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Research article

Hermite-Hadamard inequalities for generalized convex functions in interval-valued calculus

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Abstract: The importance of convex and non-convex functions in the study of optimization is widely established. The concept of convexity also plays a key part in the subject of inequalities due to the behavior of its definition. The principles of convexity and symmetry are inextricably linked. Because of the considerable association that has emerged between the two in recent years, we may apply what we learn from one to the other. In this study, first, Hermite-Hadamard type inequalities for LR-p-convex interval-valued functions (LR-p-convex-*I*-*V*-*F*) are constructed in this study. Second, for the product of p-convex various Hermite-Hadamard (*HH*) type integral inequalities are established. Similarly, we also obtain Hermite-Hadamard-Fejér (*HH*-Fejér) type integral inequality for LR-p-convex-*I*-*V*-*F*. Finally, for LR-p-convex-*I*-*V*-*F*, various discrete Schur's and Jensen's type inequalities are presented. Moreover, the results presented in this study are verified by useful nontrivial examples. Some of the results reported here for be LR-p-convex-*I*-*V*-*F* are generalizations of prior results for convex and harmonically convex functions, as well as p-convex functions.

Keywords: Interval-valued functions; LR-p-convex interval-valued functions; Hermite-Hadamard type inequality Hermite-Hadamard-Fejér inequality; Jensen's type inequality; Schur's type inequality
Mathematics Subject Classification: 26A33, 26A51, 26D10

1. Introduction

Convexity of function is a classic notion, as it is used in mathematical programming theory, game theory, mathematical economics, variational science, optimal control theory, and other domains. In the 1960s, a new branch of mathematics called convex analysis emerged. However, because the functions encountered in a large number of theoretical and practical economics problems are not classical convex functions, the generalization of function convexity has piqued the interest of many scholars in recent decades, such as h-convex functions [1–4], log-convex functions [5–9], log-h-convex functions [10], and especially coordinated convex functions [11]. Several authors [12–16] have proposed different expansions and generalizations of integral inequalities for different convex functions since 2001.

Calculation error, on the other hand, has always been a problematic issue in numerical analysis. Because the accumulation of calculation errors can render the calculation results meaningless, interval analysis as a new important tool to solve uncertainty problems has attracted a lot of attention and has also yielded fruitful results; we refer the reader to the papers [17,18]. Many writers have merged integral inequalities with interval-valued functions (*I-V-Fs*) in recent decades, and many great findings have resulted. Costa proposed Opial-type disparities for *I-V-Fs* in [19]. Chalco-Cano used the generalized Hukuhara derivative to examine Ostrowski type inequalities for *I-V-Fs* in [20,21]. The Minkowski type inequalities and Beckenbach's type inequalities for *I-V-Fs* were developed by Roman-Flores in [22].

In literature review, we have noted that most of authors used inclusion relation to obtain different types of inequalities like in 2018. Zhao et al. [23] developed h-convex interval-valued functions (h-convex IVFs) and demonstrated the following HH type inequality for h-convex IVFs.

Theorem 1. [23] Let $\psi: [\mu, \upsilon] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ be an h-convex IVF given by $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$ for all $\omega \in [\mu, \upsilon]$, with $h: [0,1] \to \mathbb{R}^+$ and $h\left(\frac{1}{2}\right) \neq 0$, where $\psi_*(\omega)$ and $\psi^*(\omega)$ are h-convex and h-concave functions, respectively. If ψ is Riemann integrable, then

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\, \psi\left(\frac{\mu+\upsilon}{2}\right) \supseteq \frac{1}{\upsilon-\mu}\,(IR)\int_\mu^\upsilon \psi(\omega)d\omega \supseteq [\psi(\mu)+\psi(\upsilon)]\int_0^1 h(\Theta)\,d\Theta, \tag{1}$$

where $\Theta \in [0,1]$. Moreover, An et al. [24] took a step forward by introducing the class of (h_1, h_2)-convex IVFs and establishing interval-valued Hermite-Hadamard (HH) type inequality for (h_1, h_2)-convex IVFs. We suggest readers to [25–27] and the references therein for more examination of literature on the applications and properties of generalized convex functions and HH type integral inequalities.

Zhang et al. [28] introduced pseudo order relation on the space of intervals and proposed the new class of convex functions in interval-valued settings by using pseudo order relation, which is known as LR-convex interval-valued functions. By using this class, they established continuous Jensen's inequalities and proved that Jensen's inequality defined by Costa and Roman-Flores [29] is a special cases of these inequalities. Khan et al. went a step further by providing new convex and extended LR-convex *I-V-F* classes, as well as new fractional HH type and HH type inequalities for LR-(h_1, h_2)-convex *I-V-F* [30], LR-p-convex *I-V-F* [31], and LR-log-h-convex *I-V-F* [32]. We refer to the readers for further analysis of literature on the applications and properties of fuzzy Riemannian integrals, and inequalities and generalized convex fuzzy mappings, see [33–40] and the references therein.

Inspired by the ongoing research work, we have used the pseudo order relation to introduce a new class of convex *I-V-F*s, which is known as *LR-p*-convex for *I-V-Fs*. We derive certain novel Schur's, Jensen's and Hermite-Hadamard type inequalities for interval-valued *LR-p*-convex functions via interval Riemannian integrals. We also provide several examples to demonstrate the applications of our key results.

2. Preliminaries

Let $\mathbb{R}_I$ the collection of all closed and bounded intervals of $\mathbb{R}$. We use $\mathbb{R}_I^+$ to represent the set of all positive intervals. The collection of all Riemann integrable real valued functions and Riemann integrable *I-V-F* is denoted by $\mathcal{R}_{[\mu,\vartheta]}$ and $\mathcal{IR}_{[\mu,\vartheta]}$, respectively. For more conceptions on *I-V·Fs*, see [23]. Moreover, we have:

For $[\mathfrak{U}_*,\mathfrak{U}^*]$, $[\mathcal{Z}_*,\mathcal{Z}^*] \in \mathbb{R}_I$, the inclusion "$\subseteq$" is defined by $[\mathfrak{U}_*,\mathfrak{U}^*] \subseteq [\mathcal{Z}_*,\mathcal{Z}^*]$, if and only if, $\mathcal{Z}_* \leq \mathfrak{U}_*$, $\mathfrak{U}^* \leq \mathcal{Z}^*$.

Remark 2.1. [28] (i) The relation "$\leq_p$" defined on $\mathbb{R}_I$ by $[\mathfrak{U}_*,\mathfrak{U}^*] \leq_p [\mathcal{Z}_*,\mathcal{Z}^*]$ if and only if $\mathfrak{U}_* \leq \mathcal{Z}_*$, $\mathfrak{U}^* \leq \mathcal{Z}^*$, for all $[\mathfrak{U}_*,\mathfrak{U}^*]$, $[\mathcal{Z}_*,\mathcal{Z}^*] \in \mathbb{R}_I$,it is an pseudo order relation. The relation $[\mathfrak{U}_*,\mathfrak{U}^*] \leq_p [\mathcal{Z}_*,\mathcal{Z}^*]$ coincident to $[\mathfrak{U}_*,\mathfrak{U}^*] \leq [\mathcal{Z}_*,\mathcal{Z}^*]$ on $\mathbb{R}_I$.
(ii) It can be easily seen that "$\leq_p$" looks like "left and right" on the real line $\mathbb{R}$, so we call "$\leq_p$" is "left and right" (or "LR" order, in short).

The concept of Riemann integral for *I-V-F* is defined as follow:

Theorem 2.2. [37]. If $\psi: [\mu,\vartheta] \subset \mathbb{R} \to \mathbb{R}_I$ is an *I-V-F* on such that $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$. Then ψ is Riemann integrable over $[\mu,\vartheta]$, if and only if, ψ_* and ψ^* both are Riemann integrable over $[\mu,\vartheta]$ such that

$$(IR)\int_\mu^\vartheta \psi(\omega)d\omega = \left[(R)\int_\mu^\vartheta \psi_*(\omega)d\omega\,, (R)\int_\mu^\vartheta \psi^*(\omega)d\omega\right].$$

Definition 2.3. [38] Let $p \in \mathbb{R}$ with $p \neq 0$. Then the interval K_p is said to be p-convex, if

$$[\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}} \in K_p, \tag{2}$$

for all $\omega, z \in K_p, \Theta \in [0,1]$, where $p = 2n+1$ and $n \in N$.

Definition 2.4. [38] Let $p \in \mathbb{R}$ with $p \neq 0$ and $K_p = [\mu,\vartheta] \subseteq \mathbb{R}$. Then, the function $\psi: [\mu,\vartheta] \to \mathbb{R}^+$ is said to be p-convex function if

$$\psi\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right) \leq \Theta\psi(\omega) + (1-\Theta)\psi(z), \tag{3}$$

for all $\omega, z \in [\mu,\vartheta], \Theta \in [0,1]$. If the inequality (3) is reversed then ψ is called p-concave function.

Definition 2.5. [28]. The *I-V-F*$\psi: [\mu,\vartheta] \to \mathbb{R}_I^+$ is said to be LR-convex-*I-V-F,* if for all $\omega, z \in [\mu,\vartheta]$ and $\Theta \in [0,1]$ we have

$$\psi(\Theta\omega + (1-\Theta)z) \leq_p \Theta\psi(\omega) + (1-\Theta)\psi(z). \tag{4}$$

If inequality (4) is reversed, then ψ is said to be LR-concave on $[\mu, \upsilon]$.

Definition 2.6. The *I-V-F* $\psi: [\mu, \upsilon] \to \mathbb{R}_I^+$ is said to be LR-p-convex-*I-V-F,* if for all $\omega, z \in [\mu, \upsilon]$ and $\Theta \in [0, 1]$, we have

$$\psi\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right) \leq_p \Theta\psi(\omega) + (1-\Theta)\psi(z), \quad (5)$$

If inequality (5) is reversed, then ψ is said to be LR-p-concave on $[\mu, \upsilon]$. ψ is LR-p-affine, if and only if, it is both LR-p-convex and LR-p-concave. The set of all LR-p-convex (LR-p-concave, LR-p-affine) *I-V-Fs* is denoted by

$$LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)\big(LRSV([\mu, \upsilon], \mathbb{R}_I^+, p), \qquad LRSA([\mu, \upsilon], \mathbb{R}_I^+, p)\big).$$

Remark 2.7. If one takes $p = 1$, then we obtain the Definitions 2.5.

Theorem 2.8. Let $\psi: [\mu, \upsilon] \to \mathbb{R}_I^+$ be an *I-V-F* defined by $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$, for all $\omega \in [\mu, \upsilon]$. Then $\psi \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$, if and only if, $\psi_*, \ \psi^* \in SX([\mu, \upsilon], \ \mathbb{R}^+, p)$.

