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Research article

Hankel determinants of a Sturmian sequence

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Abstract: Let τ be the substitution $1 \to 101$ and $0 \to 1$ on the alphabet $\{0, 1\}$. The fixed point of τ obtained starting from 1, denoted by $\mathbf{s}$, is a Sturmian sequence. We first give a characterization of $\mathbf{s}$ using f-representation. Then we show that the distribution of zeros in the determinants induces a partition of integer lattices in the first quadrant. Combining those properties, we give the explicit values of the Hankel determinants $H_{m,n}$ of $\mathbf{s}$ for all $m \geq 0$ and $n \geq 1$.

Keywords: Hankel determinants; Sturmian sequences
Mathematics Subject Classification: 11B75, 11C20

1. Introduction

Let $\mathbf{s} = (s_j)_{j\geq 0}$ be an integer sequence. For all $m \geq 0$, $n \geq 1$, the (m, n)-order Hankel matrix of $\mathbf{s}$ is

$$M_{m,n} := (s_{m+i+j})_{0\leq i,j\leq n-1} = \begin{pmatrix} s_m & s_{m+1} & \cdots & s_{m+n-1} \\ s_{m+1} & s_{m+2} & \cdots & s_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m+n-1} & s_{m+n} & \cdots & s_{m+2n-2} \end{pmatrix}.$$

The (m, n)-order Hankel determinant of $\mathbf{s}$ is $H_{m,n} = \det M_{m,n}$.

Hankel determinants of automatic sequences have been widely studied, due to its application to the study of irrationality exponent of real numbers; see for example [1, 3, 5, 6, 7, 9, 15] and references therein. In 2016, Han [10] introduced the Hankel continued fraction which is a powerful tool for evaluating Hankel determinants. By using the Hankel continued fractions, Bugeaud, Han Wen and Yao [4] characterized the irrationality exponents of values of certain degree two Mahler functions at rational points. Recently, Guo, Han and Wu [8] fully characterized apwenian sequences, that is ± 1 sequences whose Hankel determinants $H_{0,n}$ satisfying $H_{0,n}/2^{n-1} \equiv 1 \pmod 2$ for all $n \geq 1$.

However, the Hankel determinants of other low complexity sequences, such as Sturmian sequences, are rarely known. Kamae, Tamura and Wen [11] explicitly evaluated the Hankel determinants of the

Fibonacci word. Tamura [13] extended this result to infinite words generated by the substitutions $a \to a^k b, b \to a$ ($k \geq 1$). In this paper, we study the Hankel determinants of the sequence generated by the substitution

$$\tau : 1 \to 101,\ 0 \to 1.$$

Denote by $\mathbf{s} = (s_n)_{n\geq 0} = \lim_{n\to\infty} \tau^n(1)$ the fixed point of τ. Since $\tau^{n+1}(1) = \tau^n(\tau(1)) = \tau^n(1)\tau^n(0)\tau^n(1)$, the word $\tau^n(1)$ is a prefix of $\mathbf{s}$. The first values of $\mathbf{s}$ can be obtained by finding $\tau^n(1)$. For example,

$$s_0 s_1 \cdots s_6 = \tau^2(1) = \tau(101) = 1011101$$

and

$$s_0 s_1 \cdots s_{17} = \tau^3(1) = \tau(\tau^2(1)) = 101110110110111101.$$

It follows from [14, Proposition 2.1] that $\mathbf{s}$ is a Sturmian sequence. See also the sequence A104521 in [12].

We give the explicit values of Hankel determinants $H_{m,n}$ for the sequence $\mathbf{s}$ for all $m \geq 0$ and $n \geq 1$. In Figure 1, we use the color at the point (m, n) to indicate the value of $H_{m,n}$. For example, if $H_{m,n} \neq H_{m',n'}$, then points (m, n) and (m', n') will be marked by different colors. In particular, if $H_{m,n} = 0$, then the point (m, n) is marked by white. Then we can see the distribution of first values of $H_{m,n}$ from Figure 1 and the collection of all the points (m, n) with $H_{m,n} = 0$ are the union of disjoint parallelograms. These parallelograms (together with their boundaries) are divided into parallelograms of three types, labelled by $U_{k,i}$, $V_{k,i}$ and $T_{k,i}$ where $k \geq 0$ and $i \geq 1$ (for detailed definitions, see Section 3).

For all $k \geq 0$ and $i \geq 1$, Theorem 1.1 (resp. Theorem 1.2, Theorem 1.3) gives the exact value of $H_{m,n}$ for all $(m, n) \in U_{k,i}$ (resp. $V_{k,i}$, $T_{k,i}$); see Figure 2. Since for $k \geq 0$ and $i \geq 1$, the parallelograms $U_{k,i}$, $V_{k,i}$ and $T_{k,i}$ are disjoint and they tile the lattices in the first quadrant; see Proposition 3.1 in Section 3. Combing this nice property and Theorem 1.1, 1.2 and 1.3, we obtain the values of $H_{m,n}$ for all $m \geq 0$ and $n \geq 1$.

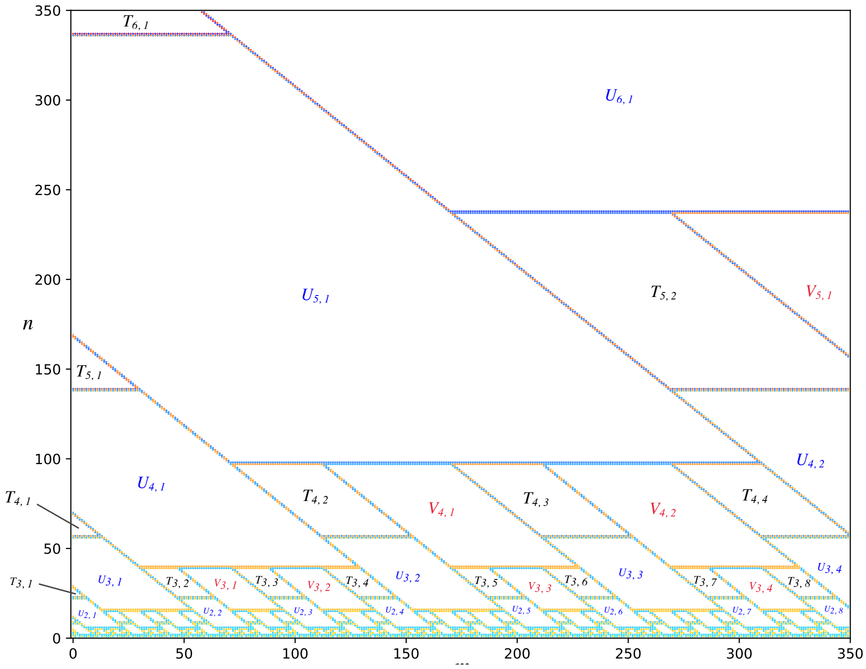

Figure 1. Visualization of the Hankel determinants $H_{n,m}$ ($0 \leq n, m \leq 350$).

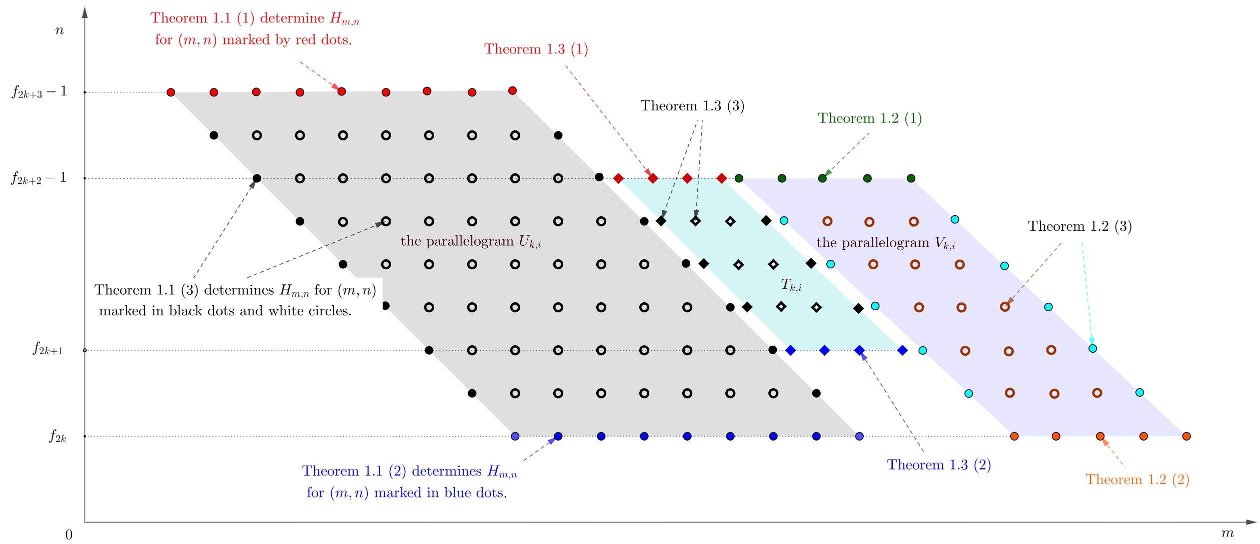

Figure 2. Theorem 1.1 (1) and (2) give the values of $H_{m,n}$ for (m,n) on the upper edge (marked by red dots) and the lower edge (marked by blue dots) of the parallelogram $U_{k,i}$, respectively. Theorem 1.1 (3) give the values of $H_{m,n}$ for (m,n) marked by black and white circles in the parallelogram $U_{k,i}$. Theorem 1.2 and 1.3 are illustrated in the same way.

To state our results, we need four (technically defined) integer sequences $(f_n)_{n\geq 0}$, $(\alpha_i)_{i\geq 1}$, $(\beta_i)_{i\geq 1}$ and $(\gamma_i)_{i\geq 1}$; see Section 3 for their definitions. These four sequences are used to locate the corners of parallelograms $U_{k,i}$, $V_{k,i}$ and $T_{k,i}$. Our main results are the following.

Theorem 1.1. *Let $k \geq 0$ and $U_k = \cup_{i\geq 1} U_{k,i}$. For all $(m,n) \in U_k$,*

1. *when $n = f_{2k+3} - 1$, $H_{m,n} = (-1)^{k+1}(-1)^{\frac{f_{2k+5}}{2} - \Phi_{k+1}(m+n)} \cdot \frac{f_{2k+1}}{2}$;*
2. *when $n = f_{2k}$, $H_{m,n} = (-1)^{k+1}(-1)^{\frac{f_{2k+2}-1}{2}} \cdot \frac{f_{2k+1}}{2}$;*
3. *when $f_{2k} < n < f_{2k+3} - 1$, if $m + n = \alpha_i - f_{2k+2} + 1$ or α_i for some $i \geq 1$, then*

$$H_{m,n} = -(-1)^{(f_{2k+3}-n)k}(-1)^{\frac{(f_{2k+3}-1-n)(f_{2k+3}-2-n)}{2}} \cdot \frac{f_{2k+1}}{2};$$

otherwise $H_{m,n} = 0$.

Theorem 1.2. *Let $k \geq 0$ and $V_k = \cup_{i\geq 1} V_{k,i}$. For all $(m,n) \in V_k$,*

1. *when $n = f_{2k+2} - 1$, $H_{m,n} = (-1)^{\frac{f_{2k+2}+f_{2k+1}-3}{2}} \cdot \frac{f_{2k+1}}{2}$;*
2. *when $n = f_{2k}$, $H_{m,n} = (-1)^{k+1}(-1)^{\frac{f_{2k+2}-1}{2}} \cdot \frac{f_{2k+1}}{2}$;*
3. *when $f_{2k} < n < f_{2k+2} - 1$, if $m + n = \beta_i + 1$ or $\beta_i + f_{2k+1}$ for some $i \geq 1$, then*

$$H_{m,n} = -(-1)^{(f_{2k+2}-n)(k+1)}(-1)^{\frac{(f_{2k+2}-1-n)(f_{2k+2}-2-n)}{2}} \cdot \frac{f_{2k+1}}{2};$$

otherwise $H_{m,n} = 0$.

Theorem 1.3. *Let $k \geq 0$ and $T_k = \cup_{i\geq 1} T_{k,i}$. For all $(m,n) \in T_k$,*

1. *when $n = f_{2k+2} - 1$, $H_{m,n} = (-1)^{\frac{f_{2k}-1}{2}} \cdot f_{2k}$;*
2. *when $n = f_{2k+1}$, $H_{m,n} = (-1)^{\Phi_k(m+n) - \frac{f_{2k+1}}{2} - 1} \cdot f_{2k}$;*
3. *when $f_{2k} < n < f_{2k+2} - 1$, if $m + n = \gamma_i - f_{2k} + 1$ or γ_i for some $i \geq 1$, then*

$$H_{m,n} = (-1)^{(f_{2k+2}-1-n)k} (-1)^{\frac{(f_{2k+2}-1-n)(f_{2k+2}-2-n)}{2}} (-1)^{\frac{f_{2k}-1}{2}} \cdot f_{2k};$$

otherwise $H_{m,n} = 0$.

Sketch of proofs of main results

The computational result indicates that the collection of points (m,n) such that the Hankel determinant $H_{m,n} = 0$ is covered by disjoint parallelograms of three different types, denoted by $U_{*,*}$, $V_{*,*}$ and $T_{*,*}$; see Figure 1. This inspires us to calculate the Hankel determinant $H_{m,n}$ in each parallelogram (according to the location of (m,n)). Then connect the values of Hankel determinants in different parallelograms.

Step 1 Locate the parallelograms.
We first find that the second coordinates of the corners of those parallelograms can be expressed in terms of an integer sequence $(f_n)_{n\geq 0}$ (see Figure 2 for example). Then we see that the first coordinates of corners of three types of parallelograms are determined by three integer sequences $(\alpha_i)_{i\geq 1}$, $(\beta_i)_{i\geq 1}$ and $(\gamma_i)_{i\geq 1}$ introduced in Section 3. Then we characterize the parallelograms (observed in Figure 1) by (3.1). Proposition 3.1 showed that parallelograms defined by (3.1) tile the first quadrant.

Step 2 Calculate $H_{m,n}$ for (m,n) inside the parallelograms $U_{k,*}$, $V_{k,*}$ and $T_{k,*}$.
Lemma 4.1 show that $H_{m,n} = 0$ for all (m,n) which are not on the boundary of those parallelograms. Now the white part in Figure 1 is clear.

Step 3 Reduction on the boundary of $U_{k,*}$, $V_{k,*}$ and $T_{k,*}$.

- By Lemma 4.2 and Lemma 4.6 (resp. Lemma 4.8, Lemma 4.10), calculating $H_{m,n}$ for all (m,n) on the boundary of $U_{k,*}$ (resp. $V_{k,*}$, $T_{k,*}$) is reduced to calculate the determinant $H_{m,n}$ for only one point (m,n) on its boundary. See Figure 3.
- Lemma 4.3 connects the values of $H_{m,n}$ for (m,n) on the lower edge of $U_{k,*}$ and $V_{k,*}$. Lemma 4.4 builds similar connections for the values of $H_{m,n}$ for (m,n) on the lower edge of $T_{k,i}$ and $T_{k,i+1}$. See Figure 4.

Therefore, to obtain the values of $H_{m,n}$ for all (m,n) on the boundary of all $U_{k,*}$ and $V_{k,*}$, we only need to calculate $H_{m,n}$ for one point (m,n) on the boundary of $U_{k,1}$.

Step 4 Reduction on k. Lemma 5.1 enable us to calculate $H_{m,n}$ for (m,n) on the boundary of $U_{k+1,*}$ by using the values of $H_{m,n}$ on the boundary of $U_{k,*}$ and $T_{k,*}$. Lemma 5.2 enable us to calculate $H_{m,n}$ for (m,n) on the boundary of $T_{k,*}$ by using the values of $H_{m,n}$ on the boundary of $U_{k,*}$ and $U_{k-1,*}$. See Figure 4.

Step 5 According to step 3 and step 4, to obtain the value of $H_{m,n}$ for all (m,n), we only need to calculate $H_{m,n}$ for (m,n) on the boundary of $U_{k,1}$, $V_{k,1}$ and $T_{k,2}$ for all k. These have been done by Theorem 5.3, Corollary 5.4 and Corollary 5.5.

