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Research article

New classes of few-weight ternary codes from simplicial complexes

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Abstract: In this article, we describe two classes of few-weight ternary codes, compute their minimum weight and weight distribution from mathematical objects called simplicial complexes. One class of codes described here has the same parameters with the binary first-order Reed-Muller codes. A class of (optimal) minimal linear codes is also obtained in this correspondence.

Keywords: simplicial complexes; ternary codes; minimal linear codes; optimal linear codes
Mathematics Subject Classification: 94A60, 94B05

1. Introduction

Recently, several infinite families of minimal and optimal linear codes are constructed via mathematical objects named simplicial complexes or down-sets by Hyun and Wu et al [3, 5, 7, 8, 12, 13]. Simplicial complexes are extremely well-behaved with the n-variable generating function, which in turn enable us to compute the exponential sum rather efficiently. Let n be a natural number and denote by $[n] = \{1, 2, \ldots, n\}$ the set of integers from 1 to n. For $\Delta \subseteq \mathbb{P}([n])$, we say Δ is a simplicial complex if $u \in \Delta$ and $v \subseteq u$ imply $v \in \Delta$, where $\mathbb{P}([n])$ denotes the power set of $[n]$. The set-inclusion defines a partial order on Δ. A maximal element of a simplicial complex Δ is an element of Δ that is not smaller than any other element in Δ. For subsets A_i of $[n]$, where $i \in [S]$, the notation $\langle A_1, A_2, \ldots, A_s\rangle$ means it is a simplicial complex generated by $\{A_1, A_2, \ldots, A_s\}$,that is $\langle A_1, A_2, \ldots, A_s\rangle = \{B : B \subseteq A_i, i \in [S]\}$. Especially, when $s = 1$, we write $\langle A_1\rangle$ simply as Δ_{A_1}.

Ternary codes of small dimension have been investigated in many literatures, see for instance [2, 6, 9–11]. A class of group character ternary codes $C_3(1, n-1)$ with parameters $[2^{n-1}, n, 2^{n-2}]$, which are the analogue of the binary first-order Reed-Muller codes RM$(1, n-1)$ are described and analyzed by Ding et al. [4]. In this paper, we describe a new class of $[2^{n-1}, n, 2^{n-2}]$ ternary codes, and determine their weigt distributions.

Minimal linear codes, though existing as special linear codes, have important applications in secret sharing and secure two-party computation. Construction of minimal linear codes with new and desirable parameters would be an interesting topic in coding theory and cryptography. We construct in this paper a family of minimal linear codes over $\mathbb{F}_3$, and compute their weight distributions. By a distance-optimal code, or simply an optimal code, we mean it has the highest minimum distance with a prescribed length and dimension. One class of these minimal codes we obtained is proved to be optimal.

2. Linear Codes and n-variable generating functions

In this paper we study a linear code with more flexible lengths as follows. Let P be a subset of $\mathbb{F}_3^n$, and we order the elements of P to fix a coordinate position of vectors. A ternary code C_P associated with P is defined to be

$$C_P = \{c_P(u) = (u \cdot x)_{x \in P} : u \in \mathbb{F}_3^n\}.$$

It is straightforward that C_P is a linear code of length $|P|$ and its dimension is at most n.

For a subset P of $\mathbb{F}_3^n$ and $u \in \mathbb{F}_3^n$, we define the exponential sum with respect to P by

$$\chi_u(P) = \sum_{v \in P} \zeta^{u \cdot v},$$

where ζ is a primitive 3-rd root of the unity. Then the Hamming weight of a codeword $c_P(u)$ in C_P is given as follows:

$$w(c_P(u)) = \left|P\right| - \sum_{v \in P} \delta_{\mathbf{0}, u \cdot v} = \left|P\right| - \frac{1}{3} \sum_{y \in \mathbb{F}_3} \sum_{v \in P} \zeta^{y(u \cdot v)} = \left|P\right| - \frac{1}{3}\Big(\left|P\right| + 2\mathrm{Re}\Big(\sum_{v \in P} \zeta^{u \cdot v}\Big)\Big) = \frac{2}{3}\Big(\left|P\right| - \mathrm{Re}(\chi_u(P))\Big) \tag{2.1}$$

where δ is the Kronecker delta function and $\mathrm{Re}(\chi_u(P))$ is the real part of $\chi_u(P)$. The main difficulty of the computation of $w(c_P(u))$ lies in the fact that it is expressed as the exponential sum with respect to a subset P which in turn is hard to compute for an arbitrary P.

