\wp},b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\frac{\eta(a+\hat{\wp})}{\Xi(a+\hat{\wp})}\int_{[a+\hat{\wp},b]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}-\int_{[a+\hat{\wp},b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\int_{[a+\hat{\wp},b]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Big(\frac{\eta(a+\hat{\wp})}{\Xi(a+\hat{\wp})}-\frac{\eta(\Im)}{\Xi(\Im)}\Big)\Delta\hat{\mu}\geq 0.$$

The proof is complete. □

Remark 2.1. In case of $\mathbb{T}=\mathbb{R}$ and related to Lebesgue measure in Theorem 2.1, we recollect [4, Theorem 1].

Theorem 2.2. Assumptions S_1, S_2, S_3, S_4, S_5 and S_{12} imply

$$\int_{[b-\hat{\wp},b]_{\mathbb{T}}}\eta(\Im)\Delta\hat{\mu}\leq\int_{[a,b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}. \tag{2.2}$$

We get the reverse of (2.2), if η/Ξ is nondecreasing.

Proof. From our hypotheses, we observe that,

$$\int_{[b-\hat{\wp},b]_{\mathbb{T}}}\eta(\Im)\Delta\hat{\mu}-\int_{[a,b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\int_{[b-\hat{\wp},b]_{\mathbb{T}}}\Xi(\Im)[1-\Upsilon(\Im)]\frac{\eta(\Im)}{\Xi(\Im)}\Delta\hat{\mu}-\int_{[a,b-\hat{\wp}]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$\leq\frac{\eta(b-\hat{\wp})}{\Xi(b-\hat{\wp})}\int_{[b-\hat{\wp},b]_{\mathbb{T}}}\Xi(\Im)[1-\Upsilon(\Im)]\Delta\hat{\mu}-\int_{[a,b-\hat{\wp}]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\frac{\eta(b-\hat{\wp})}{\Xi(b-\hat{\wp})}\Big[\int_{[b-\hat{\wp},b]_{\mathbb{T}}}\Xi(\Im)\Delta\hat{\mu}-\int_{[b-\hat{\wp},b]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}\Big]-\int_{[a,b-\hat{\wp}]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\frac{\eta(b-\hat{\wp})}{\Xi(b-\hat{\wp})}\Big[\int_{[a,b]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}-\int_{[b-\hat{\wp},b]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}\Big]-\int_{[a,b-\hat{\wp}]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\frac{\eta(b-\hat{\wp})}{\Xi(b-\hat{\wp})}\int_{[a,b-\hat{\wp}]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Delta\hat{\mu}-\int_{[a,b-\hat{\wp}]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$=\int_{[a,b-\hat{\wp}]_{\mathbb{T}}}\Xi(\Im)\Upsilon(\Im)\Big(\frac{\eta(b-\hat{\wp})}{\Xi(b-\hat{\wp})}-\frac{\eta(\Im)}{\Xi(\Im)}\Big)\Delta\hat{\mu}\leq 0.$$

□

Remark 2.2. By observing Lebesgue measure in Theorem 2.2, and $\mathbb{T}=\mathbb{R}$, we recapture [4, Theorem 2].

We will need the following lemma to prove the subsequent results.

Lemma 2.1. Let S_1, S_2, S_5 hold, such that

$$\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} = \int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu} = \int_{[b-\hat{\wp},b]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu}.$$

Then

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} &= \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Big(\eta(\Im)\Xi(\Im) - [\eta(\Im) - \eta(a+\hat{\wp})][\Xi(\Im) - \Upsilon(\Im)]\Big)\Delta\hat{\mu} \\ &\quad + \int_{[a+\hat{\wp},b]_{\mathbb{T}}} [\eta(\Im) - \eta(a+\hat{\wp})]\Upsilon(\Im)\Delta\hat{\mu}, \qquad (2.3)\end{aligned}$$

and

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} &= \int_{[a,b-\hat{\wp}]_{\mathbb{T}}} [\eta(\Im) - \eta(b-\hat{\wp})]\Upsilon(\Im)\Delta\hat{\mu} \qquad (2.4) \\ &\quad + \int_{[b-\hat{\wp},b]_{\mathbb{T}}} \Big(\eta(\Im)\Xi(\Im) - [\eta(\Im) - \eta(b-\hat{\wp})][\Xi(\Im) - \Upsilon(\Im)]\Big)\Delta\hat{\mu}.\end{aligned}$$