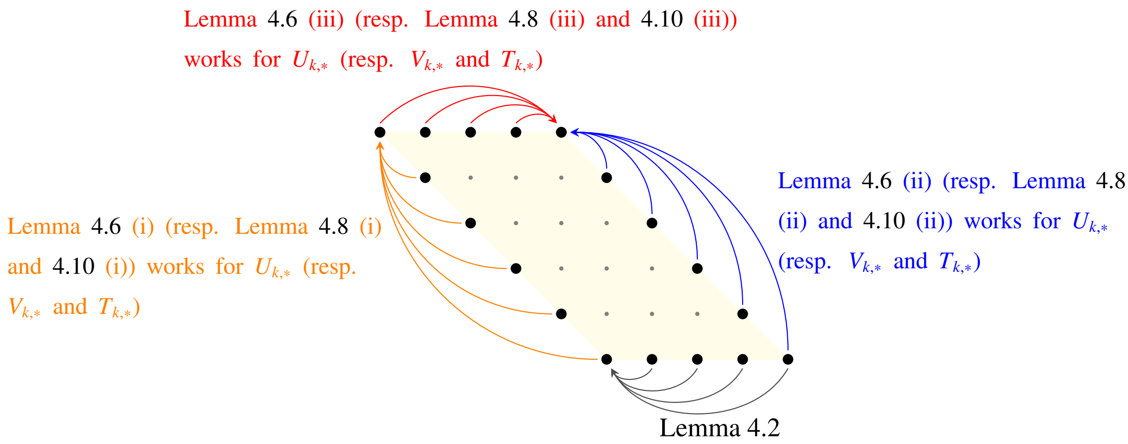

Figure 3. Illustration for Lemma 4.2, 4.6, 4.8 and 4.10.

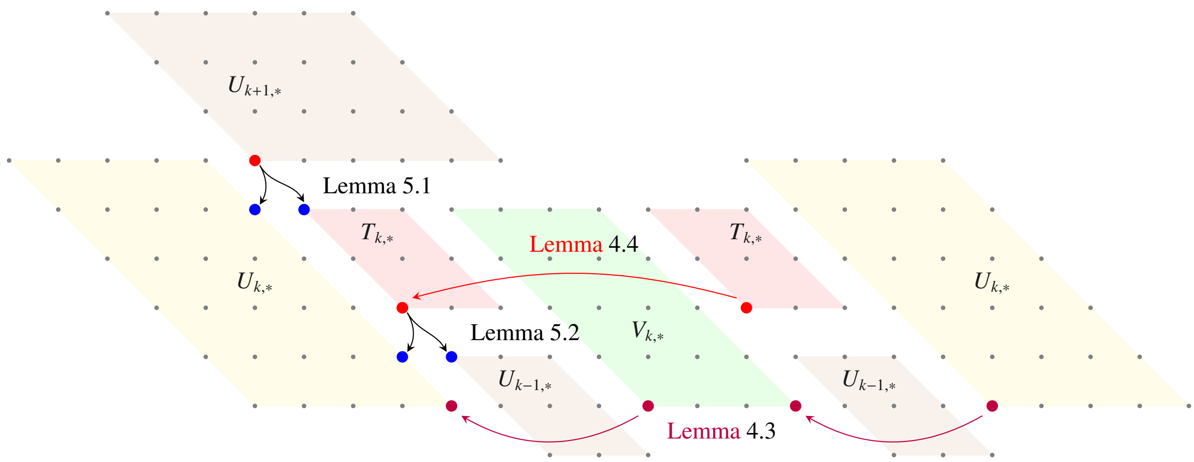

Figure 4. Illustration for Lemma 4.3, 4.4, 5.1 and 5.2.

The paper is organized as follows. In Section 2, we introduce the f-representation of positive integers and give a criterion (Proposition 2.4) to determine s_n according the f-representation of n. This criterion leads us to the key ingredient (Theorem 2.5) in calculating the Hankel determinants. Then we introduce the truncated f-representation which is essential in describing the parallelograms. In Section 3, we show that the parallelograms $U_{k,i}$, $V_{k,i}$ and $T_{k,i}$ tile all the integer points in the first quadrant. In Section 4, we first show that the Hankel determinants vanish when (m,n) is inside a parallelogram of those three types. Next we show the relations of Hankel determinants $H_{m,n}$ on the boundary of a parallelogram $U_{k,i}$ (or $V_{k,i}$, $T_{k,i}$) for a given k and i. Finally, for any $k \geq 0$, we describe the relation of values of Hankel determinants for parallelograms $U_{k,i}$ for all $i \geq 1$. In Section 5, we give the expressions for Hankel determinants on the boundary of $U_{k,i}$ (or $V_{k,i}$, $T_{k,i}$) for all $k \geq 0$. In the last section, we formulate and prove our main results.

2. Some properties of the sequence s

In this section, we first introduce the f-representation of positive integers according to the sequence $(f_n)_{n\geq 0}$. By understanding the occurrences of 0s in the sequence **s**, we prove a key result (Proposition 2.4) which can determine s_n according to the f-representation of n. Then we give the essential result (Theorem 2.5). In subsection 2.3, we introduce the truncated f-representation which is useful in determining the parallelograms. In section 2.4, we investigate to some sub-sequences of $(f_n)_{n\geq 0}$ which are need in evaluating the coefficients of the Hankel determinants. Then we characterize two sub-sequences of **s** which helps us understand $H_{m,n}$.

2.1. The occurrence of 0's in **s**.

We introduce an auxiliary sequence $(f_n)_{n\geq 0}$ to determine the positions of 0's. For all $n \geq 0$, we define

$$f_{2n} = |\tau^n(1)| \quad \text{and} \quad f_{2n+1} = |\tau^n(10)|.$$

Then $f_0 = 1$, $f_1 = 2$, and for all $n \geq 0$,

$$\begin{cases} f_{2n+2} = f_{2n} + f_{2n+1}, \\ f_{2n+3} = f_{2n} + f_{2n+2}. \end{cases} \tag{2.1}$$

The first values are

$$(f_n)_{n\geq 0} = (1, 2, 3, 4, 7, 10, 17, 24, 41, 58, 99, 140, 239, 338, 577, 816, \dots).$$

Remark 2.1. The sequence $(f_n)_{n\geq 0}$ can be expressed in terms of Pell numbers $(p_n)_{n\geq 0}$ defined by the recurrence $p_0 = 1$, $p_1 = 2$ and $p_{n+2} = 2p_{n+1} + p_n$ for all $n \geq 0$. See the sequence A000129 in [12]. Indeed, p_n (resp. p_{n-1}) is the number of 1's (resp. 0's) in $\tau^n(1)$. It is easy to verify that $f_{2n} = p_{n+1} - p_n$ and $f_{2n+1} = 2p_n$.

Since $(f_n)_{n\geq 0}$ is an increasing non-negative integer sequence, it is a numeration system in the following sense.

Lemma 2.2 (Theorem 3.1.1 [2])**.** *Let $u_0 < u_1 < u_2 < \dots$ be an increasing sequence of integers with $u_0 = 1$. Every non-negative integer N has exactly one representation of the form $\sum_{0\leq i\leq r} a_i u_i$ where $a_r \neq 0$, and for $i \geq 0$, the digits a_i are non-negative integers satisfying the inequality*

$$a_0 u_0 + a_1 u_1 + \cdots + a_i u_i < u_{i+1}.$$

Proposition 2.3. *Every integer $n \geq 0$ can be uniquely expressed as $n = \sum_{0\leq i\leq r} a_i f_i$ with $a_i \in \{0, 1\}$, $a_r \neq 0$, and*

$$\begin{cases} a_i a_{i+1} = 0 \text{ for all } 0 \leq i < r, \\ a_i a_{i+2} = 0 \text{ for all even numbers } 0 \leq i < r-1. \end{cases} \tag{2.2}$$

Proof. Suppose $a_i \in \{0, 1\}$. By Lemma 2.2, we only need to show that $a_0 f_0 + a_1 f_1 + \cdots + a_t f_t < f_{t+1}$ for all t if and only if the condition (2.2) holds.

The 'only if' part. Suppose there is an index i such that $a_i a_{i+1} = 1$. Then

$$a_0 f_0 + a_1 f_1 + \cdots + a_i f_i + a_{i+1} f_{i+1} \geq f_i + f_{i+1} \geq f_{i+2},$$

which is a contradiction for $t = i + 1$. Suppose there is an even index i such that $a_i a_{i+2} = 1$. Then

$$a_0 f_0 + a_1 f_1 + \cdots + a_i f_i + a_{i+1} f_{i+1} + a_{i+2} f_{i+2} \geq f_i + f_{i+2} = f_{i+3},$$

which is a contradiction for $t = i + 2$.

The 'if' part. Suppose the condition (2.2) holds. When t is odd, the maximum possible value of $a_0 f_0 + a_1 f_1 + \cdots + a_t f_t$ occurs when $a_t a_{t-1} \ldots a_0 = 1010 \ldots 10$, and this maximum value $f_1 + f_3 + \cdots + f_{t-2} + f_t = f_{t+1} - 1$. When t is even, the maximum possible value of $a_0 f_0 + a_1 f_1 + \cdots + a_t f_t$ occurs when $a_t a_{t-1} \ldots a_0 = 1001010 \ldots 10$. In this case, the maximum value is $f_1 + f_3 + \cdots + f_{t-5} + f_{t-3} + f_t = f_{t+1} - 1$. □

Definition (f-representation)**.** Let $n \geq 0$ be an integer. We call the representation $n = \sum_{0 \leq i \leq r} a_i f_i$ in Proposition 2.3 the *f-representation* of n. We also write $n = \sum_{i=0}^{+\infty} a_i f_i$ where $a_i = 0$ for all $i > r$. In the case that we need to emphasize that a_i depends on n, we write $a_i = a_i(n)$ as a function of n.

Proposition 2.4. *For any integer $n \geq 0$ with the f-representation $\sum_{i=0}^{r} a_i(n) f_i$, we have $s_n = 0$ if and only if $a_0(n) = 1$.*

Proof. One can verify directly that the result holds for all $n < f_4 = 7$. Assume that the result holds for $n < f_{2k}$ where $k \geq 2$. We only need to prove it for all $f_{2k} \leq n < f_{2k+2}$.

Suppose $f_{2k} \leq n < f_{2k+1}$. One has $a_{2k}(n) = 1$ and hence $a_0(n - f_{2k}) = a_0(n)$. Note that $f_{2k} = |\tau^k(1)|$ and

$$s_0 s_1 \ldots s_{f_{2k+2}-1} = \tau^{k+1}(1) = \tau^k(1)\tau^k(0)\tau^k(1). \tag{2.3}$$

We see that s_n is the $(n+1)$-th letter of $\tau^{k+1}(1)$ and it is also the $(n + 1 - f_{2k})$-th letter of $\tau^k(0) = \tau^{k-1}(1)$. Consequently, $s_n = s_{n-f_{2k}}$. Since

$$n - f_{2k} < f_{2k+1} - f_{2k} = f_{2k-2},$$

by the inductive assumption, we have $s_{n-f_{2k}} = 0$ if and only if $a_0(n - f_{2k}) = 1$. Therefore, $s_n = 0$ if and only if $a_0(n) = 1$.

Suppose $f_{2k+1} \leq n < f_{2k+2}$. In this case $a_{2k+1}(n) = 1$ and $a_0(n - f_{2k+1}) = a_0(n)$. Since $|\tau^k(10)| = f_{2k+1}$, it follows from (2.3) that $s_n = s_{n-f_{2k+1}}$. Note that

$$n - f_{2k+1} < f_{2k+2} - f_{2k+1} = f_{2k}.$$

By the inductive assumption, $s_{n-f_{2k+1}} = 0$ if and only if $a_0(n - f_{2k+1}) = 1$ which implies the result also holds for all $f_{2k+1} \leq n < f_{2k+2}$. □

2.2. *Comparing digits in the sequence* **s** *with a fixed gap*

We introduce the truncated f-representations (of positive integers) which are useful in telling two digits with a fixed gap in **s** are equal or not.

Definition. (Truncated f-representation) Let $n \geq 0$ be an integer with the f-representation $\sum_{i=0}^{+\infty} a_i(n) f_i$. For all integers $k \geq 0$, the truncated f-representation of n is

$$\Phi_k(n) := \sum_{i=0}^{2k+2} a_i(n) f_i.$$

The next lemma gives a criterion that when two digits (with a fixed gap) in **s** are equal by using their positions.

Theorem 2.5. *Let $n \geq 0$ be an integer with the f-representation $\sum_{i=0}^{+\infty} a_i(n) f_i$. Then*

(i) for all $k \geq 0$, $s_{n+f_{2k}} \neq s_n$ if and only if $\Phi_k(n) \in \{\frac{f_{2k+1}}{2}, \frac{f_{2k+1}}{2} - 1\}$;
(ii) for all $k \geq 1$, $s_{n+f_{2k+1}} \neq s_n$ if and only if $\Phi_k(n) \in \{\frac{f_{2k+3}}{2}, \frac{f_{2k+3}}{2} - 1, \frac{f_{2k+3}}{2} + f_{2k}, \frac{f_{2k+3}}{2} + f_{2k} - 1\}$.

Proof. (i) We prove by induction on k. When $k = 0$, by Proposition 2.3, there are only four possible values for $a_0(n)a_1(n)a_2(n)$. By Eq (2.1), we have

$\Phi_0(n)$	0	1	2	3
$a_0(n)a_1(n)a_2(n)$	000	100	010	001
$a_0(n+f_0)$	1	0	0	0

Then we see that $a_0(n) \neq a_0(n + f_0)$ if and only if $\Phi_0(n) = 0$ or 1. The result holds for $k = 0$.

When $k = 1$, note that $\Phi_1(n) \leq \sum_{i=0}^{4} f_i < f_5 = 10$. We see

$\Phi_1(n)$	0	1	2	3	4	5	6	7	8	9
$a_0(n)\dots a_4(n)$	00000	10000	01000	00100	00010	10010	01010	00001	10001	01001
$a_0(n+f_2)$	0	0	1	0	0	1	0	0	1	0

Then we have $a_0(n) \neq a_0(n + f_2)$ if and only if $\Phi_1(n) = 1$ or 2, that is $\frac{f_3}{2} - 1$ or $\frac{f_3}{2}$. The result also holds for $k = 1$.

Now assume that the result holds for all $0 \leq k < \ell$ with $\ell \geq 2$. We prove it for $k = \ell$. Let $w = a_{2\ell-2}(n)a_{2\ell-1}(n)\dots a_{2\ell+2}(n)$ and $v = a_{2\ell-2}(n + f_{2\ell})a_{2\ell-1}(n + f_{2\ell})\dots a_{2\ell+1}(n + f_{2\ell})$. According to Proposition 2.3, w can take only 10 different values.

While $w \neq 01000$, one can determine v directly using Eq (2.1); thus in these cases, $a_i(n+f_{2\ell}) = a_i(n)$ for all $0 \leq i \leq 2\ell - 3$; see Table 1. For instance, when $w = 10010$,

$$\begin{aligned} n + f_{2\ell} &= \left(\sum_{i=0}^{2\ell-3} a_i(n) f_i + f_{2\ell-2} + f_{2\ell+1} + \sum_{i=2\ell+3}^{+\infty} a_i(n) f_i \right) + f_{2\ell} \\ &= \left(\sum_{i=0}^{2\ell-3} a_i(n) f_i + f_{2\ell-2} \right) + \left(f_{2\ell+2} + \sum_{i=2\ell+3}^{+\infty} a_i(n) f_i \right). \end{aligned}$$

Table 1. Values of v.

w	00000	10000	01000	00100	00010	10010	01010	00001	10001	01001
v	0010	0001	?	0101	0000	1000	0100	0000	1000	0100

Hence one can see that $a_0(n + f_{2\ell}) = a_0(n)$.

