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Research article

Dynamics of fractional order delay model of coronavirus disease

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Abstract: The majority of infectious illnesses, such as HIV/AIDS, Hepatitis, and coronavirus (2019-nCov), are extremely dangerous. Due to the trial version of the vaccine and different forms of 2019-nCov like beta, gamma, delta throughout the world, still, there is no control on the transmission of coronavirus. Delay factors such as social distance, quarantine, immigration limitations, holiday extensions, hospitalizations, and isolation are being utilized as essential strategies to manage the outbreak of 2019-nCov. The effect of time delay on coronavirus disease transmission is explored using a non-linear fractional order in the Caputo sense in this paper. The existence theory of the model is investigated to ensure that it has at least one and unique solution. The Ulam-Hyres (UH) stability of the considered model is demonstrated to illustrate that the stated model's solution is stable. To determine the approximate solution of the suggested model, an efficient and reliable numerical approach (Adams-Bashforth) is utilized. Simulations are used to visualize the numerical data in order to understand the behavior of the different classes of the investigated model. The effects of time delay on dynamics of coronavirus transmission are shown through numerical simulations via MATLAB-17.

Keywords: fractional derivative; coronavirus; delay model; Adams-Bashforth method
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1. Introduction

Coronaviruses are members of the corona Miridae family of viruses. The virus's size ranges from 65 to 125 nanometers. They're 22–26 kilobases of single-stranded RNA with a single nucleus. Alpha, beta, gamma, and delta are the other members of the coronavirus family. Influenza and the Middle East Respiratory Syndrome Coronavirus (MERS-COV) are two of the most frequent pulmonary diseases that can be fatal. They have the potential to induce pulmonary failure. Because the illness's primary hosts were camels, bats, and monkeys, it was once thought to be an animal-only sickness. It was later passed on to humans, and it has now become a global disaster. The features, locations, and transmission of the coronavirus in the human population were discovered in the literature [1]. A fatal coronavirus has been spreading in Wuhan, China, since December 2019. The number of people who have died is increasing all the time. 70,000 people were infected in the first 50 days of the illness. Coronaviruses of the beta group have been identified as the source of the infection. The virus has been named Wuhan virus or new Coronavirus 2019 by Chinese experts. SARS-COV-2, COVID-19 is the designation given to the virus by the International Committee of the Red Cross (ICRC). Using mathematical techniques, Zhao et al. investigated the coronavirus outbreak in China [2]. The worst-affected regions owing to coronavirus include Africa, America, Southeast Asia, Europe, the Eastern Mediterranean, and the Western Pacific. According to the World Health Organization (WHO), around thirty million individuals have been infected with the virus. One lakh and forty-four thousand people have been reported dead, while seven lakh and sixty-five thousand people have been reported recovered. There are 11,000 2019-nCov carriers in Pakistan, with 237 fatalities. Sadly, testing kits and lab tools are only available only around 2 percent of the time. As a result, governments used delaying tactics such as social distance, self-quarantine, and travel restrictions to combat the worldwide epidemic. The 2019-nCov pandemic has now been declared a worldwide concern by the WHO. Shim et al. examined the coronavirus's transmission potential and severity in South Korea [3]. Using the mathematical model of coronavirus, Kucharski et al. discovered a few control methods [4]. Jiang et al. proposed a machine learning model for coronavirus prediction based on data [5]. Fanelli et al. studied and forecast the COVID 19 spreading in China, Italy and France via integer order mathematical model [6]. Altan introduced new model of COVID 19 consisting of 2D curvelet transform, chaotic salp swarm algorithm and deep learning technique [7]. By incorporating delay factors into the system of differential equations, the mathematical modeling of the coronavirus model is in good accord with the real events. The factor of delay has been included in this model. Quarantine, isolation, or immunization, for example, are all delay causes. In most epidemiological models, if the infection rate is kept under control, the illness will eventually settle into a stable state. Controlling infection in the current scenario of 2019-nCov is almost difficult, therefore delaying measures, such as social distance, quarantine, isolation, and others, have been used to combat the pandemic of 2019-nCov. Fortunately, the modeling's delay factors or delaying strategies are self-contained and unaffected by other transmission rates. Keeping these views, Naveed et al. proposed a new delay integer order model of the coronavirus disease [8]. The detail of the delay model is given below.

Let the whole populace is divided into five subclasses, $\mathbb{S}(t)$: the susceptible populace, $\mathbb{E}(t)$: the exposed populace, $\mathbb{I}(t)$: symptomatic populace, $\mathbb{A}(t)$: asymptomatic populace and $\mathbb{R}(t)$: is the recovered populace, then the transmission dynamics of delayed is governed by the following set of differential equations [8].

$$\begin{aligned}
\frac{d\mathbb{S}}{dt} &= \pi_s - (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))e^{-\mu\tau} - \mu\mathbb{S},\\
\frac{d\mathbb{E}}{dt} &= (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))e^{-\mu\tau} - (\omega_1+\omega_2+\omega_3+\mu)\mathbb{E},\\
\frac{d\mathbb{I}}{dt} &= \omega_1\mathbb{E} - (\omega_4+\mu)\mathbb{I},\\
\frac{d\mathbb{A}}{dt} &= \omega_2\mathbb{E} - (\omega_5+\mu)\mathbb{A},\\
\frac{d\mathbb{R}}{dt} &= \omega_3\mathbb{E} + \omega_4\mathbb{I} + \omega_5\mathbb{A} - \mu\mathbb{R}.
\end{aligned} \tag{1.1}$$

The above system involves different parameters such as: π_s is the rate of recruitment, μ is the rate of mortality due to disease infection, β_1 is the symptomatic rate of infection, β_2 is the asymptomatic rate of infection, the interaction with infected symptomatic rate is ω_1, the interaction with infected asymptomatic rate is ω_2, ω_3 is the exposed person recovered from disease, ω_4 symptomatic person recovered after quarantine and the quarantine of asymptomatic infected persons with the rate of ω_5.

From the few decades, fractional calculus (FC) has become a hot research area among researchers due to its wide range of applications in the modeling of physical processes. FC generalizes classical calculus. The first study for the generalization of the ordinary integral and differential operators into fractional derivatives (FD) is done by Riemann-Liouville and Caputo [9, 10]. Fractional order differential equation provided a natural framework to study real-world problems, such as viscoelastic system, signal processing, diffusion processes, control processing, interested reader are referred to [11–13]. The aforementioned area has been addressed from different aspects such as qualitative analysis that include existence, uniqueness and regularity, the other is numerical and analytical method to test and view the information. The operators of FC are more general than the integers order operator because they have defined nonlocal kernels. The fractional operators (FOs) preserve memory and provide the past history of any physical process. There are many FOs concerned with the convolution of the kernels. The superiority of the FOs over classical operators has been presented in many research papers. We discuss some applications of FOs in the mathematical modeling of real-world physical problems. Nisar et al. investigated the fractional-order mathematical model of the current pandemic disease (COVID-19), which was based on real data [14]. Ameen et al., analyzed the fractional-order human liver based on real clinical data [15]. In literature [16], the authors have studied the blood ethanol model under three different fractional operators. To be more specific, FOs are also used by researchers for modeling bacterial diseases. For instance, Rihan et al. used fractional operators to analyze Salmonella bacterial infection [17]. Baleanu et al. investigated the human liver model via nonsingular fractional operators [18]. Alqahtani et al. solved numerically salmonella bacterial infection under fractional operator [19]. Droghei et al. compare the fractional and empirical model to analyze the dynamics of contaminant density and fluid pressure transients in swelling shales [20]. Falcini et al. apply fractional derivatives of a function concerning another function in order to have a more general and realistic picture of nonlocal effects in morphodynamics and sediment transport models [21]. Garra et al. the authors investigate the role played by the memory effect in the propagation of nonlinear thermoelastic waves using a time-fractional approach [22]. Jahanshahi et al. proposed a fractional-order SIRD model with time-dependent memory indexes for encompassing the multi-fractional characteristics of the COVID-19 [23]. Rajagopal and his coauthors constructed a fractional order model for Italy cases in [24]. We list several other applications of FC in applied sciences in [25–28].

The new fractional version of delayed system (1.1) is proposed under the time-derivative in sense of Caputo fractional derivative operator. Due to fractional order operator, imbalance of dimension occurs on the left and right sides of equations. According to [29], we can modify the fractional operator by an auxiliary parameter Λ, having the dimension of sec., to ensure the same dimension on both sides. Now the transmission model of the infectious disease for $t > 0$ and $0 < \hbar \leq 1$ takes the form

$$\begin{aligned}
\frac{1}{\Lambda^{1-\hbar}}{}^{C}\mathbf{D}_t^{\hbar}\mathbb{S}(t) &= \pi_s - (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))e^{-\mu\tau} - \mu\mathbb{S},\\
\frac{1}{\Lambda^{1-\hbar}}{}^{C}\mathbf{D}_t^{\hbar}\mathbb{E}(t) &= (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))e^{-\mu\tau} - (\omega_1+\omega_2+\omega_3+\mu)\mathbb{E},\\
\frac{1}{\Lambda^{1-\hbar}}{}^{C}\mathbf{D}_t^{\hbar}\mathbb{I}(t) &= \omega_1\mathbb{E} - (\omega_4+\mu)\mathbb{I}, \qquad (1.2)\\
\frac{1}{\Lambda^{1-\hbar}}{}^{C}\mathbf{D}_t^{\hbar}\mathbb{A}(t) &= \omega_2\mathbb{E} - (\omega_5+\mu)\mathbb{A},\\
\frac{1}{\Lambda^{1-\hbar}}{}^{C}\mathbf{D}_t^{\hbar}\mathbb{R}(t) &= \omega_3\mathbb{E} + \omega_4\mathbb{I} + \omega_5\mathbb{A} - \mu\mathbb{R},
\end{aligned}$$

with initial conditions

$$\mathbb{S}(0) = \mathbb{S}_0,\ \ \mathbb{E}(0) = \mathbb{E}_0,\ \ \mathbb{I}(0) = \mathbb{I}_0,\ \ \mathbb{A}(0) = \mathbb{A}_0,\ \ \mathbb{R}(0) = \mathbb{R}_0.$$

The rest of paper is organized as: In Section 2, some definitions from fractional calculus are recalled. Section 3 is devoted for qualitative study of the considered model. In Section 4, the numerical solution is obtained for the system under consideration with the aid of Adams-Bashforth iterative technique. In Section 5, the Matlab software is used to perform the numerical simulation for getting the graphical representation of the considered model. In Section 6, conclusion of the paper is given.