Proof. The suppositions of the Lemma imply that

$$a \le a+\hat{\wp} \le b \qquad \text{and} \qquad a \le b-\hat{\wp} \le b.$$

Firstly, we prove the validity of the integral identity (2.3). Indeed, by direct computation, and from our hypotheses, we find that

$$\begin{aligned}&\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Big(\eta(\Im)\Xi(\Im) - [\eta(\Im) - \eta(a+\hat{\wp})][\Xi(\Im) - \Upsilon(\Im)]\Big)\Delta\hat{\mu} - \int_{[a,b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} \\ &= \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Big(\eta(\Im)\Xi(\Im) - \eta(\Im)\Upsilon(\Im) - [\eta(\Im) - \eta(a+\hat{\wp})][\Xi(\Im) - \Upsilon(\Im)]\Big)\Delta\hat{\mu} \\ &\quad + \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} - \int_{[a,b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} \\ &= \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \eta(a+\hat{\wp})[\Xi(\Im) - \Upsilon(\Im)]\Delta\hat{\mu} - \int_{[a+\hat{\wp},b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} \\ &= \eta(a+\hat{\wp})\Big(\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} - \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu}\Big) - \int_{[a+\hat{\wp},b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}. \qquad (2.5)\end{aligned}$$

Since

$$\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} = \int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu},$$

we have

$$\begin{aligned}&\eta(a+\hat{\wp})\Big(\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Xi(\Im)\Delta\hat{\mu} - \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu}\Big) - \int_{[a+\hat{\wp},b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} \\ &= \eta(a+\hat{\wp})\Big(\int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu} - \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu}\Big) - \int_{[a+\hat{\wp},b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}\end{aligned}$$

$$= \eta(a+\wp)\int_{[a+\wp,b]_{\mathbb{T}}}\Upsilon(\Im)\Delta\hat{\mu} - \int_{[a+\wp,b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}\Big)$$

$$= \int_{[a+\wp,b]_{\mathbb{T}}}[\eta(a+\wp)-\eta(\Im)]\Upsilon(\Im)\Delta\hat{\mu}. \tag{2.6}$$

Combination of (2.5) and (2.6) led to the required integral identity (2.3) asserted by the Lemma. The integral identity (2.4) can be proved similarly. The proof is done. □

Theorem 2.3. Suppose S_1, S_2, S_5, S_6, S_7 and S_{13} give

$$\begin{aligned}\int_{[b-\wp,b]_{\mathbb{T}}}\eta(\Im)\Xi(\Im)\Delta\hat{\mu} &\le \int_{[b-\wp,b]_{\mathbb{T}}}\Big(\eta(\Im)\Xi(\Im)-[\eta(\Im)-\eta(b-\wp)][\Xi(\Im)-\Upsilon(\Im)]\Big)\Delta\hat{\mu}\\ &\le \int_{[a,b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}\\ &\le \int_{[a,a+\wp]_{\mathbb{T}}}\Big(\eta(\Im)\Xi(\Im)-[\eta(\Im)-\eta(a+\wp)][\Xi(\Im)-\Upsilon(\Im)]\Big)\Delta\hat{\mu}\\ &\le \int_{[a,a+\wp]_{\mathbb{T}}}\eta(\Im)\Xi(\Im)\Delta\hat{\mu}.\end{aligned}$$

Proof. From our hypotheses. In perspective of the considerations that the function η is nonincreasing on $[a,b]$ and $0 \le \Upsilon(\Im) \le \Xi(\Im)$ for all $\Im \in [a,b]$, we infer that

$$\int_{[a,b-\wp]_{\mathbb{T}}}[\eta(\Im)-\eta(b-\wp)]\Upsilon(\Im)\Delta\hat{\mu} \ge 0, \tag{2.7}$$

and

$$\int_{[b-\wp,b]_{\mathbb{T}}}[\eta(b-\wp)-\eta(\Im)][\Xi(\Im)-\Upsilon(\Im)]\Delta\hat{\mu} \ge 0. \tag{2.8}$$

Using (2.3), (2.7) and (2.8), we find that

$$\int_{[b-\wp,b]_{\mathbb{T}}}\eta(\Im)\Xi(\Im)\Delta\hat{\mu} \le \int_{[b-\wp,b]_{\mathbb{T}}}\Big(\eta(\Im)\Xi(\Im)-[\eta(\Im)-\eta(b-\wp)][\Xi(\Im)-\Upsilon(\Im)]\Big)\Delta\hat{\mu} \le \int_{[a,b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}. \tag{2.9}$$