Proof. Assume that $\psi_*, \psi^* \in SX([\mu, \upsilon], \ \mathbb{R}^+, p)$. Then, for all $\omega, z \in [\mu, \upsilon], \Theta \in [0, 1]$, we have

$$\psi_*\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right) \leq \Theta\psi_*(\omega) + (1-\Theta)\psi_*(z),$$

and

$$\psi^*\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right) \leq \Theta\psi^*(\omega) + (1-\Theta)\psi^*(z).$$

From Definition 2.6 and pseudo order relation $\leq_p$, we have

$$\left[\psi_*\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right), \psi^*\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right)\right]$$
$$\leq_p [\Theta\psi_*(\omega) + (1-\Theta)\psi_*(z), \Theta\psi^*(\omega) + (1-\Theta)\psi^*(z)]$$
$$= \Theta[\psi_*(\omega), \psi^*(\omega)] + (1-\Theta)[\psi_*(z), \psi^*(z)],$$

that is

$$\psi\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right) \leq_p \Theta\psi(\omega) + (1-\Theta)\psi(z), \forall\, \omega, z \in [\mu, \upsilon], \Theta \in [0, 1].$$

Hence, $\psi \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$.

Conversely, let $\psi \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$. Then, for all $\omega, z \in [\mu, \upsilon]$ and $\Theta \in [0, 1]$, we have

$$\psi\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right) \leq_p \Theta\psi(\omega) + (1-\Theta)\psi(z),$$

That is,

$$\left[\psi_*\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right), \psi^*\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right)\right]$$
$$\leq_p \Theta[\psi_*(\omega), \psi^*(\omega)] + (1-\Theta)[\psi_*(z), \psi^*(z)]$$
$$= [\Theta\psi_*(\omega) + (1-\Theta)\psi_*(z), \Theta\psi^*(\omega) + (1-\Theta)\psi^*(z)].$$

It follows that

$$\psi_*\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right) \leq \Theta\psi_*(\omega) + (1-\Theta)\psi_*(z),$$

and

$$\psi^*\left([\Theta\omega^p + (1-\Theta)z^p]^{\frac{1}{p}}\right) \leq \Theta\psi^*(\omega) + (1-\Theta)\psi^*(z).$$

Hence, the result follows.

Remark 2.13. If $\psi_*(\mu) = \psi^*(\upsilon)$, then, LR-$p$-convex-*I-V-F* becomes p-convex function, see [38]. If $\psi_*(\mu) = \psi^*(\upsilon)$ with $p = 1$, then LR-p-convex-*I-V-F* becomes the classical convex function.

Example 2.14. We consider the *I-V-F*$\psi: (0, \infty) \to \mathbb{R}_I^+$ defined by $\psi(\omega) = [3\omega, 5\omega^p]$, where p is an odd number. Since $\psi_*, \psi^* \in SX([\mu, \upsilon], \mathbb{R}^+, p)$ and hence, $\psi(\omega)$ is LR-p-convex-*I-V-F*.

3. Interval Hermite-Hadamard inequalities

In view of the classical HH and HH-Fejér inequalities, we can obtain the following version of the HH-inequalities for LR-p-concave-*I-V-F*s.

Theorem 3.1. Let $\psi: [\mu, \upsilon] \to \mathbb{R}_I^+$ be an *I-V-F* such that $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$, for all $\omega \in [\mu, \upsilon]$. If $\psi \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$ and $\psi \in \mathcal{IR}_{([\mu,\upsilon])}$, then

$$\psi\left(\left[\frac{\mu^p + \upsilon^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{\upsilon^p - \mu^p}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)d\omega \leq_p \frac{\psi(\mu) + \psi(\upsilon)}{2}. \quad (6)$$

If $\psi \in LRSV([\mu, \upsilon], \mathbb{R}_I^+, p)$, then

$$\psi\left(\left[\frac{\mu^p + \upsilon^p}{2}\right]^{\frac{1}{p}}\right) \geq_p \frac{p}{\upsilon^p - \mu^p}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)d\omega \geq_p \frac{\psi(\mu) + \psi(\upsilon)}{2}.$$

Proof. Let $\psi \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$. Then, by hypothesis, we have

$$2\psi\left(\left[\frac{\mu^p + \upsilon^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \psi\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) + \psi\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right).$$

Therefore, we have

$$2\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq\psi_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)+\psi_*\big((1-\Theta)\mu^p+\Theta\upsilon^p\big),$$
$$2\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq\psi^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)+\psi^*\big((1-\Theta)\mu^p+\Theta\upsilon^p\big).$$

Then

$$2\int_0^1\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)d\Theta\leq\int_0^1\psi_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta+\int_0^1\psi_*\big((1-\Theta)\mu^p+\Theta\upsilon^p\big)d\Theta\,,$$
$$2\int_0^1\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)d\Theta\leq\int_0^1\psi^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta+\int_0^1\psi^*\big((1-\Theta)\mu^p+\Theta\upsilon^p\big)d\Theta\,.$$

It follows that

$$\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon\omega^{p-1}\psi_*(\omega)d\omega\,,$$
$$\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon\omega^{p-1}\psi^*(\omega)d\omega\,.$$

That is

$$\left[\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right),\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\right]\leq_p\frac{p}{\upsilon^p-\mu^p}\left[\int_\mu^\upsilon\omega^{p-1}\psi_*(\omega)d\omega\,,\int_\mu^\upsilon\omega^{p-1}\psi^*(\omega)d\omega\right].$$

Thus,

$$\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq_p\frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon\omega^{p-1}\psi(\omega)d\omega. \tag{7}$$

In a similar way as above, we have

$$\frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon\omega^{p-1}\psi(\omega)d\omega\leq_p\frac{\psi(\mu)+\psi(\upsilon)}{2}. \tag{8}$$

Combining (7) and (8), we have

$$\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq_p\frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon\omega^{p-1}\psi(\omega)d\omega\leq_p\frac{\psi(\mu)+\psi(\upsilon)}{2}.$$

Hence, the result is required.

Remark 3.2. If $p=1$, then Theorem 3.1 reduces to the result for LR-convex-*I-V-F*, see [30]:

$$\psi\left(\frac{\mu+\upsilon}{2}\right)\leq_p\frac{1}{\upsilon-\mu}(IR)\int_\mu^\upsilon\psi(\omega)d\omega\leq_p\frac{\psi(\mu)+\psi(\upsilon)}{2}. \tag{9}$$

If $\psi_*(\omega) = \psi^*(\omega)$, then Theorem 3.1 reduces to the result for classical p-convex function, see [39]:

$$\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon \omega^{p-1}\psi(\omega)d\omega \leq \frac{\psi(\mu)+\psi(\upsilon)}{2}. \tag{10}$$

If $\psi_*(\omega) = \psi^*(\omega)$ with $p = 1$, then Theorem 3.1 reduces to the result for classical convex-*I-V-F*:

$$\psi\left(\frac{\mu+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\mu}\int_\mu^\upsilon \psi(\omega)d\omega \leq \frac{\psi(\mu)+\psi(\upsilon)}{2}. \tag{11}$$

Example 3.3. Let p be an odd number and the *I-V-F* $\psi: [\mu, \upsilon] = [-1, 1] \to \mathbb{R}_I^+$ defined by, $\psi(\omega) = [\omega^p,\ e^{\omega^p}]$. Since end point functions $\psi_*(\omega) = \omega^p$ and $\psi^*(\omega) = e^{\omega^p}$ are both p-convex functions. Hence $\psi(\omega)$ is LR-p-convex-*I-V-F*. We now computing the following

$$\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)d\omega \leq \frac{\psi(\mu)+\ \psi(\upsilon)}{2}.$$

$$\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right) = \psi_*(0) = 0,$$

$$\frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)d\omega = \frac{1}{2}\int_{-1}^{1} \omega^{2p-1}d\omega = 0,$$

$$\frac{\psi^*(\mu)+\psi^*(\upsilon)}{2} = 0.$$

This means

$$0 \leq 0 \leq 0.$$

Similarly, it can be easily shown that

$$\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)d\omega \leq \frac{\psi^*(\mu)+\psi^*(\upsilon)}{2},$$

such that

$$\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right) = \psi^*(0) = 1,$$

$$\frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)d\omega = \frac{1}{2}\int_{-1}^{1} \omega^{p-1}e^{\omega^p}d\omega = \frac{e-e^{-1}}{2},$$

$$\frac{\psi^*(\mu)+\psi^*(\upsilon)}{2} = \frac{e+e^{-1}}{2},$$

From which, it follows that

$$1 \leq \frac{e-e^{-1}}{2} \leq \frac{e+e^{-1}}{2},$$

that is

$$[0,1] \leq_p \left[0, \frac{e-e^{-1}}{2}\right] \leq_p \left[0, \frac{e+e^{-1}}{2}\right].$$

Hence,

$$\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{\upsilon^p-\mu^p}(R)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)d\omega \leq_p \frac{\psi(\mu)+\psi(\upsilon)}{2}.$$

Theorem 3.4. Let $\psi: [\mu, \upsilon] \to \mathbb{R}_I^+$ be an *I-V-F* such that $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$, for all $\omega \in [\mu, \upsilon]$. If $\psi \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$ and $\psi \in \mathcal{IR}_{([\mu,\upsilon])}$, then

$$\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \rhd_2 \leq_p \frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)d\omega \leq_p \rhd_1 \leq_p \frac{\psi(\mu)+\psi(\vartheta)}{2},$$

where

$$\rhd_1 = \frac{\frac{\psi(\mu)+\psi(\upsilon)}{2}+\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)}{2},$$

$$\rhd_2 = \frac{\psi\left(\left[\left[\frac{3\mu^p+\upsilon^p}{4}\right]^{\frac{1}{p}}\right]\right)+\psi\left(\left[\frac{\mu^p+3\upsilon^p}{4}\right]^{\frac{1}{p}}\right)}{2},$$

and $\rhd_1 = [\rhd_{1_*}, \rhd_1{}^*]$, $\rhd_2 = [\rhd_{2_*}, \rhd_2{}^*]$.

