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Research article

Portfolio selection based on uncertain fractional differential equation

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Abstract: Portfolio selection problems are considered in the paper. The securities in the proposed problems are suggested to follow uncertain fractional differential equations which have memory characteristics. By introducing the left semi-deviation of the wealth, two problems are proposed. One is to maximize the expected value and minimize the left semi-variance of the wealth. The other is to maximize the expected value of the wealth with a chance constraint that the left semi-deviation of the wealth is not less than a given number at a confidence level. The problems are equivalent to determinant ones which will be solved by genetic algorithm. Examples are provided to show the effectiveness of the proposed methods.

Keywords: portfolio selection; uncertain process; expected value; left semi-deviation; uncertain fractional differential equation; genetic algorithm
Mathematics Subject Classification: 91G10, 34A08

1. Introduction

Portfolio selection problem is a financial problem which optimizes the wealth of an investor by allocating the asset to different securities. Markowitz pioneered portfolio selection problem in 1952. Since then, many relative literatures about portfolio selection problem appeared, for example, [1, 3–7, 20, 25]. In those study, the rewards of investment to risk securities were assumed to be random variables or stochastic processes which follow Ito's stochastic differential equations. Most of introduced portfolio selection models are mean-variance based ones. Recently, entropy based portfolio problems were studied [2, 21].

Due to the complexity of financial market. A real stock price may be impossible to follow an Ito's stochastic differential equation according to Liu [16]. Then an uncertain differential equation driven by Liu process [15] was introduced to model a stock price.

In a period of time, a stock price may have memory characteristic. For such a case, it is better to model a stock price by uncertain fractional order differential equation introduced in [27] instead of an uncertain differential equation. There are some results of investigation on uncertain fractional order differential equation, such as [10–13, 17–19, 23, 24].

Based on uncertainty theory, some portfolio selection problems were studied, such as Huang [8, 9] investigated mean-risk model for uncertain portfolio selection. In 2010, Zhu [26] considered an uncertain portfolio selection problem by uncertain optimal control approach where a security was suggested to earn an uncertain return following an uncertain differential equation. The expected value and optimistic value of the return are maximized for uncertain portfolio selection problems in Zhu [26] and Sheng and Zhu [22], respectively.

In the paper, we will present two uncertain portfolio selection problems based on uncertain fractional order differential equations. One is to maximize the expected value and minimized the left semi-variance of the wealth at a final time T. The other is to maximize the expected value of the wealth at a final time T with a chance constraint that the left semi-deviation of the wealth is not less than $\hbar$ at a confidence level β. The problems will be transformed their equivalent forms by α-path of uncertain fractional order differential equation introduced in [18]. The equivalent problems would be solved by appropriate optimization methods.

In the following section, some concepts and results on uncertainty theory and uncertain fractional order differential equation will be recalled. Then two uncertain portfolio selection problems will be introduced. Next, the problems will be transformed to their equivalent forms. Finally, a numerical example will be given to validate the effectiveness of the proposed approaches.

2. Preliminary

In the section, we first review some concepts and results in uncertainty theory [14]. Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. Set function $\mathcal{M}$ from $\mathcal{L}$ to $[0, 1]$ is called an uncertain measure if it satisfies the following three axioms: $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ, and $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. A product uncertain measure was introduced to obtain an uncertain measure of a compound event. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{i=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$, where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable ξ is defined as a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set R of real numbers such that the set $\{\xi \in B\}$ is an event in $\mathcal{L}$ for any Borel set B. The uncertainty distribution $\Phi : R \to [0, 1]$ of ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for $x \in R$. A normal uncertain variable with expected value e and variance σ^2 has the uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \ x \in R,$$

which is denoted by $\xi \sim \mathcal{N}(e, \sigma)$. The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\}dr,$$

provided that at least one of the two integrals is finite. The variance of ξ is defined by $V[\xi] = E[(\xi - E[\xi])^2]$. If ξ has (regular) inverse distribution function $\Phi^{-1}(\alpha)$, $\alpha \in (0,1)$, then

$$E[f(\xi)] = \int_0^1 f(\Phi^{-1}(\alpha))\mathrm{d}\alpha \tag{2.1}$$

for monotone (increasing or decreasing) function $f(x)$.