When $w = 01000$, set $n' = \sum_{i=0}^{2\ell-2} a_i(n) f_i$. Then $a_0(n) = a_0(n')$ and

$$\begin{aligned} n + f_{2\ell} &= \left(n' + f_{2\ell-1} + \sum_{i=2\ell+3}^{+\infty} a_i(n) f_i \right) + f_{2\ell} \\ &= (n' + f_{2\ell-4}) + f_{2\ell+1} + \sum_{i=2\ell+3}^{+\infty} a_i(n) f_i. \quad \text{(by Eq. 2.1)} \end{aligned}$$

Noticing that $n' + f_{2\ell-4} < f_{2\ell-2}$, we have $a_i(n + f_{2\ell}) = a_i(n' + f_{2\ell-4})$ for all $0 \le i \le 2\ell - 2$. In particular, $a_0(n + f_{2\ell}) = a_0(n' + f_{2\ell-4})$. By Proposition 2.4 and the inductive assumption,

$$\begin{aligned} a_0(n + f_{2\ell}) \neq a_0(n) &\iff a_0(n' + f_{2\ell-4}) \neq a_0(n') \\ &\iff n' \in \left\{ \frac{f_{2\ell-3}}{2}, \frac{f_{2\ell-3}}{2} - 1 \right\} \\ &\iff \Phi_\ell(n) = n' + f_{2\ell-1} \in \left\{ \frac{f_{2\ell-3}}{2} + f_{2\ell-1}, \frac{f_{2\ell-3}}{2} + f_{2\ell-1} - 1 \right\}. \end{aligned}$$

By Eq (2.1), we have

$$\begin{aligned} \frac{f_{2\ell+1}}{2} &= (f_{2\ell-2} + f_{2\ell})/2 \\ &= (f_{2\ell-2} + f_{2\ell-2} + f_{2\ell-1})/2 \\ &= f_{2\ell-2} + \frac{f_{2\ell-1}}{2} \\ &= f_{2\ell-2} + f_{2\ell-4} + \frac{f_{2\ell-3}}{2} \\ &= f_{2\ell-1} + \frac{f_{2\ell-3}}{2}. \end{aligned} \tag{2.4}$$

Then we obtain that

$$a_0(n + f_{2\ell}) \neq a_0(n) \iff \Phi_\ell(n) \in \left\{ \frac{f_{2\ell+1}}{2}, \frac{f_{2\ell+1}}{2} - 1 \right\}.$$

It follows from Proposition 2.4 that the result holds for $k = \ell$.

(ii) For any $k \ge 1$, let $u = a_{2k}(n) a_{2k+1}(n) a_{2k+2}(n)$. It follows from Proposition 2.4 that $u \in \{100, 010, 001\}$. The proof is divided into the following three cases.

- When $u = 001$, we also have $a_{2k+3}(n) = a_{2k+4}(n) = 0$. So

$$\begin{aligned} n + f_{2k+1} &= \left(\sum_{i=0}^{2k-1} a_i(n)f_i + f_{2k+2} + \sum_{i=2k+5}^{+\infty} a_i(n)f_i\right) + f_{2k+1} \\ &= \left(\sum_{i=0}^{2k-1} a_i(n)f_i + f_{2k-2}\right) + \left(f_{2k+3} + \sum_{i=2k+5}^{+\infty} a_i(n)f_i\right). \end{aligned}$$

Let $n' = \sum_{i=0}^{2k-1} a_i(n)f_i$. Then $a_0(n) = a_0(n')$. Since $n' < f_{2k}$ and $n' + f_{2k-2} < f_{2k+1}$, we have $a_i(n' + f_{2k-2}) = a_i(n + f_{2k+1})$ for all $0 \le i \le 2k$. Thus

$$\begin{aligned} a_0(n + f_{2k+1}) \neq a_0(n) &\iff a_i(n' + f_{2k-2}) \neq a_0(n') \\ &\iff n' = \sum_{i=0}^{2k} a_i(n')f_i \in \left\{\frac{f_{2k-1}}{2}, \frac{f_{2k-1}}{2} - 1\right\} \end{aligned}$$

where in the last step we use Theorem 2.5(i). By Eq (2.1) and Eq (2.4),

$$\frac{f_{2k-1}}{2} + f_{2k+2} = \frac{f_{2k-1}}{2} + f_{2k+1} + f_{2k} = \frac{f_{2k+3}}{2} + f_{2k}.$$

So when $u = 001$, $a_0(n + f_{2k+1}) \neq a_0(n)$ if and only if

$$\Phi_k(n) = n' + f_{2k+2} \in \left\{\frac{f_{2k+3}}{2} + f_{2k}, \frac{f_{2k+3}}{2} + f_{2k} - 1\right\}.$$

- Suppose $u = 010$. Applying Eq (2.1) twice, we obtain that

$$2f_{2k+1} = f_{2k-2} + f_{2k} + f_{2k+1} = f_{2k-2} + f_{2k+2}.$$

Then

$$\begin{aligned} n + f_{2k+1} &= \left(\sum_{i=0}^{2k-1} a_i(n)f_i + f_{2k+1} + \sum_{i=2k+3}^{+\infty} a_i(n)f_i\right) + f_{2k+1} \\ &= \left(\sum_{i=0}^{2k-1} a_i(n)f_i + f_{2k-2}\right) + \left(f_{2k+2} + \sum_{i=2k+3}^{+\infty} a_i(n)f_i\right). \end{aligned}$$

Let $n' = \sum_{i=0}^{2k-1} a_i(n)f_i$. Using Theorem 2.5(i), the same argument as in the case $u = 001$ leads us to the fact that

$$\begin{aligned} a_0(n + f_{2k+1}) \neq a_0(n) &\iff n' = \sum_{i=0}^{2k} a_i(n')f_i \in \left\{\frac{f_{2k-1}}{2}, \frac{f_{2k-1}}{2} - 1\right\} \\ &\iff \Phi_k(n) = n' + f_{2k+1} \in \left\{\frac{f_{2k+3}}{2}, \frac{f_{2k+3}}{2} - 1\right\}. \end{aligned}$$

- When $u = 100$, we have $a_{2k-2}(n) = a_{2k-1}(n) = 0$. Then

$$\begin{aligned} n + f_{2k+1} &= \left(\sum_{i=0}^{2k-3} a_i(n)f_i + f_{2k} + \sum_{i=2k+3}^{+\infty} a_i(n)f_i\right) + f_{2k+1} \\ &= \left(\sum_{i=0}^{2k-3} a_i(n)f_i\right) + \left(f_{2k+2} + \sum_{i=2k+3}^{+\infty} a_i(n)f_i\right) \end{aligned}$$

which implies that $a_0(n + f_{2k+1}) = a_0(n)$. □

2.3. Integers with the same truncated f-representation

To apply Theorem 2.5, we need to investigate the integers of the same truncated f-representations. The following two lemmas (Lemma 2.6 and Lemma 2.8) serve for this purpose.

For all $k \geq 0$, denote

$$E'_k = \left\{ x \in \mathbb{N} \,:\, \Phi_k(x) = \frac{f_{2k+3}}{2} \right\} \quad \text{and} \quad E''_k = \left\{ x \in \mathbb{N} \,:\, \Phi_k(x) = \frac{f_{2k+3}}{2} + f_{2k} \right\}.$$

Let $E_k = E'_k \cup E''_k = (x_j^{(k)})_{j\geq 1}$ where $x_1^{(k)} < x_2^{(k)} < x_3^{(k)} < \dots$. The first values of E_k are

$$(x_j^{(k)})_{j\geq 1} = \bigg(\frac{f_{2k+3}}{2}, \frac{f_{2k+3}}{2} + f_{2k}, \frac{f_{2k+3}}{2} + f_{2k+3}, \frac{f_{2k+3}}{2} + f_{2k+4}, \frac{f_{2k+3}}{2} + f_{2k+5}, \frac{f_{2k+3}}{2} + f_{2k} + f_{2k+5}, \frac{f_{2k+3}}{2} + f_{2k+3} + f_{2k+5}, \frac{f_{2k+3}}{2} + f_{2k+6}, \dots \bigg).$$

Lemma 2.6. *Let $k \geq 0$ and $x \in E'_k$ with $x = x_j^{(k)}$ for some $j \geq 2$. Then $x - f_{2k+2} = x_{j-1}^{(k)} \in E_k$.*

Proof. Let $x \in E''_k$ with $x = x_j^{(k)}$ for some $j \geq 2$. Note that $\frac{f_{2k+3}}{2} + f_{2k} = f_{2k+2} + \frac{f_{2k-1}}{2}$. By Proposition 2.3, have $a_{2k+3}(x) = a_{2k+4}(x) = 0$. When $0 < b < f_{2k} - \frac{f_{2k-1}}{2}$, we see

$$\Phi_k(x+b) = \left(\frac{f_{2k+3}}{2} + f_{2k} \right) + b < f_{2k+3}$$

which implies that $x + b \notin E_k$. When $f_{2k} - \frac{f_{2k-1}}{2} \leq b < f_{2k+2}$,

$$\Phi_k(x+b) = \left(\frac{f_{2k+3}}{2} + f_{2k} \right) + b - f_{2k+3} < \frac{f_{2k+3}}{2}.$$

So $x + b \notin E_k$. Since $\Phi_k(x + f_{2k+2}) = \frac{f_{2k+3}}{2}$, we have $x + f_{2k+2} = x_{j+1}^{(k)} \in E'_k$.

Let $x \in E'_k$ with $x = x_j^{(k)}$ for some $j \geq 1$. According to Proposition 2.3, $a_{2k+3}(x)a_{2k+4}(x) = 00$, 10 or 01, which can be divided into two sub-cases.

- $a_{2k+3}(x)a_{2k+4}(x) = 00$. For $0 < b \leq f_{2k}$, we see $\Phi_k(x+b) = \Phi_k(x) + b = \frac{f_{2k+3}}{2} + b$. Thus $x + f_{2k} = x_{j+1}^{(k)} \in E''_k$.
- $a_{2k+3}(x)a_{2k+4}(x) = 01$ or 10. For $0 < b < f_{2k}$, we see

  $$\Phi_k(x+b) = \Phi_k(x) + b = \frac{f_{2k+3}}{2} + b < \frac{f_{2k+3}}{2} + f_{2k}.$$

  Thus $x + b \notin E_k$. For $f_{2k} \leq b < f_{2k+2}$, we have

  $$\Phi_k(x+b) = \frac{f_{2k+3}}{2} + b - f_{2k+2} \in \left(\frac{f_{2k-1}}{2}, \frac{f_{2k+3}}{2} \right)$$

  which yields that $x + b \notin E_k$. Noting that $\Phi_k(x + f_{2k+2}) = \Phi_k(x) = \frac{f_{2k+3}}{2}$, we obtain that $x + f_{2k+2} = x_{j+1}^{(k)} \in E'_k$.

From the above argument, we see that if $x = x_j^{(k)} \in E_k'$ for some $j \geq 2$, then either $x - f_{2k+2} = x_{j-1}^{(k)} \in E_k''$ or $x - f_{2k+2} = x_{j-1}^{(k)} \in E_k'$ with $a_{2k+3}(x - f_{2k+2})a_{2k+4}(x - f_{2k+2}) \neq 00$. The result holds. □

Remark 2.7. From the proof of Lemma 2.6, we see the gaps between two adjacent elements in E_k are f_{2k} and f_{2k+2}. That is $x_{j+1}^{(k)} - x_j^{(k)} = f_{2k}$ or f_{2k+2} for all $j \geq 1$. Moreover, the gaps between two adjacent elements in E_k' are f_{2k+2} and f_{2k+3}.

For all $k \geq 0$, let

$$F_k = \left\{ y \in \mathbb{N} \,:\, \Phi_k(y) = \frac{f_{2k+1}}{2} \right\} = (y_j^{(k)})_{j\geq 1} \quad \text{and} \quad F_k' = \left\{ y \in \mathbb{N} \,:\, \Phi_{k+1}(y) = \frac{f_{2k+1}}{2} \right\}$$

where $y_1^{(k)} < y_2^{(k)} < y_3^{(k)} < \dots$. Write $F_k'' = F_k - F_k'$. The first values of F_k are

$$\begin{aligned}(y_j^{(k)})_{j\geq 1} = \Bigg(& \frac{f_{2k+1}}{2}, \frac{f_{2k+1}}{2} + f_{2k+3}, \frac{f_{2k+1}}{2} + f_{2k+4}, \frac{f_{2k+1}}{2} + f_{2k+5}, \\ & \frac{f_{2k+1}}{2} + f_{2k+5} + f_{2k+3}, \frac{f_{2k+1}}{2} + f_{2k+6}, \dots \Bigg).\end{aligned}$$

Lemma 2.8. *For any $y \in F_k$ with $y = y_j^{(k)}$ for some $j \geq 1$, we have*

$$y_{j+1}^{(k)} = \begin{cases} y + f_{2k+3}, & \text{if } y \in F_k'; \\ y + f_{2k+2}, & \text{if } y \in F_k''. \end{cases}$$

Proof. We prove the result by giving the construction of F_k. It clear that $y_1^{(k)} = \frac{f_{2k+1}}{2}$. Now suppose $y = y_j^{(k)} \in F_k$ where $j \geq 1$. According to Proposition 2.3, we see $a_{2k+3}(y)a_{2k+4}(y) = 00$, 01 or 01.

- $a_{2k+3}(y)a_{2k+4}(y) = 00$, i.e., $y \in F_k'$. Note that $\frac{f_{2k+1}}{2} = f_{2k-1} + \frac{f_{2k-3}}{2}$. For $0 < b < f_{2k+3} - \frac{f_{2k+1}}{2}$, we have $\Phi_k(y + b) = \frac{f_{2k+1}}{2} + b$, so $y + b \notin F_k$. For $f_{2k+3} - \frac{f_{2k+1}}{2} \leq b < f_{2k+3}$,

$$\Phi_k(y + b) = \frac{f_{2k+1}}{2} + b - f_{2k+3} < \frac{f_{2k+1}}{2},$$

so $y + b \notin F_k$. Since $\Phi_k(y + f_{2k+3}) = \Phi_k(y)$, we obtain that $y + f_{2k+3} = y_{j+1}^{(k)} \in F_k - F_k'$.
- $a_{2k+3}(y)a_{2k+4}(y) = 10$ or 01. For $0 < b < f_{2k+2} - \frac{f_{2k+1}}{2}$, we have $\Phi_k(y + b) = \frac{f_{2k+1}}{2} + b$, so $y + b \notin F_k$. For $f_{2k+2} - \frac{f_{2k+1}}{2} \leq b < f_{2k+2}$, since

$$\Phi_k(y + b) = \frac{f_{2k+1}}{2} + b - f_{2k+2} < \frac{f_{2k+1}}{2},$$

we also have $y + b \notin F_k$. It follows from $\Phi_k(y + f_{2k+2}) = \Phi_k(y)$ that $y + f_{2k+2} = y_{j+1}^{(k)} \in F_k$.

The result follows from the above two sub-cases. □

Remark 2.9. From the proof of Lemma 2.8, we see the gaps between two adjacent elements in F_k are f_{2k+2} and f_{2k+3}. That is $y^{(k)}_{j+1} - y^{(k)}_j = f_{2k+2}$ or f_{2k+3} for all $j \geq 1$. Moreover, the gaps between two adjacent elements in F''_k are f_{2k+2} and f_{2k+4}.

2.4. *Two subsequences of* **s**

The subsequences $(s_{\frac{f_{2k+1}}{2}})_{k\geq 0}$ and $(s_{\frac{f_{2k+1}}{2}-1})_{k\geq 0}$ can be determined according to the parity of k; see Lemma 2.11. We start with an auxiliary lemma which concerns the parity of $\frac{f_{2k+1}}{2}$.

Lemma 2.10. *For all* $k \geq 0$*,*

(i) $f_{2k} \equiv \begin{cases} 1, & \text{if } k \equiv 0 \text{ or } 3 \pmod 4, \\ 3, & \text{if } k \equiv 1 \text{ or } 2 \pmod 4, \end{cases} \pmod 4,$

(ii) $f_{2k+1} \equiv \begin{cases} 2, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}, \end{cases} \pmod 4.$

Proof. (i) Note that $f_0 = 1$ and $f_2 = 3$. Since f_{2n} is odd for all $n \geq 0$, using Eq. (2.1) twice, we have for all $k \geq 2$,

$$f_{2k} = f_{2k-2} + f_{2k-1} = 2f_{2k-2} + f_{2k-4} \equiv 2 + f_{2(k-2)} \pmod 4.$$

The result follows by induction on k.

(ii) The initial value is $f_1 = 2$. Using Eq. (2.1) and the previous result (i), we have for all $k \geq 1$,

$$f_{2k+1} = f_{2k} + f_{2k-2} \equiv \begin{cases} 2, & k \equiv 0, 2 \pmod 4, \\ 0, & k \equiv 1, 3 \pmod 4, \end{cases} \pmod 4$$

which is the desired result. □

In the calculation of $H_{m,n}$, we need to know s_n explicitly for some n. The next lemma determines the values of two sub-sequences **s**.