Proof. Taking $\left[\mu^p, \frac{\mu^p+\upsilon^p}{2}\right]$, we have

$$2\psi\left(\left[\frac{\Theta\mu^p+(1-\Theta)\frac{\mu^p+\upsilon^p}{2}}{2}+\frac{(1-\Theta)\mu^p+\Theta\frac{\mu^p+\upsilon^p}{2}}{2}\right]^{\frac{1}{p}}\right)$$

$$\leq_p \psi\left(\left[\Theta\mu^p+(1-\Theta)\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)+\psi\left(\left[(1-\Theta)\mu^p+\Theta\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right).$$

Therefore, we have

$$2\psi_*\left(\left[\frac{\Theta\mu^p+(1-\Theta)\frac{\mu^p+\upsilon^p}{2}}{2}+\frac{(1-\Theta)\mu^p+\Theta\frac{\mu^p+\upsilon^p}{2}}{2}\right]^{\frac{1}{p}}\right)$$

$$\leq \psi_*\left(\left[\Theta\mu^p+(1-\Theta)\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)+\psi_*\left(\left[(1-\Theta)\mu^p+\Theta\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right),$$

$$2\psi^*\left(\left[\frac{\Theta\mu^p+(1-\Theta)\frac{\mu^p+\upsilon^p}{2}}{2}+\frac{(1-\Theta)\mu^p+\Theta\frac{\mu^p+\upsilon^p}{2}}{2}\right]^{\frac{1}{p}}\right)$$

$$\leq \psi^*\left(\left[\Theta\mu^p+(1-\Theta)\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)+\psi^*\left(\left[(1-\Theta)\mu^p+\Theta\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right).$$

In consequence, we obtain

$$\frac{1}{2}\psi_*\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\leq\frac{p}{\vartheta^p-\mu^p}\int_{\mu}^{\frac{\mu^p+\vartheta^p}{2}}\omega^{p-1}\psi_*(\omega)d\omega\,,$$
$$\frac{1}{2}\psi^*\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\leq\frac{p}{\vartheta^p-\mu^p}\int_{\mu}^{\frac{\mu^p+\vartheta^p}{2}}\omega^{p-1}\psi^*(\omega)d\omega\,.$$

That is

$$\frac{1}{2}\left[\psi_*\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right),\psi^*\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\right]$$
$$\leq_p\frac{p}{\vartheta^p-\mu^p}\left[\int_{\mu}^{\frac{\mu^p+\vartheta^p}{2}}\omega^{p-1}\psi_*(\omega)d\omega\,,\int_{\mu}^{\frac{\mu^p+\vartheta^p}{2}}\omega^{p-1}\psi^*(\omega)d\omega\right].$$

It follows that

$$\frac{1}{2}\psi\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\leq_p\frac{p}{\vartheta^p-\mu^p}\int_{\mu}^{\frac{\mu^p+\vartheta^p}{2}}\omega^{p-1}\psi(\omega)d\omega. \tag{12}$$

In a similar way as above, we have

$$\frac{1}{2}\psi\left(\left[\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\leq_p\frac{p}{\vartheta^p-\mu^p}\int_{\frac{\mu^p+\vartheta^p}{2}}^{\vartheta}\omega^{p-1}\psi(\omega)d\omega. \tag{13}$$

Combining (12) and (13), we have

$$\frac{1}{2}\left[\psi\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)+\psi\left(\left[\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\right]\leq_p\frac{p}{\vartheta^p-\mu^p}\int_{\mu}^{\vartheta}\omega^{p-1}\psi(\omega)d\omega.$$

By using Theorem 3.1, we have

$$\psi\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)=\psi\left(\left[\frac{1}{2}.\frac{3\mu^p+\vartheta^p}{4}+\frac{1}{2}.\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right).$$

Therefore, we have

$$\psi_*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)=\psi_*\left(\left[\frac{1}{2}.\frac{3\mu^p+\vartheta^p}{4}+\frac{1}{2}.\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right),$$
$$\psi^*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)=\psi^*\left(\left[\frac{1}{2}.\frac{3\mu^p+\vartheta^p}{4}+\frac{1}{2}.\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right),$$
$$\leq\left[\frac{1}{2}\psi_*\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)+\frac{1}{2}\psi_*\left(\left[\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\right],$$
$$\leq\left[\frac{1}{2}\psi^*\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right)+\frac{1}{2}\psi^*\left(\left[\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\right],$$

$$
\begin{aligned}
&= \rhd_{2_*}, \\
&= \rhd_2{}^*, \\
&\leq \frac{p}{\vartheta^p - \mu^p}\int_\mu^\vartheta \omega^{p-1}\psi_*(\omega)d\omega\,, \\
&\leq \frac{p}{\vartheta^p - \mu^p}\int_\mu^\vartheta \omega^{p-1}\psi^*(\omega)d\omega\,, \\
&\leq \frac{\left[\frac{\psi_*(\mu)+\psi_*(\vartheta)}{2} + \psi_*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)\right]}{2}, \\
&\leq \frac{\left[\frac{\psi^*(\mu)+\psi^*(\vartheta)}{2} + \psi^*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)\right]}{2}, \\
&= \rhd_{1_*}, \\
&= \rhd_1{}^*, \\
&\leq \frac{\left[\frac{\psi_*(\mu)+\psi_*(\vartheta)}{2} + \frac{\psi_*(\mu)+\psi_*(\vartheta)}{2}\right]}{2}, \\
&\leq \frac{\left[\frac{\psi^*(\mu)+\psi^*(\vartheta)}{2} + \frac{\psi^*(\mu)+\psi^*(\vartheta)}{2}\right]}{2}, \\
&= \frac{\psi_*(\mu)+\psi_*(\vartheta)}{2}, \\
&= \frac{\psi^*(\mu)+\psi^*(\vartheta)}{2},
\end{aligned}
$$

that is

$$
\psi\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \rhd_2 \leq_p \frac{p}{\vartheta^p - \mu^p}(IR)\int_\mu^\vartheta \omega^{p-1}\psi(\omega)d\omega \leq_p \rhd_1 \leq_p \frac{\psi(\mu)+\psi(\vartheta)}{2},
$$

hence, the result follows.

Example 3.5. Let p be an odd number and the *I-V-F* $\psi\colon[\mu,\vartheta] = [-1,1] \to \mathbb{R}_I^+$ defined by, $\psi(\omega) = \left[\omega^p,\ e^{\omega^p}\right]$, as in Example 3.3, then $\psi(\omega)$ is p-convex *I-V-F*.

We have $\psi_*(\omega) = \omega^p$ and $\psi^*(\omega) = e^{\omega^p}$. We now compute the following

$$
\frac{\psi_*(\mu)+\psi_*(\vartheta)}{2} = 0,
$$

$$
\frac{\psi^*(\mu)+\psi^*(\vartheta)}{2} = \frac{e+e^{-1}}{2},
$$

$$\rhd_{1_*} = \frac{\frac{\psi_*(\mu)+\psi_*(\vartheta)}{2} + \psi_*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)}{2} = 0,$$

$$\rhd_1{}^* = \frac{\frac{\psi^*(\mu)+\psi^*(\vartheta)}{2} + \psi^*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)}{2} = \frac{e+e^{-1}+2}{4},$$

$$\rhd_{2_*} = \frac{1}{2}\left[\psi_*\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right) + \psi_*\left(\left[\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\right] = 0$$

$$\rhd_2{}^* = \frac{1}{2}\left[\psi^*\left(\left[\frac{3\mu^p+\vartheta^p}{4}\right]^{\frac{1}{p}}\right) + \psi^*\left(\left[\frac{\mu^p+3\vartheta^p}{4}\right]^{\frac{1}{p}}\right)\right] = \frac{e^{-\frac{1}{2}}+e^{\frac{1}{2}}}{2},$$

$$\psi_*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right) = 0,$$

$$\psi^*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right) = 1.$$

Then we obtain that

$$0 \le 0 \le 0 \le 0 \le 0,$$

$$1 \le \frac{e^{-\frac{1}{2}}+e^{\frac{1}{2}}}{2} \le \frac{e-e^{-1}}{2} \le \frac{e+e^{-1}+2}{4} \le \frac{e+e^{-1}}{2},$$

Hence, Theorem 3.4 is verified.

Now we derive some Hermite-Hadamard type inequalities for the product of two LR-p-convex-*I-V-F*.

Theorem 3.6. Let $\psi, g: [\mu, \upsilon] \to \mathbb{R}_I^+$ be two *I-V-Fs*, respectively, defined by $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$ and $g(\omega) = [g_*(\omega), g^*(\omega)]$, for all $\omega \in [\mu, \upsilon]$, respectively. If $\psi, g \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$, and $\psi g \in \mathcal{IR}_{([\mu,\upsilon])}$, then

$$\frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)g(\omega)d\omega \le_p \frac{\mathcal{M}(\mu,\upsilon)}{3} + \frac{\mathcal{N}(\mu,\upsilon)}{6}, \tag{14}$$

where $\mathcal{M}(\mu,\upsilon) = \psi(\mu)g(\mu) + \psi(\upsilon)g(\upsilon)$, $\mathcal{N}(\mu,\upsilon) = \psi(\mu)g(\upsilon) + \psi(\upsilon)g(\mu)$, and $\mathcal{M}(\mu,\upsilon) = [\mathcal{M}_*(\mu,\upsilon), \mathcal{M}^*(\mu,\upsilon)]$ and $\mathcal{N}(\mu,\upsilon) = [\mathcal{N}_*(\mu,\upsilon), \mathcal{N}^*(\mu,\upsilon)]$.

Proof. Since $\psi, g \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$ then, we have

$$\psi_*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) \le \Theta\psi_*(\mu) + (1-\Theta)\psi_*(\upsilon),$$

$$\psi^*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) \le \Theta\psi^*(\mu) + (1-\Theta)\psi^*(\upsilon).$$

Also

$$g_*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) \leq \Theta g_*(\mu) + (1-\Theta)g_*(\upsilon),$$
$$g^*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) \leq \Theta g^*(\mu) + (1-\Theta)g^*(\upsilon).$$

From the definition of LR-p-convex-*I-V-Fs* it follows that $\psi(\omega)\geq_p 0$ and $g(\omega)\geq_p 0$, so

$$\begin{aligned}
&\psi_*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) g_*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\
&\leq (\Theta\psi_*(\mu) + (1-\Theta)\psi_*(\upsilon))(\Theta g_*(\mu) + (1-\Theta)g_*(\upsilon))\\
&= \psi_*(\mu)g_*(\mu)[\Theta.\Theta] + \psi_*(\upsilon)g_*(\upsilon)[(1-\Theta)(1-\Theta)]\\
&+\psi_*(\mu)g_*(\upsilon)\Theta(1-\Theta) + \psi_*(\upsilon)g_*(\mu)\,(1-\Theta)\,\Theta,\\
&\psi^*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) g^*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\
&\leq (\Theta\psi^*(\mu) + (1-\Theta)\psi^*(\upsilon))(\Theta g^*(\mu) + (1-\Theta)g^*(\upsilon))\\
&= \psi^*(\mu)g^*(\mu)[\Theta.\Theta] + \psi^*(\upsilon)g^*(\upsilon)[(1-\Theta)(1-\Theta)]\\
&+\psi^*(\mu)g^*(\upsilon)\Theta(1-\Theta) + \psi^*(\upsilon)g^*(\mu)\,(1-\Theta)\,\Theta,
\end{aligned}$$

