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Research article

Hermite-Hadamard-Fejér type fractional inequalities relating to a convex harmonic function and a positive symmetric increasing function

Muhammad Amer Latif[1], Humaira Kalsoom[2,*] and Zareen A. Khan[3,*]

[1] Department of Basic Sciences, Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa, Saudi Arabia

[2] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

[3] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P. O. Box 84428, Riyadh 11671, Saudi Arabia

* **Correspondence:** Email: humaira87@zju.edu.cn; zakhan@pnu.edu.sa.

Abstract: The purpose of this article is to discuss some midpoint type HHF fractional integral inequalities and related results for a class of fractional operators (weighted fractional operators) that refer to harmonic convex functions with respect to an increasing function that contains a positive weighted symmetric function with respect to the harmonic mean of the endpoints of the interval. It can be concluded from all derived inequalities that our study generalizes a large number of well-known inequalities involving both classical and Riemann-Liouville fractional integral inequalities.

Keywords: harmonically symmetric function; weighted fractional operators; harmonic convex functions; Hermite-Hadamard-Fejér inequality
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1. Introduction

We recall that an interval $\chi \subset \Re$ is convex if for all $x, y \in \chi$, we have $tx + (1-t)y \in \chi$, where $t \in [0,1]$ and a function $f : \chi \to \Re$ is convex if for all $x, y \in \chi$, the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds. A function $f : \chi \to \Re$ is concave if the inequality (1.1) holds in opposite direction.

Convexity is essential to understanding and solving problems pertaining to fractional integral inequalities because of its properties and definition, and it has recently gained in importance. Convex functions have yielded several new integral inequalities, as evidenced by [2, 3, 8, 11, 14, 15, 21, 24, 43, 45–48]. Hermite-integral Hadamard's inequalities are most commonly encountered when searching for comprehensive inequalities:

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v f(x)\,dx \leq \frac{f(u)+f(v)}{2}, \tag{1.2}$$

where the function $f : \chi \to \Re$ is convex on χ and $f \in L^1([u,v])$.

There are the following two classical fractional integral inequalities which are defined by Hermite-Hadamard type inequalities:

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(v+u)^{\nu}}\left[J_{u^+}^{\nu} f(v) + J_{v^-}^{\nu} f(u)\right] \leq \frac{f(u)+f(v)}{2}, \tag{1.3}$$

where the function $f : \chi \to \Re$ is positive, convex on χ and $f \in L^1([u,v])$.

The left-sided and right-sided Riemann-Liouville fractional integrals $J_{u^+}^{\nu} f(v)$ and $J_{v^-}^{\nu} f(u)$ of order $\nu > 0$ in (1.3), are defined respectively as (see [5,22]):

$$J_{u^+}^{\nu} f(x) := \frac{1}{\Gamma(\nu)}\int_u^x (x-t)^{\nu-1} f(t)\,dt, 0 \leq u < x < v$$

and

$$J_{v^-}^{\nu} f(x) := \frac{1}{\Gamma(\nu)}\int_x^v (t-x)^{\nu-1} f(t)\,dt, 0 \leq u < x < v.$$

The extended inequalities for (1.2) and (1.5) are fractional integral inequalities of the Fejér and Hermite-Hadamard-Fejér types, and the results are as follows:

$$f\left(\frac{u+v}{2}\right)\int_u^v \zeta(x)\,dx \leq \int_u^v f(x)\zeta(x)\,dx \leq \frac{f(u)+f(v)}{2}\int_u^v \zeta(x)\,dx \tag{1.4}$$

and

$$\begin{aligned} f\left(\frac{u+v}{2}\right)\left[J_{u+}^{\nu}\zeta(v) + J_{v-}^{\nu}\zeta(u)\right] &\leq \frac{\Gamma(\nu+1)}{2(v+u)^{\nu}}\left[J_{u+}^{\nu}(f\zeta)(v) + J_{v-}^{\nu}(f\zeta)(u)\right] \\ &\leq \frac{f(u)+f(v)}{2}\left[J_{u+}^{\nu}\zeta(v) + J_{v-}^{\nu}\zeta(u)\right] \end{aligned} \tag{1.5}$$

respectively, where f is as defined above, the function $\zeta : [u,v] \to [0,\infty)$ is integrable symmetric with respect to $\frac{u+v}{2}$, that is

$$\zeta(u+v-x) = \zeta(x) \text{ for all } x \in [u,v]$$

and $\Gamma(\nu)$ is the Gamma function defined as $\Gamma(\nu) = \int_0^{\infty} x^{\nu-1}e^{-x}dx$, $\mathrm{Re}(\nu) > 0$.