In the same way as above, we can prove that

$$\int_{[a,b]_{\mathbb{T}}}\eta(\Im)\Upsilon(\Im)\Delta\hat{\mu} \le \int_{[a,a+\wp]_{\mathbb{T}}}\Big(\eta(\Im)\Xi(\Im)-[\eta(\Im)-\eta(a+\wp)][\Xi(\Im)-\Upsilon(\Im)]\Big)\Delta\hat{\mu} \le \int_{[a,a+\wp]_{\mathbb{T}}}\eta(\Im)\Xi(\Im)\Delta\hat{\mu}, \tag{2.10}$$

The confirmation is finished by joining the integral inequalities (2.9) and (2.10). □

Remark 2.3. We can reclaim [5, Theorem 1] with the use of Lebesgue measure in Theorem 2.3, and $\mathbb{T} = \mathbb{R}$.

Theorem 2.4. Assume S_1, S_2, S_5, S_6, S_8 and S_{13} be fulfilled. Then

$$\int_{[b-\wp,b]_{\mathbb{T}}}\eta(\Im)\Xi(\Im)\Delta\hat{\mu} + \int_{[a,b]_{\mathbb{T}}}\Big|[\eta(\Im)-\eta(b-\wp)]\psi(\Im)\Big|\Delta\hat{\mu}$$

$$\leq \int_{[a,b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$
$$\leq \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \eta(\Im)\Xi(\Im)\Delta\hat{\mu} - \int_{[a,b]_{\mathbb{T}}} \Big|[\eta(\Im) - \eta(a+\hat{\wp})]\psi(\Im)\Big|\Delta\hat{\mu}. \quad (2.11)$$

Proof. From our hypotheses. Clearly function η is nonincreasing on $[a,b]$ and $0 \leq \psi(\Im) \leq \Upsilon(\Im) \leq \Xi(\Im) - \psi(\Im)$ for all $\Im \in [a,b]$, we obtain

$$\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} [\eta(\Im) - \eta(a+\hat{\wp})][\Xi(\Im) - \Upsilon(\Im)]\Delta\hat{\mu} + \int_{[a+\hat{\wp},b]_{\mathbb{T}}} [\eta(a+\hat{\wp}) - \eta(\Im)]\Upsilon(\Im)\Delta\hat{\mu}$$
$$= \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Big|\eta(\Im) - \eta(a+\hat{\wp})\Big|[\Xi(\Im) - \Upsilon(\Im)]\Delta\hat{\mu} + \int_{[a+\hat{\wp},b]_{\mathbb{T}}} \Big|\eta(a+\hat{\wp}) - \eta(\Im)\Big|\Upsilon(\Im)\Delta\hat{\mu}$$
$$\geq \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \Big|\eta(\Im) - \eta(a+\hat{\wp})\Big|\psi(\Im)\Delta\hat{\mu} + \int_{[a+\hat{\wp},b]_{\mathbb{T}}} \Big|\eta(a+\hat{\wp}) - \eta(\Im)\Big|\psi(\Im)\Delta\hat{\mu}$$
$$\geq \int_{[a,b]_{\mathbb{T}}} \Big|[\eta(\Im) - \eta(a+\hat{\wp})]\psi(\Im)\Big|\Delta\hat{\mu}.$$

Also

$$\int_{[a,a+\hat{\wp}]_{\mathbb{T}}} [\eta(\Im) - \eta(a+\hat{\wp})][\Xi(\Im) - \Upsilon(\Im)]\Delta\hat{\mu} + \int_{[a+\hat{\wp},b]_{\mathbb{T}}} [\eta(a+\hat{\wp}) - \eta(\Im)]\Upsilon(\Im)\Delta\hat{\mu}$$
$$\geq \int_{[a,b]_{\mathbb{T}}} \Big|[\eta(\Im) - \eta(a+\hat{\wp})]\psi(\Im)\Big|\Delta\hat{\mu}. \quad (2.12)$$