The uncertain variables $\xi_1, \xi_2, \cdots \xi_m$ are said to be independent [15] if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i)\right\} = \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots B_m$ of real numbers. For numbers a and b, $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ if ξ and η are independent uncertain variables.

For a totally ordered set S and uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, Liu defined an uncertain process as a measurable function from $S \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

A Liu process is an uncertain process C_t which satisfies: (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) C_t has stationary and independent increments; (iii) Every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, denoted by $C_{s+t} - C_s \sim \mathcal{N}(0, t)$.

Remark 2.1. *Liu process is a counterpart of Wiener process which is used to model a stochastic differential equation. The main difference between the Wiener process and Liu process is that almost all sample paths of Wiener process are continuous (but non Lipschitz) and almost all sample paths of Liu process are Lipschitz continuous. In addition, variance of every increment $W_{s+t} - W_s$ of Wiener process W_s is t, and variance of every increment $C_{s+t} - C_s$ of Liu process C_s is t^2.*

For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1\le i\le k} |t_{i+1} - t_i|$. Then the uncertain integral of an uncertain process X_t with respect to C_t is defined by Liu [15] as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}),$$

provided that the limit exists almost surely and is finite.

Next we will recall some concepts and results about an uncertain fractional differential equation. For $0 < p \le 1$, a Caputo type of uncertain fractional differential equation driven by Liu process C_t is defined in [27] as

$${}^cD^p X_t = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t}, \quad t > 0 \tag{2.2}$$

for two given functions f and g. A solution X_t of the uncertain fractional differential equation (2.2) satisfies the following uncertain integral equation:

$$X_t = X_0 + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s, X_s)\mathrm{d}s + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} g(s, X_s)\mathrm{d}C_s, \quad t > 0,$$

where $\Gamma(p)$ is the Gamma function.

Definition 2.1. *[18] Let* $0 < \alpha < 1$. *The Caputo type of uncertain fractional differential equation (2.2) is said to be have an* α*-path* X_t^α *if it solves the corresponding fractional differential equation*

$$ {}^cD^pX_t^\alpha = f(t, X_t^\alpha) + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha), \tag{2.3}$$

where $\Phi^{-1}(\alpha)$ *is the inverse standard normal uncertainty distribution, i.e.,*

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Theorem 2.1. *[18] The solution* X_t *of (2.2) has an inverse distribution*

$$\Psi_t^{-1}(\alpha) = X_t^\alpha, \quad \alpha \in (0, 1),$$

where X_t^α *is the corresponding* α*-path which solves (2.3).*

Remark 2.2. *When we are provided empirical data by experts, we may fit the data by an uncertain (fractional) differential equation which is driven by Liu process.*

Remark 2.3. *In practice, we may have some historical or experts' empirical data. If we use an uncertain fractional differential equation to fit those data, the value of the order p and other parameters in the equation may be estimated by some methods such as moment approach or least square method.*

3. Portfolio selection problems

To begin with, we introduce an index to measure the risk of a security.

Definition 3.1. *The left semi-deviation of an uncertain variable* ξ *is defined by*

$$(\xi - E[\xi])^- = (\xi - E[\xi]) \wedge 0 = \begin{cases} \xi - E[\xi] & \text{if } \xi \le E[\xi]; \\ 0 & \text{otherwise.} \end{cases}$$

The left semi-variance of ξ *is defined by*

$$ {}_{LS}V[\xi] = E[\{(\xi - E[\xi])^-\}^2].$$

The left semi-variance may be regarded as a negative deviation from expected value for an uncertain variable.

