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Research article

Some subvarieties of semiring variety $\mathbf{COS}_3^+$

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Abstract: In this paper, we study some subvarieties of a semiring variety determined by certain additional identities. We first present alternative characterizations for equivalences $\overset{+}{\mathcal{H}}\cap\overset{\cdot}{\mathcal{L}}$, $\overset{+}{\mathcal{H}}\cap\overset{\cdot}{\mathcal{R}}$, $\overset{+}{\mathcal{H}}\cap\overset{\cdot}{\mathcal{D}}$, $\overset{+}{\mathcal{H}}\vee\overset{\cdot}{\mathcal{L}}$, $\overset{+}{\mathcal{H}}\vee\overset{\cdot}{\mathcal{R}}$, $\overset{+}{\mathcal{H}}\vee\overset{\cdot}{\mathcal{D}}$. Then we give the sufficient and necessary conditions for these equivalences to be congruence. Finally, we prove that semiring classes defined by these congruences are varieties and provide equational bases.

Keywords: semiring; Green's relation; congruence; variety of semirings; subvariety
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1. Introduction

According to [1], by a semiring we mean that an algebra $(S, +, \cdot)$ of type (2, 2) satisfies the identities:

$$(x + y) + z \approx x + (y + z), \tag{1.1}$$

$$(xy)z \approx x(yz), \tag{1.2}$$

$$x(y + z) \approx xy + xz, \tag{1.3}$$

$$(x + y)z \approx xz + yz. \tag{1.4}$$

We write just S instead of $(S, +, \cdot)$ for the sake of simplicity. The semigroup $(S, +)$ (resp., $(S, \cdot)$) is called additive (resp., multiplicative) reduct of S. A semiring S is called an additively (resp., a multiplicatively) idempotent semiring if $(S, +)$ (resp., $(S, \cdot)$) is idempotent, i.e., if it satisfies the identity $x + x \approx x$ (resp., $x^2 \approx x$). And S is called an idempotent semiring if both $(S, +)$ and $(S, \cdot)$ are idempotent, i.e., if it satisfies the identities $x + x \approx x$ and $x^2 \approx x$ (see [2]).

A variety of algebras is a class of algebras of the same type that is closed under the formation of subalgebras, homomorphic images and direct products (see [3]). It is well known (Birkhoff's theorem) that a class of algebras of the same type is a variety if and only if it is an equational class. Thus, all semirings form a variety. From [4], all idempotent semirings also form a variety, denoted by I.

By the definition of semiring, we know that a semiring can be regarded as two semigroups connected by distributive law over same nonempty set. So Green's relations on semigroups are usually used to study semirings, especially idempotent semirings. It is necessary to note that $\dot{\mathcal{L}}$ and $\overset{+}{\mathcal{L}}$ denote $\mathcal{L}$-relation on $(S, \cdot)$ and $(S, +)$, respectively. The same remark applies to other Green's relations. Pastijn and Petrich [5] studied the variety of idempotent distributive semirings and showed that both $\overset{+}{\mathcal{D}}$ and $\dot{\mathcal{D}}$ are congruences and $\dot{\mathcal{D}}\overset{+}{\mathcal{D}}\dot{\mathcal{D}} = \dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}} = \overset{+}{\mathcal{D}}\dot{\mathcal{D}}\overset{+}{\mathcal{D}}$ for such a semiring. Pastijn and Zhao [6] proved that idempotent semirings where $\dot{\mathcal{D}}$ is the congruence form a subvariety of I and gave various characterizations for idempotent semirings where $\dot{\mathcal{D}}$ is the least lattice congruence. Zhao [7] introduced a congruence ρ on the power semiring $(P(S), \cup, \circ)$ of a semigroup S and proved that $P(S)/\rho$ is an idempotent semiring if S is a band. Pastijn and Guo [8] showed that for a semiring S whose additive reduct is a completely regular semigroup, every $\overset{+}{\mathcal{H}}$-class is the maximal subring of S and hence S is the disjoint union of these maximal subrings. Pastijn and Zhao [9] proved that S in $\overset{+}{\mathrm{S}\ell}$ belongs to $(\dot{\mathrm{NB}})\cap\overset{+}{\mathrm{S}\ell}$ if $\dot{\mathcal{L}}$ and $\dot{\mathcal{R}}$ are congruences on S or $\dot{\mathcal{D}}$ is the congruence on S. Wang et al. [10] showed that the additive reduct of a band semiring is a regular band. Ghosh et al. [11] proved that $\overset{+}{\mathcal{D}}$ is the least $\overset{+}{\mathrm{S}\ell}$-congruence for a idempotent semiring and the multiplicative reduct of S is a regular band if $S\in\overset{+}{\mathrm{S}\ell}$. Pastijn [12] studied the variety U of idempotent semirings satisfying the additional identity $xy + yx \approx yx + xy$ and showed that $\overset{+}{\mathcal{D}}\cap\dot{\mathcal{D}}$ is the least U-congruence on any idempotent semiring. In general, Green's relations on reducts of a semiring are not congruences on the semiring [13]. Damljanović et al. [2] obtained the congruence openings of Green's relations on the additive reduct of a semiring and studied the variety of additively idempotent semirings. Cheng and Shao [14, 15] defined and studied several varieties of multiplicative idempotent semirings by means of congruence openings of multiplicative Green's relations on a semiring. Zhao et al. [13] defined and studied some subvarieties of I by the join and meet of Green's $\mathcal{L}$ relation and other Green's relations. Zhao et al. [16] defined and studied some subvarieties of I by the join and meet of Green's $\mathcal{D}$ relation and other Green's relations, and proved that $\dot{\mathcal{D}}$ is the congruence on an idempotent semiring if and only if both $\overset{+}{\mathcal{D}}\cap\dot{\mathcal{D}}$ and $\overset{+}{\mathcal{D}}\vee\dot{\mathcal{D}}$ are congruences on the idempotent semiring.