When P contains the zero-vector of $\mathbb{F}_3^n$, we are also interested in C_{P^c} where P^c denotes the complement of P, that is

$$C_{P^c} = \{c_{P^c}(u) = (u \cdot x)_{x \in P^c} : u \in \mathbb{F}_3^n\}.$$

Then the weight of $c_{P^c}(u)$ and that of $c_P(u)$ are related as follows:

$$w(c_{P^c}(u)) = 2 \cdot 3^{n-1}(1 - \delta_{0,u}) - w(c_P(u)). \tag{2.2}$$

For the purpose of computing the exponential sum $\chi_u(P)$, we introduce the following n-variable generating function associated with P inspired by Adamaszek [1]:

$$\mathcal{H}_P(x_1, x_2, \ldots, x_n) = \sum_{v \in P} \prod_{i=1}^{n} x_i^{v_i} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

where we denote $v = (v_1, v_2, \ldots, v_n)$ if $v \in \mathbb{F}_3^n$. By convention, we define $\mathcal{H}_P(x_1, x_2, \ldots, x_n) = 0$ if $P = \emptyset$.

Example 1. *Let* $P = \{(1, -1, -1, \ldots, -1)\}$*, then the generating function is*

$$\mathcal{H}_P(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_2 x_3 \cdots x_n}.$$

In general, one can easily obtain the following result when $P = (\mathbb{F}_3^*)^n$

$$\mathcal{H}_P(x_1, x_2, \ldots, x_n) = \frac{1}{x_1 x_2 \cdots x_n} \prod_{i=1}^{n} (1 + x_i^2).$$

3. Another class of $[2^{n-1}, n, 2^{n-2}]$ ternary codes

For the vector space $\mathbb{F}_3^n$, we consider the subset $(\mathbb{F}_3^*)^n$. We give as follows a bijection

$$\begin{aligned} \psi : (\mathbb{F}_3^*)^n &\longrightarrow \mathbb{P}([n]) \\ u = (u_1, u_2, \ldots, u_n) &\mapsto \psi(u) \end{aligned}$$

where $\psi(u) = \{i : u_i = 1\}$. Through the given map ψ, a simplicial complex Δ of $\mathbb{P}([n])$ will be regarded as the simplicial complex of $(\mathbb{F}_3^*)^n$, and be identified as a subset of $\mathbb{F}_3^n$ in this section without any real ambiguity.

Example 2. *Let* Δ *be the simplicial complex of* $(\mathbb{F}_3^*)^4$ *generated by* $\{1, 2\}$ *and* $\{3, 4\}$*. Then*

$$\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}\}$$

which is identified with

$$\{(-1, -1, -1, -1), (1, -1, -1, -1), (-1, 1, -1, -1), (-1, -1, 1, -1), (-1, -1, -1, 1), (1, 1, -1, -1), (-1, -1, 1, 1)\}.$$

The indicator function $\mathbb{1}_\Delta$ from $\mathbb{F}_3^n$ to $\mathbb{F}_2$ is defined by $\mathbb{1}_\Delta(u) = 1$ only if $u \in \Delta$. The following lemma, which is a simple consequence of the Inclusion-exclusion principle, will be used in deriving an identity involving $\mathcal{H}_\Delta(x_1, \ldots, x_n)$.

Lemma 3.1. *Let* $\Delta = \langle A_1, A_2, \ldots, A_t \rangle$ *be a simplicial complex of* $(\mathbb{F}_3^*)^n$*. Then*

$$\mathbb{1}_\Delta(u) = \sum_{k=1}^{t} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le t} \mathbb{1}_{\Delta_{A_{i_1}} \cap \cdots \cap \Delta_{A_{i_k}}}(u).$$

Proof. Since Δ is a simplicial complex of $(\mathbb{F}_3^*)^n$, we have $\Delta = \cup_{j=1}^{t} \Delta_{A_j}$. The result follows from the Inclusion–exclusion principle. □