Lemma 3.1. *Let the distribution of uncertain variable* ξ *be a regular function* $\Phi(x)$ *which is strictly increasing at point x with* $\Phi(x) > 0$. *We have*

$$ {}_{LS}V[\xi] = \int_0^1 \left\{ \left(\Phi^{-1}(\alpha) - \int_0^1 \Phi^{-1}(\theta)\mathrm{d}\theta \right)^- \right\}^2 \mathrm{d}\alpha. \tag{3.1}$$

Proof. It follows from (2.1) that

$$E[\xi] = \int_0^1 \Phi^{-1}(\theta)\mathrm{d}\theta.$$

Since left semi-deviation $(\xi - E[\xi])^-$ of ξ is negative and increasing in ξ, we know that $\{(\xi - E[\xi])^-\}^2$ is decreasing in ξ. Thus, we have

$$\begin{aligned} {}_{LS}V[\xi] &= E[\{(\xi - E[\xi])^-\}^2] \\ &= \int_0^1 \{(\Phi^{-1}(\alpha) - E[\xi])^-\}^2 \mathrm{d}\alpha \\ &= \int_0^1 \left\{\left(\Phi^{-1}(\alpha) - \int_0^1 \Phi^{-1}(\theta)\mathrm{d}\theta\right)^-\right\}^2 \mathrm{d}\alpha \end{aligned}$$

by (2.1). The lemma is proven.

As we know, a portfolio selection problem is to allocate personal wealth between investment in a risk security and investment in a risk-free asset. The risk investment is assumed to do under uncertain environment.

For the sake of convenience, we list main symbols in Table 1 used in the sequel.

Table 1. Symbols used in the paper.

Symbol	Description
p	order of fractional differential equation
X_t	wealth of an investor at time t
w	fraction of the wealth in a risk-free asset
b	rate of return in a risk-free asset
μ	draft coefficient in an uncertain fractional differential equation
σ	diffusion coefficient in an uncertain fractional differential equation
T	final time
$\hbar$	endurance level

Let X_t be the wealth of an investor at time t. The investor allocates a fraction w of the wealth in a risk-free asset and remainder in a risk asset at initial time. In the time interval $[0, T]$, the risk-free asset earns a rate of return b. The risk asset (stock) is assumed to earns an uncertain return. Since future price of a stock is dependent not only on the current price but also the previous prices, it is reasonable that an uncertain return is regarded to follow an uncertain fractional differential equation.

The wealth X_t is suggested to follow the following uncertain fractional differential equation:

$$ {}^cD^pX_t = bwX_t + \mu(1-w)X_t + \sigma(1-w)X_t\frac{\mathrm{d}C_t}{\mathrm{d}t},\ t \in [0, T], \tag{3.2}$$

where $0 < p \leq 1$, $\sigma > 0$ and $\mu \in R$.

By Theorem 2.1, the solution X_t of Eq (3.2) has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha,\ \alpha \in (0, 1),$$

where X_t^α is the corresponding α-path of Eq (3.2), which is the solution of the following fractional differential equation:

$$ {}^cD^pX_t^\alpha = bwX_t^\alpha + \mu(1-w)X_t^\alpha + \sigma(1-w)X_t^\alpha\Phi^{-1}(\alpha),\ t \in [0, T]. \tag{3.3}$$

Thus,

$$X_t^\alpha = X_0 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]t^p),\ t \in [0,T], \alpha \in (0,1), \tag{3.4}$$

where $E_{p,q}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kp+q)}$ is the Mittag-Leffler function, which is a convergent series for any p, $q > 0$ and complex number z. In approximate calculation, the value of Mittag-Leffler function at z may be given by a finite sum $E_{p,q}(z) \approx \sum_{k=1}^{N} \frac{z^k}{\Gamma(kp+q)}$ for an appropriate positive integer number N.

Now we will propose two problems for portfolio selection models based on the expected value and semi-variance subject to uncertain fractional differential equations.