In this paper, we use COS_3^+ to denote the variety of semirings satisfying the following additional identities:

$$3x \approx x, \tag{1.5}$$

$$x^2 \approx x, \tag{1.6}$$

$$2x + 2y \approx 2(x + y). \tag{1.7}$$

Obviously, COS_3^+ contains I in [13], $\mathrm{R}_2 \circ \mathrm{D}$ in [11] and $\mathscr{BLR}$ in [1]. Thus, COS_3^+ can be regarded as the generalization of these varieties of semirings. Suppose $S \in \mathrm{COS}_3^+$. Then the additive reduct of S is a completely regular semigroup and the multiplicative reduct of S is a band.

We call $\overset{+}{\mathcal{L}}$, $\overset{+}{\mathcal{R}}$, $\overset{+}{\mathcal{H}}$, $\overset{+}{\mathcal{D}}$ and $\overset{+}{\mathcal{J}}$ (resp., $\dot{\mathcal{L}}$, $\dot{\mathcal{R}}$, $\dot{\mathcal{H}}$, $\dot{\mathcal{D}}$ and $\dot{\mathcal{J}}$) additive (resp., multiplicative) Green's relations on S. By Theorem II 4.5 in [17], we have that if $S \in \mathrm{COS}_3^+$, then $\overset{+}{\mathcal{D}} = \overset{+}{\mathcal{J}}$ and $\dot{\mathcal{D}} = \dot{\mathcal{J}}$. Thus, we only need to consider one of them. It follows from [18] that for all $a, b\in S$

$$a \overset{+}{\mathcal{L}} b \Longleftrightarrow (\exists p, q\in S^0)\ a = p + b,\ b = q + a;$$

$$a \overset{+}{\mathcal{R}} b \Longleftrightarrow (\exists p, q \in S^0)\ a = b + p,\ b = a + q;$$
$$a \dot{\mathcal{L}} b \Longleftrightarrow (\exists p, q \in S^1)\ a = pb,\ b = qa;$$
$$a \dot{\mathcal{R}} b \Longleftrightarrow (\exists p, q \in S^1)\ a = bp,\ b = aq.$$

Thus, $\overset{+}{\mathcal{H}} = \overset{+}{\mathcal{L}} \cap \overset{+}{\mathcal{R}}$ and $\overset{+}{\mathcal{D}} = \overset{+}{\mathcal{L}} \vee \overset{+}{\mathcal{R}} = \overset{+}{\mathcal{L}} \circ \overset{+}{\mathcal{R}} = \overset{+}{\mathcal{R}} \circ \overset{+}{\mathcal{L}}$. In this paper, we omit the sign $\circ$ and write it directly as $\overset{+}{\mathcal{L}}\overset{+}{\mathcal{R}}$ or $\overset{+}{\mathcal{R}}\overset{+}{\mathcal{L}}$. The same is true of multiplicative Green's relations. Moreover, by Theorem II 3.6 in [17], if $S \in \mathrm{COS}_3^+$, then we have that for all $a, b \in S$,

$$a \dot{\mathcal{L}} b \Longleftrightarrow a = ab,\ b = ba; \tag{1.8}$$
$$a \dot{\mathcal{R}} b \Longleftrightarrow a = ba,\ b = ab; \tag{1.9}$$
$$a \dot{\mathcal{D}} b \Longleftrightarrow a = aba,\ b = bab; \tag{1.10}$$
$$a \overset{+}{\mathcal{L}} b \Longleftrightarrow a = a + 2b,\ b = b + 2a; \tag{1.11}$$
$$a \overset{+}{\mathcal{R}} b \Longleftrightarrow a = 2b + a,\ b = 2a + b. \tag{1.12}$$

By (1.11) and (1.12) it follows that

$$a \overset{+}{\mathcal{H}} b \Longleftrightarrow 2a = 2b. \tag{1.13}$$

According to [8], $\overset{+}{\mathcal{H}}$ is the least I-congruence on S and every $\overset{+}{\mathcal{H}}$-class is the maximal subring of S. Thus, S is the disjoint union of these maximal subrings. To go a step further, S is the disjoint union of Boolean rings.