Similarly, we find that

$$\int_{[a,b-\hat{\wp}]_{\mathbb{T}}} [\eta(\Im) - \eta(b-\hat{\wp})]\Upsilon(\Im)\Delta\hat{\mu} + \int_{[b-\hat{\wp},b]_{\mathbb{T}}} [\eta(b-\hat{\wp}) - \eta(\Im)][\Xi(\Im) - \Upsilon(\Im)]\Delta\hat{\mu}$$
$$\geq \int_{[a,b]_{\mathbb{T}}} \Big|[\eta(\Im) - \eta(b-\hat{\wp})]\psi(\Im)\Big|\Delta\hat{\mu}. \quad (2.13)$$

By combining (2.3), (2.4), (2.12) and (2.13), we arrive at the inequality (2.11) asserted by Theorem 2. □

Remark 2.4. If we take $\mathbb{T} = \mathbb{R}$, and consider the Lebesgue measure in Theorem 2.4, we recapture [5, Theorem 2].

In the following theorem, we use the additional parameters $\hat{\wp}_1, \hat{\wp}_2 \in [0,\infty)$.

Theorem 2.5. Let S_1, S_2, S_5, S_6, S_9 be satisfied, and

$$0 \leq \hat{\wp}_1 \leq \int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu} \leq \hat{\wp}_2 \leq b - a.$$

Then

$$\int_{[b-\hat{\wp}_1,b]_{\mathbb{T}}} \eta(\Im)\Delta\hat{\mu} + \eta(b)\Big(\int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu} - \hat{\wp}_1\Big) + M\int_{[a,b]_{\mathbb{T}}} \Big|\eta(\Im) - f\Big(b - \int_{[a,b]_{\mathbb{T}}} \Upsilon(\Im)\Delta\hat{\mu}\Big)\Big|\Delta\hat{\mu}$$
$$\leq \int_{[a,b]_{\mathbb{T}}} \eta(\Im)\Upsilon(\Im)\Delta\hat{\mu}$$

$$\leq \int_{[a,a+\hat{\wp}_2]_{\mathbb{T}}} \eta(\mathfrak{I})\Delta\hat{\mu} - \eta(b)\Big(\hat{\wp}_2 - \int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}\Big) - M\int_{[a,b]_{\mathbb{T}}} \Big|\eta(\mathfrak{I}) - f\Big(a + \int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}\Big)\Big|\Delta\hat{\mu}. \tag{2.14}$$

Proof. By using straightforward calculations, we have

$$\begin{aligned}
&\int_{[a,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Upsilon(\mathfrak{I})\Delta\hat{\mu} - \int_{[a,a+\hat{\wp}_2]_{\mathbb{T}}} \eta(\mathfrak{I})\Delta\hat{\mu} + \eta(b)\Big(\hat{\wp}_2 - \int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}\Big) \\
&= \int_{[a,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Upsilon(\mathfrak{I})\Delta\hat{\mu} - \int_{[a,a+\hat{\wp}_2]_{\mathbb{T}}} \eta(\mathfrak{I})\Delta\hat{\mu} + \int_{[a,a+\hat{\wp}_2]_{\mathbb{T}}} \eta(b)\Delta\hat{\mu} - \int_{[a,b]_{\mathbb{T}}} \eta(b)\Upsilon(\mathfrak{I})\Delta\hat{\mu} \\
&= \int_{[a,b]_{\mathbb{T}}} [\eta(\mathfrak{I}) - \eta(b)]\Upsilon(\mathfrak{I})\Delta\hat{\mu} - \int_{[a,a+\hat{\wp}_2]_{\mathbb{T}}} [\eta(\mathfrak{I}) - \eta(b)]\Delta\hat{\mu} \\
&\leq \int_{[a,b]_{\mathbb{T}}} [\eta(\mathfrak{I}) - \eta(b)]\Upsilon(\mathfrak{I})\Delta\hat{\mu} - \int_{[a,a+\int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}]} [\eta(\mathfrak{I}) - \eta(b)]\Delta\hat{\mu},
\end{aligned} \tag{2.15}$$

where we used the theorem's hypotheses

$$a \leq a + \hat{\wp}_1 \leq a + \int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu} \leq a + \hat{\wp}_2 \leq b,$$

and

$$\eta(\mathfrak{I}) - \eta(b) \geq 0 \quad for\ all \quad \mathfrak{I} \in [a, b].$$