Lemma 2.11. *For all* $k \geq 0$*,*

$$s_{\frac{f_{2k+1}}{2}} = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}, \end{cases} \quad \text{and} \quad s_{\frac{f_{2k+1}}{2}-1} = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ 1, & \text{if } k \text{ is even}. \end{cases}$$

Proof. By Eq (2.1), we obtain that for all $k \geq 0$,

$$\begin{aligned} \frac{f_{2k+1}}{2} &= (f_{2k-2} + f_{2k})/2 \\ &= (f_{2k-2} + f_{2k-2} + f_{2k-1})/2 \\ &= f_{2k-2} + \frac{f_{2k-1}}{2} = \cdots = \sum_{i=0}^{k-1} f_{2i} + \frac{f_1}{2}. \end{aligned} \tag{2.5}$$

When k is odd,

$$\begin{aligned}\frac{f_{2k+1}}{2} &= (f_{2k-2}+f_{2k-4})+(f_{2k-6}+f_{2k-8})+\cdots+(f_4+f_2)+f_0+\frac{f_1}{2}\\ &= f_{2k-1}+f_{2k-5}+\cdots+f_5+f_1 \quad \text{(by Eq. (2.1))}\\ &= \sum_{i=0}^{\frac{k-1}{2}} f_{4i+1}. \end{aligned} \tag{2.6}$$

When $k \geq 2$ is even,

$$\begin{aligned}\frac{f_{2k+1}}{2} &= (f_{2k-2}+f_{2k-4})+(f_{2k-6}+f_{2k-8})+\cdots+(f_2+f_0)+\frac{f_1}{2}\\ &= f_{2k-1}+f_{2k-5}+\cdots+f_3+\frac{f_1}{2} \quad \text{(by Eq. (2.1))}\\ &= \sum_{i=0}^{\frac{k-2}{2}} f_{4i+3}+\frac{f_1}{2} = \sum_{i=0}^{\frac{k-2}{2}} f_{4i+3}+f_0. \end{aligned} \tag{2.7}$$

It follows from (2.6) and (2.7) that for all $k \geq 0$,

$$a_0\left(\frac{f_{2k+1}}{2}\right) = \begin{cases} 0, & \text{if } k \text{ is odd},\\ 1, & \text{if } k \text{ is even}, \end{cases}$$

and

$$a_0\left(\frac{f_{2k+1}}{2}-1\right) = \begin{cases} 1, & \text{if } k \text{ is odd},\\ 0, & \text{if } k \text{ is even}. \end{cases}$$

Then by Proposition 2.4, the result follows. □

3. Partition of the lattice

According to the values of the Hankel determinants of **s**, we tile the integer lattice using the following parallelograms. Given a $k \geq 0$, write the elements in E'_{k+1}, F''_k and E'_k in ascending order as follows:

$$E'_{k+1} = (\alpha_i)_{i\geq 1}, \quad F''_k = (\beta'_i)_{i\geq 1}, \quad E'_k = (\gamma_i)_{i\geq 1}.$$

Moreover, let $\beta_i = \beta'_i + f_{2k}$ for all $i \geq 1$. We define three different types of parallelograms: for $i \geq 1$,

$$\begin{aligned} U_{k,i} &= \left\{(m,n)\in\mathbb{N}^2 \;:\; f_{2k}\leq n< f_{2k+3},\ \alpha_i - f_{2k+2} < n+m \leq \alpha_i\right\},\\ V_{k,i} &= \left\{(m,n)\in\mathbb{N}^2 \;:\; f_{2k}\leq n< f_{2k+2},\ \beta_i < n+m \leq \beta_i + f_{2k+1}\right\},\\ T_{k,i} &= \left\{(m,n)\in\mathbb{N}^2 \;:\; f_{2k+1}\leq n< f_{2k+2},\ \gamma_i - f_{2k} < n+m \leq \gamma_i\right\}; \end{aligned} \tag{3.1}$$

see Figure 1. Let $U_k = \cup_{i\geq 1} U_{k,i}$, $V_k = \cup_{i\geq 1} V_{k,i}$ and $T_k = \cup_{i\geq 1} T_{k,i}$.

Proposition 3.1. *The parallelograms* $\{U_{k,i}\}$, $\{V_{k,i}\}$, *and* $\{T_{k,i}\}$ *introduce a partition of pairs of positive integers. Namely,* $\mathbb{N} \times \mathbb{N}_{\geq 1} = \bigsqcup_{k\geq 0}(U_k \sqcup V_k \sqcup T_k)$ *where* $\sqcup$ *denotes the disjoint union.*

Proof. Let $m \geq 0$ and $n \geq 1$ be two integers. Since $(f_k)_{k\geq 0}$ and $(\gamma_k)_{k\geq 1}$ are two increasing unbounded non-negative integer sequences, there exist $k \geq 0$ and $\ell \geq 1$ such that $f_{2k} \leq n < f_{2k+2}$ and $\gamma_{\ell-1} < n+m \leq \gamma_\ell$ where $\gamma_0 := 0$. The result clearly holds when $\ell = 1$. Now we assume that $\ell \geq 2$. From the proof of Lemma 2.6 we see that $\gamma_\ell - \gamma_{\ell-1} = f_{2k+2}$ or f_{2k+3} for all $\ell \geq 2$. When $\gamma_\ell - f_{2k} < n + m \leq \gamma_\ell$, we have

$$(m, n) \in \begin{cases} U_{k-1}, & \text{if } f_{2k} \leq n < f_{2k+1}; \\ T_k, & \text{if } f_{2k+1} \leq n < f_{2k+2}; \end{cases}$$

see also Figure 5. When $\gamma_{\ell-1} < n + m \leq \gamma_\ell - f_{2k}$, we have the following two cases:

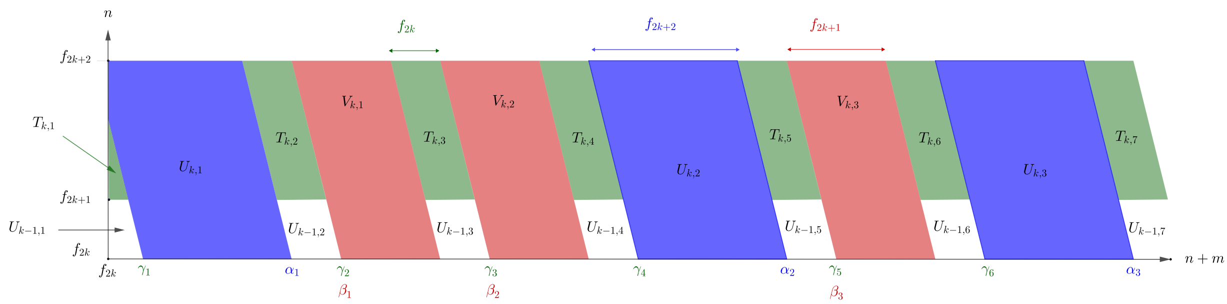

Figure 5. Partition of the strip $[0, +\infty) \times [f_{2k}, f_{2k+2})$.

Case 1: $\gamma_\ell - \gamma_{\ell-1} = f_{2k+2}$. In this case, we shall verify that $(m, n) \in V_k$. To do this, we only need to show that $\gamma_{\ell-1} - f_{2k} \in F''_k$. Since $\gamma_{\ell-1} \in E'_k$, we have $\Phi_k(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2}$ and $\Phi_k(\gamma_{\ell-1} - f_{2k}) = \frac{f_{2k+3}}{2} - f_{2k} = \frac{f_{2k+1}}{2}$. So $(\gamma_{\ell-1} - f_{2k}) \in F_k$. Suppose on the contrary that $(\gamma_{\ell-1} - f_{2k}) \in F'_k$. Then $\Phi_{k+1}(\gamma_{\ell-1} - f_{2k}) = \frac{f_{2k+1}}{2}$ and $\Phi_{k+1}(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2}$. This implies $\Phi_k(\gamma_{\ell-1} + f_{2k+2}) = \frac{f_{2k+1}}{2}$ and $(\gamma_{\ell-1} + f_{2k+2}) \notin E'_k$. Note that in this case $\gamma_\ell = \gamma_{\ell-1} + f_{2k+2}$. We conclude that $\gamma_\ell \notin E'_k$ which is a contradiction. Hence, $(\gamma_{\ell-1} - f_{2k}) \in F''_k$. The result follows.

Case 2: $\gamma_\ell - \gamma_{\ell-1} = f_{2k+3}$. We assert that, in this case, $\gamma_{\ell-1} - f_{2k} \in F'_k$. Since $\gamma_{\ell-1} \in E'_k$, we have $\Phi_k(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2}$. Consequently, $\Phi_k(\gamma_{\ell-1} - f_{2k}) = \frac{f_{2k+1}}{2}$ and $(\gamma_{\ell-1} - f_{2k}) \in F_k$. Suppose $(\gamma_{\ell-1} - f_{2k}) \in F''_k$. Then $\Phi_k(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2}$ and $\Phi_{k+1}(\gamma_{\ell-1}) \neq \frac{f_{2k+3}}{2}$. It follows that $\Phi_k(\gamma_{\ell-1} + f_{2k+3}) = \frac{f_{2k+3}}{2} + f_{2k}$. Since $\gamma_{\ell-1} + f_{2k+3} = \gamma_\ell$, we obtain that $\gamma_\ell \notin E'_k$ which is a contradiction. Now we have $\gamma_{\ell-1} - f_{2k} \in F'_k$. This yields that $\Phi_{k+1}(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2}$. Observing that $\Phi_{k+1}(\gamma_\ell - f_{2k}) = \Phi_{k+1}(\gamma_{\ell-1} + f_{2k+2}) = \frac{f_{2k+5}}{2}$, we see $\gamma_\ell - f_{2k} \in E'_{k+1}$. So $(m, n) \in U_k$. □

4. Relations of Hankel determinants

In this section, we use the Theorem 2.5 to show the determinant value inside U_k, V_k, T_k is 0. For some integer $k \geq 0$, we prove the relationship between the determinant value of the boundary of U_k, V_k, T_k. We assert that as long as we know one value of $U_k(V_k$ or $T_k)$, we can know all its values.

4.1. Inside the parallelograms

The Hankel determinant $H_{m,n}$ vanishes if (m,n) is not on the boundary of any parallelogram $U_{k,i}$, $V_{k,i}$ or $T_{k,i}$.

Lemma 4.1. *Let $m \geq 1$ and $n \geq 0$ be two integer.*

(i) If (m,n) is inside $V_{k,i}$ for some $k \geq 0$ and $i \geq 1$, i.e.,

$$\begin{cases} f_{2k} + 1 \leq n < f_{2k+2} - 1, \\ \beta_i + 1 < n + m \leq \beta_i + f_{2k+1} - 1, \end{cases}$$

then $H_{m,n} = 0$.

(ii) If (m,n) is inside $T_{k,i}$ for some $k \geq 0$ and $i \geq 1$, i.e.,

$$\begin{cases} f_{2k+1} + 1 \leq n < f_{2k+2} - 1, \\ \gamma_i - f_{2k} + 1 < n + m \leq \gamma_i - 1, \end{cases}$$

then $H_{m,n} = 0$.

(iii) If (m,n) is inside $U_{k,i}$ for some $k \geq 0$ and $i \geq 1$, i.e.,

$$\begin{cases} f_{2k} + 1 \leq n < f_{2k+3} - 1, \\ \alpha_i - f_{2k+2} + 1 < n + m \leq \alpha_i - 1, \end{cases}$$

then $H_{m,n} = 0$.

Proof. Let A_{m+i} be the i-th row of $H_{m,n}$. Then

$$H_{m,n} = \det \begin{pmatrix} s_m & s_{m+1} & \cdots & s_{m+n-1} \\ s_{m+1} & s_{m+2} & \iddots & \vdots \\ \vdots & \iddots & \iddots & \vdots \\ s_{m+n-1} & \cdots & \cdots & s_{m+2n-2} \end{pmatrix} = \det \begin{pmatrix} A_m \\ A_{m+1} \\ \vdots \\ A_{m+n-1} \end{pmatrix}.$$

(i) When $m \leq \beta'_i + 1$, recall that $\beta'_i = \beta_i - f_{2k} \in F''_k$. Since $n \leq f_{2k+2} - 2$, by Lemma 2.8, we have $\Phi_k(\beta'_k + j) \neq \frac{f_{2k+1}}{2}$ or $\frac{f_{2k+1}}{2} - 1$ for all $1 \leq j \leq n$. Then it follows from Theorem 2.5(i) that

$$\begin{aligned} A_{\beta'_i+1} &= (s_{\beta'_i+1},\ s_{\beta'_i+2},\ \ldots,\ s_{\beta'_i+n}) \\ &= (s_{\beta_i+1},\ s_{\beta_i+2},\ \ldots,\ s_{\beta_i+n}) = A_{\beta_i+1}, \end{aligned}$$

which gives $H_{m,n} = 0$. When $m > \beta'_i + 1$, note that $n + m \leq \beta'_k + f_{2k+2} - 1$. By Lemma 2.8, we have $\Phi_k(m + j) \neq \frac{f_{2k+1}}{2}$ or $\frac{f_{2k+1}}{2} - 1$ for all $1 \leq j \leq n$. Then it follows from Theorem 2.5(i) that

$$\begin{aligned} A_m &= (s_m,\ s_{m+1},\ \ldots,\ s_{m+n-1}) \\ &= (s_{m+f_{2k}},\ s_{m+f_{2k}+1},\ \ldots,\ s_{m+f_{2k}+n-1}) = A_{m+f_{2k}}. \end{aligned}$$

So $H_{m,n} = 0$.

(ii) Recall that $\gamma_i \in E'_k$ and by Lemma 2.6, γ_i and $\gamma_i - f_{2k+2}$ are adjacent elements in E_k. Let

$$r = \begin{cases} \gamma_i - f_{2k+2} + 1 - m, & \text{if } m \le \gamma_i - f_{2k+2} + 1, \\ 0, & \text{if } m > \gamma_i - f_{2k+2} + 1. \end{cases}$$

Combining Lemma 2.6 and Theorem 2.5(ii), we have $A_{m+r} = A_{m+r+f_{2k+1}}$ which means $H_{m,n} = 0$.

(iii) Recall that $\alpha_i \in E'_{k+1}$ and by Lemma 2.6, α_i and $\alpha_i - f_{2k+4}$ are adjacent elements in E_{k+1}. When $m \le \alpha_i - f_{2k+4} + 1$, note that $\alpha_i - f_{2k+4} + n < \alpha_i - f_{2k+2} - 1$. By Theorem 2.5(ii), we have

$$\begin{aligned} A_{\alpha_i - f_{2k+4}+1} &= (s_{\alpha_i - f_{2k+4}+1},\ s_{\alpha_i - f_{2k+4}+2},\ \ldots,\ s_{\alpha_i - f_{2k+4}+n}) \\ &= (s_{\alpha_i - f_{2k+2}+1},\ s_{\alpha_i - f_{2k+2}+2},\ \ldots,\ s_{\alpha_i - f_{2k+2}+n}) = A_{\alpha_i - f_{2k+2}+1}. \end{aligned}$$

Thus $H_{m,n} = 0$. When $m > \alpha_i - f_{2k+4} + 1$, since $n + m - 1 \le \alpha_i - 2$, by Theorem 2.5(ii), we obtain that

$$\begin{aligned} A_m &= (s_m,\ s_{m+1},\ \ldots,\ s_{m+n-1}) \\ &= (s_{m+f_{2k+3}},\ s_{m+f_{2k+3}+1},\ \ldots,\ s_{m+f_{2k+3}+n-1}) = A_{m+f_{2k+3}} \end{aligned}$$

which also implies $H_{m,n} = 0$. □

4.2. Determinants on the horizontal edges of the parallelograms

We first deal with the Hankel determinants $H_{m,n}$ on the horizontal edges with $n = f_{2k}$ and f_{2k+1} where $k \ge 0$.

Lemma 4.2. *Let $k \ge 0$ and $i \ge 1$.*

(i) (Bottom edge of $V_{k,i}$) $H_{\beta'_i+r,\, f_{2k}} = H_{\beta'_i+1,\, f_{2k}}$ *for all* $1 \le r \le f_{2k+1}$.
(ii) (Bottom edge of $U_{k,i}$) $H_{\alpha_i - f_{2k+3}+r,\, f_{2k}} = H_{\alpha_i - f_{2k},\, f_{2k}}$ *for all* $1 \le r \le f_{2k+2}$.
(iii) (Bottom edge of $T_{k,i}$) $H_{\gamma_i - f_{2k+2}+r,\, f_{2k+1}} = (-1)^{r+1} H_{\gamma_i - f_{2k+1},\, f_{2k+1}}$ *for all* $1 \le r \le f_{2k}$ *with* $\gamma_i - f_{2k+2} + r \ge 0$.