The content is arranged as follows: In Section 2 we give characterizations for $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{L}}$ (resp., $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{R}}$, $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{D}}$) as a congruence on S and prove that the class of semirings defined by this congruence is a subvariety of COS_3^+. In Section 3 we give characterizations for $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}}$ (resp., $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}}$, $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{D}}$) as a congruence on S and prove that the class of semirings defined by this congruence is a subvariety of COS_3^+. These conclusions are further extensions of several results in [13, 16]. Throughout this paper, unless otherwise stated, S always belongs to COS_3^+ when we refer to S. Let $Con(S)$ denote the collection of all congruences on S, $\vartriangle$ the equality relation on S, and $\triangledown$ the universal relation on S.

2. Some subvarieties related to $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{L}}$, $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{R}}$ and $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{D}}$

In this section, we prove that the classes of semirings related to $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{L}}$, $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{R}}$ and $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{D}}$ are all subvarieties of COS_3^+, and give equational bases.

We have the characterizations for $\overset{+}{\mathcal{H}}$, $\dot{\mathcal{L}}$, $\dot{\mathcal{R}}$ and $\dot{\mathcal{D}}$ by (1.8)–(1.10) and (1.13), respectively.

Lemma 2.1. *Suppose $S \in COS_3^+$. Then for all $a, b \in S$,*

(i) $a \overset{+}{\mathcal{H}} b \Longleftrightarrow (\exists u, v \in S)\ a = u + 2v,\ b = 2u + v$;

(ii) $a \dot{\mathcal{L}} b \Longleftrightarrow (\exists u, v \in S)\ a = uvu,\ b = vu$;

(iii) $a\,\dot{\mathcal{R}}\,b \Longleftrightarrow (\exists u, v\in S)\ a = uvu,\ b = uv$;
(iv) $a\,\dot{\mathcal{D}}\,b \Longleftrightarrow (\exists u, v\in S)\ a = uv,\ b = vu$.

Proof. We only prove (i), and in a similar manner, we can also prove (ii)–(iv). If $a\,\overset{+}{\mathcal{H}}\,b$ and let $u = a$ and $v = b$, then $u + 2v = a + 2b = a$ by (1.11) and $2u + v = 2a + b = b$ by (1.12). Conversely, if $u, v\in S$ and let $a = u + 2v$ and $b = 2u + v$, then $2a = 2u + 2v = 2b$. Thus, by (1.13), $a\,\overset{+}{\mathcal{H}}\,b$. □

The conclusions with respect to join and meet between above Green's relations are given by the following lemma.

Lemma 2.2. *Suppose $S\in COS_3^+$. Then for all $a, b\in S$,*
(i) $a(\overset{+}{\mathcal{H}}\cap\dot{\mathcal{L}})b \Longleftrightarrow (\exists u, v\in S)\ a = uvu + 2vu,\ b = 2uvu + vu$;
(ii) $a(\overset{+}{\mathcal{H}}\cap\dot{\mathcal{R}})b \Longleftrightarrow (\exists u, v\in S)\ a = uvu + 2uv,\ b = 2uvu + uv$;
(iii) $a(\overset{+}{\mathcal{H}}\cap\dot{\mathcal{D}})b \Longleftrightarrow (\exists u, v\in S)\ a = uv + 2vu,\ b = 2uv + vu$.

Proof. We now prove (i), and the others can be proved similarly. If $a(\overset{+}{\mathcal{H}}\cap\dot{\mathcal{L}})b$ and let $u = ab$ and $v = ba$. Then

$$\begin{aligned} uvu + 2vu &= ab + 2bab && \text{(by (1.6))}\\ &= a + 2b && \text{(by (1.8))}\\ &= a, && \text{(by (1.11))}\\ 2uvu + vu &= 2ab + bab && \text{(by (1.6))}\\ &= 2a + b && \text{(by (1.8))}\\ &= b. && \text{(by (1.12))} \end{aligned}$$

Conversely, if $u, v\in S$ and let $a = uvu + 2vu$ and $b = 2uvu + vu$. Then $a\,\overset{+}{\mathcal{H}}\,b$ and

$$\begin{aligned} ab &= (uvu + 2vu)(2uvu + vu) \\ &= 2uvu + 2vu + uvu + 2vu && \text{(by (1.5), (1.6))}\\ &= uvu + 2vu && \text{(by (1.5))}\\ &= a,\\ ba &= (2uvu + vu)(uvu + 2vu)\\ &= 2uvu + vu + 2uvu + 2vu && \text{(by (1.5), (1.6))}\\ &= 2uvu + vu && \text{(by (1.5))}\\ &= b. \end{aligned}$$

i.e., $a\,\dot{\mathcal{L}}\,b$. Thus, $a(\overset{+}{\mathcal{H}}\cap\dot{\mathcal{L}})b$. □

By replacing above Green's relations with other Green's relations, we can deduce more similar characterizations, just as Lemma 2.1 in [13] and Lemma 1.3 in [16].

Now we can establish the following theorem.