Problem 1: We want to maximize the expected value and minimize the left semi-variance of the wealth at the final time T. That is

$$\begin{cases} \max\limits_{w\in[0,1]} E[X_T] - \lambda \cdot {}_{LS}V[X_T] \\ \text{subject to} \\ \quad {}^cD^pX_t = bwX_t + \mu(1-w)X_t + \sigma(1-w)X_t\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \quad t \in [0,T], \\ \quad X_0 = x_0, \end{cases} \tag{3.5}$$

where λ is a multiplier.

Problem 2: We want to maximize the expected value of the wealth at the final time T with a chance constraint that the left semi-deviation of the wealth is not less than an endurance level $\hbar$ at a confidence level β. That is

$$\begin{cases} \max\limits_{w\in[0,1]} E[X_T] \\ \text{subject to} \\ \quad \mathcal{M}\{(X_T - E[X_T])^- \geq \hbar\} \geq \beta \\ \quad {}^cD^pX_t = bwX_t + \mu(1-w)X_t + \sigma(1-w)X_t\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \quad t \in [0,T], \\ \quad X_0 = x_0. \end{cases} \tag{3.6}$$

Next we will discuss the equivalent forms of problems (3.5) and (3.6).

Theorem 3.1. *The problem (3.5) and the following optimization problem are equivalent.*

$$\begin{aligned} \max_{w\in[0,1]} \quad & x_0 \int_0^1 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p)\mathrm{d}\alpha \\ & -\lambda x_0^2 \int_0^1 \Big\{\Big(E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p) \\ & \quad - \int_0^1 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p)\mathrm{d}\alpha\Big)^-\Big\}^2 \mathrm{d}\alpha, \end{aligned} \tag{3.7}$$

where $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$.

Proof. It follows from (2.1) and (3.4) that

$$E\left[X_T\right] = \int_0^1 X_T^\alpha \mathrm{d}\alpha = x_0 \int_0^1 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p)\mathrm{d}\alpha. \tag{3.8}$$

It follows from Lemma 3.1 that

$$\begin{aligned} {}_{LS}V[X_T] &= \int_0^1 \left\{(X_T^\alpha - E[X_T])^-\right\}^2 \mathrm{d}\alpha \\ &= x_0^2 \int_0^1 \left\{\Big(E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p) \right. \\ &\quad \left. - \int_0^1 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p)\mathrm{d}\alpha\Big)^-\right\}^2 \mathrm{d}\alpha. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) deduces the conclusion. The theorem is proved.

For problem (3.6), we get its equivalent form as follows.

Theorem 3.2. *For $\hbar < 0$, the problem (3.6) and the following optimization problem are equivalent.*

$$\begin{cases} \max\limits_{w\in[0,1]} x_0 \displaystyle\int_0^1 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p)\mathrm{d}\alpha \\ \textit{subject to} \\ \quad \Big(x_0 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(1-\beta)]T^p) \\ \quad -x_0 \displaystyle\int_0^1 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p)\mathrm{d}\alpha\Big)^- \geq \hbar, \end{cases} \tag{3.10}$$

where $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$.

Proof. It follows from (3.8) that

$$E[X_T] = x_0 \int_0^1 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p)\mathrm{d}\alpha. \tag{3.11}$$

Since the inverse distribution of X_T is X_T^α, we know that $(X_T - E[X_T])^-$ has the inverse distribution $(X_T^\alpha - E[X_T])^-$. Thus, the constraint $\mathcal{M}\{(X_T - E[X_T])^- \geq \hbar\} \geq \beta$ is equivalent to

$$(X_T^{1-\beta} - E[X_T])^- \geq \hbar.$$

That is

$$\begin{aligned} &\Big(x_0 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(1-\beta)]T^p) \\ &-x_0 \int_0^1 E_{p,1}([bw + \mu(1-w) + \sigma(1-w)\Phi^{-1}(\alpha)]T^p)\mathrm{d}\alpha\Big)^- \geq \hbar. \end{aligned} \tag{3.12}$$

The theorem is proved.