Integrating both sides of above inequality over [0, 1] we get

$$\begin{aligned}
&\int_0^1 \psi_*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) g_*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\
&\quad = \frac{p}{\upsilon^p - \mu^p}\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)g_*(\omega)\,d\omega\\
&\quad \leq \left(\psi_*(\mu)g_*(\mu) + \psi_*(\upsilon)g_*(\upsilon)\right)\int_0^1 \Theta.\Theta\,d\Theta\\
&\quad +\left(\psi_*(\mu)g_*(\upsilon) + \psi_*(\upsilon)g_*(\mu)\right)\int_0^1 \Theta(1-\Theta)d\Theta,\\
&\int_0^1 \psi^*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) g^*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\
&\quad = \frac{p}{\upsilon^p - \mu^p}\int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)g^*(\omega)\,d\omega\\
&\quad \leq \left(\psi^*(\mu)g^*(\mu) + \psi^*(\upsilon)g^*(\upsilon)\right)\int_0^1 \Theta.\Theta d\Theta\\
&\quad +\left(\psi^*(\mu)g^*(\upsilon) + \psi^*(\upsilon)g^*(\mu)\right)\int_0^1 \Theta(1-\Theta)d\Theta.
\end{aligned}$$

It follows that,

$$\frac{p}{\upsilon^p - \mu^p}\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)g_*(\omega)\,d\omega \leq \frac{\mathcal{M}_*\big((\mu,\upsilon)\big)}{3} + \frac{\mathcal{N}_*\big((\mu,\upsilon)\big)}{6},$$
$$\frac{p}{\upsilon^p - \mu^p}\int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)g^*(\omega)\,d\omega \leq \frac{\mathcal{M}^*\big((\mu,\upsilon)\big)}{3} + \frac{\mathcal{N}^*\big((\mu,\upsilon)\big)}{6},$$

that is,

$$\frac{p}{\upsilon^p-\mu^p}\left[\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)g_*(\omega)d\omega\,,\int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)g^*(\omega)\,d\omega\right]$$
$$\leq_p \left[\frac{\mathcal{M}_*\big((\mu,\upsilon)\big)}{3},\frac{\mathcal{M}^*\big((\mu,\upsilon)\big)}{3}\right]+\left[\frac{\mathcal{N}_*\big((\mu,\upsilon)\big)}{6},\frac{\mathcal{N}^*\big((\mu,\upsilon)\big)}{6}\right].$$

Thus,

$$\frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)g(\omega)d\omega \leq_p \frac{\mathcal{M}(\mu,\upsilon)}{3}+\frac{\mathcal{N}(\mu,\upsilon)}{6},$$

and the theorem has been established.

Theorem 3.7. Let $\psi, g\colon[\mu,\upsilon]\to\mathbb{R}_I^+$ be two *I-V-Fs*, respectively, defined by $\psi(\omega)=[\psi_*(\omega),\psi^*(\omega)]$ and $g(\omega)=[g_*(\omega),g^*(\omega)]$, for all $\omega\in[\mu,\upsilon]$. If $\psi, g\in LRSX([\mu,\upsilon],\mathbb{R}_I^+,p)$ and $\psi g\in \mathcal{IR}_{([\mu,\upsilon])}$, then

$$2\,\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq_p \frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)g(\omega)d\omega\ +\frac{\mathcal{M}(\mu,\upsilon)}{6}+\frac{\mathcal{N}(\mu,\upsilon)}{3}, \tag{15}$$

where

$$\mathcal{M}(\mu,\upsilon)=\psi(\mu)g(\mu)+\ \psi(\upsilon)g(\upsilon), \mathcal{N}(\mu,\upsilon)=\psi(\mu)g(\upsilon)+\ \psi(\upsilon)g(\mu),$$

and

$$\mathcal{M}(\mu,\upsilon)=[\mathcal{M}_*(\mu,\upsilon),\mathcal{M}^*(\mu,\upsilon)]\ \text{ and }\ \mathcal{N}(\mu,\upsilon)=[\mathcal{N}_*(\mu,\upsilon),\mathcal{N}^*(\mu,\upsilon)].$$

Proof. By hypothesis, we have

$$\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)g_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{1}{4}\begin{bmatrix}\psi_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)g_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\+\psi_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)g_*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\end{bmatrix}$$
$$+\frac{1}{4}\begin{bmatrix}\psi_*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)g_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\+\psi_*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)g_*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\end{bmatrix},$$
$$\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)g^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{1}{4}\begin{bmatrix}\psi^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)g^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\+\psi^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)g^*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\end{bmatrix}$$
$$+\frac{1}{4}\begin{bmatrix}\psi^*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)g^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\+\psi^*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)g^*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\end{bmatrix},$$

$$\leq \frac{1}{4}\begin{bmatrix} \psi_*\left([\Theta\mu^p + (1-\Theta)\vartheta^p]^{\frac{1}{p}}\right) g_*\left([\Theta\mu^p + (1-\Theta)\vartheta^p]^{\frac{1}{p}}\right) \\ +\psi_*\left([(1-\Theta)\mu^p + \Theta\vartheta^p]^{\frac{1}{p}}\right) g_*\left([(1-\Theta)\mu^p + \Theta\vartheta^p]^{\frac{1}{p}}\right) \end{bmatrix}$$

$$+\frac{1}{4}\begin{bmatrix} \left(\Theta\psi_*(\mu) + (1-\Theta)\psi_*(\vartheta)\right) \\ \left((1-\Theta)g_*(\mu) + \Theta g_*(\vartheta)\right) \\ +\left((1-\Theta)\psi_*(\mu) + \Theta\psi_*(\vartheta)\right) \\ \left(\Theta g_*(\mu) + (1-\Theta)g_*(\vartheta)\right) \end{bmatrix},$$

$$\leq \frac{1}{4}\begin{bmatrix} \psi^*\left([\Theta\mu^p + (1-\Theta)\vartheta^p]^{\frac{1}{p}}\right) g^*\left([\Theta\mu^p + (1-\Theta)\vartheta^p]^{\frac{1}{p}}\right) \\ +\psi^*\left([(1-\Theta)\mu^p + \Theta\vartheta^p]^{\frac{1}{p}}\right) g^*\left([(1-\Theta)\mu^p + \Theta\vartheta^p]^{\frac{1}{p}}\right) \end{bmatrix}$$

$$+\frac{1}{4}\begin{bmatrix} \left(\Theta\psi^*(\mu) + (1-\Theta)\psi^*(\vartheta)\right) \\ \left((1-\Theta)g^*(\mu) + \Theta g^*(\vartheta)\right) \\ +\left((1-\Theta)\psi^*(\mu) + \Theta\psi^*(\vartheta)\right) \\ \left(\Theta g^*(\mu) + (1-\Theta)g^*(\vartheta)\right) \end{bmatrix},$$

$$= \frac{1}{4}\begin{bmatrix} \psi_*\left([\Theta\mu^p + (1-\Theta)\vartheta^p]^{\frac{1}{p}}\right) g_*\left([\Theta\mu^p + (1-\Theta)\vartheta^p]^{\frac{1}{p}}\right) \\ +\psi_*\left([(1-\Theta)\mu^p + \Theta\vartheta^p]^{\frac{1}{p}}\right) g_*\left([(1-\Theta)\mu^p + \Theta\vartheta^p]^{\frac{1}{p}}\right) \end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix} \{\Theta.\Theta + (1-\Theta)(1-\Theta)\}\mathcal{N}_*\left((\mu,\vartheta)\right) \\ +\{2\Theta(1-\Theta)\}\mathcal{M}_*\left((\mu,\vartheta)\right) \end{bmatrix},$$

$$= \frac{1}{4}\begin{bmatrix} \psi^*\left([\Theta\mu^p + (1-\Theta)\vartheta^p]^{\frac{1}{p}}\right) g^*\left([\Theta\mu^p + (1-\Theta)\vartheta^p]^{\frac{1}{p}}\right) \\ +\psi^*\left([(1-\Theta)\mu^p + \Theta\vartheta^p]^{\frac{1}{p}}\right) g^*\left([(1-\Theta)\mu^p + \Theta\vartheta^p]^{\frac{1}{p}}\right) \end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix} \{\Theta.\Theta + (1-\Theta)(1-\Theta)\}\mathcal{N}^*\left((\mu,\vartheta)\right) \\ +\{\{2\Theta(1-\Theta)\}\}\mathcal{M}^*\left((\mu,\vartheta)\right) \end{bmatrix},$$

Integrating over $[0,1]$, we have

$$2\psi_*\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) g_*\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\vartheta^p - \mu^p}(R)\int_\mu^\vartheta \omega^{p-1}\psi_*(\omega)g_*(\omega)d\omega + \frac{\mathcal{M}_*\left((\mu,\vartheta)\right)}{6} + \frac{\mathcal{N}_*\left((\mu,\vartheta)\right)}{3},$$

$$2\psi^*\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) g^*\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\vartheta^p - \mu^p}(R)\int_\mu^\vartheta \omega^{p-1}\psi^*(\omega)g^*(\omega)d\omega + \frac{\mathcal{M}^*\left((\mu,\vartheta)\right)}{6} + \frac{\mathcal{N}^*\left((\mu,\vartheta)\right)}{3},$$

that is,

$$2\psi\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) \leq_p \frac{p}{\vartheta^p - \mu^p}(IR)\int_\mu^\vartheta \omega^{p-1}\psi(\omega)g(\omega)d\omega + \frac{\mathcal{M}(\mu,\vartheta)}{6} + \frac{\mathcal{N}(\mu,\vartheta)}{3},$$

hence, the result is required.

Example 3.8. Let p be an odd number, and the LR-p-convex *I-V-Fs*$\psi, g\colon [\mu,\vartheta] = [2,3] \to \mathbb{R}_I^+$ are, respectively defined by, $\psi(\omega) = \left[2 - \omega^{\frac{p}{2}}, 2\left(2 - \omega^{\frac{p}{2}}\right)\right]$, and $g(\omega) = [\omega^p, 2\omega^p]$.

Since $\psi_*(\omega) = 2 - \omega^{\frac{p}{2}}$, $\psi^*(\omega) = 2\left(2 - \omega^{\frac{p}{2}}\right)$, and $g_*(\omega) = \omega^p$, $g^*(\omega) = 2\omega^p$, then we compute the following

$$\frac{p}{\vartheta^p - \mu^p}\int_{\mu}^{\vartheta} \omega^{p-1}\psi_*(\omega) \times g_*(\omega)d\omega = 1,$$
$$\frac{p}{\vartheta^p - \mu^p}\int_{\mu}^{\vartheta} \omega^{p-1}\psi^*(\omega) \times g^*(\omega)d\omega = 4,$$
$$\frac{\mathcal{M}_*(u,\vartheta)}{3} = \frac{1}{2}\left(10 - 2\sqrt{2} - 3\sqrt{3}\right),$$
$$\frac{\mathcal{M}^*(u,\vartheta)}{3} = \left(20 - 4\sqrt{2} - 6\sqrt{3}\right),$$
$$\frac{\mathcal{N}_*(u,\vartheta)}{6} = \left(10 - 3\sqrt{2} - 2\sqrt{3}\right)\frac{1}{2},$$
$$\frac{\mathcal{N}^*(u,\vartheta)}{6} = \left(20 - 6\sqrt{2} - 4\sqrt{3}\right),$$

that means,

$$1 \leq \left(20 - 5\sqrt{2} - 5\sqrt{3}\right)\frac{1}{2},$$
$$4 \leq \left(40 - 10\sqrt{2} - 10\sqrt{3}\right).$$

Hence, Theorem 3.6 is demonstrated.