Proof. (i) Let $A_j = (s_{\beta'_i+j},\ s_{\beta'_i+j+1},\ \ldots,\ s_{\beta'_i+j+f_{2k}-1})$. Then for $1 \le j < f_{2k+1}$,

$$H_{\beta'_i+j,\, f_{2k}} = \det\begin{pmatrix} A_j \\ A_{j+1} \\ \vdots \\ A_{f_{2k}+j-1} \end{pmatrix} \quad \text{and} \quad H_{\beta'_i+j+1,\, f_{2k}} = \det\begin{pmatrix} A_{j+1} \\ A_{j+2} \\ \vdots \\ A_{f_{2k}+j} \end{pmatrix}.$$

Recall that $\beta'_i \in F''_k$. By Lemma 2.8, since $j + f_{2k} - 1 \le f_{2k+2} - 2$, we see $\Phi_k(\beta'_i + \ell) \ne \frac{f_{2k+1}}{2}$ or $\frac{f_{2k+1}}{2} - 1$ for all $1 \le \ell \le f_{2k+2} - 2$. Applying Theorem 2.5(i), we have

$$\begin{aligned} A_j &= (s_{\beta'_i+j},\ s_{\beta'_i+j+1},\ \ldots,\ s_{\beta'_i+j+f_{2k}-1}) \\ &= (s_{\beta'_i+j+f_{2k}},\ s_{\beta'_i+j+1+f_{2k}},\ \ldots,\ s_{\beta'_i+j+2f_{2k}-1}) = A_{f_{2k}+j}. \end{aligned}$$

Therefore, for $1 \le j < f_{2k+1}$,

$$H_{\beta'_i+j,\, f_{2k}} = \begin{vmatrix} A_j \\ A_{j+1} \\ \vdots \\ A_{f_{2k}+j-1} \end{vmatrix} = \begin{vmatrix} A_{f_{2k}+j} \\ A_{j+1} \\ \vdots \\ A_{f_{2k}+j-1} \end{vmatrix} = (-1)^{f_{2k}-1} \begin{vmatrix} A_{j+1} \\ A_{j+2} \\ \vdots \\ A_{f_{2k}+j} \end{vmatrix} = H_{\beta'_i+j+1,\, f_{2k}}$$

where the last equality follows from Lemma 2.10(i).

(ii) Recall that $\alpha_i \in E'_{k+1}$ and $\Phi_{k+1}(\alpha_i) = \frac{f_{2k+5}}{2}$. Let $y = \alpha_i - f_{2k+3}$. Then $\Phi_{k+1}(y) = \frac{f_{2k+1}}{2}$ and $y \in F'_k$. Let $B_j = (s_{y+j},\ s_{y+j+1},\ \ldots,\ s_{y+j+f_{2k}-1})$. Then for $1 \le j < f_{2k+2}$,

$$H_{y+j,\,f_{2k}} = \det\begin{pmatrix} B_j \\ B_{j+1} \\ \vdots \\ B_{f_{2k}+j-1} \end{pmatrix} \quad \text{and} \quad H_{y+j+1,\,f_{2k}} = \det\begin{pmatrix} B_{j+1} \\ B_{j+2} \\ \vdots \\ B_{f_{2k}+j} \end{pmatrix}.$$

Since $j + f_{2k} - 1 \le f_{2k+3} - 2$, by Lemma 2.8 and Theorem 2.5(i),

$$\begin{aligned} B_j &= (s_{y+j},\ s_{y+j+1},\ \ldots,\ s_{y+j+f_{2k}-1}) \\ &= (s_{y+j+f_{2k}},\ s_{y+j+1+f_{2k}},\ \ldots,\ s_{y+j+2f_{2k}-1}) = B_{f_{2k}+j}. \end{aligned}$$

Therefore, for $1 \le j < f_{2k+2}$,

$$H_{y+j,\,f_{2k}} = (-1)^{f_{2k}-1} H_{y+j+1,\,f_{2k}} = H_{y+j+1,\,f_{2k}}$$

where the last equality follows from Lemma 2.10(i).

(iii) Recall that $\gamma_i \in E'_k$. By Lemma 2.6, $g := \gamma_i - f_{2k+2} \in E_k$. Write

$$A_{g+j} = (s_{g+j},\ s_{g+j+1},\ \ldots,\ s_{g+j+f_{2k+1}-1}).$$

For $1 \le r < f_{2k}$,

$$H_{g+r,\,f_{2k+1}} = \begin{vmatrix} A_{g+r} \\ A_{g+r+1} \\ \vdots \\ A_{g+r+f_{2k+1}-1} \end{vmatrix} \quad \text{and} \quad H_{g+r+1,\,f_{2k+1}} = \begin{vmatrix} A_{g+r+1} \\ A_{g+r+2} \\ \vdots \\ A_{g+r+f_{2k+1}} \end{vmatrix}.$$

By Theorem 2.5, $A_{g+r} = A_{g+r+f_{2k+1}}$. Then using Lemma 2.10, for all $1 \le r < f_{2k}$,

$$H_{g+r,\,f_{2k+1}} = (-1)^{f_{2k+1}-1} H_{g+r+1,\,f_{2k+1}} = -H_{g+r+1,\,f_{2k+1}}$$

and $H_{g+r,\,f_{2k+1}} = (-1)^{f_{2k}-r} H_{g+r+f_{2k},\,f_{2k+1}} = (-1)^{1+r} H_{g+r+f_{2k},\,f_{2k+1}}$. □

In fact, for all $i \ge 1$, the Hankel determinants on the bottom of $U_{k,i}$ and $V_{k,i}$ take the same value which depends only on k. The following lemma helps us to connect the determinants on the bottom of $U_{k,*}$ and $V_{k,*}$.

Lemma 4.3. *Let $k \ge 0$ and $i \ge 1$. If $\gamma_{i+1} - \gamma_i = f_{2k+3}$, then $H_{\gamma_i+f_{2k}+1,\,f_{2k}} = H_{\gamma_{i+1}-f_{2k}+1,\,f_{2k}}$. If $\gamma_{i+1} - \gamma_i = f_{2k+2}$, then $H_{\gamma_i+1,\,f_{2k}} = H_{\gamma_{i+1}-f_{2k}+1,\,f_{2k}}$.*

Proof. Suppose $\gamma_{i+1} - \gamma_i = f_{2k+3}$. Then $\Phi_k(\gamma_i + f_{2k}) = \frac{f_{2k+3}}{2} + f_{2k}$. Since $3f_{2k} = f_{2k+2} + f_{2k-1} < f_{2k+3}$, by Theorem 2.5(ii), we have

$$\begin{aligned} \begin{pmatrix} s_{\gamma_i+f_{2k}+1} & \cdots & s_{\gamma_i+3f_{2k}-1} \end{pmatrix} &= \begin{pmatrix} s_{\gamma_i+f_{2k}+1+f_{2k+1}} & \cdots & s_{\gamma_i+3f_{2k}-1+f_{2k+1}} \end{pmatrix} \\ &= \begin{pmatrix} s_{\gamma_{i+1}-f_{2k}+1} & \cdots & s_{\gamma_{i+1}+f_{2k}-1} \end{pmatrix}. \end{aligned}$$

Therefore

$$H_{\gamma_i+f_{2k}+1,\,f_{2k}}=\begin{vmatrix} s_{\gamma_i+f_{2k}+1} & \cdots & s_{\gamma_i+2f_{2k}} \\ \vdots & \vdots & \vdots \\ s_{\gamma_i+2f_{2k}} & \cdots & s_{\gamma_i+3f_{2k}-1} \end{vmatrix}=\begin{vmatrix} s_{\gamma_{i+1}-f_{2k}+1} & \cdots & s_{\gamma_{i+1}} \\ \vdots & \vdots & \vdots \\ s_{\gamma_{i+1}} & \cdots & s_{\gamma_{i+1}+f_{2k}-1} \end{vmatrix}$$
$$=H_{\gamma_{i+1}-f_{2k}+1,\,f_{2k}}.$$

When $\gamma_{i+1}-\gamma_i=f_{2k+2}$, we have $\Phi_k(\gamma_i)=\frac{f_{2k+3}}{2}$. By Theorem 2.5(ii),

$$\begin{aligned}\begin{pmatrix} s_{\gamma_i+1} & \cdots & s_{\gamma_i+2f_{2k}-1}\end{pmatrix} &= \begin{pmatrix} s_{\gamma_i+1+f_{2k+1}} & \cdots & s_{\gamma_i+2f_{2k}-1+f_{2k+1}}\end{pmatrix}\\ &= \begin{pmatrix} s_{\gamma_{i+1}-f_{2k}+1} & \cdots & s_{\gamma_{i+1}+f_{2k}-1}\end{pmatrix}.\end{aligned}$$

So $H_{\gamma_i+1,\,f_{2k}}=H_{\gamma_{i+1}-f_{2k}+1,\,f_{2k}}$. □

Next we give the connection between $T_{k,i}$ and $T_{k,i+1}$.

Lemma 4.4. *For all* $i\geq 1$, $H_{\gamma_i-f_{2k+1},\,f_{2k+1}}=H_{\gamma_{i+1}-f_{2k+1},\,f_{2k+1}}$.

Proof. If $\gamma_{i+1}-\gamma_i=f_{2k+3}$, then $\Phi_{k+1}(\gamma_i)=\frac{f_{2k+3}}{2}$ and $\Phi_{k+1}(\gamma_i+f_{2k+1})<\frac{f_{2k+5}}{2}$. By Theorem 2.5(ii), we have

$$\begin{aligned}\begin{pmatrix} s_{\gamma_i-f_{2k+1}} & \cdots & s_{\gamma_i+f_{2k+1}-2}\end{pmatrix} &= \begin{pmatrix} s_{\gamma_i-f_{2k+1}+f_{2k+3}} & \cdots & s_{\gamma_i+f_{2k+1}-2+f_{2k+3}}\end{pmatrix}\\ &= \begin{pmatrix} s_{\gamma_{i+1}-f_{2k+1}} & \cdots & s_{\gamma_{i+1}+f_{2k+1}-2}\end{pmatrix}.\end{aligned}$$

Consequently, $H_{\gamma_i-f_{2k+1},\,f_{2k+1}}=H_{\gamma_{i+1}-f_{2k+1},\,f_{2k+1}}$.

If $\gamma_{i+1}-\gamma_i=f_{2k+2}$, then $\Phi_{k+1}(\gamma_i)=\frac{f_{2k+3}}{2}+f_{2k+3}$ or $\frac{f_{2k+3}}{2}+f_{2k+4}$. By Theorem 2.5(i), we have

$$\begin{aligned}\begin{pmatrix} s_{\gamma_i-f_{2k+1}} & \cdots & s_{\gamma_i+f_{2k+1}-2}\end{pmatrix} &= \begin{pmatrix} s_{\gamma_i-f_{2k+1}+f_{2k+2}} & \cdots & s_{\gamma_i+f_{2k+1}-2+f_{2k+2}}\end{pmatrix}\\ &= \begin{pmatrix} s_{\gamma_{i+1}-f_{2k+1}} & \cdots & s_{\gamma_{i+1}+f_{2k+1}-2}\end{pmatrix}.\end{aligned}$$

Consequently, $H_{\gamma_i-f_{2k+1},\,f_{2k+1}}=H_{\gamma_{i+1}-f_{2k+1},\,f_{2k+1}}$. □

According to Lemma 4.3 and Lemma 4.4, the values of the determinants on the bottom edges of $U_{k,i}$ and $V_{k,i}$ only depends on k. We improve Lemma 4.2 to the following proposition.

Proposition 4.5. *Let* $k\geq 0$. *For all* $i\geq 1$,

(i) (Bottom edges of $U_{k,i}$ and $V_{k,i}$) *for all* $1\leq r\leq f_{2k+1}$ *and* $1\leq r'\leq f_{2k+2}$,

$$H_{\alpha_i-f_{2k+3}+r',\,f_{2k}}=H_{\beta'_i+r,\,f_{2k}}=H_{\alpha_1-f_{2k},\,f_{2k}};$$

(ii) (Bottom edge of $T_{k,i}$) $H_{\gamma_i-f_{2k+2}+r,\,f_{2k+1}}=(-1)^{r+1}H_{\gamma_1-f_{2k+1},\,f_{2k+1}}$ *for all* $1\leq r\leq f_{2k}$ *with* $\gamma_i-f_{2k+2}+r\geq 0$.

Proof. Since $\alpha_i-f_{2k+2}\in E'_k$ and $\beta'_i+f_{2k}\in E'_k$, Lemma 4.3 shows that the values of two determinants on the bottom edge of two adjacent parallelograms in $\{U_{k,j}\}_{j\geq 1}\cup\{V_{k,j}\}_{j\geq 1}$ are the same. Then Lemma 4.2 implies the result (i). The result (ii) follows from Lemma 4.2(iii) and Lemma 4.4. □

4.3. On the boundary of $U_{k,i}$

Lemma 4.6. *Let $k \geq 0$ and $i \geq 1$. For all $0 \leq r \leq f_{2k+2} - 1$ with $\alpha_i - f_{2k+4} + 2 + r \geq 0$,*

(i) (Right edge of $U_{k,i}$) $H_{\alpha_i - f_{2k+3}+1+r, f_{2k+3}-1-r} = (-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{\alpha_i - f_{2k+3}+1, f_{2k+3}-1}$,

(ii) (Left edge of $U_{k,i}$) $H_{\alpha_i - f_{2k+4}+2+r, f_{2k+3}-1-r} = (-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{\alpha_i - f_{2k+3}+1, f_{2k+3}-1}$,

(iii) (Upper edge of $U_{k,i}$) $H_{\alpha_i - f_{2k+4}+2+r, f_{2k+3}-1} = (-1)^{r} H_{\alpha_i - f_{2k+3}+1, f_{2k+3}-1}$.

Proof. Write $y = \alpha_i - f_{2k+3}$. Recall that $\alpha_i \in E'_{k+1}$. So $\Phi_{k+1}(y) = \frac{f_{2k+1}}{2}$ and $y \in F'_k$.

(i) For $0 \leq r < f_{2k+2}$, let A_{y+r+j} be the j-th column of $M_{y+1+r, f_{2k+3}-1-r}$. Applying Lemma 2.8 and Theorem 2.5(i), we see $s_{y+r+\ell} = s_{y+r+\ell+f_{2k}}$ for $1 \leq \ell \leq f_{2k+3} - r - 2$ and $s_{y+f_{2k+3}-1} \neq s_{y+f_{2k+3}-1+f_{2k}}$. Then Proposition 2.4 and Lemma 2.11 yields $s_{y+f_{2k+3}-1} - s_{y+f_{2k+3}-1+f_{2k}} = (-1)^k$. Therefore,

$$A_{y+r+1} - A_{y+r+f_{2k}} = \begin{pmatrix} s_{y+r+1} \\ s_{y+r+2} \\ \vdots \\ s_{y+f_{2k+3}-2} \\ s_{y+f_{2k+3}-1} \end{pmatrix} - \begin{pmatrix} s_{y+r+1+f_{2k}} \\ s_{y+r+2+f_{2k}} \\ \vdots \\ s_{y+f_{2k+3}-2+f_{2k}} \\ s_{y+f_{2k+3}-1+f_{2k}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^k \end{pmatrix}$$

and

$$\begin{aligned} H_{y+1+r, f_{2k+3}-1-r} &= \left| A_{y+r+1} \quad A_{y+r+2} \quad \ldots \quad A_{y+f_{2k+3}-1} \right| \\ &= \left| (A_{y+r+1} - A_{y+r+f_{2k}}) \quad A_{y+r+2} \quad \ldots \quad A_{y+f_{2k+3}-1} \right| \\ &= \begin{vmatrix} \mathbf{0}_{f_{2k+3}-2-r,1} & M_{y+r+2, f_{2k+3}-r-2} \\ (-1)^k & * \end{vmatrix} \\ &= (-1)^k(-1)^{1+f_{2k+3}-1-r} H_{y+r+2, f_{2k+3}-r-2} \\ &= (-1)^{k+r} H_{y+r+2, f_{2k+3}-r-2} \end{aligned} \tag{4.1}$$

where in the last equality we apply Lemma 2.10 and $\mathbf{0}_{i,j}$ denotes the $i \times j$ zero matrix. It follows from Eq (4.1) that

$$\begin{aligned} H_{y+1, f_{2k+3}-1} &= (-1)^k H_{y+2, f_{2k+3}-2} = (-1)^k(-1)^{k+1} H_{y+3, f_{2k+3}-3} \\ &= (-1)^k(-1)^{k+1} \ldots (-1)^{k+r-1} H_{y+1+r, f_{2k+3}-1-r} \\ &= (-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{y+1+r, f_{2k+3}-1-r}. \end{aligned}$$