Proposition 3.2. *Let* Δ *be a simplicial complex of* $(\mathbb{F}_3^*)^n$ *with* $\mathcal{F}$ *the set of maximal elements of* Δ*. Then we have*

$$\mathcal{H}_\Delta(x_1, \ldots, x_n) = \frac{1}{x_1 x_2 \cdots x_n} \sum_{\emptyset \ne S \subseteq \mathcal{F}} (-1)^{|S|+1} \prod_{i \in \cap S} (1 + x_i^2)$$

where we define $\prod_{i \in \emptyset} (1 + x_i^2) = 1$ *by convention.*

Proof. Let $\Delta = \langle F_1, F_2, \ldots, F_t \rangle$, where $F_i \in \mathcal{F}$. Then we see that, by Lemma 3.1,

$$\begin{aligned}
\mathcal{H}_\Delta(x_1, \ldots, x_n) &= \sum_{u \in \Delta} \mathbb{1}_\Delta(u) \prod_{i=1}^{n} x_i^{u_i} \\
&= \sum_{u \in \Delta} \sum_{k=1}^{t} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le t} \mathbb{1}_{\Delta_{F_{i_1}} \cap \cdots \cap \Delta_{F_{i_k}}}(u) \prod_{i=1}^{n} x_i^{u_i} \\
&= \sum_{k=1}^{t} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le t} \mathcal{H}_{\Delta_{F_{i_1}} \cap \cdots \cap \Delta_{F_{i_k}}}(x_1, \ldots, x_n) \\
&= \sum_{k=1}^{t} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le t} \frac{1}{x_1 x_2 \cdots x_n} \prod_{i \in \cap_{j=1}^{k} F_{i_j}} (1 + x_i^2) \\
&= \frac{1}{x_1 x_2 \cdots x_n} \sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|+1} \prod_{i \in \cap S} (1 + x_i^2).
\end{aligned}$$

□

Example 3. *Let Δ be a simplicial complex of $(\mathbb{F}_3^*)^3$ with the set of maximal element $\mathcal{F} = \{\{1, 2\}, \{3\}\}$. Proposition 3.2 shows that*

$$\mathcal{H}_\Delta(x_1, x_2, x_3) = \frac{1}{x_1 x_2 x_3}\Big(1 + x_1^2 + x_2^2 + x_3^2 + x_1^2 x_2^2\Big) = \frac{1}{x_1 x_2 x_3}\Big((1 + x_1^2)(1 + x_2^2) + (1 + x_3^2) - 1\Big).$$

Lemma 3.3. *Let Δ be a simplicial complex of $(\mathbb{F}_3^*)^n$ with $\mathcal{F}$ the set of maximal elements of Δ. For $u \in \mathbb{F}_3^n$, we have that*

$$\mathrm{Re}(\chi_u(\Delta)) = \sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|+1} \prod_{i \in \cap S} (\zeta^{u_i} + \zeta^{-u_i}) \cdot \mathrm{Re}\Big(\prod_{i \notin \cap S} \zeta^{-u_i}\Big)$$

where we define $\prod_{i \in \emptyset} (\zeta^{u_i} + \zeta^{-u_i}) = \prod_{i \notin [n]} \zeta^{-u_i} = 1$ by convention.

Proof. According to Proposition 3.2, we get that

$$\begin{aligned}
\chi_u(\Delta) &= \mathcal{H}_\Delta(\zeta^{u_1}, \ldots, \zeta^{u_n}) \\
&= \frac{1}{\zeta^{\sum u_i}} \sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|+1} \prod_{i \in \cap S} (1 + \zeta^{2u_i}) \\
&= \sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|+1} \prod_{i \notin \cap S} \zeta^{-u_i} \prod_{i \in \cap S} (\zeta^{u_i} + \zeta^{-u_i}).
\end{aligned}$$

Since $\zeta^{u_i} + \zeta^{-u_i}$ is a real number for $u_i \in \mathbb{F}_3$, it follows that

$$\mathrm{Re}(\chi_u(\Delta)) = \sum_{\emptyset \neq S \subseteq \mathcal{F}} (-1)^{|S|+1} \prod_{i \in \cap S} (\zeta^{u_i} + \zeta^{-u_i}) \cdot \mathrm{Re}\Big(\prod_{i \notin \cap S} \zeta^{-u_i}\Big).$$

□

Theorem 3.4. *Let Δ be a simplicial complex of $(\mathbb{F}_3^*)^n$ with one maximal element $\{A\}$. If $|A|= n-1$, where $n \geq 2$, there are $\binom{n}{m}2^m$ codewords in the code C_Δ which have the same Hamming weight*

$$W(m) := 2^{n-m}\,\frac{2^m-(-1)^m}{3}$$

for any integer $0 \leq m \leq n$. Moreover, the minimum distance of C_Δ is $W(2)$, which is 2^{n-2}.