The function $\eta(\mathfrak{I}) - \eta(b)$ is nonincreasing and integrable on $[a, b]$ and by applying Theorem 2 with $\Xi(\mathfrak{I}) = 1$, $\psi(\mathfrak{I}) = M$ and $\eta(\mathfrak{I})$ replaced by $\eta(\mathfrak{I}) - \eta(b)$, hence

$$\begin{aligned}
&\int_{[a,b]_{\mathbb{T}}} [\eta(\mathfrak{I}) - \eta(b)]\Upsilon(\mathfrak{I})\Delta\hat{\mu} - \int_{[a,a+\int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}]} [\eta(\mathfrak{I}) - \eta(b)]\Delta\hat{\mu} \\
&\qquad \leq -M\int_{[a,b]_{\mathbb{T}}} \Big|\eta(\mathfrak{I}) - f\Big(a + \int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}\Big)\Big|\Delta\hat{\mu}.
\end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we obtain

$$\begin{aligned}
&\int_{[a,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Upsilon(\mathfrak{I})\Delta\hat{\mu} - \int_{[a,a+\hat{\wp}_2]_{\mathbb{T}}} \eta(\mathfrak{I})\Delta\hat{\mu} + \eta(b)(\hat{\wp}_2 - \int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}) \\
&\qquad \leq -M\int_{[a,b]_{\mathbb{T}}} \Big|\eta(\mathfrak{I}) - f\Big(a + \int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}\Big)\Big|\Delta\hat{\mu},
\end{aligned} \tag{2.17}$$

which is the right-hand side inequality in (2.14).

Similarly, one can show that

$$\begin{aligned}
&\int_{[a,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Upsilon(\mathfrak{I})\Delta\hat{\mu} - \int_{[b-\hat{\wp}_1,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Delta\hat{\mu} + \eta(b)\Big(\int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu} - \hat{\wp}_2\Big) \\
&\geq \int_{[a,b]_{\mathbb{T}}} [\eta(\mathfrak{I}) - \eta(b)]\Upsilon(\mathfrak{I})\Delta\hat{\mu} + \int_{[b-\int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu},b]} [\eta(b) - \eta(\mathfrak{I})]\Delta\hat{\mu} \\
&\geq M\int_{[a,b]_{\mathbb{T}}} \Big|\eta(\mathfrak{I}) - f\Big(b - \int_{[a,b]_{\mathbb{T}}} \Upsilon(\mathfrak{I})\Delta\hat{\mu}\Big)\Big|\Delta\hat{\mu},
\end{aligned} \tag{2.18}$$

which is the left-hand side inequality in (2.14). □

Remark 2.5. [5, Theorem 3] can be obtained if $\mathbb{T} = \mathbb{R}$ and Lebesgue measure in Theorem 2.5.
Theorem 2.6. If S_1, S_2, S_5, S_6, S_7 and S_{14} hold. Then

$$\int_{[a,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Upsilon(\mathfrak{I})\Delta\hat{\mu} \leq \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \eta(\mathfrak{I})\Xi(\mathfrak{I})\Delta\hat{\mu} - \int_{[a,b]_{\mathbb{T}}} \Big|(\eta(\mathfrak{I}) - \eta(a+\hat{\wp}))\psi(\mathfrak{I})\Big|\Delta\hat{\mu}. \tag{2.19}$$

Proof. Follows similar to the proof of the right-hand side inequality in Theorem 2. □

Remark 2.6. If we take $\mathbb{T} = \mathbb{R}$, and consider the Lebesgue measure in Theorem 2.6, we recapture [6, Theorem 2.12].
Corollary 2.1. Hypotheses S_1, S_2, S_3, S_{10} and S_{11} yield

$$\int_{[a,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Upsilon(\mathfrak{I})\Delta\hat{\mu} \leq \int_{[a,a+\hat{\wp}]_{\mathbb{T}}} \eta(\mathfrak{I})\Delta\hat{\mu} - \int_{[a,b]_{\mathbb{T}}} \Big|\Big(\frac{\eta(\mathfrak{I})}{\Xi(\mathfrak{I})} - \frac{\eta(a+\hat{\wp})}{\Xi(a+\hat{\wp})}\Big)\Xi(\mathfrak{I})\psi(\mathfrak{I})\Big|\Delta\hat{\mu}. \tag{2.20}$$

Proof. Insert $\Upsilon(\mathfrak{I}) \mapsto \Xi(\mathfrak{I})\Upsilon(\mathfrak{I})$, $\eta(\mathfrak{I}) \mapsto \eta(\mathfrak{I})/\Xi(\mathfrak{I})$ and $\psi(\mathfrak{I}) \mapsto \Xi(\mathfrak{I})\psi(\mathfrak{I})$ in Theorem 2. □