(ii) Let $B_{y-f_{2k+2}+1+r+j}$ be the j-th row of $M_{y-f_{2k+2}+2+r, f_{2k+3}-1-r}$. Combining Lemma 2.8, Theorem 2.5(i), Proposition 2.4 and Lemma 2.11, a similar argument as above yields

$$
\begin{aligned}
H_{y-f_{2k+2}+2+r,f_{2k+3}-1-r} &= \begin{vmatrix} B_{y-f_{2k+2}+1+r+1} \\ \vdots \\ B_y \\ \vdots \\ B_{y+f_{2k}} \end{vmatrix} = \begin{vmatrix} B_{y-f_{2k+2}+1+r+1} \\ \vdots \\ B_y \\ \vdots \\ B_{y+f_{2k}} - B_y \end{vmatrix} \\
&= \begin{vmatrix} * & M_{y-f_{2k+2}+2+(r+1),f_{2k+3}-1-(r+1)} \\ (-1)^k & \mathbf{0}_{1,f_{2k+3}-2-r} \end{vmatrix} \\
&= (-1)^{k+r} H_{y-f_{2k+2}+2+(r+1),f_{2k+3}-1-(r+1)}. \qquad (4.2)
\end{aligned}
$$

Applying Eq (4.2), we have

$$
\begin{aligned}
H_{y-f_{2k+2}+2+r,f_{2k+3}-1-r} &= (-1)^{k+r}(-1)^{k+r+1}\cdots(-1)^{k+f_{2k+2}-2} H_{y+1,\, f_{2k}} \\
&= (-1)^{\frac{(2k+r+f_{2k+2}-2)(f_{2k+2}-r-1)}{2}} H_{y+1,\, f_{2k}} \\
&= (-1)^{\frac{(2k+r+f_{2k+2}-2)(f_{2k+2}-r-1)}{2}} H_{y+f_{2k+2},\, f_{2k}} && \text{(by Lemma 4.2(ii))} \\
&= (-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{y+1,\, f_{2k+3}-1}. && \text{(by Lemma 4.6(i))}
\end{aligned}
$$

(iii) Let C_j be the j-th column of $M_{y-f_{2k+2}+2+r,f_{2k+3}-1}$. Then

$$
\begin{aligned}
H_{y-f_{2k+2}+2+r,f_{2k+3}-1} &= \det(C_1, C_2, \dots, C_{f_{2k+3}-1}) \\
&= \det(C_1, C_2, \dots, C_{f_{2k+3}-1-r}, C_1', C_2', \dots, C_r')
\end{aligned}
$$

where $C_p' = C_{f_{2k+3}-1-r+p} - C_{f_{2k+3}-1-r+p-f_{2k}}$ for $1 \le p \le r$. According to Lemma 2.8, Theorem 2.5(i), we have $s_{y+\ell} = s_{y+\ell+f_{2k}}$ for all $1 \le \ell \le f_{2k+3} - 2$ and $f_{2k+3} + 1 \le \ell \le f_{2k+3} + r - 2$. By Proposition 2.4 and Lemma 2.11, we obtain that $s_{y+f_{2k+3}-1+f_{2k}} - s_{y+f_{2k+3}-1} = (-1)^{k+1}$ and $s_{y+f_{2k+3}+f_{2k}} - s_{y+f_{2k+3}} = (-1)^k$. Thus

$$
(C_1', C_2', \dots, C_r') = \begin{pmatrix} \mathbf{0}_{f_{2k+3}-1-r,r} \\ X \end{pmatrix}
$$

where X is the $r \times r$ matrix

$$
\begin{pmatrix} 0 & \cdots & 0 & (-1)^{k+1} \\ \vdots & \iddots & \iddots & (-1)^k \\ \vdots & \iddots & \iddots & \vdots \\ (-1)^{k+1} & (-1)^k & \cdots & 0 \end{pmatrix}.
$$

Expanding by the last r columns, we have

$$
\begin{aligned}
H_{y-f_{2k+2}+2+r,f_{2k+3}-1} &= \det\begin{pmatrix} M_{y-f_{2k+2}+2+r,f_{2k+3}-1-r} & \mathbf{0}_{f_{2k+3}-1-r,r} \\ * & X \end{pmatrix} \\
&= (-1)^{(k+1)r}(-1)^{\frac{(r-1)r}{2}} H_{y-f_{2k+2}+2+r,f_{2k+3}-1-r} \\
&= (-1)^{(k+1)r}(-1)^{\frac{(r-1)r}{2}}(-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{y+1,f_{2k+3}-1} && \text{(by Lemma 4.6(ii))} \\
&= (-1)^r H_{y+1,f_{2k+3}-1}.
\end{aligned}
$$

□

Remark 4.7. From Proposition 4.5(i) and Lemma 4.6, Hankel determinants on the boundary of $U_{k,i}$ can be determined by $H_{\alpha_1-f_{2k+3}+1,f_{2k+3}-1} = H_{\frac{f_{2k+1}}{2}+1,f_{2k+3}-1}$ (the upper right corner of $U_{k,1}$).

4.4. On the boundary of $V_{k,i}$

Lemma 4.8. *Let $k \geq 0$ and $i \geq 1$. For all $0 \leq r \leq f_{2k+1} - 1$,*

(i) (Left edge of $V_{k,i}$) $H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} = (-1)^{rk}(-1)^{\frac{r(r+1)}{2}} H_{\beta'_i-f_{2k+1}+2,f_{2k+2}-1}$,

(ii) (Right edge of $V_{k,i}$) $H_{\beta'_i+1+r,f_{2k+2}-1-r} = (-1)^{rk}(-1)^{\frac{r(r+1)}{2}} H_{\beta'_i+1,f_{2k+2}-1}$,

(iii) (Upper edge of $V_{k,i}$) $H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1} = H_{\beta'_i+1,f_{2k+2}-1}$.

Proof. (i) Denote by $A_{\beta'_i-f_{2k+1}+1+r+j}$ the j-th row of $M_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r}$. Then

$$H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} = \det\begin{pmatrix} A_{\beta'_i-f_{2k+1}+2+r} \\ A_{\beta'_i-f_{2k+1}+2+r+1} \\ \vdots \\ A_{\beta'_i+f_{2k}} \end{pmatrix} = \det\begin{pmatrix} A_{\beta'_i-f_{2k+1}+2+r} \\ A_{\beta'_i-f_{2k+1}+2+r+1} \\ \vdots \\ A_{\beta'_i+f_{2k}} - A_{\beta'_i} \end{pmatrix}.$$

From Lemma 2.8, Theorem 2.5 and Lemma 2.11, we have

$$A_{\beta'_i+f_{2k}} - A_{\beta'_i} = ((-1)^k, 0, \cdots, 0).$$

For $0 \leq r \leq f_{2k+1} - 1$,

$$\begin{aligned} H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} &= \begin{pmatrix} * & M_{\beta'_i-f_{2k+1}+2+(r+1),f_{2k+2}-1-(r+1)} \\ (-1)^k & \mathbf{0}_{1,f_{2k+2}-2-r} \end{pmatrix} \\ &= (-1)^k(-1)^{1+f_{2k+2}-1-r} H_{\beta'_i-f_{2k+1}+2+(r+1),f_{2k+2}-1-(r+1)} \\ &= (-1)^{k+1+r} H_{\beta'_i-f_{2k+1}+2+(r+1),f_{2k+2}-1-(r+1)}. \qquad \text{(by Lemma 2.10(i))} \end{aligned}$$

Thus

$$\begin{aligned} H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} &= (-1)^{k+1+r-1}(-1)^{k+1+r-2}\cdots(-1)^{k+1} H_{\beta'_i+2-f_{2k+1},f_{2k+2}-1} \\ &= (-1)^{r(k+1)}(-1)^{\frac{r(r-1)}{2}} H_{\beta'_i+2-f_{2k+1},f_{2k+2}-1}. \end{aligned}$$

(ii) Let $B_{\beta'_i+r+j}$ be the j-th column of $M_{\beta'_i+1+r,f_{2k+2}-1-r}$.

$$\begin{aligned} H_{\beta'_i+1+r,f_{2k+2}-1-r} &= \det\begin{pmatrix} s_{\beta'_i+1+r} & s_{\beta'_i+2+r} & \cdots & s_{\beta'_i+f_{2k+2}-1} \\ \vdots & \vdots & \iddots & \vdots \\ s_{\beta'_i+f_{2k+2}-1} & s_{\beta'_i+f_{2k+2}} & \cdots & s_{\beta'_i+2f_{2k+2}-3-r} \end{pmatrix} \\ &= \det\begin{pmatrix} B_{\beta'_i+1+r} & B_{\beta'_i+2+r} & \cdots & B_{\beta'_i+f_{2k+2}-1} \end{pmatrix} \\ &= \det\begin{pmatrix} B_{\beta'_i+1+r} - B_{\beta'_i+1+r+f_{2k}} & B_{\beta'_i+2+r} & \cdots & B_{\beta'_i+f_{2k+2}-1} \end{pmatrix}. \end{aligned}$$

Recall that $\beta'_i \in F''_k$. By Lemma 2.8, β'_i and $\beta'_i + f_{2k+2}$ are adjacent elements in F_k. It follows from Theorem 2.5(i) and Lemma 2.11 that

$$
\begin{aligned}
H_{\beta'_i+1+r,f_{2k+2}-1-r} &= \det\begin{pmatrix} \mathbf{0}_{f_{2k+2}-2-r,1} & M_{\beta'_i+1+(r+1),f_{2k+2}-1-(r+1)} \\ (-1)^k & * \end{pmatrix} \\
&= (-1)^k(-1)^{1+f_{2k+2}-1-r}H_{\beta'_i+1+(r+1),f_{2k+2}-1-(r+1)} \\
&= (-1)^{k+1+r}H_{\beta'_i+1+(r+1),f_{2k+2}-1-(r+1)}. \qquad \text{(by Lemma 2.10(i))}
\end{aligned}
$$

Hence $H_{\beta'_i+1+r,f_{2k+2}-1-r} = (-1)^{r(k+1)}(-1)^{\frac{r(r-1)}{2}}H_{\beta'_i+1,f_{2k+2}-1}$.

(iii) Let $C_{\beta'_i-f_{2k+1}+1+r+j}$ be the j-th column of $M_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1}$. Then

$$
\begin{aligned}
H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1} &= \det\begin{pmatrix} C_{\beta'_i-f_{2k+1}+2+r} & C_{\beta'_i-f_{2k+1}+2+r+1} & \cdots & C_{\beta'_i+f_{2k}+r} \end{pmatrix} \\
&= \det\begin{pmatrix} C_{\beta'_i-f_{2k+1}+2+r} & \cdots & C_{\beta'_i+f_{2k}} & C'_1 & \cdots & C'_r \end{pmatrix}
\end{aligned}
$$

where $C'_p = C_{\beta'_i+f_{2k}+p} - C_{\beta'_i+p}$ for $1 \le p \le r$. By Lemma 2.8 and Theorem 2.5(i), we have $s_{\beta'_i+\ell} = s_{\beta'_i+\ell+f_{2k}}$ for $1 \le \ell \le f_{2k+2} - 2$ and $f_{2k+2} + 1 \le \ell \le r + f_{2k+2} - 1$. Moreover, by Proposition 2.4 and Lemma 2.11, we have $s_{\beta'_i+f_{2k}+f_{2k+2}-1} - s_{\beta'_i+f_{2k+2}-1} = (-1)^{k+1}$ and $s_{\beta'_i+f_{2k}+f_{2k+2}} - s_{\beta'_i+f_{2k+2}} = (-1)^k$. Thus

$$
\begin{pmatrix} C'_1 & \cdots & C'_r \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{f_{2k+2}-1-r,\, r} \\ X \end{pmatrix}
$$

where X is the $r \times r$ matrix

$$
\begin{pmatrix}
0 & \cdots & 0 & (-1)^{k+1} \\
\vdots & \iddots & \iddots & (-1)^k \\
\vdots & \iddots & \iddots & \vdots \\
(-1)^{k+1} & (-1)^k & \cdots & 0
\end{pmatrix}.
$$

Now expanding $H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1}$ by its last r columns, we obtain that for $0 \le r \le f_{2k+1} - 1$,

$$
\begin{aligned}
H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1} &= \det\begin{pmatrix} M_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} & \mathbf{0}_{f_{2k+2}-1-r,\, r} \\ * & X \end{pmatrix} \\
&= (-1)^{(k+1)r}(-1)^{\frac{(r-1)r}{2}}H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} \\
&= H_{\beta'_i-f_{2k+1}+2,f_{2k+2}-1}. \qquad \text{(by Lemma 4.8(i))}
\end{aligned}
$$

□

Remark 4.9. From Proposition 4.5(i) and Lemma 4.8, Hankel determinants on the boundary of $V_{k,i}$ can be determined by $H_{\frac{f_{2k+1}}{2}+f_{2k+3}+1,f_{2k}}$ (the lower left corner of $V_{k,1}$).

4.5. On the boundary of $T_{k,i}$

Lemma 4.10. *Let $k \ge 0$ and $i \ge 1$.*

(i) (Left edge of $T_{k,i}$) *For all $0 \le r \le f_{2k} - 1$ with $\gamma_i - f_{2k+3} + 2 + r \ge 0$,*

$$H_{\gamma_i - f_{2k+3}+2+r,\, f_{2k+2}-1-r} = (-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{\gamma_i - f_{2k+3}+2,\, f_{2k+2}-1}.$$

(ii) (Right edge of $T_{k,i}$) *For all* $0 \le r \le f_{2k} - 1$ *with* $\gamma_i - f_{2k+2} + 1 + r \ge 0$*,*

$$H_{\gamma_i - f_{2k+2}+1+r,\, f_{2k+2}-1-r} = (-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{\gamma_i - f_{2k+2}+1,\, f_{2k+2}-1}.$$

(iii) (Upper edge of $T_{k,i}$) *For all* $0 \le r \le f_{2k} - 1$ *with* $\gamma_i - f_{2k+3} + 2 + r \ge 0$*,*

$$H_{\gamma_i - f_{2k+3}+2+r,\, f_{2k+2}-1} = H_{\gamma_i - f_{2k+3}+2,\, f_{2k+2}-1}.$$

Proof. To shorten the notation, write $x = \gamma_i - f_{2k+3} + 2$ and $x' = \gamma_i - f_{2k+2} + 1$.