In addition to being able to be generalized, convexity and convex functions have several generalizations. One of those generalizations is the concept of harmonic convexity and harmonic convex functions, which can be defined as follows:

Definition 1.1. Define $\chi \subseteq \Re\backslash\{0\}$ as an interval of real numbers. A function f from χ to the real numbers is considered to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.6}$$

for all $x, y \in \chi$ and $t \in [0,1]$. Harmonically concave f is defined as the inequality in (1.6) reversed.

Using harmonic-convexity, the Hermite-Hadamard type yields the following result.

Theorem 1.2. *Let $f : \chi \subseteq \Re \backslash \{0\} \to \Re$ be a harmonically convex function and $u, v \in \chi$ with $u < v$. If $f \in L([u, v])$ then the following inequalities hold:*

$$f\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} \int_v^u \frac{f(x)}{x^2} dx \leq \frac{f(u)+f(v)}{2}. \tag{1.7}$$

Harmonic symmetricity of a function is given in the definition below.

Definition 1.3. A function $\zeta : [u, v] \subseteq \Re \backslash \{0\} \to \Re$ is harmonically symmetric with respect to $\frac{2uv}{u+v}$ if

$$\zeta(x) = \zeta\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)$$

holds for all $x \in [u, v]$.

Fejér type inequalities using harmonic convexity and the notion of harmonic symmetricity were presented in Chan and Wu [7].

Theorem 1.4. *Let $f : \chi \subseteq \Re \backslash \{0\} \to \Re$ be a harmonically convex function and $u, v \in \chi$ with $u < v$. If $f \in L([u, v])$ and $\zeta : [u, v] \subseteq \Re \backslash \{0\} \to \Re$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2uv}{u+v}$, then*

$$f\left(\frac{2uv}{u+v}\right) \int_v^u \zeta(x)\, dx \leq \frac{uv}{v-u} \int_v^u \frac{f(x)\zeta(x)}{x^2} dx \leq \frac{f(u)+f(v)}{2} \int_v^u \zeta(x)\, dx. \tag{1.8}$$

Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral form were proved in [19].

Theorem 1.5. *Let $f : \chi \subseteq (0, \infty) \to \Re$ be a function such that $f \in L([u, v])$, where $u, v \in \chi$ with $u < v$. If f is a harmonically convex function on $[u, v]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2uv}{u+v}\right) \leq \frac{\Gamma(\nu+1)}{2}\left(\frac{uv}{v-u}\right)^{\nu} \left\{ J^{\nu}_{\frac{1}{u}-}(f \circ g)\left(\frac{1}{v}\right) + J^{\nu}_{\frac{1}{v}+}(f \circ g)\left(\frac{1}{u}\right) \right\} \leq \frac{f(u)+f(v)}{2} \tag{1.9}$$

with $\nu > 0$ and $g(x) = \frac{1}{x}$.

Hermite-Hadamard-Fejér inequality for harmonically convex function in fractional integral form were obtained by İşcan et al. in [18].

Theorem 1.6. *Let $f : [u, v] \to \Re$ be a harmonically convex function with $u < v$ and $f \in L([u, v])$. If $\zeta : [u, v] \to \Re$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2uv}{u+v}$, then the following inequalities for fractional integrals holds:*

$$\begin{aligned} f\left(\frac{2uv}{u+v}\right) \left\{ J^{\nu}_{\frac{1}{u}-}(\zeta \circ g)\left(\frac{1}{v}\right) + J^{\nu}_{\frac{1}{v}+}(f \circ g)\left(\frac{1}{u}\right) \right\} &\leq J^{\nu}_{\frac{1}{u}-}(f\zeta \circ g)\left(\frac{1}{v}\right) + J^{\nu}_{\frac{1}{v}+}(f\zeta \circ g)\left(\frac{1}{u}\right) \\ &\leq \frac{f(u)+f(v)}{2} \left\{ J^{\nu}_{\frac{1}{u}-}(\zeta \circ g)\left(\frac{1}{v}\right) + J^{\nu}_{\frac{1}{v}+}(f \circ g)\left(\frac{1}{u}\right) \right\} \end{aligned} \tag{1.10}$$

with $\nu > 0$ and $g(x) = \frac{1}{x}$, $x \in \left[\frac{1}{v}, \frac{1}{u}\right]$.