For Theorem 3.7, we have

$$2\psi_*\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) \times g_*\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) = \frac{20 - 5\sqrt{10}}{4},$$
$$2\psi^*\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) \times g^*\left(\left[\frac{\mu^p + \vartheta^p}{2}\right]^{\frac{1}{p}}\right) = 20 - 5\sqrt{10},$$
$$\frac{\mathcal{M}_*(u,\vartheta)}{6} = \frac{1}{2}\left(10 - 2\sqrt{2} - 3\sqrt{3}\right),$$
$$\frac{\mathcal{M}^*(u,\vartheta)}{6} = \left(20 - 4\sqrt{2} - 6\sqrt{3}\right),$$
$$\frac{\mathcal{N}_*(u,\vartheta)}{3} = \frac{1}{2}\left(10 - 3\sqrt{2} - 2\sqrt{3}\right),$$
$$\frac{\mathcal{N}^*(u,\vartheta)}{3} = \left(20 - 6\sqrt{2} - 4\sqrt{3}\right),$$

that means,

$$\frac{20-5\sqrt{10}}{4} \leq \left(1+\frac{20-5\sqrt{2}-5\sqrt{3}}{2}\right),$$
$$20-5\sqrt{10} \leq \left(1+\frac{20-5\sqrt{2}-5\sqrt{3}}{2}\right)4.$$

Hence, Theorem 3.7 is verified.

Theorem 3.9. (Second HH-Fejér inequality for LR-p-convex-*I-V-F*)

Let $\psi: [\mu, \upsilon] \to \mathbb{R}_I^+$ be an *I-V-F* with $\mu < \upsilon$, such that $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$ for all $\omega \in [\mu, \upsilon]$ and $\psi \in \mathcal{IR}_{([\mu,\upsilon])}$.

If $\psi \in LRSX([\mu, \upsilon], \mathbb{R}_I^+, p)$, then $\mathcal{W}: [\mu, \upsilon] \to \mathbb{R}, \mathcal{W}(\omega) \geq 0$, p-symmetric with respect to $\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}$, and

$$\frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)\mathcal{W}(\omega)d\omega \leq_p [\psi(\mu)+\psi(\upsilon)]\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta. \quad (16)$$

If $\psi \in LRSV([\mu, \upsilon], \mathbb{R}_I^+, p)$ then, inequality (16) is reversed.

Proof. Let $\psi \in LRSX\left([\mu, \upsilon], \mathbb{R}_I{}^+, p\right)$. Then

$$\begin{aligned}
&\psi_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\
&\leq \left(\Theta\psi_*(\mu)+(1-\Theta)\psi_*(\upsilon)\right)\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right),\\
&\psi^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\\
&\leq \left(\Theta\psi^*(\mu)+(1-\Theta)\psi^*(\upsilon)\right)\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right).
\end{aligned} \quad (17)$$

Also

$$\begin{aligned}
&\psi_*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\\
&\leq \left((1-\Theta)\psi_*(\mu)+\Theta\psi_*(\upsilon)\right)\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right),\\
&\psi^*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\\
&\leq \left((1-\Theta)\psi^*(\mu)+\Theta\psi^*(\upsilon)\right)\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right).
\end{aligned} \quad (18)$$

Adding (17), (18), and integrating over $[0, 1]$, we get

$$\int_0^1 \psi_*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta$$
$$+\int_0^1 \psi_*\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta$$
$$\leq \int_0^1 \begin{bmatrix} \psi_*(\mu)\left\{\Theta\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) + (1-\Theta)\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)\right\} \\ +\psi_*(\upsilon)\left\{(1-\Theta)\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) + \Theta\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)\right\} \end{bmatrix} d\Theta,$$
$$\int_0^1 \psi^*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta$$
$$+\int_0^1 \psi^*\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta$$
$$\leq \int_0^1 \begin{bmatrix} \psi^*(\mu)\left\{\Theta\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) + (1-\Theta)\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)\right\} \\ +\psi^*(\upsilon)\left\{(1-\Theta)\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right) + \Theta\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)\right\} \end{bmatrix} d\Theta.$$

$$= 2\psi_*(\mu)\int_0^1 \Theta\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta + 2\psi_*(\upsilon)\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta,$$
$$= 2\psi^*(\mu)\int_0^1 \Theta\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta + 2\psi^*(\upsilon)\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta.$$

Since $\mathcal{W}$ is symmetric, so

$$\begin{aligned} &= 2[\psi_*(\mu) + \psi_*(\upsilon)]\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta, \\ &= 2[\psi^*(\mu) + \psi^*(\upsilon)]\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta. \end{aligned} \tag{19}$$

Since

$$\begin{aligned} &\int_0^1 \psi_*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta \\ &\quad = \int_0^1 \psi_*\big((1-\Theta)\mu + \Theta\upsilon\big)\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta \\ &\quad = \frac{p}{\upsilon^p - \mu^p}\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega, \\ &\int_0^1 \psi^*\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p + (1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta \\ &\quad = \int_0^1 \psi^*\big((1-\Theta)\mu + \Theta\upsilon\big)\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta \\ &\quad = \frac{p}{\upsilon^p - \mu^p}\int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega. \end{aligned} \tag{20}$$

From (20), we have

$$\frac{p}{\upsilon^p - \mu^p}\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega \leq [\psi_*(\mu) + \psi_*(\upsilon)]\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta,$$
$$\frac{p}{\upsilon^p - \mu^p}\int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega \leq [\psi^*(\mu) + \psi^*(\upsilon)]\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p + \Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta,$$

that is,

$$\frac{p}{\upsilon^p-\mu^p}\left[\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega, \qquad \int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega\right]$$

$$\leq_p [\psi_*(\mu)+\psi_*(\upsilon), \psi^*(\mu)+\psi^*(\upsilon)]\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta.$$

Hence

$$\frac{p}{\upsilon^p-\mu^p}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)\mathcal{W}(\omega)d\omega \leq_p [\psi(\mu)+\psi(\upsilon)]\int_0^1 \Theta\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta.$$

Theorem 3.10. (First HH-Fejér inequality for LR-p-convex-*I-V-F*)

Let $\psi:[\mu,\upsilon]\to\mathbb{R}_I^+$ be an *I-V-F* with $\mu<\upsilon$, such that $\psi(\omega)=[\psi_*(\omega),\psi^*(\omega)]$ for all $\omega\in[\mu,\upsilon]$ and $\psi\in\mathcal{IR}_{([\mu,\upsilon])}$. If $\psi\in LRSX([\mu,\upsilon],\mathbb{R}_I^+,p)$ and $\mathcal{W}:[\mu,\upsilon]\to\mathbb{R}, \mathcal{W}(\omega)\geq 0, p$-symmetric with respect to $\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}$ and $\int_\mu^\upsilon \omega^{p-1}\mathcal{W}(\omega)d\omega>0$, then

$$\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\leq_p \frac{1}{\int_\mu^\upsilon \omega^{p-1}\mathcal{W}(\omega)d\omega}(IR)\int_\mu^\upsilon \omega^{p-1}\psi(\omega)\mathcal{W}(\omega)d\omega. \tag{21}$$

If $\psi\in LRSV([\mu,\upsilon],\mathbb{R}_I^+,p)$ then, inequality (21) is reversed.

Proof. Since $\psi\in LRSX([\mu,\upsilon],\mathbb{R}_I^+,p)$ then

$$\begin{aligned}\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)&\leq\frac{1}{2}\left(\psi_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)+\psi_*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\right),\\ \psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)&\leq\frac{1}{2}\left(\psi^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)+\psi^*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\right).\end{aligned} \tag{22}$$

By multiplying (22) by $\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)=\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)$ and integrate it by Θ over $[0,1]$, we obtain

$$\begin{aligned}&\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta\\ &\leq\frac{1}{2}\left(\begin{array}{c}\int_0^1\psi_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta\\ +\int_0^1\psi_*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta\end{array}\right),\\ &\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta\\ &\leq\frac{1}{2}\left(\begin{array}{c}\int_0^1\psi^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta\\ +\int_0^1\psi^*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta\end{array}\right).\end{aligned} \tag{23}$$

Since

$$\begin{aligned}
&\int_0^1 \psi_*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta\\
&=\int_0^1 \psi_*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta\\
&=\frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon \omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega,\\
&\int_0^1 \psi^*\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([\Theta\mu^p+(1-\Theta)\upsilon^p]^{\frac{1}{p}}\right)d\Theta\\
&=\int_0^1 \psi^*\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\upsilon^p]^{\frac{1}{p}}\right)d\Theta\\
&=\frac{p}{\upsilon^p-\mu^p}\int_\mu^\upsilon \omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega,
\end{aligned} \tag{24}$$

From (24), we have

$$\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\le\frac{1}{\int_\mu^\upsilon\omega^{p-1}\mathcal{W}(\omega)d\omega}\int_\mu^\upsilon\omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega\,,$$
$$\psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\le\frac{1}{\int_\mu^\upsilon\omega^{p-1}\mathcal{W}(\omega)d\omega}\int_\mu^\upsilon\omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega\,,$$

From which, we have

$$\left[\psi_*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right),\quad \psi^*\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\right]$$
$$\le_p\frac{1}{\int_\mu^\upsilon\omega^{p-1}\mathcal{W}(\omega)d\omega}\left[\int_\mu^\upsilon\omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega,\ \int_\mu^\upsilon\omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega\right],$$

that is,

$$\psi\left(\left[\frac{\mu^p+\upsilon^p}{2}\right]^{\frac{1}{p}}\right)\le_p\frac{1}{\int_\mu^\upsilon\omega^{p-1}\mathcal{W}(\omega)d\omega}(IR)\int_\mu^\upsilon\omega^{p-1}\psi(\omega)\mathcal{W}(\omega)d\omega.$$

This completes the proof.

Remark 3.11.

If $p=1$, then combining Theorems 3.9 and 3.10, we get LR-convex-*I-V-F*, see [30].

If $f_*(\mu)=f^*(\mu)$, then, Theorems 3.9 and 3.10 reduce to classical first and second HH-Fejér inequality for p-convex function.

If $f_*(\mu)=f^*(\mu)$, with $p=1$ then, Theorems 3.9 and 3.10 reduce to classical first and second HH-Fejér inequality for classical convex function.

If $\mathcal{W}(x)=1$, then combining Theorems 3.9 and 3.10, we get Theorem 3.1.