Remark 2.7. [6, Corollary 2.3] can be recovered with the help of $\mathbb{T} = \mathbb{R}$, and Lebesgue measure in Corollary 2.1.
Theorem 2.7. If S_1, S_2, S_5, S_6, S_7 and S_{15} hold. Then

$$\int_{[b-\hat{\wp},b]_{\mathbb{T}}} \eta(\mathfrak{I})\Xi(\mathfrak{I})\Delta\hat{\mu} + \int_{[a,b]_{\mathbb{T}}} \Big|(\eta(\mathfrak{I}) - \eta(b-\hat{\wp}))\psi(\mathfrak{I})\Big|\Delta\hat{\mu} \leq \int_{[a,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Upsilon(\mathfrak{I})\Delta\hat{\mu}. \tag{2.21}$$

Proof. Carry out the same proof of the left-hand side inequality in Theorem 2. □

Remark 2.8. If we take $\mathbb{T} = \mathbb{R}$, and consider the Lebesgue measure in Theorem 2.7, we recapture [6, Theorem 2.13].
Corollary 2.2. Let S_1, S_2, S_3, S_9 and S_{12}, be fulfilled. Then

$$\int_{[b-\hat{\wp},b]_{\mathbb{T}}} \eta(\mathfrak{I})\Delta\hat{\mu} + \int_{[a,b]_{\mathbb{T}}} \Big|\Big(\frac{\eta(\mathfrak{I})}{\Xi(\mathfrak{I})} - \frac{\eta(b-\hat{\wp})}{\Xi(b-\hat{\wp})}\Big)\Xi(\mathfrak{I})\psi(\mathfrak{I})\Big|\Delta\hat{\mu} \leq \int_{[a,b]_{\mathbb{T}}} \eta(\mathfrak{I})\Upsilon(\mathfrak{I})\Delta\hat{\mu}. \tag{2.22}$$

Proof. Proof can be completed by taking $\Upsilon(\mathfrak{I}) \mapsto \Xi(\mathfrak{I})\Upsilon(\mathfrak{I})$, $\eta(\mathfrak{I}) \mapsto \eta(\mathfrak{I})/\Xi(\mathfrak{I})$ and $\psi(\mathfrak{I}) \mapsto \Xi(\mathfrak{I})\psi(\mathfrak{I})$ in Theorem 2. □

Remark 2.9. By letting $\mathbb{T} = \mathbb{R}$, and consider the Lebesgue measure in Corollary 2.2, we recapture [6, Corollary 2.4].

3. Conclusions

In this article, we explore new generalizations of the integral Steffensen inequality given in [4–6] via general time scale measure space with a positive σ-finite measure, we generalize a number of those inequalities to a general time scale measure space. Besides that, in order to obtain some new inequalities as special cases, we also extend our inequalities to discrete and constant calculus.

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Conflict of interest

The authors declare that there is no competing interest.