Left-sided and right-sided Riemann-Liouville fractional integrals are generalized in the definition given below:

Definition 1.7. [20] Let $\tau(x)$ be an increasing positive and monotonic function on the interval $(u, v]$ with a continuous derivative $\tau'(x)$ on the interval (u, v) with $\tau(x) = 0$, $0 \in [u, v]$. Then, the left-side and right-side of the weighted fractional integrals of a function f with respect to another function $\tau(x)$ on $[u, v]$ of order $\nu > 0$ are defined by:

$$\left({}_{\zeta}J_{u+}^{\nu;\tau}f\right)(x) = \frac{\zeta^{-1}(x)}{\Gamma(\nu)} \int_u^x \tau'(t)(\tau(x) - \tau(t))^{\nu-1} f(t)\zeta(t)\,dt \tag{1.11}$$

and

$$\left({}_{\zeta}J_{v-}^{\nu;\tau}f\right)(x) = \frac{\zeta^{-1}(x)}{\Gamma(\nu)} \int_x^v \tau'(t)(\tau(t) - \tau(x))^{\nu-1} f(t)\zeta(t)\,dt, \tag{1.12}$$

where $\zeta^{-1}(x) = \frac{1}{\zeta(x)}, \zeta(x) \neq 0$.

The following observations are obvious from the above definition:

- If $\tau(x) = x$ and $\zeta(x) = 1$, then the weighted fractional integral operators in the Definition 1.7 reduce to the classical Riemann-Liouville fractional integral operators.
- If $\zeta(x) = 1$, we get the fractional integral operators of a function f with respect to another function $\tau(x)$ of order $\nu > 0$, defined in [1, 38] as follows:

$$(J_{u+}^{\nu;\tau}f)(x) = \frac{1}{\Gamma(\nu)} \int_u^x \tau'(t)(\tau(x) - \tau(t))^{\nu-1} f(t)\,dt$$

and

$$(J_{v-}^{\nu;\tau}f)(x) = \frac{1}{\Gamma(\nu)} \int_x^v \tau'(t)(\tau(t) - \tau(x))^{\nu-1} f(t)\,dt.$$

The study analyzes several inequalities of the Hermite-Hadamard-Fejér type through weighted fractional operators with positive symmetric weight function in the kernel.

2. Main results

Throughout this paper, $\Re$ denotes the set of all real numbers, $\chi \subset \Re$ denotes an interval. In addition. for $u, v \in J^{\circ}$ with $u < v$, the functions $\theta_{u,v}, \theta_{v,u} : [0, 1] \to \Re$ are defined as:

$$\theta_{u,v}(t) = \frac{uv}{tu + (1-t)v}, \theta_{v,u}(t) = \frac{uv}{tv + (1-t)u}.$$

We start this section with the following Lemma which will be used repeatedly in the sequel.

Lemma 2.1. *The following results hold:*

(i) If $\zeta : [u, v] \subset (0, \infty) \to [0, \infty)$ be an integrable function and symmetric with respect to $\frac{2uv}{u+v}$, then we have

$$\zeta(\theta_{u,v}(t)) = \zeta(\theta_{v,u}(t)) \tag{2.1}$$

for each $t \in [0, 1]$,

(ii) If $\zeta : [u,v] \subset (0,\infty) \to [0,\infty)$ be an integrable and symmetric function with respect to $\frac{2uv}{u+v}$, then we have for $\nu > 0$

$$J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right) = J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)$$
$$= \frac{1}{2}\left[J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right) + J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right]. \quad (2.2)$$

where $h(x) = \frac{1}{x}$, $x \in \left[\frac{1}{v}, \frac{1}{u}\right]$.

Proof. (i) Let $x = \theta_{u,v}(t) = \frac{uv}{tu+(1-t)v}$. It is clear that $x \in [u,v]$ for each $t \in [0,1]$ and then

$$\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}} = \frac{uv}{tv+(1-t)u} = \theta_{v,u}(t).$$

Hence by the definition of harmonic symmetry, we obtain

$$\zeta(\theta_{u,v}(t)) = \zeta(x) = \zeta\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right) = \zeta(\theta_{v,u}(t)).$$

(ii) By using the harmonic symmetric property of ζ, we have

$$(\zeta\circ\tau)(t) = \zeta(\tau(t)) = \zeta\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{\tau(t)}}\right),$$

for all $t \in \left[\tau^{-1}\left(\frac{1}{v}\right), \tau^{-1}\left(\frac{1}{u}\right)\right]$.
From this and by setting $\frac{1}{\tau(x)} = \frac{1}{\frac{1}{u}+\frac{1}{v}-\tau(t)}$, it follows that

$$\begin{aligned}
&J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\\
&= \frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\tau(x)-\frac{1}{v}\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,\tau'(x)\,dx\\
&= \frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\tau(x)-\frac{1}{v}\right)^{\nu-1}(\zeta\circ h\circ\tau)\left(\frac{1}{u}+\frac{1}{v}-t\right)\tau'(t)\,dt\\
&= \frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\frac{1}{u}-\tau(t)\right)^{\nu-1}(\zeta\circ h\circ\tau)(t)\,\tau'(t)\,dt\\
&= J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)-}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right).
\end{aligned}$$