Theorem 2.3. *Suppose $S \in COS_3^+$. Then*

(*i*) $\overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{L}} \in Con(S)$ *if and only if S satisfies the identities*

$$z(xyx + 2yx)z(2xyx + yx) \approx z(xyx + 2yx), \tag{2.1}$$

$$(z + xyx + 2yx)(z + 2xyx + yx) \approx z + xyx + 2yx, \tag{2.2}$$

$$(xyx + 2yx + z)(2xyx + yx + z) \approx xyx + 2yx + z; \tag{2.3}$$

(*ii*) $\overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{R}} \in Con(S)$ *if and only if S satisfies the identities*

$$(2xyx + xy)z(xyx + 2xy)z \approx (xyx + 2xy)z, \tag{2.4}$$

$$(z + 2xyx + xy)(z + xyx + 2xy) \approx z + xyx + 2xy, \tag{2.5}$$

$$(2xyx + xy + z)(xyx + 2xy + z) \approx xyx + 2xy + z; \tag{2.6}$$

(*iii*) $\overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{D}} \in Con(S)$ *if and only if S satisfies the identities*

$$(z + xy + 2yx)(z + 2xy + yx)(z + xy + 2yx) \approx z + xy + 2yx, \tag{2.7}$$

$$(xy + 2yx + z)(2xy + yx + z)(xy + 2yx + z) \approx xy + 2yx + z. \tag{2.8}$$

Proof. Here we prove (*i*), and the proofs of (*ii*) and (*iii*) are similar. If $\overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{L}} \in Con(S)$, then, by (*i*) in Lemma 2.2, $(aba + 2ba)(\overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{L}})(2aba + ba)$ for all $a, b, c \in S$. Hence $c(aba + 2ba)\ \mathcal{L}\ c(2aba + ba)$, $(c + aba + 2ba)\ \overset{\cdot}{\mathcal{L}}\ (c + 2aba + ba)$ and $(aba + 2ba + c)\ \overset{\cdot}{\mathcal{L}}\ (2aba + ba + c)$, i.e.,

$$c(aba + 2ba)c(2aba + ba) = c(aba + 2ba),$$

$$(c + aba + 2ba)(c + 2aba + ba) = c + aba + 2ba,$$

$$(aba + 2ba + c)(2aba + ba + c) = aba + 2ba + c.$$

Thus, S satisfies (2.1)–(2.3).

Conversely, if S satisfies (2.1)–(2.3), let $a, b \in S$ and $a(\overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{L}})b$. Then, by (*i*) in Lemma 2.2, there exist $u, v \in S$ such that $a = uvu + 2vu$ and $b = 2uvu + vu$. Thus, for all $c \in S$,

$$c(uvu + 2vu)c(2uvu + vu) = c(uvu + 2vu),$$

$$(c + uvu + 2vu)(c + 2uvu + vu) = c + uvu + 2vu,$$

$$(uvu + 2vu + c)(2uvu + vu + c) = uvu + 2vu + c.$$

i.e., $(ca)(cb) = ca$, $(c+a)(c+b) = c+a$ and $(a+c)(b+c) = a+c$. Dually, $(cb)(ca) = cb$, $(c+b)(c+a) = c+b$ and $(b + c)(a + c) = b + c$. That is to say, $ca\ \mathcal{L}\ cb$, $(c + a)\overset{\cdot}{\mathcal{L}}(c + b)$ and $(a + c)\overset{\cdot}{\mathcal{L}}(b + c)$. Hence $\overset{\cdot}{\mathcal{L}}$ is the left congruence on $(S, \cdot)$ and the congruence on $(S, +)$. Since $\overset{\cdot}{\mathcal{L}}$ is the right congruence on $(S, \cdot)$, it follows that $\overset{\cdot}{\mathcal{L}}$ is the congruence on S. Thus, $\overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{L}} \in Con(S)$. □

By replacing above Green's relations with other Green's relations, we can deduce more similar conclusions, just as Lemma 2.2 in [13] and Theorem 2.6 in [16].

From Theorem 2.3, the classes of semirings $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{L}} \in Con(S)\}$, $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{R}} \in Con(S)\}$ and $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \overset{\cdot}{\mathcal{D}} \in Con(S)\}$ are all varieties.

A useful further specialization of this result is provided by the following corollary.

Corollary 2.4. *Suppose* $S \in COS_3^+$. *Then*

(*i*) $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{L}} = \triangle$ *if and only if* S *satisfies the identity* $xyx + 2yx \approx 2xyx + yx$;

(*ii*) $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{R}} = \triangle$ *if and only if* S *satisfies the identity* $xyx + 2xy \approx 2xyx + xy$;

(*iii*) $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{D}} = \triangle$ *if and only if* S *satisfies the identity* $xy + 2yx \approx 2xy + yx$.

Thus, the classes of semirings $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \dot{\mathcal{L}} = \triangle\}$, $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \dot{\mathcal{R}} = \triangle\}$ and $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \dot{\mathcal{D}} = \triangle\}$ are all varieties.

We can immediately deduce the following corollary by (1.8)–(1.10) and (1.13).

Corollary 2.5. *Suppose* $S \in COS_3^+$. *Then*

(*i*) $\overset{+}{\mathcal{H}} = \triangledown$ *if and only if* S *satisfies the identity* $2x \approx 2y$;

(*ii*) $\dot{\mathcal{L}} = \triangledown$ *if and only if* S *satisfies the identity* $xy \approx x$;

(*iii*) $\dot{\mathcal{R}} = \triangledown$ *if and only if* S *satisfies the identity* $yx \approx x$;

(*iv*) $\dot{\mathcal{D}} = \triangledown$ *if and only if* S *satisfies the identity* $xyx \approx x$.