References

1. J. F. Steffensen, On certain inequalities between mean values and their application to actuarial problems, *Scand. Actuar. J.*, **1918** (1918), 82–97. https://doi.org/10.1080/03461238.1918.10405302
2. J. C. Evard, H. Gauchman, Steffensen type inequalities over general measure spaces, *Analysis*, **17** (1997), 301–322. https://doi.org/10.1524/anly.1997.17.23.301
3. J. Jakšetiĉ, J. Pečariĉ, K. S. Kalamir, Extension of Cerones generalizations of Steffensens inequality, *Jordan J. Math. Stat.*, **8** (2015), 179–194.
4. Pečarić, Josip E. Notes on some general inequalities, *Publ. I. Math. Beograd*, **32** (1982), 131–135.
5. S. H. Wu, H. M. Srivastava, Some improvements and generalizations of Steffensens integral inequality, *Appl. Math. Comput.*, **192** (2007), 422–428. https://doi.org/10.1016/j.amc.2007.03.020
6. J. Pečarić, A. Perušić, K. Smoljak, Mercer and Wu-Srivastava generalisations of Steffensens inequality, *Appl. Math. Comput.*, **219** (2013), 10548–10558. https://doi.org/10.1016/j.amc.2013.04.028
7. S. Hilger, Analysis on measure chains-A unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18–56. https://doi.org/10.1007/BF03323153
8. F. M. KH, A. A. El-Deeb, A. Abdeldaim, Z. A. Khan, On some generalizations of dynamic Opial-type inequalities on time scales, *Adv. Differ, Equ.*, **2019** (2019), 323. https://doi.org/10.1186/s13662-019-2268-0
9. M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Birkhäuser Boston, 2001.
10. R. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: A survey, *Math. Inequal. Appl.*, **4** (2001), 535–557. http://scholarbank.nus.edu.sg/handle/10635/103416
11. R. Agarwal, D. ORegan, S. Saker. *Dynamic inequalities on time scales*, Cham: Springer, 2014.
12. S. H. Saker, A. A. El-Deeb, H. M. Rezk, R. P. Agarwal, On Hilberts inequality on time scales, *Appl. Anal. Discrete Math.*, **11** (2017), 399–423. https://doi.org/10.2298/AADM170428001S
13. Y. Tian, A. A. El-Deeb, F. Meng, Some nonlinear delay Volterra-Fredholm type dynamic integral inequalities on time scales, *Discrete Dyn. Nat. Soc.*, **2018** (2018), 5841985. https://doi.org/10.1155/2018/5841985
14. M. U. Awan, N. Akhtar, S. Iftikhar, M. A. Noor, Y. M. Chu, New Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions, *J. Inequ. Appl.*, **2020** (2020), 125. https://doi.org/10.1186/s13660-020-02393-x
15. H. J. Hu, L. z. Liu, Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hormanders condition, *Math. Notes*, **101** (2017), 830–840. https://doi.org/10.1134/S0001434617050091

16. C. X. Huang, G. Sheng, L. Z. Liu, Boundedness on Morrey space for Toeplitz type operator associated to singular integral operator with variable Caldern-Zygmund kernel, *J. Math. Inequal.*, **3** (2014), 453–464. https://doi.org/10.7153/jmi-08-33

17. Y. M. Chu, H. Wang, T. H. Zhao, Sharp bounds for the Neuman mean in terms of the quadratic and second Seiffert means, *J. Inequal. Appl.*, **2014** (2014), 299. https://doi.org/10.1186/1029-242X-2014-299

18. C. X. Huang, L. Z. Liu, Sharp function inequalities and boundness for Toeplitz type operator related to general fractional singular integral operator, *Publ. I. Math. Beograd*, **92** (2012), 165–176. https://doi.org/10.2298/PIM1206165H

19. G. A. Anastassiou, Foundations of nabla fractional calculus on time scales and inequalities, *Comput. Math. Appl.*, **59** (2010), 3750–3762. https://doi.org/10.1016/j.camwa.2010.03.072

20. G. A. Anastassiou, Principles of delta fractional calculus on time scales and inequalities, *Math. Comput. Model.*, **52** (2010), 556–566. https://doi.org/10.1016/j.mcm.2010.03.055

21. G. A. Anastassiou, Integral operator inequalities on time scales, *Int. J. Differ. Equ.*, **7** (2012), 111–137.

22. W. N. Li, Some new dynamic inequalities on time scales, *J. Math. Anal. Appl.*, **319** (2006), 802–814. https://doi.org/10.1016/j.jmaa.2005.06.065

23. M. Sahir, Dynamic inequalities for convex functions harmonized on time scales, *J. Appl. Math. Phys.*, **5** (2017), 2360–2370. https://doi.org/10.4236/jamp.2017.512193

24. A. A. M. El-Deeb, O. Bazighifan, J. Awrejcewicz, A variety of dynamic Steffensen-type inequalities on a general time scale, *Symmetry*, **13** (2021), 1738. https://doi.org/10.3390/sym13091738

25. A. A. El-Deeb, S. D. Makharesh, D. Baleanu, Dynamic Hilbert-type inequalities with Fenchel-Legendre transform, *Symmetry*, **12** (2020), 582. https://doi.org/10.3390/sym12040582

26. A. A. El-Deeb, O. Bazighifan, J. Awrejcewicz, On some new weighted Steffensen-type inequalities on time scales, *Mathematics*, **9** (2021), 2670. https://doi.org/10.3390/math9212670

27. A. A. El-Deeb, D. Baleanu, New weighted Opial-type inequalities on time scales for convex functions, *Symmetry*, **12** (2020), 842. https://doi.org/10.3390/sym12050842

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