Proof. If $x \in \mathbb{F}_3$, then

$$\zeta^x + \zeta^{-x} = \begin{cases} 2, & \text{if } x = 0, \\ -1, & \text{otherwise.} \end{cases}$$

and

$$\mathrm{Re}(\zeta^{-x}) = \begin{cases} 1, & \text{if } x = 0, \\ -\frac{1}{2}, & \text{otherwise.} \end{cases}$$

Since $|A|= n-1$, denote $i_0 \in [n] \setminus A$. By Lemma 3.3, for a non-zero vector $u = (u_1, u_2, \ldots, u_n)$ in $\mathbb{F}_3^n$,

$$\mathrm{Re}(\chi_u(\Delta)) = \mathrm{Re}(\zeta^{-u_{i_0}}) \cdot \prod_{i\in A}(\zeta^{u_i} + \zeta^{-u_i}) = \begin{cases} (-1)^{n-1-k}2^k, & \text{if } u_{i_0} = 0, \\ (-1)^{n-k}2^{k-1}, & \text{otherwise.} \end{cases}$$

where $k = \#\{i : u_i = 0, i \in A\}$. According to equality (2.1), we obtain the Hamming weight of codeword $c_\Delta(u)$ as follows

$$w(c_\Delta(u)) = \begin{cases} 2^{k+1}\,\frac{2^{n-k-1}-(-1)^{n-k-1}}{3}, & \text{if } u_{i_0} = 0, \\ 2^k\,\frac{2^{n-k}-(-1)^{n-k}}{3}, & \text{otherwise.} \end{cases}$$

Let $m = \#\{i : u_i \neq 0, 1 \leq i \leq n\}$, then there are $\binom{n}{m}2^m$ codewords which have the Hamming weight

$$w(c_\Delta(u)) = W(m) := 2^{n-m}\,\frac{2^m-(-1)^m}{3}. \tag{3.1}$$

The nonzero weights $W(m)$ in (3.1) are pairwise distinct and satisfy

$$\begin{aligned} W(2) < W(4) < \cdots < W(2\lfloor n/2\rfloor) &< W(2\lfloor (n-1)/2\rfloor + 1) \\ &< W(2\lfloor (n-1)/2\rfloor - 1) < \cdots < W(3) < W(1). \end{aligned}$$

Hence, the minimum distance of C_Δ is $W(2)$. □

Example 4. *Let C_Δ be a linear code defined in Theorem 3.4. If $n = 5$, the weight distribution of the corresponding code is given in Table 1.*

Table 1. Weight distribution of C_Δ for $n = 5$ in Example 4.

Weight	Frequency
0	1
8	40
10	80
11	32
12	80
16	10

Corollary 3.5. *Let Δ be a simplicial complex of $(\mathbb{F}_3^*)^n$ with one maximal element $\{A\}$. If $|A|= n-1$, where $n \geq 2$, then C_Δ is a $[2^{n-1}, n, 2^{n-2}]$-code over $\mathbb{F}_3$.*

Proof. Since $|A|= n-1$, the length of C_Δ is 2^{n-1}. It then remains to prove the dimension is n. Let $\mathbf{e}_i$ be the vector of $\mathbb{F}_3^n$ whose i-th coordinate is 1 and other coordinates are all zero, $\mathbf{w}_i$ be the vector of $\mathbb{F}_3^n$ whose i-th coordinate is 1 and other coordinates are all -1, where $1 \leq i \leq n$. We denote by $A = \{i_1, i_2, \ldots, i_{n-1}\}$. Since Δ considered as a subset of $\mathbb{F}_3^n$ contains $\mathbf{w}_{i_1}, \mathbf{w}_{i_2}, \ldots, \mathbf{w}_{i_{n-1}}$, the codewords $c_\Delta(\mathbf{e}_i)$ of C_Δ are all nonzero. To finish the proof, we notice that $c_\Delta(\mathbf{e}_i)$ are linearly independent which generate any codeword of C_Δ. □