Example 3.12. We consider the *I-V-F*ψ: $[1,4]\to\mathbb{F}_I(\mathbb{R})$ defined by,

$$\psi(\omega)=\left[e^{\omega^p},2e^{\omega^p}\right]. \tag{25}$$

Since end point functions $\psi_*(\omega),\psi^*(\omega)$ both are p-convex functions, then $\mathcal{T}(\omega)$ is LR-p-convex-*I-V-F*. If

$$\mathcal{W}(\omega)=\begin{cases}\omega^p-1, & \sigma\in\left[1,\frac{5}{2}\right],\\ 4-\omega^p, & \sigma\in\left(\frac{5}{2},4\right],\end{cases} \tag{26}$$

for $p=1$. Then

$$\begin{aligned}
&\frac{p}{\vartheta^p-\mu^p}\int_1^4\omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega=\frac{1}{3}\int_1^4\omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega\\
&=\frac{1}{3}\int_1^{\frac{5}{2}}\omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega+\frac{1}{3}\int_{\frac{5}{2}}^4\omega^{p-1}\psi_*(\omega)\mathcal{W}(\omega)d\omega,\\
&=\frac{1}{3}\int_1^{\frac{5}{2}}e^{\omega}(\omega-1)d\omega+\frac{1}{3}\int_{\frac{5}{2}}^4e^{\omega}(4-\omega)d\omega\approx 11,\\
&\frac{p}{\vartheta^p-\mu^p}\int_1^4\omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega=\frac{1}{3}\int_1^4\omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega\\
&=\frac{1}{3}\int_1^{\frac{5}{2}}\omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega+\frac{1}{3}\int_{\frac{5}{2}}^4\omega^{p-1}\psi^*(\omega)\mathcal{W}(\omega)d\omega,\\
&=\frac{2}{3}\int_1^{\frac{5}{2}}e^{\omega}(\omega-1)d\omega+\frac{2}{3}\int_{\frac{5}{2}}^4e^{\omega}(4-\omega)d\omega\approx 22,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&[\psi_*(\mu)+\psi_*(\vartheta)]\int_0^1\Theta\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\vartheta^p]^{\frac{1}{p}}\right)d\Theta\\
&[\psi^*(\mu)+\psi^*(\vartheta)]\int_0^1\Theta\mathcal{W}\left([(1-\Theta)\mu^p+\Theta\vartheta^p]^{\frac{1}{p}}\right)d\Theta\\
&=[e+e^4]\left[\int_0^{\frac{1}{2}}3\Theta^2d\omega+\int_{\frac{1}{2}}^1\Theta(3-3\Theta)d\Theta\right]\approx\frac{43}{2}.\\
&=2[e+e^4]\left[\int_0^{\frac{1}{2}}3\Theta^2d\omega+\int_{\frac{1}{2}}^1\Theta(3-3\Theta)d\Theta\right]\approx 43.
\end{aligned} \tag{28}$$

From (27) and (28), we have

$$[11,22]\leq_p\left[\frac{43}{2},43\right].$$

Hence, Theorem 3.9 is verified.

For Theorem 3.10, we have

$$\begin{aligned}
&\psi_*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)\approx\frac{61}{5},\\
&\psi^*\left(\left[\frac{\mu^p+\vartheta^p}{2}\right]^{\frac{1}{p}}\right)\approx\frac{122}{5},
\end{aligned} \tag{29}$$

$$\int_\mu^\vartheta\omega^{p-1}\mathcal{W}(\omega)d\omega=\int_1^{\frac{5}{2}}\omega^{p-1}(\omega-1)d\omega+\int_{\frac{5}{2}}^4\omega^{p-1}(4-\omega)d\omega=\frac{9}{4},$$

$$\frac{1}{\int_{\mu}^{\vartheta} \omega^{p-1}\mathcal{W}(\omega)d\omega}\int_{1}^{4} \omega^{p-1}\psi_{*}(\omega)\mathcal{W}(\omega)d\omega \approx \frac{73}{5},$$
$$\frac{1}{\int_{\mu}^{\vartheta} \omega^{p-1}\mathcal{W}(\omega)d\omega}\int_{1}^{4} \omega^{p-1}\psi^{*}(\omega)\mathcal{W}(\omega)d\omega \approx \frac{146}{5}. \tag{30}$$

From (29) and (30), we have

$$\left[\frac{61}{5},\frac{122}{5}\right] \leq_p \left[\frac{73}{5},\frac{293}{10}\right].$$

Hence, Theorem 3.10 is demonstrated.

4. Discrete Jensen's and Schur's type inequalities

In this section, we propose the concept of discrete Jensen's type inequality for LR-p-convex-*I*-*V*-*F*. Some refinements of discrete Jensen's and Schur's type inequality are also obtained. We begin by presenting the discrete Jensen type inequality for LR-p-convex-*I*-*V*-*F* in the following result.

Theorem 4.1. (Discrete Jensen type inequality for LR-p-convex-*I*-*V*-*F*)
Let $\mathcal{w}_j \in \mathbb{R}^+$, $\mu_j \in [\mu, \mathcal{v}], (j = 1, 2, 3, \ldots k, k \geq 2)$ and let $\psi: [\mu, \mathcal{v}] \to \mathbb{R}_I^+$ be an *I*-*V*-*F* such that $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$ for all $\omega \in [\mu, \mathcal{v}]$. If $\psi \in LRSX([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$, then

$$\psi\left(\left[\frac{1}{W_k}\sum_{j=1}^{k} \mathcal{w}_j\, {\mu_j}^p\right]^{\frac{1}{p}}\right) \leq_p \sum_{j}^{k} \frac{\mathcal{w}_j}{W_k}\psi(\mu_j), \tag{31}$$

where $W_k = \sum_{j=1}^{k} \mathcal{w}_j$.

If $\psi \in LRSV([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$ then, Eq (31) is reversed.

Proof. For $k = 2$ Eq (31) is true. Consider Eq (31) is true for $k = n - 1$, then

$$\psi\left(\left[\frac{1}{W_{n-1}}\sum_{j=1}^{n-1} \mathcal{w}_j\, {\mu_j}^p\right]^{\frac{1}{p}}\right) \leq_p \sum_{j=1}^{n-1} \frac{\mathcal{w}_j}{W_{n-1}}\psi(\mu_j),$$

Now, let us prove that Eq (31) holds for $k = n$.

$$\psi\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathcal{w}_j\, {\mu_j}^p\right]^{\frac{1}{p}}\right)$$
$$= \psi\left(\left[\frac{W_{n-2}}{W_n}\frac{1}{W_{n-2}}\sum_{j=1}^{n-2} \mathcal{w}_j\, {\mu_j}^p + \frac{\mathcal{w}_{n-1} + \mathcal{w}_n}{W_n}\left(\frac{\mathcal{w}_{n-1}}{\mathcal{w}_{n-1} + \mathcal{w}_n}{\mu_{n-1}}^p + \frac{\mathcal{w}_n}{\mathcal{w}_{n-1} + \mathcal{w}_n}{\mu_n}^p\right)\right]^{\frac{1}{p}}\right).$$

Therefore, we have

$$\psi_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathcal{w}_j\ \mu_j{}^p\right]^{\frac{1}{p}}\right) = \psi_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2} \mathcal{w}_j\ \mu_j{}^p + \frac{\mathcal{w}_{n-1}+\mathcal{w}_n}{W_n}(\frac{\mathcal{w}_{n-1}}{\mathcal{w}_{n-1}+\mathcal{w}_n}\mu_{n-1}{}^p + \frac{\mathcal{w}_n}{\mathcal{w}_{n-1}+\mathcal{w}_n}\mu_n{}^p\right]^{\frac{1}{p}}\right),$$

$$\psi^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathcal{w}_j\ \mu_j{}^p\right]^{\frac{1}{p}}\right) = \psi^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2} \mathcal{w}_j\ \mu_j{}^p + \frac{\mathcal{w}_{n-1}+\mathcal{w}_n}{W_n}\left(\frac{\mathcal{w}_{n-1}}{\mathcal{w}_{n-1}+\mathcal{w}_n}\mu_{n-1}{}^p + \frac{\mathcal{w}_n}{\mathcal{w}_{n-1}+\mathcal{w}_n}\mu_n{}^p\right)\right]^{\frac{1}{p}}\right),$$

$$\leq \sum_{j=1}^{n-2}\frac{\mathcal{w}_j}{W_n}\psi_*(\mu_j) + \frac{\mathcal{w}_{n-1}+\mathcal{w}_n}{W_n}\psi_*\left(\left[\frac{\mathcal{w}_{n-1}}{\mathcal{w}_{n-1}+\mathcal{w}_n}\mu_{n-1}{}^p + \frac{\mathcal{w}_n}{\mathcal{w}_{n-1}+\mathcal{w}_n}\mu_n{}^p\right]^{\frac{1}{p}}\right),$$

$$\leq \sum_{j=1}^{n-2}\frac{\mathcal{w}_j}{W_n}\psi^*(\mu_j) + \frac{\mathcal{w}_{n-1}+\mathcal{w}_n}{W_n}\psi^*\left(\left[\frac{\mathcal{w}_{n-1}}{\mathcal{w}_{n-1}+\mathcal{w}_n}\mu_{n-1}{}^p + \frac{\mathcal{w}_n}{\mathcal{w}_{n-1}+\mathcal{w}_n}\mu_n{}^p\right]^{\frac{1}{p}}\right),$$

$$\leq \sum_{j=1}^{n-2}\frac{\mathcal{w}_j}{W_n}\psi_*(\mu_j) + \frac{\mathcal{w}_{n-1}+\mathcal{w}_n}{W_n}\left[\frac{\mathcal{w}_{n-1}}{\mathcal{w}_{n-1}+\mathcal{w}_n}\psi_*(\mu_{n-1}) + \frac{\mathcal{w}_n}{\mathcal{w}_{n-1}+\mathcal{w}_n}\psi_*(\mu_n)\right],$$

$$\leq \sum_{j=1}^{n-2}\frac{\mathcal{w}_j}{W_n}\psi^*(\mu_j) + \frac{\mathcal{w}_{n-1}+\mathcal{w}_n}{W_n}\left[\frac{\mathcal{w}_{n-1}}{\mathcal{w}_{n-1}+\mathcal{w}_n}\psi^*(\mu_{n-1}) + \frac{\mathcal{w}_n}{\mathcal{w}_{n-1}+\mathcal{w}_n}\psi^*(\mu_n)\right],$$

$$\leq \sum_{j=1}^{n-2}\frac{\mathcal{w}_j}{W_n}\psi_*(\mu_j) + \left[\frac{\mathcal{w}_{n-1}}{W_n}\psi_*(\mu_{n-1}) + \frac{\mathcal{w}_n}{W_n}\psi_*(\mu_n)\right],$$

$$\leq \sum_{j=1}^{n-2}\frac{\mathcal{w}_j}{W_n}\psi^*(\mu_j) + \left[\frac{\mathcal{w}_{n-1}}{W_n}\psi^*(\mu_{n-1}) + \frac{\mathcal{w}_n}{W_n}\psi^*(\mu_n)\right],$$

$$= \sum_{j=1}^{n}\frac{\mathcal{w}_j}{W_n}\psi_*(\mu_j),$$

$$= \sum_{j=1}^{n}\frac{\mathcal{w}_j}{W_n}\psi^*(\mu_j).$$

From which, we have

$$\left[\psi_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathcal{w}_j\ \mu_j{}^p\right]^{\frac{1}{p}}\right), \psi^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathcal{w}_j\ \mu_j{}^p\right]^{\frac{1}{p}}\right)\right] \leq_p \left[\sum_{j=1}^{n}\frac{\mathcal{w}_j}{W_n}\psi_*(\mu_j), \sum_{j=1}^{n}\frac{\mathcal{w}_j}{W_n}\psi^*(\mu_j)\right],$$

that is,

$$\psi\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathcal{w}_j\ \mu_j{}^p\right]^{\frac{1}{p}}\right) \leq_p \sum_{j=1}^{n}\frac{\mathcal{w}_j}{W_n}\psi(\mu_j),$$

and the result follows.