2. Preliminaries

This section gives some basic definitions and preliminaries of the fractional calculus theory, which are used in this study [30]. For the sake of simplicity we use RL, FI, and FOD for Reimann-Liouville, fractional integral and fractional order derivative, respectively.

Definition 2.1. *Let $\mathscr{V} \in L^1(\mathbb{R}^+)$ be a function, the RL FI of order $\hbar$ is defined by*

$$\mathbb{I}_t^{\hbar}\mathscr{V}(t) = \frac{1}{\Gamma(\hbar)}\int_0^t (t-\wp)^{1-\hbar}\mathscr{V}(\wp)d\wp. \qquad (2.1)$$

Definition 2.2. *The Caputo FOD of $\mathscr{V}$ is defined as*

$${}^{\mathscr{C}}\mathbb{D}_t^{\hbar}\mathscr{V}(t) = \frac{1}{\Gamma(n-\hbar)}\int_0^t (t-\wp)^{n-\hbar-1}\mathscr{V}^n(\wp)d\wp,$$

provided that integral part exists, and $n-1 < \hbar \leq n$ $n = [\hbar]+1$. If $\hbar \in (0,1)$, then one has

$${}^{\mathscr{C}}\mathbb{D}_t^{\hbar}\mathscr{V}(t) = \frac{1}{\Gamma(1-\hbar)}\int_0^t \frac{\mathscr{V}'(\wp)}{(t-\wp)^{\hbar}}d\wp.$$

Lemma 2.2.1. *In the case of fractional differential equations, the following results are valid:*

$$\mathbb{I}^{\hbar}[{}^{\mathscr{C}}\mathbb{D}^{\hbar}q](t) = q(t) + \eta_0 + \eta_1 t + \eta_2 t^2 + ... + \eta_{n-1}t^{n-1}.$$

3. Theoretical analysis of model (3.1)

3.1. Positivity and the basic reproduction number

In this part, we deal with the theory related to positive solution, invariant feasible region and boundedness of solution of the considered model. We find the basic reproduction number (BRN) and two equilibria states. One is disease free equilibrium (DFE) state and the other is endemic equilibrium (EE) state. For further discussion, let

$$\mathbb{R}^5_+ = \Big\{(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}) \in \mathscr{R}^5 : \mathbb{S}(0) > 0, \mathbb{E}(0) \geq 0, \mathbb{I}(0) \geq 0, \mathbb{A}(0) \geq 0, \mathbb{R} \geq 0\Big\},$$

and $Int\mathbb{R}^5_+$ denotes the interior of $\mathbb{R}^5_+$.

Lemma 3.0.2 (Positivity). *If* $(\mathbb{S}(0), \mathbb{E}(0), \mathbb{I}(0), \mathbb{A}(0), \mathbb{R}(0))^T \in Int\mathbb{R}^5_+$, *then for any* $t \geq 0$, *the solution of the considered mode are non-negative.*

Proof. From the first equation of considered model (2), we have

$$\begin{aligned} \frac{1}{\Lambda^{1-\boldsymbol{h}}}\,{}^{C}\mathbf{D}_t^{\boldsymbol{h}}\mathbb{S}(t)|_{\mathbb{S}=0} &= \pi_S \\ {}^{C}\mathbf{D}_t^{\boldsymbol{h}}\mathbb{S}(t)|_{\mathbb{S}=0} &= \Lambda^{1-\boldsymbol{h}}\pi_S \geq 0. \end{aligned}$$

Now, from the 2nd equation of (2), we get

$$\begin{aligned} \frac{1}{\Lambda^{1-\boldsymbol{h}}}\,{}^{C}\mathbf{D}_t^{\boldsymbol{h}}\mathbb{E}(t)|_{\mathbb{E}=0} &= (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))\, e^{-\mu\tau} \\ {}^{C}\mathbf{D}_t^{\boldsymbol{h}}\mathbb{E}(t)|_{\mathbb{E}=0} &= \Lambda^{1-\boldsymbol{h}}\, (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))\, e^{-\mu\tau} \geq 0. \end{aligned}$$

Similarly

$$\begin{aligned} {}^{C}\mathbf{D}_t^{\boldsymbol{h}}\mathbb{I}(t)|_{\mathbb{I}=0} &= \Lambda^{1-\boldsymbol{h}}\omega_1\mathbb{E} \geq 0, \\ {}^{C}\mathbf{D}_t^{\boldsymbol{h}}\mathbb{A}(t)|_{\mathbb{A}=0} &= \Lambda^{1-\boldsymbol{h}}\omega_2\mathbb{E} \geq 0, \\ {}^{C}\mathbf{D}_t^{\boldsymbol{h}}\mathbb{R}(t)|_{\mathbb{R}=0} &= \omega_3\mathbb{E} + \omega_4\mathbb{I} + \omega_5\mathbb{A} \geq 0. \end{aligned}$$

Thus, the all solutions of the model (2) are non-negative. □

3.1.1. Positively invariant region

Suppose ß = $\{\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R} \in \mathbf{R}^5 : \mathbb{S} + \mathbb{E} + \mathbb{I} + \mathbb{A} + \mathbb{R} \leq \frac{\pi_s}{\mu}\}$. Now to prove the feasibility of system (3.1), we need to show that the closed set ß is the region.

Theorem 3.1. *The closed set* ß *is positively invariant with respect to fractional system.*

Proof. First of all we prove that $\mathrm{N} = \mathbb{S} + \mathbb{E} + \mathbb{I} + \mathbb{A} + \mathbb{R}$, is bounded and non-negative functions. By adding all the relation of the system (3.1), the total population with the fractional derivative is given as

$$\frac{1}{\Lambda^{1-\hbar}}\,{}^{C}\mathbf{D}_t^{\hbar}\mathrm{N}(t) = \pi_s - \mu\mathrm{N}(t),$$

where $\mathrm{N} = \mathbb{S} + \mathbb{E} + \mathbb{I} + \mathbb{A} + \mathbb{R}$. Now using the Laplace transform to obtain the population as

$$\mathrm{N}(t) = \mathrm{N}(0)\mathbf{E}_{\hbar}(-\mu\Lambda^{1-\hbar}t^{\hbar}) + \int_0^t \pi_s\Lambda^{1-\hbar}\wp^{\hbar-1}\mathbf{E}_{\hbar,\hbar}(-\mu\Lambda^{1-\hbar}\wp^{\hbar})d\wp.$$

The term N(0) denotes size of human population at $t = 0$ and $\mathbf{E}_{\hbar}(.)$ denotes a Mittag-Leffler function. After doing some basic algebra, one can write

$$\begin{aligned} \mathrm{N}(t) &= \mathrm{N}(0)\mathbf{E}_{\hbar}(-\mu\Lambda^{1-\hbar}t^{\hbar}) + \int_0^t \pi_s\Lambda^{1-\hbar}\wp^{\hbar-1}\sum_{j=0}^{\infty}\frac{(-1)^j\mu^j\Lambda^{j(1-\hbar)}\wp^{j\hbar}}{\Gamma(j\hbar+\hbar)}d\wp \\ &= \frac{\pi_s\Lambda^{1-\hbar}}{\mu\Lambda^{1-\hbar}} + \mathbf{E}_{\hbar}(-\mu\Lambda^{1-\hbar}t^{\hbar})\Big(\mathrm{N}(0) - \frac{\pi_s\Lambda^{1-\hbar}}{\mu\Lambda^{1-\hbar}}\Big) \\ &= \frac{\pi_s}{\mu} + \mathbf{E}_{\hbar}(-d\Lambda^{1-\hbar}t^{\hbar})\Big(\mathrm{N}(0) - \frac{\pi_s}{\mu}\Big). \end{aligned}$$

Thus if $\mathrm{N}(0) \leq \frac{\pi_s}{\mu}$ then for $t > 0$, $\mathrm{N}(t) \leq \frac{\pi_s}{\mu}$. So, the closed set ß is positively invariant in the frame of fractional derivative. □

3.1.2. Steady states and the basic reproduction number

Here we discuss the steady states and the basic reproduction number. The model (2) has two equilibrium points. By solving $\frac{1}{\Lambda^{1-\hbar}}\,{}^{C}\mathbf{D}_t^{\hbar}(.) = 0$,the model (2) two equilibrium points, which satisfies the system as:

$$\begin{aligned} \pi_s - (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))e^{-\mu\tau} - \mu\mathbb{S} &= 0, \\ (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))e^{-\mu\tau} - (\omega_1+\omega_2+\omega_3+\mu)\mathbb{E} &= 0, \\ \omega_1\mathbb{E} - (\omega_4+\mu)\mathbb{I} &= 0, \\ \omega_2\mathbb{E} - (\omega_5+\mu)\mathbb{A} &= 0, \\ \omega_3\mathbb{E} + \omega_4\mathbb{I} + \omega_5\mathbb{A} - \mu\mathbb{R} &= 0. \end{aligned} \tag{3.1}$$