4. Solution methods

Based on the analysis of the previous section, problems (3.5) and (3.6) are, respectively, equivalent to the optimization problems (3.7) and (3.10), which may be solved by genetic algorithm (GA). To solve problems (3.7) and (3.10), we have to calculate an integral of a function in $\alpha \in (0, 1)$ at first. Let $\epsilon > 0$ be small enough and $h = (1 - 2\epsilon)/n$. Divide the interval $[\epsilon, 1 - \epsilon]$ by $\epsilon = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = 1 - \epsilon$, where $\alpha_i = \epsilon + ih$ for $i = 1, 2, \cdots, n - 1$. For a function $f(\alpha)$, $\alpha \in (0, 1)$, the integral $\int_0^1 f(\alpha)\mathrm{d}\alpha$ is approximately calculated based on the Simpson's rule by

$$\int_0^1 f(\alpha)\mathrm{d}\alpha \approx \frac{h}{6}\sum_{i=0}^{n-1}\{f(\alpha_i) + 4f(\alpha_i + h/2) + f(\alpha_{i+1})\}. \tag{4.1}$$

5. Numerical examples

Example 5.1. For the problem (3.5), its equivalent form is the problem (3.7). Let $x_0 = 2$, $p = 0.7$, $b = 0.03$, $\mu = 0.08$, $\sigma = 0.03$, $T = 2$. We employ genetic algorithm (GA) to solve problem (3.7) and then get the optimal solution.

Table 2 shows the optimal allocations for different multipliers. The optimal allocations increase as the multipliers increase. That means that more allocation in a sure asset will be allowed if we wish to have less left semi-variance of the wealth.

Table 2. The optimal allocations relative to the multipliers.

Multiplier (λ)	Optimal allocation (w)	Multiplier (λ)	Optimal allocation (w)
10	0.00000029	60	0.7761
13	0.0030	80	0.8317
14	0.0710	100	0.8651
15	0.1303	120	0.8875
20	0.3410	140	0.9035
30	0.5564	160	0.9155
40	0.6657	200	0.9323

Example 5.2. For the problem (3.6), its equivalent form is the problem (3.10). Let $x_0 = 1$, $p = 0.7$, $b = 0.04$, $\mu = 0.008$, $\sigma = 0.03$, $T = 2$, $\hbar = -0.07$. We employ genetic algorithm (GA) to solve problem (3.10) and then get the optimal solution.

Table 3 shows the optimal allocations for different confidence levels. The optimal allocations increase as the confidence levels increase. That means that more allocation in a sure asset will be allowed if we wish to have more confidence level at which the left semi-deviation of the wealth is not less than $\hbar$.

Table 3. The optimal allocations relative to the confidence level.

Confidence level (β)	Optimal allocation (w)	Confidence level (β)	Optimal allocation (w)
0.7	2.9×10^{-7}	0.95	0.2933
0.8	2.9×10^{-7}	0.96	0.3412
0.88	2.9×10^{-7}	0.97	0.3936
0.89	0.0392	0.98	0.4543
0.90	0.0803	0.99	0.5334
0.91	0.1214	0.999	0.6841
0.92	0.1629	0.9999	0.7611
0.93	0.2050	0.99999	0.8080
0.94	0.2483	0.999999	0.8394

6. Conclusions

Two uncertain portfolio selection problems were established based on expected value criterion, where the risk security is suggested to follow an uncertain fractional differential equation. The problems are transformed to equivalent forms by the integrals of α-paths while the relative integrals are approximated by the compound Simpson formula. Genetic algorithm was employed to find the optimal solutions of the equivalent optimization problems. Numerical examples showed the effectiveness of the proposed methods. Also, in the examples, the relations of the solutions of the proposed problems and relative parameters were discussed. The main advantage of the proposed method is that it is suitable to deal with a portfolio selection problem in uncertain fractional cases. Of course, there may be a disadvantage that in practice, it would be not easy to fit empirical data by using an uncertain fractional differential equation. It is an interesting work in the future.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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