If $\mathcal{w}_1 = \mathcal{w}_2 = \mathcal{w}_3 = \cdots = \mathcal{w}_k = 1$, then Theorem 4.1 reduces to the following result:

Corollary 4.2. Let $\mu_j \in [\mu, \mathcal{v}], (j = 1, 2, 3, \ldots k, k \geq 2)$ and let $\psi: [\mu, \mathcal{v}] \to \mathbb{R}_I^+$ be an *I-V-F* such that $\psi(\omega) = [\psi_*(\omega),\ \psi^*(\omega)]$ for all $\omega \in [\mu, \mathcal{v}]$. If $\psi \in LRSX([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$, then

$$\psi\left(\left[\frac{1}{W_k}\sum_{j=1}^{k}\mu_j{}^p\right]^{\frac{1}{p}}\right) \leq_p \sum_{j=1}^{k}\frac{1}{k}\psi(\mu_j), \tag{32}$$

If $\psi \in LRSV([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$ then, inequality (32) is reversed.

To obtain a refinement of Jensen type inequality for LR-p-convex-*I-V-Fs* firstly, we prove the following the result:

Theorem 4.3. Let $\psi: [\mu, \mathcal{v}] \to \mathbb{R}_I^+$ be an *I-V-F* such that $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$ for all $\omega \in [\mu, \mathcal{v}]$. If $\psi \in LRSX([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$, then for $\mu_1, \mu_2, \mu_3 \in [\mu, \mathcal{v}]$, such that $\mu_1 < \mu_2 < \mu_3$ and ${\mu_3}^p - {\mu_1}^p$, ${\mu_3}^p - {\mu_2}^p, {\mu_2}^p - {\mu_1}^p \in \mathcal{L}$, we have

$$({\mu_3}^p - {\mu_1}^p)\psi(\mu_2) \leq_p ({\mu_3}^p - {\mu_2}^p)\psi(\mu_1) + ({\mu_2}^p - {\mu_1}^p)\psi(\mu_3). \tag{33}$$

If $\psi \in LRSV([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$ then, inequalit (33) is reversed.

Proof. Let $\mu_1, \mu_2, \mu_3 \in [\mu, \mathcal{v}]$ and $({\mu_3}^p - {\mu_1}^p) > 0$.

Consider$\Theta = \frac{{\mu_3}^p - {\mu_2}^p}{{\mu_3}^p - {\mu_1}^p}$, then ${\mu_2}^p = \Theta {\mu_1}^p + (1 - \Theta){\mu_3}^p$. Since $\psi \in LRSX([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$ then, by hypothesis, we have

$$\begin{aligned} \psi_*(\mu_2) &\leq \frac{{\mu_3}^p - {\mu_2}^p}{{\mu_3}^p - {\mu_1}^p}\psi_*(\mu_1) + \frac{{\mu_2}^p - {\mu_1}^p}{{\mu_3}^p - {\mu_1}^p}\psi_*(\mu_3), \\ \psi^*(\mu_2) &\leq \frac{{\mu_3}^p - {\mu_2}^p}{{\mu_3}^p - {\mu_1}^p}\psi^*(\mu_1) + \frac{{\mu_2}^p - {\mu_1}^p}{{\mu_3}^p - {\mu_1}^p}\psi^*(\mu_3). \end{aligned} \tag{34}$$

From (34), we have

$$\begin{aligned} ({\mu_3}^p - {\mu_1}^p)\psi_*(\mu_2) &\leq ({\mu_3}^p - {\mu_2}^p)\psi_*(\mu_1) + ({\mu_2}^p - {\mu_1}^p)\psi_*(\mu_3), \\ ({\mu_3}^p - {\mu_1}^p)\psi^*(\mu_2) &\leq ({\mu_3}^p - {\mu_2}^p)\psi^*(\mu_1) + ({\mu_2}^p - {\mu_1}^p)\psi^*(\mu_3), \end{aligned}$$

That is,

$$[({\mu_3}^p - {\mu_1}^p)\psi_*(\mu_2), ({\mu_3}^p - {\mu_1}^p)\psi^*(\mu_2)]$$

$$\leq_p [({\mu_3}^p - {\mu_2}^p)\psi_*(\mu_1) + ({\mu_2}^p - {\mu_1}^p)\psi_*(\mu_3), ({\mu_3}^p - {\mu_2}^p)\psi^*(\mu_1) + ({\mu_2}^p - {\mu_1}^p)\psi^*(\mu_3)],$$

Hence

$$({\mu_3}^p - {\mu_1}^p)\psi(\mu_2) \leq_p ({\mu_3}^p - {\mu_2}^p)\psi(\mu_1) + ({\mu_2}^p - {\mu_1}^p)\psi(\mu_3).$$

Now we obtain a refinement of Jensen's inequality for LR-p-convex-*I-V-F* which is given in the following results.

Theorem 4.4. Let $\mathcal{w}_j \in \mathbb{R}^+$, $\mu_j \in [\mu, \mathcal{v}], (j = 1, 2, 3, \dots k, k \geq 2), \psi: [\mu, \mathcal{v}] \to \mathbb{R}_I^+$ be an *I-V-F* such that $\psi(\omega) = [\psi_*(\omega), \psi^*(\omega)]$ for all $\omega \in [\mu, \mathcal{v}]$. If $\psi \in LRSX([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$ and $\mu_1, \mu_2, \dots, \mu_j \in (L, U) \subseteq [\mu, \mathcal{v}]$, then,

$$\sum_{j=1}^{k} \frac{\mathcal{w}_j}{W_k}\psi(\mu_j) \leq_p \sum_{j=1}^{k} \left(\left(\frac{U^p - {\mu_j}^p}{U^p - L^p}\right)\left(\frac{\mathcal{w}_j}{W_k}\right)\psi(L) + \left(\frac{{\mu_j}^p - L^p}{U^p - L^p}\right)\left(\frac{\mathcal{w}_j}{W_k}\right)\psi(U) \right), \tag{35}$$

where $W_k = \sum_{j=1}^{k} \mathcal{w}_j$. If $\psi \in LRSV([\mu, \mathcal{v}], \mathbb{R}_I^+, p)$, then, Eq (35) is reversed.

Proof. Consider $= \mu_1, \mu_j = \mu_2, (j = 1, 2, 3, \dots k)$, $U = \mu_3$. Then, by hypothesis and Eq (4.4), we have

$$\psi_*(\mu_j) \le \frac{U^p - \mu_j{}^p}{U^p - L^p}\psi_*(L) + \frac{\mu_j{}^p - L^p}{U^p - L^p}\psi_*(U),$$
$$\psi^*(\mu_j) \le \frac{U - \mu_j{}^p}{U^p - L^p}\psi^*(L) + \frac{\mu_j{}^p - L^p}{U^p - L^p}\psi^*(U).$$

Above inequality can be written as,

$$\begin{aligned}\frac{w_j}{W_k}\psi_*(\mu_j) &\le \left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi_*(L) + \left(\frac{\mu_j{}^p - L^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi_*(U),\\ \frac{w_j}{W_k}\psi^*(\mu_j) &\le \left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi^*(L) + \left(\frac{\mu_j{}^p - L^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi^*(U).\end{aligned} \tag{36}$$

Taking sum of all inequalities (36) for $j = 1, 2, 3, \dots k$, we have

$$\sum_{j=1}^{k}\frac{w_j}{W_k}\psi_*(\mu_j) \le \sum_{j=1}^{k}\left(\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi_*(L) + \left(\frac{\mu_j{}^p - L^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi_*(U)\right),$$
$$\sum_{j=1}^{k}\frac{w_j}{W_k}\psi^*(\mu_j) \le \sum_{j=1}^{k}\left(\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi^*(L) + \left(\frac{\mu_j{}^p - L^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi^*(U)\right).$$

That is,

$$\begin{aligned}\sum_{j=1}^{k}\frac{w_j}{W_k}\psi(\mu_j) &= \left[\sum_{j=1}^{k}\frac{w_j}{W_k}\psi_*(\mu_j), \quad \sum_{j=1}^{k}\frac{w_j}{W_k}\psi^*(\mu_j)\right]\\ &\le_p \left[\sum_{j=1}^{k}\left(\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi_*(L) + \left(\frac{\mu_j{}^p - L^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi_*(U)\right),\right.\\ &\left.\sum_{j=1}^{k}\left(\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi^*(L) + \left(\frac{\mu_j{}^p - L^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi^*(U)\right)\right],\\ &\le_p \sum_{j=1}^{k}\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)[\psi_*(L), \psi^*(L)] + \sum_{j=1}^{k}\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)[\psi_*(U), \psi^*(U)].\\ &= \sum_{j=1}^{k}\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi(L) + \sum_{j=1}^{k}\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi(U).\end{aligned}$$

Thus,

$$\sum_{j=1}^{k}\frac{w_j}{W_k}\psi(\mu_j) \le_p \sum_{j=1}^{k}\left(\left(\frac{U^p - \mu_j{}^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi(L) + \left(\frac{\mu_j{}^p - L^p}{U^p - L^p}\right)\left(\frac{w_j}{W_k}\right)\psi(U)\right)$$

this completes the proof.

We now consider some special cases of Theorems 4.1 and 4.4.

If $\psi_*(\omega) = \psi^*(\omega)$, then Theorem 4.1 and 4.4 reduce to the following results:

Corollary 4.5. [35] (Jensen's type inequality for LR-p-convex function)

Let $w_j \in \mathbb{R}^+$, $\mu_j \in [\mu, v], (j = 1, 2, 3, \dots k, k \ge 2)$ and let $\psi: [\mu, v] \to \mathbb{R}^+$ be a non-negative real-valued function. If $\psi \in SX([\mu, v], \mathbb{R}_I^+, p)$, then

$$\psi\left(\left[\frac{1}{W_k}\sum_{j=1}^{k} w_j\ \mu_j{}^p\right]^{\frac{1}{p}}\right) \le \sum_{j=1}^{k}\frac{w_j}{W_k}\psi(\mu_j), \tag{37}$$

where $W_k = \sum_{j=1}^{k} w_j$. If $\psi \in SV([\mu, v], \mathbb{R}^+, p)$ then, Eq (37) is reversed.