(i) Let $\max\{0, -x\} \le \ell \le f_{2k} - 1$ and let A_j be the jth row of $H_{x+\ell, f_{2k+2}-1-\ell}$. By Theorem 2.5(ii) and Lemma 2.11, we see

$$A_{f_{2k+2}-1-\ell} - A_{f_{2k}-1-\ell} = \begin{pmatrix} (-1)^{k+1} & 0 & \dots & 0 \end{pmatrix}.$$

Then for $\max\{0, -x\} \le \ell \le f_{2k} - 1$,

$$\begin{aligned}
H_{x+\ell, f_{2k+2}-1-\ell} &= \det\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{f_{2k+2}-2-\ell} \\ A_{f_{2k+2}-1-\ell} \end{pmatrix} = \det\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{f_{2k+2}-2-\ell} \\ A_{f_{2k+2}-1-\ell} - A_{f_{2k}-1-\ell} \end{pmatrix} \\
&= \begin{vmatrix} * & M_{x+(\ell+1), f_{2k+2}-1-(\ell+1)} \\ (-1)^{k+1} & \mathbf{0}_{1, f_{2k+2}-2-\ell} \end{vmatrix} \\
&= (-1)^{k+1}(-1)^{1+f_{2k+2}-1-\ell} H_{x+(\ell+1), f_{2k+2}-1-(\ell+1)} \\
&= (-1)^{k+\ell} H_{x+(\ell+1), f_{2k+2}-1-(\ell+1)}. \qquad \text{(by Lemma 2.10(i))}
\end{aligned}$$

Applying the above equality r times, one has

$$\begin{aligned}
H_{x+r, f_{2k+2}-1-r} &= (-1)^{k+r-1}(-1)^{k+r-2} \cdots (-1)^{k} H_{x, f_{2k+2}-1} \\
&= (-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{x, f_{2k+2}-1}.
\end{aligned}$$

(ii) Let $\max\{0, -x'\} \le \ell \le f_{2k} - 1$ and let B_j be the jth column of $H_{x'+\ell, f_{2k+2}-1-\ell}$. By Theorem 2.5(ii) and Lemma 2.11, we see

$$B_1 - B_{1+f_{2k+1}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{k+1} \end{pmatrix}.$$

Therefore, for $\max\{0, -x'\} \le \ell \le f_{2k} - 1$,

$$\begin{aligned}
H_{x'+\ell, f_{2k+2}-1-\ell} &= \det\begin{pmatrix} B_1 & B_2 & \cdots & B_{f_{2k+2}-1-\ell} \end{pmatrix} \\
&= \det\begin{pmatrix} B_1 - B_{1+f_{2k+1}} & B_2 & \cdots & B_{f_{2k+2}-1-\ell} \end{pmatrix} \\
&= \det\begin{pmatrix} \mathbf{0}_{f_{2k+2}-2-\ell,1} & M_{x'+(\ell+1), f_{2k+2}-1-(\ell+1)} \\ (-1)^{k+1} & * \end{pmatrix} \\
&= (-1)^{k+1}(-1)^{1+f_{2k+2}-1-\ell} H_{x'+(\ell+1), f_{2k+2}-1-(\ell+1)} \\
&= (-1)^{k+\ell} H_{x'+(\ell+1), f_{2k+2}-1-(\ell+1)}. && \text{(by Lemma 2.10(i))}
\end{aligned}$$

Applying the above equality r times, one has

$$\begin{aligned}
H_{x'+r, f_{2k+2}-1-r} &= (-1)^{k+r-1}(-1)^{k+r-2}\cdots(-1)^{k} H_{x', f_{2k+2}-1} \\
&= (-1)^{rk}(-1)^{\frac{r(r-1)}{2}} H_{x', f_{2k+2}-1}.
\end{aligned}$$

(iii) Let $\max\{0, -x\} \le r \le f_{2k} - 1$ and let C_j be the jth column of $M_{x+r, f_{2k+2}-1}$. Then

$$\begin{aligned}
H_{x+r, f_{2k+2}-1} &= \det\begin{pmatrix} C_1 & C_2 & \cdots & C_{f_{2k+2}-1} \end{pmatrix} \\
&= \det\begin{pmatrix} C_1 & C_2 & \cdots & C_{f_{2k+2}-r-1} & C'_1 & \cdots & C'_r \end{pmatrix}
\end{aligned}$$

where $C'_p = C_{f_{2k+2}-r-1+p} - C_{f_{2k}-r-1+p}$ for $1 \le p \le r$. Note that

$$\begin{pmatrix} C'_1 & \cdots & C'_r \end{pmatrix} = \begin{pmatrix} s_{\gamma_i - f_{2k}+1} & \cdots & s_{\gamma_i - f_{2k}+r} \\ \vdots & \ddots & \vdots \\ s_{\gamma_i + f_{2k+1}-1} & \cdots & s_{\gamma_i + f_{2k+1}+r-2} \end{pmatrix} - \begin{pmatrix} s_{\gamma_i - f_{2k+2}+1} & \cdots & s_{\gamma_i - f_{2k+2}+r} \\ \vdots & \ddots & \vdots \\ s_{\gamma_i - 1} & \cdots & s_{\gamma_i + r-2} \end{pmatrix}.$$

By Lemma 2.6 and Theorem 2.5(ii), for $1 \le q \le f_{2k+2} - 2$ and $f_{2k+2} + 1 \le q \le f_{2k+2} + r - 2$,

$$s_{\gamma_i - f_{2k}+q} = s_{\gamma_i - f_{2k+2}+q}.$$

Moreover, by Lemma 2.11, $s_{\gamma_i + f_{2k+1}-1} - s_{\gamma_i - 1} = (-1)^k$ and $s_{\gamma_i + f_{2k+1}} - s_{\gamma_i} = (-1)^{k+1}$. Then

$$\begin{pmatrix} C'_1 & \cdots & C'_r \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{f_{2k+2}-1-r, r} \\ X \end{pmatrix}$$

where X is the $r \times r$ matrix

$$\begin{pmatrix} 0 & \cdots & 0 & (-1)^k \\ \vdots & \iddots & \iddots & (-1)^{k+1} \\ \vdots & \iddots & \iddots & \vdots \\ (-1)^k & (-1)^{k+1} & \cdots & 0 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
H_{x+r, f_{2k+2}-1} &= \det\begin{pmatrix} M_{x+r, f_{2k+2}-1-r} & \mathbf{0}_{f_{2k+2}-1-r, r} \\ * & X \end{pmatrix} \\
&= (-1)^{kr}(-1)^{\frac{(r-1)r}{2}} H_{x+r, f_{2k+2}-1-r} \\
&= H_{x, f_{2k+2}-1}. && \text{(by Lemma 4.10(i))}
\end{aligned}$$

□

Remark 4.11. From Proposition 4.5(ii) and Lemma 4.10, Hankel determinants on the boundary of $T_{k,i}$ can be determined by $H_{\frac{f_{2k+3}}{2}+f_{2k}+1,f_{2k+1}}$(the lower left corner of $T_{k,2}$).

5. Evaluating the Hankel determinants

In section 4, we show that for any $k \geq 0$, to know all the determinants on the boundary of $U_{k,i}$ (resp. $V_{k,i}$, $T_{k,i}$) for all i, it is enough to know the value of one determinant on the boundary $U_{k,i}$ (resp. $V_{k,i}$ or $T_{k,i}$) for some i. In this section, for certain i, we shall give the expression of a determinant on the boundary $U_{k,i}$ (resp. $V_{k,i}$ or $T_{k,i}$) for all k.

The next result allows us to determine the determinant on the lower left corner of $U_{k,i}$ by using the determinants on the boundary of $U_{k-1,*}$ and $T_{k-1,*}$.

Lemma 5.1. (Lower left corner of $U_{k,i}$) *For all $k \geq 1$ and $i \geq 1$,*

$$H_{\alpha_i-f_{2k+3}+1,\,f_{2k}} = (-1)^k(H_{\alpha_i-f_{2k+3}+2,\,f_{2k}-1} - H_{\alpha_i-f_{2k+3}+1,\,f_{2k}-1}).$$

Proof. Let $y = \alpha_i - f_{2k+3}$ and let A_j be the jth column of $H_{y+1,f_{2k}}$. Then

$$H_{y+1,f_{2k}} = \begin{vmatrix} s_{y+1} & s_{y+2} & \cdots & s_{y+f_{2k}} \\ \vdots & \vdots & \iddots & \vdots \\ s_{y+f_{2k}} & s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \end{vmatrix} = \det\begin{pmatrix} A_1 & A_2 & \cdots & A_{f_{2k}-1} \end{pmatrix}.$$

Recall that $\alpha_i \in E'_{k+1}$. Then $\Phi_{k+1}(y) = \frac{f_{2k+1}}{2}$ and $\Phi_{k-1}(y+f_{2k}) = \frac{f_{2k-1}}{2}$. This implies $y \in F'_k$ and $y + f_{2k} \in F_{k-1}$. By Lemma 2.8 and Theorem 2.5(i), the fact $y + f_{2k} \in F_{k-1}$ yields that $s_{y+\ell} = s_{y+f_{2k-2}+\ell}$ for $1 \leq \ell \leq f_{2k} - 2$. By Lemma 2.11, $s_{y+f_{2k}-1} - s_{y+f_{2k}+f_{2k-2}-1} = (-1)^{k+1}$ and $s_{y+f_{2k}} - s_{y+f_{2k}+f_{2k-2}} = (-1)^k$. So

$$\begin{aligned} H_{y+1,f_{2k}} &= \det\begin{pmatrix} A_1 - A_{1+f_{2k-2}} & A_2 & \cdots & A_{f_{2k}-1} \end{pmatrix} \\ &= \begin{vmatrix} 0 & s_{y+2} & \cdots & s_{y+f_{2k}} \\ \vdots & \vdots & \iddots & \vdots \\ 0 & s_{y+f_{2k}-1} & \cdots & s_{y+2f_{2k}-3} \\ (-1)^{k+1} & s_{y+f_{2k}} & \cdots & s_{y+2f_{2k}-2} \\ (-1)^k & s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \end{vmatrix} \\ &= (-1)^k(-1)^{1+f_{2k}}H_{y+2,f_{2k}-1} + (-1)^{k+1}(-1)^{f_{2k}}X \end{aligned} \tag{5.1}$$

where

$$X = \begin{vmatrix} s_{y+2} & \cdots & s_{y+f_{2k}} \\ \vdots & \iddots & \vdots \\ s_{y+f_{2k}-1} & \cdots & s_{y+2f_{2k}-3} \\ s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \end{vmatrix} = (-1)^{f_{2k}-2}\begin{vmatrix} s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \\ s_{y+2} & \cdots & s_{y+f_{2k}} \\ \vdots & \iddots & \vdots \\ s_{y+f_{2k}-1} & \cdots & s_{y+2f_{2k}-3} \end{vmatrix}.$$

Since $y + f_{2k} \in F'_k$, by Lemma 2.8 and Theorem 2.5(i), we see

$$\begin{pmatrix} s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \end{pmatrix} = \begin{pmatrix} s_{y+1} & \cdots & s_{y+f_{2k}-1} \end{pmatrix}$$

and $X = (-1)^{f_{2k}-2}H_{y+1,f_{2k}-1}$. Then the result follows from Eq (5.1) and Lemma 2.10. □

Now we show how to obtain the determinant on the lower left corner of $T_{k,i}$ by using determinants on the boundary of $U_{k,i-1}$ and $U_{k-1,i+1}$.

Lemma 5.2. (Lower left corner of $T_{k,i}$) *For all $k \geq 1$ and $i \geq 2$,*

$$H_{\gamma_i - f_{2k+2}+1,\, f_{2k+1}} = (-1)^k (H_{\gamma_i - f_{2k+2}+2,\, f_{2k+1}-1} + H_{\gamma_i - f_{2k+2}+1,\, f_{2k+1}-1}).$$

Proof. Let $y = \gamma_i - f_{2k+2}$ and let A_j be the jth column of $H_{y+1,f_{2k+1}}$. Then

$$H_{y+1,f_{2k+1}} = \begin{vmatrix} s_{y+1} & s_{y+2} & \cdots & s_{y+f_{2k+1}} \\ \vdots & \vdots & \iddots & \vdots \\ s_{y+f_{2k+1}} & s_{y+f_{2k+1}+1} & \cdots & s_{y+2f_{2k+1}-1} \end{vmatrix} = \det\begin{pmatrix} A_1 & A_2 & \cdots & A_{f_{2k+1}-1} \end{pmatrix}.$$

Recall that $\gamma_i \in E_k'$. By Lemma 2.6, $y \in E_k$ and $\Phi_k(y + f_{2k+1}) = \frac{f_{2k+1}}{2}$. This implies $y + f_{2k+1} \in F_k$. By Lemma 2.8 and Theorem 2.5(i), the fact $y + f_{2k+1} \in F_k$ yields that $s_{y+\ell} = s_{y+f_{2k}+\ell}$ for $1 \leq \ell \leq f_{2k+1} - 2$. By Lemma 2.11, $s_{y+f_{2k+1}-1} - s_{y+f_{2k+1}+f_{2k}-1} = (-1)^k$ and $s_{y+f_{2k+1}} - s_{y+f_{2k+1}+f_{2k}} = (-1)^{k+1}$. So

$$\begin{aligned} H_{y+1,f_{2k+1}} &= \det\begin{pmatrix} A_1 - A_{1+f_{2k}} & A_2 & \cdots & A_{f_{2k+1}-1} \end{pmatrix} \\ &= \begin{vmatrix} 0 & s_{y+2} & \cdots & s_{y+f_{2k+1}} \\ \vdots & \vdots & \iddots & \vdots \\ 0 & s_{y+f_{2k+1}-1} & \cdots & s_{y+2f_{2k+1}-3} \\ (-1)^k & s_{y+f_{2k+1}} & \cdots & s_{y+2f_{2k+1}-2} \\ (-1)^{k+1} & s_{y+f_{2k+1}+1} & \cdots & s_{y+2f_{2k+1}-1} \end{vmatrix} \\ &= (-1)^{k+1}(-1)^{1+f_{2k+1}} H_{y+2,f_{2k+1}-1} + (-1)^k (-1)^{f_{2k+1}} X \end{aligned} \tag{5.2}$$

where

$$X = \begin{vmatrix} s_{y+2} & \cdots & s_{y+f_{2k+1}} \\ \vdots & \iddots & \vdots \\ s_{y+f_{2k+1}-1} & \cdots & s_{y+2f_{2k+1}-3} \\ s_{y+f_{2k+1}+1} & \cdots & s_{y+2f_{2k+1}-1} \end{vmatrix} = (-1)^{f_{2k+1}-2} \begin{vmatrix} s_{y+f_{2k+1}+1} & \cdots & s_{y+2f_{2k+1}-1} \\ s_{y+2} & \cdots & s_{y+f_{2k+1}} \\ \vdots & \iddots & \vdots \\ s_{y+f_{2k+1}-1} & \cdots & s_{y+2f_{2k+1}-3} \end{vmatrix}.$$

Since $y \in E_k$, by Lemma 2.6 and Theorem 2.5(ii), we see

$$\begin{pmatrix} s_{y+f_{2k+1}+1} & \cdots & s_{y+2f_{2k+1}-1} \end{pmatrix} = \begin{pmatrix} s_{y+1} & \cdots & s_{y+f_{2k+1}-1} \end{pmatrix}$$

and $X = (-1)^{f_{2k+1}-2} H_{y+1,f_{2k+1}-1}$. Then the result follows from Eq (5.2) and Lemma 2.10. □

Now We are able to give the exact value of the Hanker determinant on the upper right corner of $U_{k,i}$, and hence we know all the determinants on the boundary of $U_{k,i}$.

Theorem 5.3. (Upper right corner of $U_{k,1}$) *Let $k \geq 1$. Then*

$$H_{\frac{f_{2k+1}}{2}+1,\, f_{2k+3}-1} = (-1)^{k+1} \frac{f_{2k+1}}{2}.$$

Proof. We can check directly that the result holds for $k = 1, 2$. Now suppose $k \geq 3$. Let $h_k = H_{\frac{f_{2k+1}}{2}+1, f_{2k+3}-1}$. Then

$$\begin{aligned}
h_k &= (-1)^{(f_{2k+2}-1)k}(-1)^{\frac{(f_{2k+2}-1)(f_{2k+2}-2)}{2}} H_{\frac{f_{2k+5}}{2}-f_{2k}, f_{2k}} && \text{(by Lemma 4.6(i))}\\
&= (-1)^{\frac{f_{2k+2}-1}{2}} H_{\frac{f_{2k+1}}{2}+1, f_{2k}} && \text{(by Lemma 2.10 and Lemma 4.2(ii))}\\
&= (-1)^{\frac{f_{2k+2}-1}{2}}(-1)^k (H_{\frac{f_{2k+1}}{2}+2, f_{2k}-1} - H_{\frac{f_{2k+1}}{2}+1, f_{2k}-1}). && \text{(by Lemma 5.1)}
\end{aligned}$$

Applying Lemma 4.6(i) for $k-1$* and $r = f_{2k-2}$,

$$\begin{aligned}
H_{\frac{f_{2k+1}}{2}+1, f_{2k}-1} &= (-1)^{f_{2k-2}(k-1)}(-1)^{\frac{f_{2k-2}(f_{2k-2}-1)}{2}} H_{\frac{f_{2k-1}}{2}+1, f_{2k+1}-1}\\
&= (-1)^{k-1}(-1)^{\frac{f_{2k-2}-1}{2}} h_{k-1}. && \text{(by Lemma 2.10)}
\end{aligned}$$

Applying Lemma 4.10(i) for $k-1$ and $r = f_{2k-2}$,

$$\begin{aligned}
H_{\frac{f_{2k+1}}{2}+2, f_{2k}-1} &= (-1)^{(f_{2k-2}-1)(k-1)}(-1)^{\frac{(f_{2k-2}-1)(f_{2k-2}-2)}{2}} H_{\frac{f_{2k+1}}{2}+f_{2k-2}+1, f_{2k-1}}\\
&= (-1)^{\frac{f_{2k-2}-1}{2}} H_{\frac{f_{2k+1}}{2}+f_{2k-2}+1, f_{2k-1}} && \text{(by Lemma 2.10)}\\
&= (-1)^{\frac{f_{2k-2}-1}{2}}(-1)^{k-1}\left(H_{\frac{f_{2k+1}}{2}+f_{2k-2}+2, f_{2k-1}-1} + H_{\frac{f_{2k+1}}{2}+f_{2k-2}+1, f_{2k-1}-1}\right) && \text{(by Lemma 5.2)}\\
&= (-1)^{\frac{f_{2k-2}-1}{2}}(-1)^{k-1}\left(h_{k-2} + H_{\frac{f_{2k+1}}{2}+f_{2k-2}+1, f_{2k-1}-1}\right) && \text{(by Lemma 4.6(iii))}\\
&= (-1)^{\frac{f_{2k-2}-1}{2}}(-1)^{k-1}\left(h_{k-2} - h_{k-1}\right). && \text{(by Lemma 4.6(i) and Lemma 2.10)}
\end{aligned}$$