The DFE point F_0 can be easily obtained by putting all states variables equals to zero except susceptible state variable. Thus,

$$\mathrm{F}_0 = \left(\frac{\pi_S}{\mu}, 0, 0, 0, 0\right).$$

Also, the EE point F^* is given by

$$\mathrm{F}_1 = (\mathbb{S}^*, \mathbb{E}^*, \mathbb{I}^*, \mathbb{A}^*, \mathbb{R}^*),$$

where

$$\mathbb{S}^* = \frac{(\omega_4+\mu)(\mu+\omega_1+\omega_2+\omega_3)(\mu+\omega_5)}{\beta_1 e^{-\mu\tau}\omega_1(\omega_5+\mu) + \omega_2 e^{-\mu\tau}\beta_1(\mu+\omega_4)},$$

$$\begin{aligned}
\mathbb{E}^* &= \frac{\pi_s - \mu\mathbb{S}^*}{\mu + \omega_1 + \omega_2 + \omega_3}, \\
\mathbb{I}^* &= \frac{\omega_1\mathbb{E}^*}{\mu + \omega_4}, \\
\mathbb{A}^* &= \frac{\omega_2\mathbb{E}^*}{\mu + \omega_5}, \\
\mathbb{R}^* &= \frac{\omega_5\mathbb{A}^* + \omega_3\mathbb{E}^* + \omega_4\mathbb{I}^*}{\mu + \omega_5}.
\end{aligned}$$

The basic Reproduction number (BRN): The basic reproduction number of an infection is the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection. The BRN $\mathcal{R}$ can be obtained by using next generation matrix. From the considered model (1.1), we take the infectious and recovered class of the human populace as

$$F = \begin{pmatrix} 0 & \frac{\beta_1\pi_S e^{-\mu\tau}}{\mu} & \frac{\beta_2\pi_S e^{-\mu\tau}}{\mu} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} \omega_1 + \omega_2 + \omega_3 + \mu & 0 & 0 & 0 \\ -\omega_1 & \omega_4 + \mu & 0 & 0 \\ -\omega_2 & 0 & \omega_5 + \mu & 0 \\ -\omega_3 & -\omega_4 & -\omega_5 & \mu \end{pmatrix}.$$

Now $\mathcal{R}$ is the spectral radius of FV^{-1} is given by

$$FV^{-1} = \begin{pmatrix} \frac{\pi_s e^{-\mu\tau}(\beta_1\omega_1(\mu+\omega_5)+\beta_2\omega_2(\mu+\omega_4))}{\mu(\omega_1+\omega_2+\omega_3+\mu)(\mu+\omega_4)(\mu+\omega_5)} & \frac{\beta_1\pi_S e^{-\mu\tau}}{\mu(\omega_4+\mu)} & \frac{\beta_2\pi_S e^{-\mu\tau}}{\mu(\omega_5+\mu)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $\mathcal{R}$ is given by

$$\mathcal{R} = \frac{\pi_s e^{-\mu\tau}\left(\beta_1\omega_1(\mu + \omega_5) + \beta_2\omega_2(\mu + \omega_4)\right)}{(\mu + \omega_4)(\mu + \omega_5)(\mu + \omega_3 + \omega_2 + \omega_1)\mu}.$$

3.2. Analysis of local stability

Here, we will show the local stability of the BRN.

Theorem 3.2. *If $\mathcal{R} < 1$, then the DFE of the said problem is locally asymptotically stable, otherwise unstable for $\mathcal{R} > 1$.*

Proof. The Jacobian matrix for the considered model at DFE may be calculated as

$$J(F_0) = \begin{bmatrix} -\mu & 0 & -\frac{\beta_1\pi_S e^{-\mu\tau}}{\mu} & -\frac{\beta_2\pi_S e^{-\mu\tau}}{\mu} & 0 \\ 0 & -(\mu + \omega_1 + \omega_2 + \omega_3) & \frac{\beta_1\pi_S e^{-\mu\tau}}{\mu} & \frac{\beta_2\pi_S e^{-\mu\tau}}{\mu} & 0 \\ 0 & \omega_1 & -\omega_4 - \mu & 0 & 0 \\ 0 & \omega_2 & 0 & -\omega_5 - \mu & 0 \\ 0 & \omega_3 & \omega_4 & \omega_5 & -\mu \end{bmatrix}.$$

From the Jacobian matrix we have obtained the following eigenvalues: $\lambda_1 = -\mu < 0$, $\lambda_2 = -\mu < 0$ and

$$|J(F_0) - \lambda I| = \begin{vmatrix} -(\omega_1+\omega_2+\omega_3+\mu)-\lambda & \frac{\beta_1\pi_S e^{-\mu\tau}}{\mu} & \frac{\beta_2\pi_S e^{-\mu\tau}}{\mu} \\ \omega_1 & -(\omega_4+\mu)-\lambda & 0 \\ \omega_2 & 0 & -(\omega_5+\mu)-\lambda \end{vmatrix} = 0.$$

Put $c_1 = (\omega_1+\omega_2+\omega_3+\mu) > 0$, $c_2 = \frac{\beta_1\pi_S e^{-\mu\tau}}{\mu} > 0$, $c_3 = \frac{\beta_2\pi_S e^{-\mu\tau}}{\mu} > 0$, $c_4 = \omega_4+\mu > 0$ and $c_5 = \omega_5+\mu > 0$.

$$|J(F_0) - \lambda I| = \begin{vmatrix} -c_1-\lambda & c_2 & c_3 \\ \omega_1 & -c_4-\lambda & 0 \\ \omega_2 & 0 & -c_5-\lambda \end{vmatrix} = 0.$$

We get $\lambda^3 + \lambda^2[c_5 + c_1 + c_4] + \lambda[c_5(c_1 + c_4) + c_1c_4 - c_2\omega_1 - c_3\omega_2] + (c_1c_4c_5 - \omega_2c_3c_4 - c_5c_2\omega_1) = 0$.

Applying the criteria of Routh-Hurwitz of third order polynomial as $[c_5 + c_1 + c_4] > 0$, $(c_1c_4c_5 - \omega_2c_3c_4 - c_5c_2\omega_1) > 0$, if $\frac{(c_1c_4c_5-\omega_2c_3c_4+c_5c_2\omega_1)}{c_1c_4c_5} < 1$ substituting the values, we have $\mathcal{R} = \frac{\pi_s e^{-\mu\tau}(\beta_1\omega_1(\mu+\omega_5)+\beta_2\omega_2(\mu+\omega_4))}{(\mu+\omega_4)(\mu+\omega_5)(\mu+\omega_3+\omega_2+\omega_1)\mu} < 1$ and $[c_5 + c_1 + c_4] + \lambda[c_5(c_1 + c_4) + c_1c_4 - c_2\omega_1 - c_3\omega_2] > (c_1c_4c_5 - \omega_2c_3c_4 - c_5c_2\omega_1)$, if $\mathcal{R} < 1$. Therefore, all eigenvalues are non=positive. Thus F_0 is locally asymptotically stable. □

Theorem 3.3. *If $\mathcal{R} > 1$, then the EE of F_1 is locally asymptotically stable, otherwise unstable if $\mathcal{R} < 1$.*

Proof. The Jacobian matrix of F_1 can be calculated as

$$J(F_1) = \begin{bmatrix} -(\beta_1\mathbb{I}^* + \beta_2\mathbb{A}^*)e^{-\mu\tau} - \mu & 0 & -\frac{\beta_1\pi_S e^{-\mu\tau}}{\mu} & -\frac{\beta_2\pi_S e^{-\mu\tau}}{\mu} & 0 \\ (\beta_1\mathbb{I}^* + \beta_2\mathbb{A}^*)e^{-\mu\tau} & -(\mu+\omega_1+\omega_2+\omega_3) & \frac{\beta_1\pi_S e^{-\mu\tau}}{\mu} & \frac{\beta_2\pi_S e^{-\mu\tau}}{\mu} & 0 \\ 0 & \omega_1 & -\omega_4-\mu & 0 & 0 \\ 0 & \omega_2 & 0 & -\omega_5-\mu & 0 \\ 0 & \omega_3 & \omega_4 & \omega_5 & -\mu \end{bmatrix}.$$

We have obtained the following eigenvalues from the Jacobian matrix $J(F_1)$:
$\lambda_1 = -\mu < 0$ and

$$|J(F_1) - \lambda I| = \begin{vmatrix} -d_1-\mu-\lambda & 0 & -d_2 & -d_3 \\ b_1 & -b_4-\lambda & b_2 & b_3 \\ 0 & \omega_1 & -b_5-\lambda & 0 \\ 0 & \omega_2 & 0 & -d_6-\lambda \end{vmatrix} = 0,$$

where

$$d_1 = (\beta_1\mathbb{I}^* + \beta_2\mathbb{A}^*)e^{-\mu\tau} > 0,\ \ d_2 = \beta_1\mathbb{S}^*e^{-\mu\tau} > 0,\ \ d_3 = \beta_2\mathbb{S}^*e^{-\mu\tau} > 0,$$
$$d_4 = \omega_1+\omega_2+\omega_3+\mu > 0,\quad d_5 = \omega_4+\mu > 0,\quad d_6 = \omega_5+\mu > 0.$$