4. Minimal ternary codes

For the set $[n]$, we define

$$\mathcal{C}_2([n]) = \{(A, B) : A \subseteq [n], B \subseteq [n], A \cap B = \emptyset\}$$

to be the set of pairs of disjoint subsets of $[n]$. When Δ_1 and Δ_2 are two disjoint simplicial complexes of $\mathbb{P}([n])$, we consider the set

$$\mathcal{C}_2(\Delta_1, \Delta_2) = \{(A, B) : A \in \Delta_1, B \in \Delta_2\}.$$

Since $\Delta_1 \cap \Delta_2 = \emptyset$, we have $\mathcal{C}_2(\Delta_1, \Delta_2) \subseteq \mathcal{C}_2([n])$. Considering the vector space $\mathbb{F}_3^n$, there is a bijection

$$\begin{array}{rcl} \varphi = (\varphi_1, \varphi_2) : \mathbb{F}_3^n & \longrightarrow & \mathcal{C}_2([n]) \\ u = (u_1, u_2, \ldots, u_n) & \mapsto & (\varphi_1(u), \varphi_2(u)) \end{array}$$

where $\varphi_1(u) = \{i : u_i = 1\}$ and $\varphi_2(u) = \{j : u_j = -1\}$. The set $\mathcal{C}_2(\Delta_1, \Delta_2)$ given by two disjoint simplicial complexes, under the map φ, will be then identified with the subset of $\mathbb{F}_3^n$ without any real ambiguity.

Example 5. *Let Δ_1, Δ_2 be simplicial complexes of $\mathbb{P}([4])$ generated by $\{1, 2\}$ and $\{3, 4\}$. Then $\mathcal{C}_2(\Delta_1, \Delta_2)$ consists of elements*

$$(\emptyset, \emptyset)\ \ (\emptyset, \{3\})\ \ (\emptyset, \{4\})\ \ (\emptyset, \{3, 4\})\ \ (\{1\}, \emptyset)\ \ (\{1\}, \{3\})\ \ (\{1\}, \{4\})\ \ (\{1\}, \{3, 4\})$$

$$(\{2\}, \emptyset)\ \ (\{2\}, \{3\})\ \ (\{2\}, \{4\})\ \ (\{2\}, \{3, 4\}),\ \ (\{1, 2\}, \emptyset)\ \ (\{1, 2\}, \{3\})\ \ (\{1, 2\}\ \{4\})\ \ (\{1, 2\}, \{3, 4\})$$

which are identified with elements of $\mathbb{F}_3^n$ as follows

$$(0, 0, 0, 0)\ \ (0, 0, -1, 0)\ \ (0, 0, 0, -1)\ \ (0, 0, -1, -1)\ \ (1, 0, 0, 0)\ \ (1, 0, -1, 0)\ \ (1, 0, 0, -1)\ \ (1, 0, -1, -1)$$

$$(0, 1, 0, 0)\ \ (0, 1, -1, 0)\ \ (0, 1, 0, -1)\ \ (0, 1, -1, -1)\ \ (1, 1, 0, 0)\ \ (1, 1, -1, 0)\ \ (1, 1, 0, -1)\ \ (1, 1, -1, -1)$$

Proposition 4.1. *Let Δ_1, Δ_2 be simplicial complexes of $\mathbb{P}([n])$ with the family of maximal elements $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. If $\Delta_1 \cap \Delta_2 = \emptyset$, then we have*

$$\mathcal{H}_{\mathcal{C}_2(\Delta_1,\Delta_2)}(x_1,\ldots,x_n) = \sum_{\emptyset\neq S\subseteq\mathcal{F}_1}\sum_{\emptyset\neq T\subseteq\mathcal{F}_2}(-1)^{|S|+|T|+2}\prod_{i\in\cap S}(1+x_i)\cdot\prod_{j\in\cap T}(1+x_j^{-1})$$

where we define $\prod_{i\in\emptyset}(1+x_i) = \prod_{j\in\emptyset}(1+x_j^{-1}) = 1$.