Thus, the classes of semirings $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} = \triangledown\}$, $\{S \in \mathrm{COS}_3^+ : \dot{\mathcal{L}} = \triangledown\}$, $\{S \in \mathrm{COS}_3^+ : \dot{\mathcal{R}} = \triangledown\}$ and $\{S \in \mathrm{COS}_3^+ : \dot{\mathcal{D}} = \triangledown\}$ are all varieties. According to [4, 8], these varieties happen to be R_2, $\dot{\mathrm{L}}\mathrm{z}$, $\dot{\mathrm{R}}\mathrm{z}$ and $\dot{R}$, respectively.

Since $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{L}} = \triangledown$ (resp., $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{R}} = \triangledown$, $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{D}} = \triangledown$) means that $\overset{+}{\mathcal{H}} = \dot{\mathcal{L}} = \triangledown$ (resp., $\overset{+}{\mathcal{H}} = \dot{\mathcal{R}} = \triangledown$, $\overset{+}{\mathcal{H}} = \dot{\mathcal{D}} = \triangledown$), it follows the following corollary.

Corollary 2.6. *Suppose* $S \in COS_3^+$. *Then*

(*i*) $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{L}} = \triangledown$ *if and only if* S *satisfies the identities* $2x \approx 2y,\ xy \approx x$;

(*ii*) $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{R}} = \triangledown$ *if and only if* S *satisfies the identities* $2x \approx 2y,\ yx \approx x$;

(*iii*) $\overset{+}{\mathcal{H}} \cap \dot{\mathcal{D}} = \triangledown$ *if and only if* S *satisfies the identities* $2x \approx 2y,\ xyx \approx x$.

Thus, the classes of semirings $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \dot{\mathcal{L}} = \triangledown\}$, $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \dot{\mathcal{R}} = \triangledown\}$ and $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \cap \dot{\mathcal{D}} = \triangledown\}$ are all varieties.

3. Some subvarieties related to $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}}$, $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}}$ and $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{D}}$

In this section, we prove that the classes of semirings related to $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}}$, $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}}$ and $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{D}}$ are all subvarieties of COS_3^+ and give equational bases.

We first show the characterizations for $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}}$, $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}}$ and $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{D}}$, respectively.

Lemma 3.1. *Suppose* $S \in COS_3^+$. *Then*

(*i*) $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}} = \overset{+}{\mathcal{H}} \dot{\mathcal{L}} \overset{+}{\mathcal{H}}$;

(*ii*) $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}} = \overset{+}{\mathcal{H}} \dot{\mathcal{R}} \overset{+}{\mathcal{H}}$;

(*iii*) $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{D}} = \overset{+}{\mathcal{H}} \dot{\mathcal{D}} \overset{+}{\mathcal{H}}$.

Proof. We only prove (i), and the proofs of (ii) and (iii) are similar. It is obvious that $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}} \supseteq \overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}$. To show that $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}} \subseteq \overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}$, we need verify that $\dot{\mathcal{L}}\overset{+}{\mathcal{H}}\dot{\mathcal{L}} \subseteq \overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}$. If $a, b \in S$ and $a\ \dot{\mathcal{L}}\overset{+}{\mathcal{H}}\dot{\mathcal{L}}\ b$, then there exist $c, d \in S$ such that $a\ \dot{\mathcal{L}}\ c\ \overset{+}{\mathcal{H}}\ d\ \dot{\mathcal{L}}\ b$. Certainly, $a = ac\ \overset{+}{\mathcal{H}}\ ad$ and $b = bd\ \overset{+}{\mathcal{H}}\ bc$, and so $a = ac\ \overset{+}{\mathcal{H}}\ adc$ by $a\ \overset{+}{\mathcal{H}}\ ad$. Since $\dot{\mathcal{L}}$ is the right congruence on $(S, \cdot)$, it follows that $adc\ \dot{\mathcal{L}}\ cdc$ by $a\ \dot{\mathcal{L}}\ c$ and $bc\ \dot{\mathcal{L}}\ dc$ by $b\ \dot{\mathcal{L}}\ d$. Thereby $a\ \overset{+}{\mathcal{H}}\ adc\ \dot{\mathcal{L}}\ cdc\ \dot{\mathcal{L}}\ dc\ \dot{\mathcal{L}}\ bc\ \overset{+}{\mathcal{H}}\ b$. i.e., $a\ \overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}\ b$. This implies that $\dot{\mathcal{L}}\overset{+}{\mathcal{H}}\dot{\mathcal{L}} \subseteq \overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}$. Since $\overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}$ is the equivalence containing $\overset{+}{\mathcal{H}}$ and $\dot{\mathcal{L}}$, and so $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}} \subseteq \overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}$. Thus, $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}} = \overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}$. □

Further characterizations are available.