Now

$$\lambda^4 + (d_1+d_4+d_5+d_6+\mu)\lambda^3 + (d_3\omega_2 - d_1d_6 - d_6\mu - d_5d_6 - d_4d_6 + d_2\omega_1 - d_1d_5 - d_1d_4 - d_5\mu - d_4\mu - d_4d_5)\lambda^2$$

$$+(d_3d_5\omega_2 + d_3\mu\omega_2 + d_2d_6\omega_1 - d_1d_5d_6 - d_1d_4d_6 - d_5d_6\mu - d_4d_6\mu - d_4d_5d_6 + d_2\mu\omega_1 - d_1d_4d_5 - d_4d_5\mu)\lambda+$$

$$(d_3d_5\mu\omega_2 + d_2d_6\mu\omega_1 + d_1d_4d_5d_6 + d_4d_5d_6\mu) = 0.$$

Therefore, $n_0\lambda^4 + n_1\lambda^3 + n_2\lambda^2 + n_3\lambda + n_4 = 0$, where

$$\begin{aligned}
n_1 &= (d_1 + d_4 + d_5 + d_6 + \mu),\\
n_2 &= (d_3\omega_2 - d_1d_6 - d_6\mu - d_5d_6 - d_4d_6 + d_2\omega_1 - d_1d_5 - d_1d_4 - d_5\mu - d_4\mu - d_4d_5),\\
n_3 &= (d_3d_5\omega_2 + d_3\mu\omega_2 + d_2d_6\omega_1 - d_1d_5d_6 - d_1d_4d_6 - d_5d_6\mu - d_4d_6\mu - d_4d_5d_6\\
&\quad +d_2\mu\omega_1 - d_1d_4d_5 - d_4d_5\mu),\\
n_4 &= (d_3d_5\mu\omega_2 + d_2d_6\mu\omega_1 + d_1d_4d_5d_6 + d_4d_5d_6\mu).
\end{aligned}$$

Now applying the critera of Routh-Hurwitz for fourth order polynomial as $n_0 > 0$, $n_1 > 0$, $n_1n_2 - n_0n_3 > 0$, $(n_1n_2 - n_0n_3)n_3 - n_1^2n_4 > 0$ and $n_4 > 0$ only if $\mathcal{R} > 1$. Hence, all of the eigenvalues are non-positive. Therefore, by Routh Hurwitz criteria F_1 is locally asymptotically stable. □

3.2.1. Existence theory

Beside the study of non-integer-order differential equations numerically, the qualitative study of such system s of equations has remained the interest of many researchers [31]. Using of a fixed point theory, we construct the existence and uniqueness of proposed model (2). For this, we rewrite the system (2) as

$$\begin{cases} \frac{1}{\Lambda^{1-\hbar}}{}^{C}\mathbf{D}_t^{\hbar}\mathcal{G} = \Psi(t, \mathcal{G}(t)), \quad \text{for} \quad t \in [0, T], \quad 0 < \hbar \leq 1, \\ \mathcal{G}(t)|_{t=0} = \mathcal{G}_0 \geq 0. \end{cases} \tag{3.2}$$

only if

$$\begin{cases} \mathcal{G}(t) = (\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R})^T, \\ \Psi(t, \mathcal{G}(t)) = (\mathcal{Y}_i(t, \mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R})), \end{cases} \tag{3.3}$$

for $i = 1, 2, 3, 4, 5$, and each $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5$, are defined as

$$\begin{cases} \mathcal{Y}_1 = \pi_s - (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))e^{-\mu\tau} - \mu\mathbb{S}, \\ \mathcal{Y}_2 = (\beta_1\mathbb{I}(t-\tau)\mathbb{S}(t-\tau) + \beta_2\mathbb{A}(t-\tau)\mathbb{S}(t-\tau))e^{-\mu\tau} - (\omega_1 + \omega_2 + \omega_3 + \mu)\mathbb{E}, \\ \mathcal{Y}_3 = \omega_1\mathbb{E} - (\omega_4 + \mu)\mathbb{I}, \\ \mathcal{Y}_4 = \omega_2\mathbb{E} - (\omega_5 + \mu)\mathbb{A}, \\ \mathcal{Y}_5 = \omega_3\mathbb{E} + \omega_4\mathbb{I} + \omega_5\mathbb{A} - \mu\mathbb{R}. \end{cases}$$

On applying FI on both side (3.2), we get

$$\mathcal{G}(t) - \mathcal{G}(0) = \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t \Psi(\wp, \mathcal{G}(\wp))(t-\wp)^{\hbar-1}d\wp. \tag{3.4}$$

In particular for each class we can rewrite (3.4) as

$$\begin{aligned}
\mathbb{S}(t)-\mathbb{S}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{\hbar-1}\mathcal{Y}_1 d\wp,\\
\mathbb{E}(t)-\mathbb{E}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{\hbar-1}\mathcal{Y}_2 d\wp,\\
\mathbb{I}(t)-\mathbb{I}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{\hbar-1}\mathcal{Y}_3 d\wp,\\
\mathbb{A}(t)-\mathbb{A}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{\hbar-1}\mathcal{Y}_4 d\wp,\\
\mathbb{R}(t)-\mathbb{R}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{\hbar-1}\mathcal{Y}_5 d\wp.
\end{aligned} \tag{3.5}$$

In order to prove the kernel $\Psi(\wp, \mathcal{G}(\wp))$ satisfies the Lipschitz condition and contraction, it is sufficient to prove that all $\mathcal{Y}_i$ satisfy Lipschitz condition and contraction.

Theorem 3.4. *The kernel $\mathcal{Y}_i$ satisfy the Lipschitz condition and contraction for bounded function $\mathcal{G}(t)$, if the given inequality holds*

$$1 > \Upsilon_i \geq 0,$$

for $i = 1, 2, 3, 4, 5$.

Proof. We will start from $\mathbb{S}$ and $\mathbb{S}_1$, one can get

$$\begin{aligned}
\|\mathcal{Y}_1(t,\mathbb{S}) - \mathcal{Y}_1(t,\mathbb{S}_1)\| &= \|(\beta_1\mathbb{I}\mathbb{S} + \beta_2\mathbb{A}\mathbb{S})e^{-\mu\tau} - \mu\mathbb{S}\|\\
&\leq e^{-\pi\tau}\beta_1\|\mathbb{I}\|\|\mathbb{S}(t) - \mathbb{S}_1(t)\| + e^{-\pi\tau}\beta_2\|\mathbb{A}\|\|\mathbb{S}(t) - \mathbb{S}_1(t)\| - \mu\|\mathbb{S}(t) - \mathbb{S}_1(t)\|\\
&\leq [e^{-\pi\tau}(\|\beta_1\mathbb{I} + \beta_2\mathbb{A}\|)]\|\mathbb{S}(t) - \mathbb{S}_1(t)\|\\
&\leq [e^{-\pi\tau}(\|\beta_1 a_3 + \beta_2 a_4\|) - \mu]\|\mathbb{S} - \mathbb{S}_1\|.
\end{aligned}$$

Let $\Upsilon_1 = e^{-\pi\tau}(\|\beta_1 a_3 + \beta_2 a_4\| - \mu)$, where $\mathbb{I} \leq a_3$ and $\mathbb{A} \leq a_4$ is a bounded function, thus

$$\|\mathcal{Y}_1(t,\mathbb{S}) - \mathcal{Y}_1(t,\mathbb{S}_1)\| \leq \Upsilon_1\|\mathbb{S} - \mathbb{S}_1\|. \tag{3.6}$$

Hence, for $\mathcal{Y}_1$ the Lipschitz condition is obtained and if $1 > e^{-\pi\tau}(\|\beta_1 a_3 + \beta_2 a_4\|) \geq 0$, then $\mathcal{Y}_1$ is a contraction. Similarly, one can show that $\mathcal{Y}_i,\ \ i = 2, 3, 4, 5$ also fulfills the Lipschitz condition which is given as

$$\begin{cases}
\|\mathcal{Y}_2(t,\mathbb{E}) - \mathcal{Y}_2(t,\mathbb{E}_1)\| \leq \Upsilon_2\|\mathbb{E} - \mathbb{E}_1\|,\\
\|\mathcal{Y}_3(t,\mathbb{I}) - \mathcal{Y}_3(t,\mathbb{I}_1)\| \leq \Upsilon_3\|\mathbb{I} - \mathbb{I}_1\|,\\
\|\mathcal{Y}_4(t,\mathbb{A}) - \mathcal{Y}_4(t,\mathbb{A}_1)\| \leq \Upsilon_4\|\mathbb{A} - \mathbb{A}_1\|,\\
\|\mathcal{Y}_5(t,\mathbb{R}) - \mathcal{Y}_5(t,\mathbb{R}_1)\| \leq \Upsilon_5\|\mathbb{R} - \mathbb{R}_1\|,
\end{cases} \tag{3.7}$$

where $\Upsilon_2 = \omega_1 + \omega_2 + \omega_3 + \mu$, $\Upsilon_3 = \omega_4 + \mu$, $\Upsilon_4 = \omega_5 + \mu$ and $\Upsilon_5 = \mu$. □