Proof.

$$\begin{aligned}\mathcal{H}_{\mathcal{C}_2(\Delta_1,\Delta_2)}(x_1,\ldots,x_n) &= \sum_{(A,B)\in\mathcal{C}_2(\Delta_1,\Delta_2)}\prod_{i\in A}x_i\prod_{j\in B}x_j^{-1}\\ &= \Big(\sum_{A\in\Delta_1}\prod_{i\in A}x_i\Big)\cdot\Big(\sum_{B\in\Delta_2}\prod_{j\in B}x_j^{-1}\Big)\\ &= \sum_{\emptyset\neq S\subseteq\mathcal{F}_1}\sum_{\emptyset\neq T\subseteq\mathcal{F}_2}(-1)^{|S|+|T|+2}\prod_{i\in\cap S}(1+x_i)\cdot\prod_{j\in\cap T}(1+x_j^{-1})\end{aligned}$$

where the last equality is derived from [3, Theorem 1]. □

Example 6. *Let Δ_1, Δ_2 be simplicial complexes of $\mathbb{P}([3])$ with $\mathcal{F}_1 = \{\{1\}\}$ and $\mathcal{F}_2 = \{\{2\}\}$. Proposition 4.1 shows that $\mathcal{H}_{\mathcal{C}_2(\Delta_1,\Delta_2)}(x_1,\ldots,x_n) = (1+x_1)(1+x_2^{-1})$. Let $u = (u_1, u_2, \ldots, u_n)$, we have*

$$\begin{aligned}\mathrm{Re}(\chi_u(\mathcal{C}_2(\Delta_1,\Delta_2))) &= \mathrm{Re}((1+\zeta^{u_1})(1+\zeta^{-u_2}))\\ &= \begin{cases}4, & \text{if } u_1 = u_2 = 0,\\ -\frac{1}{2}, & \text{if } u_1 = -u_2 \neq 0,\\ 1, & \text{otherwise.}\end{cases}\end{aligned}$$

It then follows from (2.1) *that*

$$\begin{aligned}w(c_{\mathcal{C}_2(\Delta_1,\Delta_2)}(u)) &= \frac{2}{3}\Big(|\mathcal{C}_2(\Delta_1,\Delta_2)|-\mathrm{Re}(\chi_u(\mathcal{C}_2(\Delta_1,\Delta_2)))\Big)\\ &= \begin{cases}0, & \text{if } u_1 = u_2 = 0,\\ 3, & \text{if } u_1 = -u_2 \neq 0,\\ 2, & \text{otherwise.}\end{cases}\end{aligned}$$

It follows from (2.2) *that for $u \in (\mathbb{F}_3^n)^*$,*

$$w(c_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}(u)) = \begin{cases}2\cdot 3^{n-1}, & \text{if } u_1 = u_2 = 0,\\ 2\cdot 3^{n-1}-3, & \text{if } u_1 = -u_2 \neq 0,\\ 2\cdot 3^{n-1}-2, & \text{otherwise.}\end{cases}$$

Theorem 4.2. *Let $\Delta_1 = \langle\{r\},\{s\}\rangle$ and $\Delta_2 = \langle\{t\}\rangle$ be simplicial complexes of $\mathbb{P}([n])$, where $1 \le r, s, t \le n$ are pairwise distinct and $n \geq 3$. Then $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is a $[3^n - 6, n, 3^n - 3^{n-1} - 5]$-code and its weight distribution is given in Table 2.*

Table 2. Weight distribution of $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ in Theorem 4.2.

Weight	Frequency
0	1
$3^n - 3^{n-1}$	$3^{n-3} - 1$
$3^n - 3^{n-1} - 2$	$4 \cdot 3^{n-3}$
$3^n - 3^{n-1} - 3$	$8 \cdot 3^{n-3}$
$3^n - 3^{n-1} - 4$	$2 \cdot 3^{n-3}$
$3^n - 3^{n-1} - 5$	$12 \cdot 3^{n-3}$

Proof. The length of $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is $|\mathcal{C}_2(\Delta_1,\Delta_2)^c| = 3^n - 6$ and its dimension is n according to the proof of [5, Lemma 3.6-(ii)]. Since $\Delta_1 = \langle\{r\},\{s\}\rangle$ and $\Delta_2 = \langle\{t\}\rangle$, by Proposition 4.1, the generating function is

$$\mathcal{H}_{\mathcal{C}_2(\Delta_1,\Delta_2)}(x_1,\ldots,x_n) = (1+x_r)(1+x_t^{-1}) + (1+x_s)(1+x_t^{-1}) - (1+x_t^{-1}) = (1+x_t^{-1})(1+x_r+x_s).$$