Lemma 3.2. *Suppose $S \in COS_3^+$. Then for $a, b \in S$,*

(i) $a(\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}})b \Longleftrightarrow \overset{+}{\mathcal{H}}_{ab} = \overset{+}{\mathcal{H}}_a, \overset{+}{\mathcal{H}}_{ba} = \overset{+}{\mathcal{H}}_b;$

(ii) $a(\overset{+}{\mathcal{H}}\vee\dot{\mathcal{R}})b \Longleftrightarrow \overset{+}{\mathcal{H}}_{ba} = \overset{+}{\mathcal{H}}_a, \overset{+}{\mathcal{H}}_{ab} = \overset{+}{\mathcal{H}}_b;$

(iii) $a(\overset{+}{\mathcal{H}}\vee\dot{\mathcal{D}})b \Longleftrightarrow \overset{+}{\mathcal{H}}_{aba} = \overset{+}{\mathcal{H}}_a, \overset{+}{\mathcal{H}}_{bab} = \overset{+}{\mathcal{H}}_b.$

Proof. We only prove (i), and in a similar manner, we can also prove (ii) and (iii). If $a, b \in S$ and $a(\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}})b$, then, by ($i$) in Lemma 3.1, there exist $c, d \in S$ such that $a\ \overset{+}{\mathcal{H}}\ c\ \dot{\mathcal{L}}\ d\ \overset{+}{\mathcal{H}}\ b$. So $ab\ \overset{+}{\mathcal{H}}\ cb$ and $c = cd\ \overset{+}{\mathcal{H}}\ cb$. These imply that $a\ \overset{+}{\mathcal{H}}\ c = cd\ \overset{+}{\mathcal{H}}\ cb\ \overset{+}{\mathcal{H}}\ ab$. Thus, $\overset{+}{\mathcal{H}}_a = \overset{+}{\mathcal{H}}_{ab}$, and similarly $\overset{+}{\mathcal{H}}_b = \overset{+}{\mathcal{H}}_{ba}$. Conversely, if $\overset{+}{\mathcal{H}}_{ab} = \overset{+}{\mathcal{H}}_a$ and $\overset{+}{\mathcal{H}}_{ba} = \overset{+}{\mathcal{H}}_b$, then $a\ \overset{+}{\mathcal{H}}\ ab\ \dot{\mathcal{L}}\ bab\ \overset{+}{\mathcal{H}}\ b$ by $bab\ \overset{+}{\mathcal{H}}\ b$., i.e., $a\ \overset{+}{\mathcal{H}}\dot{\mathcal{L}}\overset{+}{\mathcal{H}}\ b$. Thus, by ($i$) in Lemma 3.1, $a(\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}})b$. □

In the case where $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}}$, $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{R}}$ and $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{D}}$ are congruences on S, the arguments can be made by the following lemma.

Lemma 3.3. *Suppose $S \in COS_3^+$. Then*

(i) $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{L}} \in Con(S)$ *if and only if* $\dot{\mathcal{L}}_{S/\overset{+}{\mathcal{H}}}$ *is the congruence on* $S/\overset{+}{\mathcal{H}}$;

(ii) $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{R}} \in Con(S)$ *if and only if* $\dot{\mathcal{R}}_{S/\overset{+}{\mathcal{H}}}$ *is the congruence on* $S/\overset{+}{\mathcal{H}}$;

(iii) $\overset{+}{\mathcal{H}}\vee\dot{\mathcal{D}} \in Con(S)$ *if and only if* $\dot{\mathcal{D}}_{S/\overset{+}{\mathcal{H}}}$ *is the congruence on* $S/\overset{+}{\mathcal{H}}$.

The equational characterization for $\dot{\mathcal{L}}$ (resp., $\dot{\mathcal{R}}$, $\dot{\mathcal{D}}$) are given by the following lemma.

Lemma 3.4. *Suppose $S \in COS_3^+$. Then*

(i) $\dot{\mathcal{L}} \in Con(S)$ *if and only if S satisfies the identities:*

$$z(xyx)z(yx) \approx z(xyx), \tag{3.1}$$

$$(z + xyx)(z + yx) \approx z + xyx, \tag{3.2}$$

$$(xyx + z)(yx + z) \approx xyx + z; \tag{3.3}$$

(ii) $\dot{\mathcal{R}} \in Con(S)$ *if and only if S satisfies the identities:*

$$(xy)z(xyx)z \approx (xyx)z, \tag{3.4}$$

$$(z + xy)(z + xyx) \approx z + xyx, \tag{3.5}$$
$$(xy + z)(xyx + z) \approx xyx + z; \tag{3.6}$$

(*iii*) $\dot{\mathcal{D}} \in Con(S)$ *if and only if* S *satisfies the identities:*

$$(z + xy)(z + yx)(z + xy) \approx z + xy, \tag{3.7}$$
$$(xy + z)(yx + z)(xy + z) \approx xy + z. \tag{3.8}$$

Proof. We now prove (*i*), and the others can be proved similarly. By (*ii*) in Lemma 2.1, $aba\ \dot{\mathcal{L}}\ ba$ for all $a, b \in S$. If $\dot{\mathcal{L}} \in Con(S)$, then $c(aba)\ \dot{\mathcal{L}}\ c(ba)$, $(c + aba)\ \dot{\mathcal{L}}\ (c + ba)$ and $(aba + c)\ \dot{\mathcal{L}}\ ba + c$ for all $c \in S$, i.e.,

$$c(aba)c(ba) = c(aba),$$
$$(c + aba)(c + ba) = c + aba,$$
$$(aba + c)(ba + c) = aba + c.$$

Thus, S satisfies (3.1)–(3.3).