According to system (3.5), consider the following recursive form:

$$\Psi_{1n}=\mathbb{S}_n(t)-\mathbb{S}_{n-1}(t)=\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t(\mathcal{Y}_1(\wp,\mathbb{S}_{n-1})-\mathcal{Y}_1(\wp,\mathbb{S}_{n-2}))(t-\wp)^{\hbar-1}d\wp,$$
$$\Psi_{2n}=\mathbb{E}_n(t)-\mathbb{E}_{n-1}(t)=\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t(\mathcal{Y}_2(\wp,\mathbb{E}_{n-1})-\mathcal{Y}_2(\wp,\mathbb{E}_{n-2}))(t-\wp)^{\hbar-1}d\wp,$$
$$\Psi_{3n}=\mathbb{I}_n(t)-\mathbb{I}_{n-1}(t)=\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t(\mathcal{Y}_3(\wp,\mathbb{I}_{n-1})-\mathcal{Y}_3(\wp,\mathbb{I}_{n-2}))(t-\wp)^{\hbar-1}d\wp,$$
$$\Psi_{4n}=\mathbb{A}_n(t)-\mathbb{A}_{n-1}(t)=\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t(\mathcal{Y}_4(\wp,\mathbb{A}_{n-1})-\mathcal{Y}_4(\wp,\mathbb{A}_{n-2}))(t-\wp)^{\hbar-1}d\wp,$$
$$\Psi_{5n}=\mathbb{R}_n(t)-\mathbb{R}_{n-1}(t)=\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t(\mathcal{Y}_5(\wp,\mathbb{R}_{n-1})-\mathcal{Y}_5(\wp,\mathbb{R}_{n-2}))(t-\wp)^{\hbar-1}d\wp,$$

with initial conditions $\mathbb{S}(0)=\mathbb{S}_0$, $\mathbb{E}(0)=\mathbb{E}_0$, $\mathbb{I}(0)=\mathbb{I}_0$, $\mathbb{A}(0)=\mathbb{A}_0$ and $\mathbb{R}(0)=\mathbb{R}_0$. Applying the norm to the first equation of the above system, we obtain

$$\begin{aligned}\|\Psi_{1n}\| &= \|\mathbb{S}_n(t)-\mathbb{S}_{n-1}(t)\| = \|\tfrac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t(\mathcal{Y}_1(\wp,\mathbb{S}_{n-1})-\mathcal{Y}_1(\wp,\mathbb{S}_{n-2}))(t-\wp)^{\hbar-1}d\wp\|,\\ &\le \tfrac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t\|(\mathcal{Y}_1(\wp,\mathbb{S}_{n-1})-\mathcal{Y}_1(\wp,\mathbb{S}_{n-2}))(t-\wp)^{\hbar-1}\|d\wp,\end{aligned}$$

using Lipchitz condition, we have

$$\|\Psi_{1n}\|\le\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_1\int_0^t\|\Psi_{1(n-1)}(\wp)\|d\wp. \tag{3.8}$$

Similarly, we obtained

$$\begin{aligned}\|\Psi_{2n}\|&\le\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_2\int_0^t\|\Psi_{2(n-1)}(\wp)\|d\wp,\\ \|\Psi_{3n}\|&\le\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_3\int_0^t\|\Psi_{3(n-1)}(\wp)\|d\wp,\\ \|\Psi_{4n}\|&\le\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_4\int_0^t\|\Psi_{4(n-1)}(\wp)\|d\wp,\\ \|\Psi_{5n}\|&\le\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_5\int_0^t\|\Psi_{5(n-1)}(\wp)\|d\wp.\end{aligned} \tag{3.9}$$

Hence, we can write

$$\mathbb{S}_n(t)=\sum_{k=1}^n\Psi_{1k}(t),\quad \mathbb{E}_n(t)=\sum_{k=1}^n\Psi_{2k}(t),\quad \mathbb{I}_n(t)=\sum_{k=1}^n\Psi_{3k}(t),\quad \mathbb{A}_n(t)=\sum_{k=1}^n\Psi_{4k}(t),\mathbb{R}_n(t)=\sum_{k=1}^n\Psi_{5k}(t).$$

Next, we will show the existence of solution of the model (2).

Theorem 3.5. *The solution of system (2) is exists for finite time* t_1, *if*

$$\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}t_1\Upsilon_i<1.$$

Proof. Using recursive technique and from Eq (3.8) and Eq (3.9), we deduce that

$$\begin{aligned} ||\Psi_{1n}|| &\le ||\mathbb{S}_n(0)||\Big[\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_1 t\Big]^n, \quad ||\Psi_{2n}|| \le ||\mathbb{E}_n(0)||\Big[\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_2 t\Big]^n, \\ ||\Psi_{3n}|| &\le ||\mathbb{I}_n(0)||\Big[\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_3 t\Big]^n, \quad ||\Psi_{4n}|| \le ||\mathbb{A}_n(0)||\Big[\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_4 t\Big]^n, \\ ||\Psi_{5n}|| &\le ||\mathbb{R}_n(0)||\Big[\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_5 t\Big]^n. \end{aligned} \tag{3.10}$$

Hence, the given system possess at least one solution and also it is continuous. Further, we will prove that the above function construct solution for the system (2), assume that

$$\begin{aligned} \mathbb{S}(t) - \mathbb{S}(0) &= \mathbb{S}_n(t) - \mathbb{J}_{1n}(t), \\ \mathbb{E}(t) - \mathbb{E}(0) &= \mathbb{E}_n(t) - \mathbb{J}_{2n}(t), \\ \mathbb{I}(t) - \mathbb{I}(0) &= \mathbb{I}_n(t) - \mathbb{J}_{3n}(t), \\ \mathbb{A}(t) - \mathbb{A}(0) &= \mathbb{A}_n(t) - \mathbb{J}_{4n}(t), \\ \mathbb{R}(t) - \mathbb{R}(0) &= \mathbb{R}_n(t) - \mathbb{J}_{5n}(t). \end{aligned}$$

Now

$$\begin{aligned} ||\mathbb{J}_{1n}(t)|| &= ||\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (\mathcal{Y}_1(\wp, \mathbb{S}) - \mathcal{Y}_1(\wp, \mathbb{S}_{n-1}))d\wp|| \\ &\le \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t ||(\mathcal{Y}_1(\wp, \mathbb{S}) - \mathcal{Y}_1(\wp, \mathbb{S}_{n-1}))||d\wp \\ &\le \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_1 ||\mathbb{S} - \mathbb{S}_{n-1}||t. \end{aligned}$$

After repeating the method again, we have

$$||\mathbb{J}_{1n}(t)|| \le \Big[\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}t\Big]^{n+1}\Upsilon_1^{n+1}h.$$

At t_1, we obtain

$$||\mathbb{J}_{1n}(t)|| \le \Big[\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}t_1\Big]^{n+1}\Upsilon_1^{n+1}h.$$

Applying limit to the above equation as n goes to ∞, we get $||\mathbb{J}_{1n}(t)|| \to 0$. On the same fashion, we can also show that $||\mathbb{J}_{in}(t)||, i = 2, 3, 4, 5$. Thus, we complete the proof. □

Next we show the uniqueness of the solution of the model (2). Let us suppose that $\mathbb{S}_1(t), \mathbb{E}_1(t), \mathbb{I}_1(t), \mathbb{A}_1(t)$, and $\mathbb{R}_1(t)$ be another solution of the model (2). Consider

$$\mathbb{S}(t) - \mathbb{S}_1(t) = \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (\mathcal{Y}_1(\wp, \mathbb{S}) - \mathcal{Y}_1(\wp, \mathbb{S}_1))d\wp, \tag{3.11}$$

taking norm of the Eq (3.11), we have

$$\|\mathbb{S}(t)-\mathbb{S}_1(t)\| = \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t \|(\mathcal{Y}_1(\wp,\mathbb{S})-\mathcal{Y}_1(\wp,\mathbb{S}_1))\|d\wp.$$

On using Lipschitz condition, we obtain

$$\|\mathbb{S}(t)-\mathbb{S}_1(t)\| \leq \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_1 t\|\mathbb{S}(t)-\mathbb{S}_1(t)\|.$$

Hence,

$$\|\mathbb{S}(t)-\mathbb{S}_1(t)\|\Big(1-\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_1 t\Big)\leq 0. \tag{3.12}$$

Theorem 3.6. *The considered system possess a unique solution if the condition given below is satisfied.*

$$1-\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_1 t > 0.$$

Proof. Let the following condition holds

$$\|\mathbb{S}(t)-\mathbb{S}_1(t)\|\Big(1-\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\Upsilon_1 t\Big)\leq 0.$$

Then, $\|\mathbb{S}(t)-\mathbb{S}_1(t)\| = 0$. Thus, we get $\mathbb{S}(t)=\mathbb{S}_1(t)$. Similarly, we can show the same equality for $\mathbb{E},\mathbb{I},\mathbb{A},\mathbb{R}$. □

Next, we show the Ulam-Hyers (UH) stability for the considered system .

Definition 3.7. *The considered system is said to be UH stable if* $\exists$ *some constants* $\delta_i > 0$, $i\in\mathbb{N}^5$ *and for each* $\mho_i > 0$, $i\in\mathbb{N}^5$, *for*

$$\Big|\mathbb{S}(t)-\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{1-\hbar}\mathcal{Y}_1(\wp,\mathbb{S}(\wp))d\wp\Big| \leq \mho_1,$$
$$\Big|\mathbb{E}(t)-\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{1-\hbar}\mathcal{Y}_2(\wp,\mathbb{E}(\wp))d\wp\Big| \leq \mho_2,$$
$$\Big|\mathbb{I}(t)-\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{1-\hbar}\mathcal{Y}_3(\wp,\mathbb{I}(\wp))d\wp\Big| \leq \mho_3,$$
$$\Big|\mathbb{A}(t)-\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{1-\hbar}\mathcal{Y}_4(\wp,\mathbb{A}(\wp))d\wp\Big| \leq \mho_4,$$
$$\Big|\mathbb{R}(t)-\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)}\int_0^t (t-\wp)^{1-\hbar}\mathcal{Y}_5(\wp,\mathbb{R}(\wp))d\wp\Big| \leq \mho_5,$$

and there exist $\{\dot{\mathbb{S}}, \dot{\mathbb{E}}, \dot{\mathbb{I}}, \dot{\mathbb{A}}, \dot{\mathbb{R}}\}$ *satisfy the following*

$$\begin{aligned}
\dot{\mathbb{S}}(t) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \mathcal{Y}_1(\wp, \dot{\mathbb{S}}(\wp)) d\wp, \\
\dot{\mathbb{E}}(t) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \mathcal{Y}_2(\wp, \dot{\mathbb{E}}(\wp)) d\wp, \\
\dot{\mathbb{I}}(t) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \mathcal{Y}_3(\wp, \dot{\mathbb{I}}(\wp)) d\wp, \\
\dot{\mathbb{A}}(t) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \mathcal{Y}_4(\wp, \dot{\mathbb{A}}(\wp)) d\wp, \\
\dot{\mathbb{R}}(t) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \mathcal{Y}_5(\wp, \dot{\mathbb{R}}(\wp)) d\wp,
\end{aligned} \tag{3.13}$$

such that

$$|\mathbb{S} - \dot{\mathbb{S}}| \le \delta_1 \mho_1, \;\; |\mathbb{E} - \dot{\mathbb{E}}| \le \delta_2 \mho_2, \;\; |\mathbb{I} - \dot{\mathbb{I}}| \le \delta_3 \mho_3, \;\; |\mathbb{A} - \dot{\mathbb{A}}| \le \delta_4 \mho_4, \;\; |\mathbb{R} - \dot{\mathbb{R}}| \le \delta_5 \mho_5.$$