Set $\mathcal{B}_i := \{(u_r,u_s,i) : u_r,u_s \in \mathbb{F}_3 \setminus \{-i\}\}$. Let $u = (u_1,u_2,\ldots,u_n)$, we have

$$\begin{aligned} \mathrm{Re}(\chi_u(\mathcal{C}_2(\Delta_1,\Delta_2))) &= \mathrm{Re}((1+\zeta^{-u_t})(1+\zeta^{u_r}+\zeta^{u_s})) \\ &= \begin{cases} 6, & \text{if } u_r = u_s = u_t = 0, \\ 3, & \text{if } u_r + u_s \neq 0, u_r u_s = u_t = 0, \\ \frac{3}{2}, & \text{if } (u_r,u_s,u_t) \in \mathcal{B}_{-1} \cup \mathcal{B}_1, \\ -\frac{3}{2}, & \text{if } u_r = u_s = -u_t \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It then follows from (2.1) that

$$\begin{aligned} w(c_{\mathcal{C}_2(\Delta_1,\Delta_2)}(u)) &= \frac{2}{3}\Big(|\mathcal{C}_2(\Delta_1,\Delta_2)| - \mathrm{Re}(\chi_u(\mathcal{C}_2(\Delta_1,\Delta_2)))\Big) \\ &= \begin{cases} 0, & \text{if } u_r = u_s = u_t = 0, \\ 2, & \text{if } u_r + u_s \neq 0, u_r u_s = u_t = 0, \\ 3, & \text{if } (u_r,u_s,u_t) \in \mathcal{B}_{-1} \cup \mathcal{B}_1, \\ 5, & \text{if } u_r = u_s = -u_t \neq 0, \\ 4, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows from (2.2) that for $u \in (\mathbb{F}_3^n)^*$,

$$w(c_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}(u)) = \begin{cases} 3^n - 3^{n-1}, & \text{if } u_r = u_s = u_t = 0, \\ 3^n - 3^{n-1} - 2, & \text{if } u_r + u_s \neq 0, u_r u_s = u_t = 0, \\ 3^n - 3^{n-1} - 3, & \text{if } (u_r,u_s,u_t) \in \mathcal{B}_{-1} \cup \mathcal{B}_1, \\ 3^n - 3^{n-1} - 5, & \text{if } u_r = u_s = -u_t \neq 0, \\ 3^n - 3^{n-1} - 4, & \text{otherwise.} \end{cases}$$

The frequency of each codeword of $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is computed by counting the vector u on its dimension. □

Remark 1. *Let $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ be a linear code defined in Theorem* 4.2*.*

1). Since $n \geq 3$, then

$$\frac{d}{d_{max}} = \frac{2 \cdot 3^{n-1} - 5}{2 \cdot 3^{n-1}} > \frac{2}{3}$$

where d and d_{max} are the minimum and maximum weights. Hence, $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is minimal.

2). In [5, Theorem 4.7], for instance, if $p = 3$ and $r = 1$, they obtain a linear code with the same parameters as $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ but with different weight distribution.

Theorem 4.3. *Let $\Delta_1 = \langle\{r, s\}\rangle$ and $\Delta_2 = \langle\{t\}\rangle$ be simplicial complexes of $\mathbb{P}([n])$, where $1 \leq r, s, t \leq n$ are pairwise distinct and $n \geq 3$. Then $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is an optimal $[3^n - 8, n, 3^n - 3^{n-1} - 6]$-code and its weight distribution is given in Table 3.*

Table 3. Weight distribution of $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ in Theorem 4.3.

Weight	Frequency
0	1
$3^n - 3^{n-1}$	$3^{n-3} - 1$
$3^n - 3^{n-1} - 4$	$12 \cdot 3^{n-3}$
$3^n - 3^{n-1} - 5$	$6 \cdot 3^{n-3}$
$3^n - 3^{n-1} - 6$	$8 \cdot 3^{n-3}$

Proof. The length of $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is $|\mathcal{C}_2(\Delta_1, \Delta_2)^c| = 3^n - 8$ and its dimension is n according to the proof of [5, Lemma 3.6-(ii)]. Since $\Delta_1 = \langle\{r, s\}\rangle$ and $\Delta_2 = \langle\{t\}\rangle$, by Proposition 4.1, the generating function is

$$\mathcal{H}_{\mathcal{C}_2(\Delta_1,\Delta_2)}(x_1, \ldots, x_n) = (1 + x_r)(1 + x_s)(1 + x_t^{-1}) = (1 + x_t^{-1})(1 + x_r + x_s + x_r x_s).$$