Conversely, if S satisfies (3.1)–(3.3), let $a, b \in S$ and $a\ \dot{\mathcal{L}}\ b$, then, by (*ii*) in Lemma 2.1, there exist $u, v \in S$ such that $a = uvu$ and $b = vu$. Also, for all $c \in S$,

$$c(uvu)c(vu) = c(uvu),$$
$$(c + uvu)(c + vu) = c + uvu,$$
$$(uvu + c)(vu + c) = uvu + c.$$

i.e., $(ca)(cb) = (ca)$, $(c + a)(c + b) = c + a$ and $(a + c)(b + c) = a + c$. Dually, $(cb)(ca) = cb$, $(c + b)(c + a) = c + b$ and $(b + c)(a + c) = b + c$. That is to say, $ca\ \dot{\mathcal{L}}\ cb$, $(c + a)\ \dot{\mathcal{L}}\ (c + b)$ and $(a + c)\ \dot{\mathcal{L}}\ (b + c)$. Hence $\dot{\mathcal{L}}$ is the left congruence on $(S, \cdot)$ and the congruence on $(S, +)$. Since $\dot{\mathcal{L}}$ is the righ congruence on $(S, \cdot)$, it follows that $\dot{\mathcal{L}}$ is the congruence on S. □

Interestingly, Lemma 3.4 also holds for $S \in I$ in [16].

In view of Lemma 3.4, the classes of semirings $\{S \in \mathbf{COS}_3^+ \colon \dot{\mathcal{L}} \in Con(S)\}$, $\{S \in \mathbf{COS}_3^+ \colon \dot{\mathcal{R}} \in Con(S)\}$ and $\{S \in \mathbf{COS}_3^+ \colon \dot{\mathcal{D}} \in Con(S)\}$ are all varieties.

By Lemmas 3.3 and 3.4, we obtain the following theorem.

Theorem 3.5. *Suppose* $S \in \mathbf{COS}_3^+$. *Then*

(*i*) $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}} \in Con(S)$ *if and only if* S *satisfies the identities:*

$$2z(xyx)z(yx) \approx 2z(xyx), \tag{3.9}$$
$$2(z + xyx)(z + yx) \approx 2(z + xyx), \tag{3.10}$$
$$2(xyx + z)(yx + z) \approx 2(xyx + z); \tag{3.11}$$

(*ii*) $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}} \in Con(S)$ *if and only if* S *satisfies the identities:*

$$2(xy)z(xyx)z \approx 2(xyx)z, \tag{3.12}$$

$$2(z+xy)(z+xyx) \approx 2(z+xyx), \tag{3.13}$$
$$2(xy+z)(xyx+z) \approx 2(xyx+z); \tag{3.14}$$

(*iii*) $\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{D}} \in Con(S)$ *if and only if* S *satisfies the identities:*

$$2(z+xy)(z+yx)(z+xy) \approx 2(z+xy), \tag{3.15}$$
$$2(xy+z)(yx+z)(xy+z) \approx 2(xy+z). \tag{3.16}$$

Proof. We only prove (*i*), and we can also prove (*ii*) and (*iii*) in a similar manner. If $\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}} \in Con(S)$, then, by (*i*) in Lemma 3.3, $\overset{+}{\mathcal{H}}_{c(aba)c(ba)} = \overset{+}{\mathcal{H}}_{c(aba)}$, $\overset{+}{\mathcal{H}}_{(c+aba)(c+ba)} = \overset{+}{\mathcal{H}}_{(c+aba)}$ and $\overset{+}{\mathcal{H}}_{(aba+c)(ba+c)} = \overset{+}{\mathcal{H}}_{(aba+c)}$ for all $a, b, c \in S$, i.e.,

$$2c(aba)c(ba) = 2c(aba),$$
$$2(c+aba)(c+ba) = 2(c+aba),$$
$$2(aba+c)(ba+c) = 2(aba+c).$$

Thus, S satisfies (3.9)–(3.11).

Conversely, if S satisfies (3.9)–(3.11), then $\overset{+}{\mathcal{H}}_{c(aba)c(ba)} = \overset{+}{\mathcal{H}}_{c(aba)}$, $\overset{+}{\mathcal{H}}_{(c+aba)(c+ba)} = \overset{+}{\mathcal{H}}_{(c+aba)}$ and $\overset{+}{\mathcal{H}}_{(aba+c)(ba+c)} = \overset{+}{\mathcal{H}}_{(aba+c)}$ for all $a, b, c \in S$. Hence, by (*i*) in Lemma 3.4, $\overset{\cdot}{\mathcal{L}}$ is the congruence on $S/\overset{+}{\mathcal{H}}$. Also, by (*i*) in Lemma 3.3, $\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}} \in Con(S)$. □

By replacing $\overset{+}{\mathcal{H}}$ with $\overset{+}{\mathcal{D}}$, we can deduce similar conclusion, just as Theorem 2.7 in [16].