Assumption: Let us assume a Banach space on a real valued function $\mathscr{B}(\mathcal{U})$ and $\mathcal{U} = [0, b]$ and $\mathcal{U} = \mathscr{B}(\mathcal{U}) \times \mathscr{B}(\mathcal{U}) \times \mathscr{B}(\mathcal{U}) \times \mathscr{B}(\mathcal{U}) \times \mathscr{B}(\mathcal{U})$ prescribe a norm $\|\dot{\mathbb{S}}, \dot{\mathbb{E}}, \dot{\mathbb{I}}, \dot{\mathbb{A}}, \dot{\mathbb{R}}\| = \sup_{t \in \mathcal{U}} |\dot{\mathbb{S}}| + \sup_{t \in \mathcal{U}} |\dot{\mathbb{E}}| + \sup_{t \in \mathcal{U}} |\dot{\mathbb{I}}| + \sup_{t \in \mathcal{U}} |\dot{\mathbb{A}}| + \sup_{t \in \mathcal{U}} |\dot{\mathbb{R}}|$.

Theorem 3.8. *The considered system is UH stable with the above assumption.*

Proof. The system has a unique solution, we have

$$\begin{aligned}
\|\mathbb{S} - \dot{\mathbb{S}}\| &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \|\mathcal{Y}_1(\wp, \mathbb{S}(\wp)) - \mathcal{Y}_1(\wp, \dot{\mathbb{S}}(\wp))\| d\wp \\
&\le \psi_1 \|\mathbb{S} - \dot{\mathbb{S}}\|.
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\|\mathbb{E} - \dot{\mathbb{E}}\| &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \|\mathcal{Y}_2(\wp, \mathbb{E}(\wp)) - \mathcal{Y}_2(\wp, \dot{\mathbb{E}}(\wp))\| d\wp \\
&\le \psi_2 \|\mathbb{E} - \dot{\mathbb{E}}\|.
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\|\mathbb{I} - \dot{\mathbb{I}}\| &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \|\mathcal{Y}_3(\wp, \mathbb{I}(\wp)) - \mathcal{Y}_3(\wp, \dot{\mathbb{I}}(\wp))\| d\wp \\
&\le \psi_3 \|\mathbb{I} - \dot{\mathbb{I}}\|.
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\|\mathbb{A} - \dot{\mathbb{A}}\| &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \|\mathcal{Y}_4(\wp, \mathbb{A}(\wp)) - \mathcal{Y}_4(\wp, \dot{\mathbb{A}}(\wp))\| d\wp \\
&\le \psi_4 \|\mathbb{A} - \dot{\mathbb{A}}\|.
\end{aligned} \tag{3.17}$$

$$\|\mathbb{R} - \dot{\mathbb{R}}\| = \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{1-\hbar} \|\mathcal{Y}_5(\wp, \mathbb{R}(\wp)) - \mathcal{Y}_5(\wp, \dot{\mathbb{R}}(\wp))\| d\wp$$

$$\leq \quad \psi_5 ||\mathbb{R} - \dot{\mathbb{R}}||. \tag{3.18}$$

Using $\mho_i = \psi_i$ and $\frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} = \delta_i$, we have

$$||\mathbb{S} - \dot{\mathbb{S}}|| \leq \mho_1 \delta_1. \tag{3.19}$$

Similarly, for the rest of the classes, we have the following

$$\begin{aligned} ||\mathbb{E} - \dot{\mathbb{E}}|| &\leq \mho_2 \delta_2, \\ ||\mathbb{I} - \dot{\mathbb{I}}|| &\leq \mho_3 \delta_3, \\ ||\mathbb{A} - \dot{\mathbb{A}}|| &\leq \mho_3 \delta_4, \\ ||\mathbb{R} - \dot{\mathbb{R}}|| &\leq \mho_5 \delta_5. \end{aligned} \tag{3.20}$$

Thus, we have completed the proof. □

4. Numerical scheme

In this part, we will utilize the Adams-Bashforth approach to develop a general algorithm for solving the considered system. Consider (3.5)

$$\begin{aligned} \mathbb{S}(t) - \mathbb{S}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{\hbar-1} \mathcal{Y}_1(\wp, \mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}) d\wp, \\ \mathbb{E}(t) - \mathbb{E}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{\hbar-1} \mathcal{Y}_2(\wp, \mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}) d\wp, \\ \mathbb{I}(t) - \mathbb{I}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{\hbar-1} \mathcal{Y}_3(\wp, \mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}) d\wp, \\ \mathbb{A}(t) - \mathbb{A}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{\hbar-1} \mathcal{Y}_4(\wp, \mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{E}) d\wp, \\ \mathbb{R}(t) - \mathbb{R}(0) &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{\hbar-1} \mathcal{Y}_5(\wp, \mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}) d\wp. \end{aligned} \tag{4.1}$$

First, we construct the numerical scheme for the first equation of the above system. For this we take

$$\mathbb{S}(t) - \mathbb{S}(0) = \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^t (t-\wp)^{\hbar-1} \mathcal{Y}_1(\wp, \mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}) d\wp. \tag{4.2}$$

Let us denote $t_v = v\Delta, \quad v = 0, 1, 2, \cdots, J$, where $\Delta = \frac{T}{J}$ is the step size and J is a positive integer and $T > 0$. At $t = t_{v+1},\ v = 0, 1, 2, \cdots$, the Eq (4.2) becomes

$$\mathbb{S}(t_{v+1}) - \mathbb{S}(0) = \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^{t_{v+1}} (t_{v+1} - t)^{\hbar-1} \, \mathcal{Y}_1(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}, t) dt, \tag{4.3}$$

and

$$\mathbb{S}(t_v) - \mathbb{S}(0) = \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^{t_v} (t_v - t)^{\hbar-1} \, \mathcal{Y}_1(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}, t) dt. \tag{4.4}$$

By subtracting (4.4) from (4.3), we obtain

$$\begin{aligned}\mathbb{S}(t_{v+1}) &= \mathbb{S}(t_v) + \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^{t_{v+1}} (t_{v+1} - t)^{\hbar-1}\, \mathcal{Y}_1(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}, t)dt \\ &+ \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^{t_v} (t_v - t)^{\hbar-1}\, \mathcal{Y}_1(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}, t)dt.\end{aligned}$$

Let us write the above equation into the following form

$$\mathbb{S}(t_{v+1}) = \mathbb{S}(t_v) + \mathscr{H}_{\hbar,1} + \mathscr{H}_{\hbar,2}, \tag{4.5}$$

where

$$\begin{aligned}\mathscr{H}_{\hbar,1} &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^{t_{v+1}} (t_{v+1} - t)^{\hbar-1}\, \mathcal{Y}_1(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}, t)dt, \\ \mathscr{H}_{\hbar,2} &= \frac{\Lambda^{1-\hbar}}{\Gamma(\hbar)} \int_0^{t_v} (t_v - t)^{\hbar-1}\, \mathcal{Y}_1(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}, t)dt.\end{aligned}$$

On using the Lagrangian polynomial interpolation, one can get the approximation the function $\mathcal{Y}_1(\mathbb{S}, \mathbb{E}, \mathbb{I}, \mathbb{A}, \mathbb{R}, t)$ as

$$\begin{aligned}\mathbf{Z}(t) &= \frac{t - t_{v-1}}{t_v - t_{v-1}} \mathcal{Y}_1(\mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v, \mathbb{A}_v, \mathbb{R}_v, t_v) + \frac{t - t_v}{t_{v-1} - t_v} \mathcal{Y}_1(\mathbb{S}_{v-1}, \mathbb{E}_{v-1}, \mathbb{I}_{v-1}, \mathbb{A}_{v-1}, \mathbb{R}_{v-1}, t_{v-1}) \\ &= \frac{\mathcal{Y}_1(\mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v, \mathbb{A}_v, \mathbb{R}_v, t_v)}{\Delta}(t - t_{v-1}) - \frac{\mathcal{Y}_1(\mathbb{S}_{v-1}, \mathbb{E}_{v-1}, \mathbb{I}_{v-1}, \mathbb{A}_{v-1}, \mathbb{R}_{v-1}, t_{v-1})}{\Delta}(t - t_v),\end{aligned} \tag{4.6}$$

thus,

$$\begin{aligned}\mathscr{H}_{\hbar,1} &= \frac{\mathcal{Y}_1(\mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v, \mathbb{A}_v, \mathbb{R}_v, t_v)}{\Delta\Gamma(\hbar)} \int_0^{t_{v+1}} (t_{v+1} - t)^{\hbar-1} (t - t_{v-1})\, dt \\ &- \frac{\mathcal{Y}_1(\mathbb{S}_{v-1}, \mathbb{E}_{v-1}, \mathbb{I}_{v-1}, \mathbb{A}_{v-1}, \mathbb{R}_{v-1}, t_{v-1})}{\Delta\Gamma(\hbar)} \int_0^{t_{v+1}} (t_{v+1} - t) (t - t_v)\, dt.\end{aligned}$$