Set $\mathcal{M}_i = \{(u_r, u_s, i) : u_r + u_s \neq 0, u_r, u_s \in \mathbb{F}_3 \setminus \{i\}\}$. Let $u = (u_1, u_2, \ldots, u_n)$, we have

$$\begin{aligned} \mathrm{Re}(\chi_u(\mathcal{C}_2(\Delta_1, \Delta_2))) &= \mathrm{Re}((1 + \zeta^{-u_t})(1 + \zeta^{u_r} + \zeta^{u_s} + \zeta^{u_r+u_s})) \\ &= \begin{cases} 8, & \text{if } u_r = u_s = u_t = 0, \\ \frac{1}{2}, & \text{if } u_r = -u_s \neq 0, u_t \neq 0 \text{ or } u_r = u_s = u_t \neq 0, \\ -1, & \text{if } (u_r, u_s, u_t) \in \mathcal{M}_{-1} \cap \mathcal{M}_0 \cap \mathcal{M}_1, \\ 2, & \text{otherwise.} \end{cases} \end{aligned}$$

It then follows from (2.1) that

$$\begin{aligned} w(c_{\mathcal{C}_2(\Delta_1,\Delta_2)}(u)) &= \frac{2}{3}\Big(|\mathcal{C}_2(\Delta_1, \Delta_2)| - \mathrm{Re}(\chi_u(\mathcal{C}_2(\Delta_1, \Delta_2)))\Big) \\ &= \begin{cases} 0, & \text{if } u_r = u_s = u_t = 0, \\ 5, & \text{if } u_r = -u_s \neq 0, u_t \neq 0 \text{ or } u_r = u_s = u_t \neq 0, \\ 6, & \text{if } (u_r, u_s, u_t) \in \mathcal{M}_{-1} \cap \mathcal{M}_0 \cap \mathcal{M}_1, \\ 4, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows from (2.2) that for $u \in (\mathbb{F}_3^n)^*$,

$$w(c_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}(u)) = \begin{cases} 3^n - 3^{n-1}, & \text{if } u_r = u_s = u_t = 0, \\ 3^n - 3^{n-1} - 5, & \text{if } u_r = -u_s \neq 0, u_t \neq 0 \text{ or } u_r = u_s = u_t \neq 0, \\ 3^n - 3^{n-1} - 6, & \text{if } (u_r, u_s, u_t) \in \mathcal{M}_{-1} \cap \mathcal{M}_0 \cap \mathcal{M}_1, \\ 3^n - 3^{n-1} - 4, & \text{otherwise.} \end{cases}$$

The frequency of each codeword of $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is computed by counting the vector u on its dimension. To check the optimality, we assume that there is a $[3^n - 8, n, 3^n - 3^{n-1} - 5]$-code. Applying the Griesmer bound, we get that

$$3^n - 8 \geq \sum_{i=0}^{n-1} \left\lceil \frac{3^n - 3^{n-1} - 5}{3^i} \right\rceil = 3^n - 7,$$

which is a contradiction, so $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is optimal. □

Remark 2. *Let $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ be a linear code defined in Theorem* 4.3.

1). Since $n \geq 3$, then

$$\frac{d}{d_{max}} = \frac{2 \cdot 3^{n-1} - 3}{2 \cdot 3^{n-1}} > \frac{2}{3}$$

where d and d_{max} are the minimum and maximum weights. Hence, $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ is minimal.

2). The codes produced by our construction and the codes in [5] for $p = 3$ have totally different parameters. Meanwhile, with a slight change of Δ_1, the codes here and the codes in Theorem 4.2 are different.

5. Conclusions

The ternary codes C_Δ described in Theorem 3.4 have the same parameters and weight distributions as the group character codes $C_3(1, n-1)$. Thus, the ternary codes C_Δ may be viewed as the analogue of the group character codes $C_3(1, n-1)$. As a result, the codes C_Δ is good for practical error detection. As pointed in [4], the weight distribution of the codes C_Δ is given by the eigenvalues of the Hamming scheme. It may be interesting to investigate the relationship between these codes and the Hamming scheme.

The ternary codes $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ described in Theorem 4.2 and 4.3 have few weights and are minimal. Thus, the dual codes of $C_{\mathcal{C}_2(\Delta_1,\Delta_2)^c}$ may be utilized to construct secret sharing schemes.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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