From Theorem 3.5, the classes of semirings $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}} \in Con(S)\}$, $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{R}} \in Con(S)\}$ and $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{D}} \in Con(S)\}$ are all varieties.

Considering that the universal relation is a special congruence, we immediately have the following corollary.

Corollary 3.6. *Suppose* $S \in COS_3^+$. *Then*

(*i*) $\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}} = \triangledown$ *if and only if* S *satisfies the identity* $2xy \approx 2x$*;*

(*ii*) $\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{R}} = \triangledown$ *if and only if* S *satisfies the identity* $2yx \approx 2x$*;*

(*iii*) $\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{D}} = \triangledown$ *if and only if* S *satisfies the identity* $2xyx \approx 2x$.

Proof. We only prove (*i*), and in the same manner, we can also prove (*ii*) and (*iii*). If $\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}} = \triangledown$, then, by (*i*) in Lemma 3.2, $a(\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}})b$ for all $a, b \in S$ and $\overset{+}{\mathcal{H}}_{ab} = \overset{+}{\mathcal{H}}_a$, i.e., $2ab = 2a$. Thus, S satisfies identity $2xy \approx 2x$. Conversely, if S satisfies the former identity, then $a \overset{+}{\mathcal{H}} ab \overset{\cdot}{\mathcal{L}} bab \overset{+}{\mathcal{H}} b$ for all $a, b \in S$, and so $a(\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}})b$ by Lemma 3.1. Thus, $\overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}} = \triangledown$. □

Thus, the classes of semirings $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{L}} = \triangledown\}$, $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{R}} = \triangledown\}$ and $\{S \in \mathrm{COS}_3^+ : \overset{+}{\mathcal{H}} \vee \overset{\cdot}{\mathcal{D}} = \triangledown\}$ are all varieties.

The following corollary now follows directly from Lemma 2.1.

Corollary 3.7. *Suppose $S \in COS_3^+$. Then*

(*i*) $\overset{+}{\mathcal{H}} = \triangle$ *if and only if S satisfies the identity $x + 2y \approx 2x + y$;*

(*ii*) $\dot{\mathcal{L}} = \triangle$ *if and only if S satisfies the identity $xyx \approx yx$;*

(*iii*) $\dot{\mathcal{R}} = \triangle$ *if and only if S satisfies the identity $xyx \approx xy$;*

(*iv*) $\dot{\mathcal{D}} = \triangle$ *if and only if S satisfies the identities $xy \approx yx$.*

Thus, the classes of semirings $\{S \in \mathrm{COS}_3^+ \colon \overset{+}{\mathcal{H}} = \triangle\}$, $\{S \in \mathrm{COS}_3^+ \colon \dot{\mathcal{L}} = \triangle\}$, $\{S \in \mathrm{COS}_3^+ \colon \dot{\mathcal{R}} = \triangle\}$ and $\{S \in \mathrm{COS}_3^+ \colon \dot{\mathcal{D}} = \triangle\}$ are all varieties. According to [13], the first variety of semirings happens to be I.

Since $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}} = \triangle$ (resp., $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}} = \triangle$, $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{D}} = \triangle$) means that $\overset{+}{\mathcal{H}} = \dot{\mathcal{L}} = \triangle$ (resp., $\overset{+}{\mathcal{H}} = \dot{\mathcal{R}} = \triangle$, $\overset{+}{\mathcal{H}} = \dot{\mathcal{D}} = \triangle$), it follows the following corollary.

Corollary 3.8. *Suppose $S \in COS_3^+$. Then*

(*i*) $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}} = \triangle$ *if and only if S satisfies the identities $x + 2y \approx 2x + y$, $xyx \approx yx$;*

(*ii*) $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}} = \triangle$ *if and only if S satisfies the identities $x + 2y \approx 2x + y$, $xyx \approx xy$;*

(*iii*) $\overset{+}{\mathcal{H}} \vee \dot{\mathcal{D}} = \triangle$ *if and only if S satisfies the identities $x + 2y \approx 2x + y$, $xy \approx yx$.*

Thus, the classes of semirings $\{S \in \mathrm{COS}_3^+ \colon \overset{+}{\mathcal{H}} \vee \dot{\mathcal{L}} = \triangle\}$, $\{S \in \mathrm{COS}_3^+ \colon \overset{+}{\mathcal{H}} \vee \dot{\mathcal{R}} = \triangle\}$ and $\{S \in \mathrm{COS}_3^+ \colon \overset{+}{\mathcal{H}} \vee \dot{\mathcal{D}} = \triangle\}$ are all varieties. According to [8], these varieties happen to be $\dot{\mathrm{R}}\mathrm{r}$, $\dot{\mathrm{L}}\mathrm{r}$ and $S\dot{\ell}$, respectively.

4. Conclusions

In this paper, we consider a semiring variety COS_3^+, which is determined by additional identity $x^2 \approx x$. We first give the sufficient and necessary conditions for some equivalences to be semiring congruences. Then we prove that the semiring classes defined by these congruences are subvarieties of COS_3^+. In the future, we will consider COS_n^+, which denotes the semiring variety determined by additional identity $x^n \approx x$. And we will extend the results in this paper.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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