Simplifying the integral of the above equation, one can obtain

$$\begin{aligned}\mathscr{H}_{\hbar,1} &= \frac{\mathcal{Y}_1(\mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v, \mathbb{A}_v, \mathbb{R}_v, t_v)}{\Delta\Gamma(\hbar)} \left\{ \frac{2\Delta}{\hbar} t_{v+1}^{\hbar} - \frac{t_{v+1}^{\hbar+1}}{\hbar + 1} \right\} \\ &- \frac{\mathcal{Y}_1(\mathbb{S}_{v-1}, \mathbb{E}_{v-1}, \mathbb{I}_{v-1}, \mathbb{A}_{v-1}, \mathbb{R}_{v-1}, t_{v-1})}{\Delta\Gamma(\hbar)} \left\{ \frac{\Delta}{\hbar} t_{v+1}^{\hbar} - \frac{t_{v+1}^{\hbar+1}}{\hbar + 1} \right\}.\end{aligned}$$

Similarly

$$\begin{aligned}\mathscr{H}_{\hbar,2} &= \frac{\mathcal{Y}_1(\mathbb{S}_v, \mathbb{E}_v, \mathbb{I}_v, \mathbb{A}_v, \mathbb{R}_v, t_v)}{\Delta\Gamma(\hbar)} \left\{ \frac{\Delta}{\hbar} t_v^{\hbar} - \frac{t_v^{\hbar+1}}{\hbar + 1} \right\} \\ &+ \frac{\mathcal{Y}_1(\mathbb{S}_{v-1}, \mathbb{E}_{v-1}, \mathbb{I}_{v-1}, \mathbb{A}_{v-1}, \mathbb{R}_{v-1}, t_{v-1})}{\Delta\Gamma(\hbar + 1)} t_v^{\hbar+1}.\end{aligned}$$

Putting the values of $\mathscr{H}_{\hbar,1}$ and $\mathscr{H}_{\hbar,2}$ in Eq (4.5), we obtain the following approximate solution for the first equation as

$$\begin{cases}\mathbb{S}(t_{v+1}) & = \mathbb{S}(t_v) + \Lambda^{1-\hbar}\Bigg[\frac{\mathcal{Y}_1(\mathbb{S}_v,\mathbb{E}_v,\mathbb{I}_v,\mathbb{A}_v,\mathbb{R}_v,t_v)}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\left\{\frac{2(v+1)^{\hbar}+v^{\hbar}}{\hbar} - \frac{(v+1)^{\hbar+1}+v^{\hbar+1}}{\hbar+1}\right\} \\ & +\frac{\mathcal{Y}_1(\mathbb{S}_{v-1},\mathbb{E}_{v-1},\mathbb{I}_{v-1},\mathbb{A}_{v-1},\mathbb{R}_{v-1},t_{v-1})}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\times\left\{\frac{(v+1)^{\hbar+1}}{\hbar} - \frac{(v+1)^{\hbar+1}}{\hbar+1} + \frac{v^{\hbar}}{\hbar+1}\right\}\Bigg].\end{cases} \tag{4.7}$$

Using the above procedure, one can get the numerical scheme for other compartments as

$$\begin{cases}\mathbb{E}(t_{v+1}) & = \mathbb{E}(t_v) + \Lambda^{1-\hbar}\Bigg[\frac{\mathcal{Y}_2(\mathbb{S}_v,\mathbb{E}_v,\mathbb{I}_v,\mathbb{A}_v,\mathbb{R}_v,t_v)}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\left\{\frac{2(v+1)^{\hbar}+v^{\hbar}}{\hbar} - \frac{(v+1)^{\hbar+1}+v^{\hbar+1}}{\hbar+1}\right\} \\ & +\frac{\mathcal{Y}_2(\mathbb{S}_{v-1},\mathbb{E}_{v-1},\mathbb{I}_{v-1},\mathbb{A}_{v-1},\mathbb{R}_{v-1},t_{v-1})}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\times\left\{\frac{(v+1)^{\hbar+1}}{\hbar} - \frac{(v+1)^{\hbar+1}}{\hbar+1} + \frac{v^{\hbar}}{\hbar+1}\right\}\Bigg].\end{cases} \tag{4.8}$$

$$\begin{cases}\mathbb{I}(t_{v+1}) & = \mathbb{I}(t_v) + \Lambda^{1-\hbar}\Bigg[\frac{\mathcal{Y}_3(\mathbb{S}_v,\mathbb{E}_v,\mathbb{I}_v,\mathbb{A}_v,\mathbb{R}_v,t_v)}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\left\{\frac{2(v+1)^{\hbar}+v^{\hbar}}{\hbar} - \frac{(v+1)^{\hbar+1}+v^{\hbar+1}}{\hbar+1}\right\} \\ & +\frac{\mathcal{Y}_3(\mathbb{S}_{v-1},\mathbb{E}_{v-1},\mathbb{I}_{v-1},\mathbb{A}_{v-1},\mathbb{R}_{v-1},t_{v-1})}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\times\left\{\frac{(v+1)^{\hbar+1}}{\hbar} - \frac{(v+1)^{\hbar+1}}{\hbar+1} + \frac{v^{\hbar}}{\hbar+1}\right\}\Bigg].\end{cases} \tag{4.9}$$

$$\begin{cases}\mathbb{A}(t_{v+1}) & = \mathbb{A}(t_v) + \Lambda^{1-\hbar}\Bigg[\frac{\mathcal{Y}_4(\mathbb{S}_v,\mathbb{E}_v,\mathbb{I}_v,\mathbb{A}_v,\mathbb{R}_v,t_v)}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\left\{\frac{2(v+1)^{\hbar}+v^{\hbar}}{\hbar} - \frac{(v+1)^{\hbar+1}+v^{\hbar+1}}{\hbar+1}\right\} \\ & +\frac{\mathcal{Y}_4(\mathbb{S}_{v-1},\mathbb{E}_{v-1},\mathbb{I}_{v-1},\mathbb{A}_{v-1},\mathbb{R}_{v-1},t_{v-1})}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\times\left\{\frac{(v+1)^{\hbar+1}}{\hbar} - \frac{(v+1)^{\hbar+1}}{\hbar+1} + \frac{v^{\hbar}}{\hbar+1}\right\}\Bigg].\end{cases} \tag{4.10}$$

$$\begin{cases}\mathbb{R}(t_{v+1}) & = \mathbb{R}(t_v) + \Lambda^{1-\hbar}\Bigg[\frac{\mathcal{Y}_5(\mathbb{S}_v,\mathbb{E}_v,\mathbb{I}_v,\mathbb{A}_v,\mathbb{R}_v,t_v)}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\left\{\frac{2(v+1)^{\hbar}+v^{\hbar}}{\hbar} - \frac{(v+1)^{\hbar+1}+v^{\hbar+1}}{\hbar+1}\right\} \\ & +\frac{\mathcal{Y}_5(\mathbb{S}_{v-1},\mathbb{E}_{v-1},\mathbb{I}_{v-1},\mathbb{A}_{v-1},\mathbb{R}_{v-1},t_{v-1})}{\Delta\Gamma(\hbar)}\Delta^{\hbar+1}\times\left\{\frac{(v+1)^{\hbar+1}}{\hbar} - \frac{(v+1)^{\hbar+1}}{\hbar+1} + \frac{v^{\hbar}}{\hbar+1}\right\}\Bigg].\end{cases} \tag{4.11}$$

5. Numerical simulation and discussion

Here we visualize the effect of fractional order and delay term on the transmission of coronavirus disease. In the given part, we provide the graphical representation by numerical simulation for verification of our considered fractional numerical scheme. For this, we chose some initial values for all the three compartments of our said problem having fractional order $\hbar$. The initial conditions used for numerical simulations are: $\mathbb{S}(0) = 0.5$, $\mathbb{E}(0) = 0.2$, $\mathbb{I}(0) = 0.1$, $\mathbb{A}(0) = 0.1$, $\mathbb{R}(0) = 0.1$. We have taken the parameter values as: $\Lambda = 0.99$, $\omega_4 = 0.0987$, $\beta_1 = 1.05$, $\mu = 0.5$, $\beta_2 = 1.05$, $\omega_1 = 0.4787$, $\omega_3 = 0.0854$, $\pi_S = 0.5$, $\omega_5 = 0.1234$, $\omega_2 = 1.0004$. For 2019-nCov, we displayed each compartment of the delayed model without the time delay factor in Figure 1 for various values of $\hbar$. From Figure 1 (a)–(e), it is observed the susceptible population increases with the passage of time. Then exposed population must be decreases because all population remains in the susceptible class and will not expose to the infection. However, there is a little increase in the symptomatic and Asymptomatic population in a very short interval of time, then it goes down decreases and become stable. Also, recovered population decreases with passage of time. The impact of fractional order on the dynamical behaviour of proposed model is clearly observed in the Figure 1. All the compartment fatly attain their maximum or minimum at lower fractional order. At $\hbar = 1$, the black color curves represent the dynamics of the integer model (1). Thus, fractional order model provide fast history about all compartments of the proposed model. Thus, our model (2) generalizes the model (1).