Corollary 4.6. Let $\mathcal{w}_j \in \mathbb{R}^+$, $\mu_j \in [\mu, \upsilon], (j = 1, 2, 3, \ldots k, k \geq 2)$, and $\psi: [\mu, \upsilon] \to \mathbb{R}^+$ be an non-negative real-valued function. If $\psi \in SX([\mu, \upsilon], \mathbb{R}_I^+, p)$ and $\mu_1, \mu_2, \ldots, \mu_j \in (L, U) \subseteq [\mu, \upsilon]$ then,

$$\sum_{j=1}^{k} \frac{\mathcal{w}_j}{W_k} \psi(\mu_j) \leq \sum_{j=1}^{k} \left(\left(\frac{U^p - {\mu_j}^p}{U^p - L^p} \right) \left(\frac{\mathcal{w}_j}{W_k} \right) \psi(L) + \left(\frac{{\mu_j}^p - L^p}{U^p - L^p} \right) \left(\frac{\mathcal{w}_j}{W_k} \right) \psi(U) \right), \tag{38}$$

where $W_k = \sum_{j=1}^{k} \mathcal{w}_j$. If $\psi \in SV([\mu, \upsilon], \mathbb{R}^+, p)$ then, Eq (38) is reversed.

5. Conclusions

In this study, we established several novel HH-, HH-Fejér, Schur's, Jensen's type inequalities for LR-p-convex-*I-V-Fs,* and HH-Inequalities are true for this concept of LR-p-convex-*I-V-Fs*. We also obtained some related discrete and integral inequalities as exceptional cases. Some useful examples are also provided to prove the validity of our main results. As a future research, we try to explore this concept for Katugampola fractional integral operator, fuzzy Katugampola fractional integral operator and some applications in interval and fuzzy interval nonlinear programing. By using these concepts, the new direction of study can be found in convex optimization theory and fuzzy convex analysis. We hope that this concept will be helpful for other authors to pay their roles in different fields of sciences.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. M. Z. Sarikaya, A. Saglam, H. Yildirim, On some Hadamard-type inequalities for *h*-convex functions, *J. Math. Inequal.*, **2** (2008), 335–341. https://doi.org/10.7153/jmi-02-30
2. M. Bombardelli, S. Varosanec, Properties of *h*-convex functions related to the Hermite-Hadamard-Fejer inequalities, *Comput. Math. Appl.*, **58** (2009), 1869–1877. https://doi.org/10.1016/j.camwa.2009.07.073
3. M. A. Noor, K. I. Noor, M. U. Awan, A new Hermite-Hadamard type inequality for *h*-convex functions, *Comput. Math. Inform.*, **24** (2015), 191–197.
4. S. S. Dragomir, On the Hadamard's inequlality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwan. J. Math.*, **5** (2001), 775–788.

5. S. S. Dragomir, B. Mond, Integral inequalities of Hadamard type for log-convex functions, *Demonstr. Math*., **31** (1998), 354–364. https://doi.org/10.1515/dema-1998-0214
6. S. S. Dragomir, Refinements of the Hermite-Hadamard integral inequality for log-convex functions, *J. Inequal. Pure Appl. Math.*, **6** (2000), 129–134.
7. S. S. Dragomir, A survey of Jensen type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces, *Commun. Math. Anal*., **10** (2011), 82–104.
8. C. P. Niculescu, The Hermite-Hadamard inequality for log-convex functions, *Nonlinear Anal-Theor*., **75** (2012), 662–669. https://doi.org/10.1016/j.na.2011.08.066
9. S. S. Dragomir, New inequalities of Hermite-Hadamard type for log-convex functions, *Khayyam J. Math*., **3** (2017), 98–115. https://doi.org/10.22034/kjm.2017.47458
10. M. A. Noor, F. Qi, M. U. Awan, Some Hermite-Hadamard type inequalities for log-*h*-convex functions, *Analysis*, **33** (2013), 367–375. https://doi.org/10.1524/anly.2013.1223
11. S. S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwan. J. Math*., **5** (2001), 775–788. https://doi.org/10.11650/twjm/1500574995
12. M. Alomari, M. Darus, On the Hadamard's inequality for log-convex functions on the coordinates, *J. Inequal. Appl*., **2009** (2009), 283147. https://doi.org/10.1155/2009/283147
13. M. Z. Sarikaya, On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals, *Integr. Transf. Spec. F*., **25** (2014), 134–147. https://doi.org/10.1080/10652469.2013.824436
14. M. E. Ozdemir, E. Set, M. Z. Sarikaya, Some new Hadamard type inequalities for co-ordinated *m*-convex and (α, m)-convex functions, *Hacet. J. Math. Stat*., **40** (2011), 219–229.
15. M. Alomari, M. Darus, Co-ordinated *s*-convex function in the first sense with some Hadamard-type inequalities, *Int. J. Contemp. Math. Sci.*, **3** (2008), 1557–1567.
16. M. A. Latif, M. Alomari, Hadamard-type inequalities for product two convex functions on the coordinates, *Int. Math. Forum*., **4** (2009), 2327–2338.
17. D. Singh, B. A. Dar, D. S. Kim, Sufficiency and duality in non-smooth interval valued programming problems, *J. Ind. Manag. Optim*., **15** (2019), 647–665. https://doi.org/10.3934/jimo.2018063
18. I. Ahmad, D. Singh, B. A. Dar, Optimality conditions in multiobjective programming problems with interval valued objective functions, *Control Cybern*., **44** (2015), 19–45.
19. T. M. Costa, H. Roman-Flores, Y. Chalco-Cano, Opial-type inequalities for interval-valued functions, *Fuzzy Set. Syst*., **358** (2019), 48–63. https://doi.org/10.1016/j.fss.2018.04.012
20. Y. Chalco-Cano, A. Flores-Franulic, H. Roman-Flores, Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative, *Comput. Appl. Math*., **31** (2012), 475–472.
21. A. Flores-Franulic, Y. Chalco-Cano, H. Roman-Flores, An Ostrowski type inequality for interval-valued functions, In*: IFSA World Congress and NAFIPS Annual Meeting IEEE*, **35** (2013), 1459–1462. https://doi.org/10.1109/ifsa-nafips.2013.6608617
22. H. Roman-Flores, Y. Chalco-Cano, W. A. Lodwick, Some integral inequalities for interval-valued functions, *Comput. Appl. Math*., **37** (2016), 1306–1318. https://doi.org/10.1007/s40314-016-0396-7

23. D. Zhao, T. An, G. Ye, W. Liu, New Jensen and Hermite-Hadamard type inequalities for *h*-convex interval-valued functions, *J. Inequal. Appl.*, **2018** (2018), 302. https://doi.org/10.1186/s13660-018-1896-3
24. Y. An, G. Ye, D. Zhao, W. Liu, Hermite-Hadamard type inequalities for interval (*h1*, *h2*)-convex functions. *Mathematics*, **7** (2019), 436. https://doi.org/10.3390/math7050436
25. H. Budak, T. Tunç, M. Z. Sarikaya, Fractional Hermite-Hadamard-type inequalities for interval-valued functions, *Proc. Amer. Math. Soc.*, **148** (2020), 705–718. https://doi.org/10.1090/proc/14741
26. H. Budak, T. Tunç, M. Z. Sarikaya, Fractional Hermite-Hadamard-type inequalities for interval-valued functions, *Proc. Am. Math. Soc.*, **148** (2020), 705–718.
27. L. Chen, M. S. Saleem, M. S. Zahoor, R. Bano, Some inequalities related to interval-valued-convex functions, *J. Math.*, **2021** (2021), 6617074. https://doi.org/10.1155/2021/6617074
28. D. Zhang, C. Guo, D. Chen, G. Wang, Jensen's inequalities for set-valued and fuzzy set-valued functions, *Fuzzy Sets Syst.*, **404** (2021), 178–204. https://doi.org/10.1016/j.fss.2020.06.003
29. T. M. Costa, H. Roman-Flores, Some integral inequalities for fuzzy-interval-valued functions, *Inform. Sci.*, **420** (2017), 110–125. https://doi.org/10.1016/j.ins.2017.08.055
30. M. B. Khan, M. A. Noor, K. I. Noor, K. S. Nisar, K. A. Ismail, A. Elfasakhany, Some inequalities for LR-(*h1*, *h2*)-convex interval-valued functions by means of pseudo order relation, *Int. J. Comput. Intell. Syst.*, **14** (2021), 1–15. https://doi.org/10.1007/s44196-021-00032-x
31. M. B. Khan, P. O. Mohammed, M. A. Noor, D. Baleanu, J. L. G. Guirao, Some new fractional estimates of inequalities for LR-*p*-convex interval-valued functions by means of pseudo order relation, *Axioms*, **10** (2021), 1–18. https://doi.org/10.3390/axioms10030175
32. M. B. Khan, M. A. Noor, H. M. Al-Bayatti, K. I. Noor, Some new inequalities for LR-log-*h*-convex interval-valued functions by means of pseudo order relation, *Appl. Math. Inf. Sci*, **15** (2021), 459–470. https://doi.org/10.18576/amis/150408
33. M. Kunt, İ. İşcan, Hermite-Hadamard-Fejer type inequalities for *p*-convex functions, *Arab J. Math. Sci.*, **23** (2017), 215–230. https://doi.org/10.1016/j.ajmsc.2016.11.001
34. L. Fejer, Uber die Fourierreihen II, *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369–390.
35. M. A. Noor, K. I. Noor, M. U. Awan, S. Costache, Some integral inequalities for harmonically *h*-convex functions, *UPB Sci. Bull. Ser. A, Appl. Math. Phys.*, **77** (2015), 5–16.
36. V. Lupulescu, Fractional calculus for interval-valued functions, *Fuzzy Sets Syst.*, **265** (2015), 63–85. https://doi.org/10.1016/j.fss.2014.04.005
37. R. E, Moore, *Interval analysis*. Prentice Hall, Englewood Cliffs, 1966.
38. J. P. Wan, *p*-convex functions and their properties, *Pure Appl. Math.*, **23** (2007), 130–133.
39. I. Iscan, Hermite-Hadamard type inequalities for *p*-convex functions, *Int. J. Anal. Appl.*, **11** (2016), 137–145.
40. M. Kunt, İ. İşcan, Hermite-Hadamard-Fejer type inequalities for *p*-convex functions, *Arab J. Math. Sci.*, **23** (2017), 215–230. https://doi.org/10.1016/j.ajmsc.2016.11.001