Combing previous equations, we have

$$\begin{aligned}
h_k &= (-1)^{\frac{f_{2k+2}-1}{2}}(-1)^k (H_{\alpha_1-f_{2k+3}+2, f_{2k}-1} - H_{\alpha_1-f_{2k+3}+1, f_{2k}-1})\\
&= (-1)^{\frac{f_{2k+2}-1}{2}}(-1)^k(-1)^{\frac{f_{2k-2}-1}{2}}(-1)^{k-1}(h_{k-2} - 2h_{k-1})\\
&= h_{k-2} - 2h_{k-1}. && \text{(by Lemma 2.10)}
\end{aligned}$$

The initial values are $h_1 = 2$, $h_2 = -5$. The result follows from the recurrence relation of h_k and its initial values. □

Corollary 5.4. (Lower left corner of $V_{k,1}$) *For all $k \geq 1$,*

$$H_{\frac{f_{2k+1}}{2}+f_{2k+3}+1, f_{2k}} = (-1)^{k+\frac{f_{2k+2}+1}{2}} \frac{f_{2k+1}}{2}.$$

Proof. By Proposition 4.5,

*We need to mention that α_i's depend also on k. For example, $\alpha_1^{(k-1)} = \frac{f_{2k+3}}{2}$ and $\alpha_1^{(k)} = \frac{f_{2k+5}}{2}$.

$$\begin{aligned}
H_{\frac{f_{2k+1}}{2}+f_{2k+3}+1,\,f_{2k}} &= H_{\frac{f_{2k+5}}{2}-f_{2k},\,f_{2k}} \\
&= (-1)^{(f_{2k+2}-1)k}(-1)^{\frac{(f_{2k+2}-1)(f_{2k+2}-2)}{2}} H_{\frac{f_{2k+1}}{2}+1,f_{2k+3}-1} && \text{(by Lemma 4.6(i))} \\
&= (-1)^{\frac{f_{2k+2}-1}{2}} H_{\frac{f_{2k+1}}{2}+1,f_{2k+3}-1} && \text{(by Lemma 2.10)} \\
&= (-1)^{k+\frac{f_{2k+2}+1}{2}} \frac{f_{2k+1}}{2}. && \text{(by Theorem 5.3)}
\end{aligned}$$

□

Corollary 5.5. (Lower left corner of $T_{k,2}$) *For all* $k \geq 1$, $H_{\frac{f_{2k+3}}{2}+f_{2k}+1,f_{2k+1}} = f_{2k}$.

Proof. From Lemma 5.2, we have

$$H_{\frac{f_{2k+3}}{2}+f_{2k}+1,f_{2k+1}} = (-1)^k (H_{\frac{f_{2k+3}}{2}+f_{2k}+2,f_{2k+1}-1} + H_{\frac{f_{2k+3}}{2}+f_{2k}+1,f_{2k+1}-1}). \tag{5.3}$$

Note that $H_{\frac{f_{2k+3}}{2}+f_{2k}+2,f_{2k+1}-1}$ is on the upper left corner of $U_{k-1,2}$. By Lemma 4.6(iii),

$$H_{\frac{f_{2k+3}}{2}+f_{2k}+2,f_{2k+1}-1} = H_{\frac{f_{2k-1}}{2}+f_{2k+3}+1,f_{2k+1}-1}.$$

According to Proposition 4.5 and Lemma 4.6, the determinants on the upper left corner of $U_{k-1,1}$ and $U_{k-1,1}$ are equal. Namely, $H_{\frac{f_{2k-1}}{2}+f_{2k+3}+1,f_{2k+1}-1} = H_{\frac{f_{2k-1}}{2}+1,f_{2k+1}-1}$. Therefore,

$$H_{\frac{f_{2k+3}}{2}+f_{2k}+2,f_{2k+1}-1} = H_{\frac{f_{2k-1}}{2}+1,f_{2k+1}-1}. \tag{5.4}$$

It follows from Lemma 4.6(i) and Lemma 2.10 that

$$\begin{aligned}
H_{\frac{f_{2k+3}}{2}+f_{2k}+1,,\,f_{2k+1}-1} &= (-1)^{2f_{2k}k}(-1)^{\frac{2f_{2k}(2f_{2k}-1)}{2}} H_{\frac{f_{2k+1}}{2}+1,\,f_{2k+3}-1} \\
&= -H_{\frac{f_{2k+1}}{2}+1,\,f_{2k+3}-1}.
\end{aligned} \tag{5.5}$$

Using Eq (5.3), Eq (5.4) and Eq (5.5), we have

$$\begin{aligned}
H_{\frac{f_{2k+3}}{2}+f_{2k}+1,f_{2k+1}} &= (-1)^k \left(H_{\frac{f_{2k-1}}{2}+1,f_{2k+1}-1} - H_{\frac{f_{2k+1}}{2}+1,\,f_{2k+3}-1} \right) \\
&= (-1)^k \left((-1)^k \frac{f_{2k-1}}{2} - (-1)^{k+1} \frac{f_{2k+1}}{2} \right) && \text{(by Theorem 5.3)} \\
&= f_{2k}.
\end{aligned}$$

□

6. Proof of Theorem 1.1, 1.2 and 1.3

Proof of Theorem 1.1. Suppose $(m,n) \in U_{k,i}$ for some i. Then $\alpha_i - f_{2k+2} < n + m \le \alpha_i$ and $m = \alpha_i - f_{2k+2} + 1 - n + r$ where $0 \le r < f_{2k+2}$.

Case 1: $n = f_{2k+3} - 1$. Applying Lemma 4.6(i), Proposition 4.5 and then Lemma 4.6(i) again, we have

$$H_{\alpha_i - f_{2k+3}+1, f_{2k+3}-1} = H_{\alpha_1 - f_{2k+3}+1, f_{2k+3}-1} = (-1)^{k+1}\frac{f_{2k+1}}{2}. \tag{6.1}$$

where the last equality follows from Theorem 5.3. Since $r = \alpha_i - m - n = \frac{f_{2k+5}}{2} - \Phi_{k+1}(m+n)$, by Lemma 4.6(iii),

$$\begin{aligned} H_{m,n} &= H_{\alpha_i - f_{2k+4}+2+r, f_{2k+3}-1} \\ &= (-1)^r H_{\alpha_i - f_{2k+3}+1, f_{2k+3}-1} \\ &= (-1)^{\frac{f_{2k+5}}{2} - \Phi_{k+1}(m+n)} H_{\alpha_i - f_{2k+3}+1, f_{2k+3}-1} \\ &= (-1)^{k+1}(-1)^{\frac{f_{2k+5}}{2} - \Phi_{k+1}(m+n)}\frac{f_{2k+1}}{2} \end{aligned}$$

where the last equality follows from Eq (6.1).

Case 2: $n = f_{2k}$. By Proposition 4.5, we have

$$\begin{aligned} H_{m,n} &= H_{\alpha_1 - f_{2k}, f_{2k}} \\ &= (-1)^{\ell k}(-1)^{\frac{\ell(\ell-1)}{2}} H_{\alpha_1 - f_{2k+3}+1, f_{2k+3}-1} && \text{(by Lemma 4.6(iii))} \\ &= (-1)^{k+1}(-1)^{\frac{f_{2k+2}-1}{2}} \cdot \frac{f_{2k+1}}{2} && \text{(by Theorem 5.3)} \end{aligned}$$

where $\ell = f_{2k+3} - 1 - f_{2k}$ is even by Lemma 2.10.

Case 3: $f_{2k} < n < f_{2k+3} - 1$. If $m + n = \alpha_i$ (or $\alpha_i - f_{2k+2} + 1$), then applying Lemma 4.6(i) (or Lemma 4.6(ii)) and then Eq (6.1), we have

$$\begin{aligned} H_{m,n} &= (-1)^{\ell k}(-1)^{\frac{\ell(\ell-1)}{2}} H_{\alpha_i - f_{2k+3}+1, f_{2k+3}-1} \\ &= (-1)^{\ell k}(-1)^{\frac{\ell(\ell-1)}{2}}(-1)^{k+1}\frac{f_{2k+1}}{2} \end{aligned}$$

where $\ell = f_{2k+3} - 1 - n$. If $\alpha_i - f_{2k+2} + 1 < m + n < \alpha_i$, then Lemma 4.1 yields $H_{m,n} = 0$. □

Proof of Theorem 1.2. Suppose $(m,n) \in V_{k,i}$ for some i. Then $\beta_i < n + m \le \beta_i + f_{2k+1}$.

Case 1: $n = f_{2k+2} - 1$. By Lemma 4.8(iii) & (i), we have

$$\begin{aligned} H_{m,n} &= H_{\beta_i' - f_{2k+1}+2, f_{2k+2}-1} \\ &= (-1)^{(f_{2k+1}-1)k}(-1)^{\frac{(f_{2k+1}-1)f_{2k+1}}{2}} H_{\beta_i'+1, f_{2k}}. \end{aligned} \tag{6.2}$$

According to Proposition 4.5(i) and Corollary 5.4,

$$H_{\beta_i'+1, f_{2k}} = H_{\beta_1'+1, f_{2k}} = (-1)^{k+\frac{f_{2k+2}+1}{2}}\frac{f_{2k+1}}{2}. \tag{6.3}$$

The result follows from Eq (6.2) and Eq (6.3).

Case 2: $n = f_{2k}$. By Proposition 4.5, $H_{m,n} = H_{\alpha_1 - f_{2k}, f_{2k}}$. Then the result follows from Theorem 1.1(ii).

Case 3: $f_{2k} < n < f_{2k+2} - 1$. If $m + n = \beta_i + 1$ (or $\beta_i + f_{2k+1}$), then by Lemma 4.8(i) (or Lemma 4.8(ii)), we have

$$H_{m,n} = -(-1)^{(f_{2k+2}-n)(k+1)}(-1)^{\frac{(f_{2k+2}-1-n)(f_{2k+2}-2-n)}{2}} \frac{f_{2k+1}}{2}.$$

If $\beta_i + 1 < m + n < \beta_i + f_{2k+1}$, then Lemma 4.1 shows $H_{m,n} = 0$. □

Proof of Theorem 1.3. Suppose $(m, n) \in T_{k,i}$ for some i. Then $\gamma_i - f_{2k} < n + m \leq \gamma_i$.

Case 1: $n = f_{2k+2} - 1$. By Lemma 4.10(iii) & (i),

$$\begin{aligned} H_{m,n} &= H_{\gamma_i - f_{2k+3}+2, f_{2k+2}-1} \\ &= (-1)^{(f_{2k}-1)k}(-1)^{\frac{(f_{2k}-1)(f_{2k}-2)}{2}} H_{\gamma_i - f_{2k+2}+1, f_{2k+1}} \\ &= (-1)^{\frac{f_{2k}-1}{2}} H_{\gamma_i - f_{2k+2}+1, f_{2k+1}}. \end{aligned} \tag{6.4}$$

where the last equality follows from Lemma 2.10(i). Using Proposition 4.5(ii) and Corollary 5.5,

$$H_{\gamma_i - f_{2k+2}+1, f_{2k+1}} = H_{\gamma_2 - f_{2k+2}+1, f_{2k+1}} = f_{2k}. \tag{6.5}$$

The result follows from Eq (6.4) and Eq (6.5).

Case 2: $n = f_{2k+1}$. Write $m = \gamma_i - f_{2k+2} + r$ with $1 \leq r \leq f_{2k}$. By Proposition 4.5(ii) and Corollary 5.5,

$$H_{m,n} = (-1)^{r+1} H_{\gamma_2 - f_{2k+2}+1, f_{2k+1}} = (-1)^{r+1} f_{2k}.$$

Note that $\gamma_i = m + n + f_{2k} - r$ and $1 \leq r \leq f_{2k}$. Consequently,

$$\frac{f_{2k+3}}{2} = \Phi_k(\gamma_i) = \Phi_k(m + n + f_{2k} - r) = \Phi_k(m + n) + f_{2k} - r$$

which gives $r = \Phi_k(m + n) - \frac{f_{2k+1}}{2}$. Then the result follows.

Case 3: $f_{2k} < n < f_{2k+2} - 1$. If $m + n = \gamma_i - f_{2k} + 1$ (or γ_i), then by Lemma 4.10(i) (or Lemma 4.10(ii)),

$$H_{m,n} = (-1)^{(f_{2k+2}-1-n)k}(-1)^{\frac{(f_{2k+2}-1-n)(f_{2k+2}-2-n)}{2}}(-1)^{\frac{f_{2k}-1}{2}} \cdot f_{2k}.$$

If $\gamma_i - f_{2k} + 1 < m + n < \gamma_i$, then Lemma 4.1 yields $H_{m,n} = 0$. □

7. Conclusions

In this paper, we study the Hankel determinants $H_{m,n}$ of the Sturmian sequence $\mathbf{s} = \tau^\infty(1)$. In Theorem 1.1, 1.2 and 1.3, we give the closed form of the Hankel determinants $H_{m,n}$ for all $m \geq 0$ and $n \geq 1$. To extend the results to other Sturmian sequences, the difficulty is to locate the parallelograms that are composed by (m, n)'s such that $H_{m,n} = 0$. This will need further effort.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. J. P. Allouche, J. Peyrière, Z. X. Wen, Z. Y. Wen, Hankel determinants of the Thue-Morse sequence, *Ann. Inst. Fourier*, Grenoble, **48** (1998), 1–27.
2. J. P. Allouche, J. Shallit, Automatic sequences: Theory and Applications, *Cambridge University Press, 2003.*
3. Y. Bugeaud, On the rational approximation to the Thue-Morse-Mahler numbers, *Ann. Inst. Fourier*, **61** (2011), 2065–2076. https://doi.org/10.5802/aif.2666
4. Y. Bugeaud, G. N. Han, Z. Y. Wen, J. Y. Yao, Hankel Determinants, Padé Approximations, and Irrationality Exponents, *Int. Math. Res. Notices* (IMRN), **5** (2016), 1467–1496.
5. M. Coons, On the rational approximation of the sum of the reciprocals of the Fermat numbers, *Ramanujan J.*, **30** (2013), 39–65. https://doi.org/10.1007/s11139-012-9410-x
6. H. Fu, G. N. Han, Computer assisted proof for Apwenian sequences, *Proc. ISSAC 2016 Conference*, Waterloo, Ontario, Canada, 2016.
7. R. J. Fokkink, C. Kraaikamp, J. Shallit. Hankel matrices for the period-doubling sequence. *Indagationes Mathematicae*, **28**(1) (2017), 108-119. https://doi.org/10.1016/j.indag.2016.11.008
8. Y. J. Guo, G. N. Han, W. Wu. Criterions for apwenian sequences, *Adv. Math.*, **389** (2021), 107899. https://doi.org/10.1016/j.aim.2021.107899
9. Y. J. Guo, Z. X. Wen, W. Wu, On the irrationality exponent of the regular paperfolding numbers, *Linear Algebra Appl,*, **446** (2014), 237–264. https://doi.org/10.1016/j.laa.2013.12.023
10. G. N. Han, Hankel continued fraction and its applications, *Adv. Math.*, **303** (2016), 295–321.
11. T. Kamae, J. Tamura, Z. Y. Wen, Hankel determinants for the Fibonacci word and Padé approximation, *Acta Arithmetica*, **89** (1999), 123–161. https://doi.org/10.4064/aa-89-2-123-161
12. N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, (2009). Available from: `https://oeis.org/`.
13. J. Tamura, Padé approximation for the infinite words generated by certain substitutions and Hankel determinants, *Number Theory and Its Applications*, Kluwer Academic Publishers. (1990), 309–346.

14. B. Tan, Z. Y. Wen, Invertible substitutions and Sturmian sequences, *European J. Combin.*, **24** (2003), 983–1002. https://doi.org/10.1016/S0195-6698(03)00105-7

15. K. Väänänen, On rational approximations of certain Mahler functions with a connection to the Thue-Morse sequence, *Int. J. Number Theory*, **11** (2015), 487–493. https://doi.org/10.1142/S1793042115500244

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