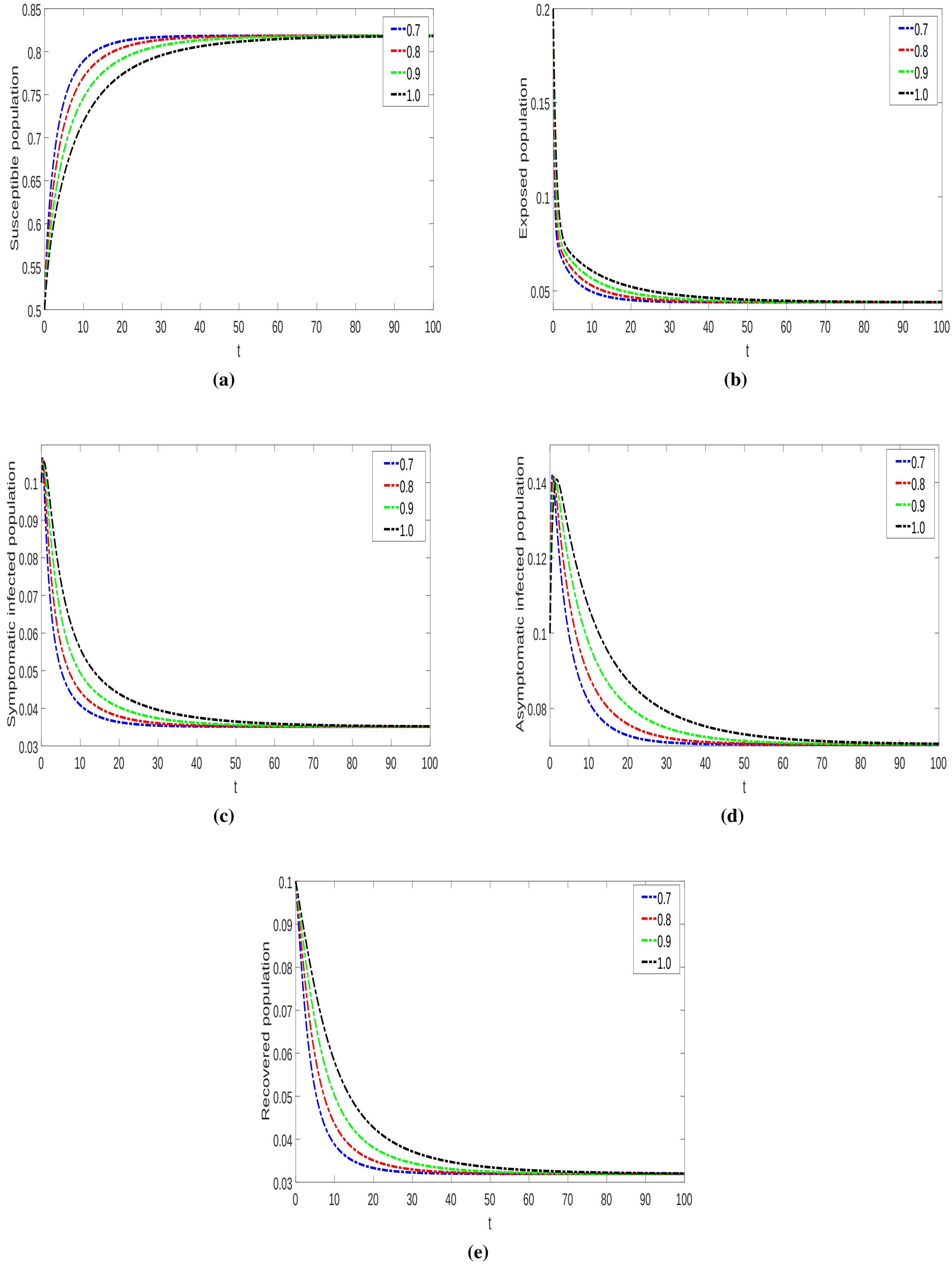

Figure 1. Graphical illustration of the proposed model for different fractional orders with out delay.

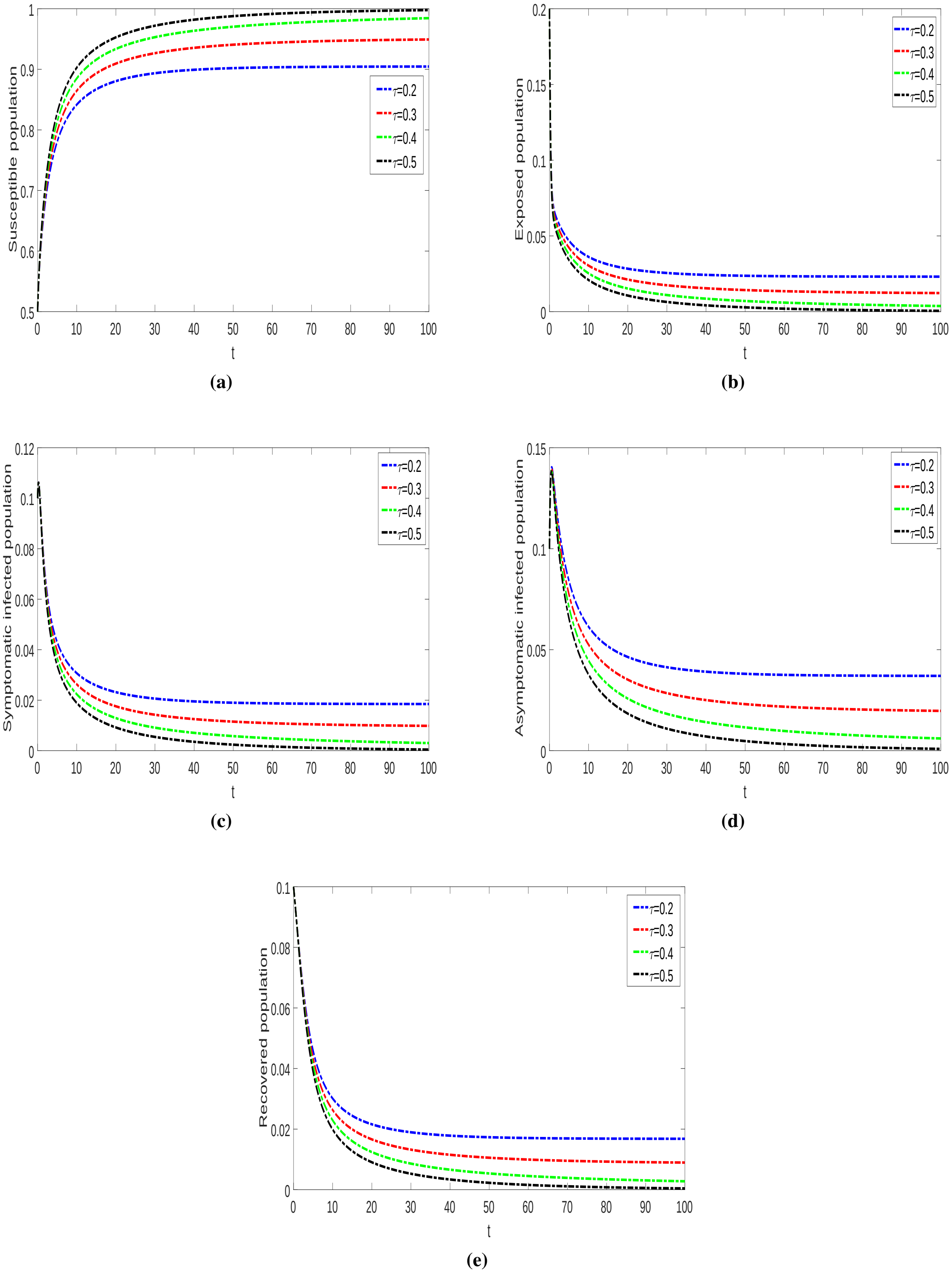

Figure 2. Graphical illustration of the proposed model for different delay term τ and fixed fractional order 0.7

In Figure 2 we have simulated each compartment of the proposed model with fractional order $\hbar = 0.7$, and at different values of the delay term τ. When the delay time increases, the populace of the susceptible class grows while the populace of the exposed class decreases. Increased delaying strategies leads to a reduction in the infected population without a shift in the rate of transmission, according to our results. Despite the fact that the increase in delaying techniques may be seen as an exponential decrease in the number of symptomatic infected people. When $\tau = 0.5$, the number of symptomatic infected people drops to zero, as seen in Figure 2(c). According to the graphs, we may overcome the pandemic of 2019-nCov by using delaying strategies such as social distance, quarantine, travel limitations, holiday extensions, hospitalization, and isolation for roughly 100 days.

6. Conclusions

Mathematical analysis of epidemic illnesses with a time delay effect is a useful tool for studying disease behavior. We've noticed important techniques for overcoming 2019-nCov's present crisis, as well as delaying strategies and delayed elements, all over the world. Reduce the 2019-nCov quickly by making the most of delaying techniques. Quarantine, isolation, social distance, and immigration limitations are the most effective strategies for delaying variables. However, based on the information provided, we may employ delaying techniques for a total of 100 days to achieve the desired results. An integer order model of the delay coronavirus was studied in literature [8]. Due to superiority and advantages over local operators, fractional operators have seen tremendous growth over time. In the literature of fractional calculus, there exist several nonlocal operators. We utilized a nonlocal operator with a power-law kernel in this study. This research examined an essential model of coronavirus with a time delay in the human population. We have developed the existence and uniqueness theory using fixed point theory, which assures that the model has at least one and unique solution. The Ulam-Hyres stability of the model was discussed using the idea of functional analysis. A reliable numerical method (Adams-Basforth technique) was used to find a numerical solution to the problem under consideration. The numerical simulations were used to show the theoretical conclusions of the model and to investigate the influence of fractional order on the dynamics of the model. Numerical simulations were used to investigate the influence of time delay on coronavirus transmission decrease. It is noticed that the delay model of coronavirus produces good results in the reduction of the diseases. People may adopt the delay strategies like social distancing, quarantine, strengthening the immune system to save themselves from getting the infection. Different nonlocal operators can be used to study the present work in the future.

Conflict of interest

There exist no conflict of interest regarding to this research work.

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