□

Theorem 2.2. *Let* $f : I \subseteq (0,\infty) \to \Re$ *be an* L^1 *convex function with* $0 < u < v$, $u,v \in I$ *and* $\zeta : [u,v] \to \Re$ *be an integrable, positive and weighted symmetric function with respect to* $\frac{2uv}{u+v}$. *If* τ *is an increasing and positive function on* $[u,v)$ *and* $\tau(x)$ *is continuous on* (u,v), *then, we have for* $\nu > 0$:

$$\begin{aligned}
f\left(\frac{2uv}{u+v}\right)&\left[\left(J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right.\\
&\left.\quad+\left(J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right]\\
&\le \zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right)\\
&\quad+\zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right)\\
&\le \frac{f(u)+f(v)}{2}\left[J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right.\\
&\left.\quad+J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right].
\end{aligned}\tag{2.3}$$

Proof. Since f is a harmonic convex function on $[u,v]$, we have

$$f\left(\frac{2xy}{x+y}\right)\le\frac{f(x)+f(y)}{2},\quad \forall x,y\in[u,v].$$

Choosing $x=\zeta(\theta_{u,v}(t))$ and $y=\zeta(\theta_{v,u}(t))$, we obtain

$$2f\left(\frac{2uv}{u+v}\right)\le f(\zeta(\theta_{u,v}(t)))+f(\zeta(\theta_{v,u}(t))).$$

Multiplying both the sides by $t^{\nu-1}\zeta(\theta_{u,v}(t))$ and then integrating the resultant with respect to "t" over $[0,1]$, we obtain

$$\begin{aligned}
2f\left(\frac{2uv}{u+v}\right)\int_0^1 t^{\nu-1}\zeta(\theta_{u,v}(t))\,dt &\le \int_0^1 t^{\nu-1}f(\zeta(\theta_{u,v}(t)))\,\zeta(\theta_{u,v}(t))\,dt\\
&\quad+\int_0^1 t^{\nu-1}f(\zeta(\theta_{v,u}(t)))\,\zeta(\theta_{u,v}(t))\,dt.
\end{aligned}\tag{2.4}$$

To prove the first inequality in (2.3), we need to use (2.2)

$$\begin{aligned}
&\frac{\Gamma(\nu)}{2\left(\frac{1}{u}-\frac{1}{v}\right)^{\nu}}\left[J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)+J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right]\\
&=\Gamma(\nu)\left(\frac{uv}{v-u}\right)^{\nu}\left(J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right)\\
&=\left(\frac{uv}{v-u}\right)^{\nu}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,\tau'(x)\,dx
\end{aligned}$$

$$= \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left(\frac{\frac{1}{u} - \tau(x)}{\frac{1}{u} - \frac{1}{v}}\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\, \tau'(x) \frac{dx}{\frac{1}{u} - \frac{1}{v}}$$

$$= \int_0^1 t^{\nu-1} \zeta(\theta_{u,v}(t))\, dt, \quad (2.5)$$

where $t = \frac{\frac{1}{u} - \tau(x)}{\frac{1}{u} - \frac{1}{v}}$.

By evaluating the weighted fractional operator, one can observe that

$$\zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f \circ h \circ \tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right)$$

$$+\zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f \circ h \circ \tau)\right)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)$$

$$= \zeta\left(\frac{1}{u}\right) \frac{(\zeta \circ \tau)^{-1}\left(\tau^{-1}\left(\frac{1}{u}\right)\right)}{\Gamma(\nu)}$$

$$\times \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left(\frac{1}{u} - \tau(x)\right)^{\nu-1} (f \circ h \circ \tau)(x)(\zeta \circ h \circ \tau)(x)\, \tau'(x)\, dx$$

$$+\zeta\left(\frac{1}{v}\right) \frac{(\zeta \circ \tau)^{-1}\left(\tau^{-1}\left(\frac{1}{v}\right)\right)}{\Gamma(\nu)}$$

$$\times \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left(\tau(x) - \frac{1}{v}\right)^{\nu-1} (f \circ h \circ \tau)(x)(\zeta \circ h \circ \tau)(x)\, \tau'(x)\, dx$$

$$= \frac{1}{\Gamma(\nu)} \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left(\frac{1}{u} - \tau(x)\right)^{\nu-1} (f \circ h \circ \tau)(x)(\zeta \circ h \circ \tau)\, \tau'(x)\, dx$$

$$+\frac{1}{\Gamma(\nu)} \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left(\tau(x) - \frac{1}{v}\right)^{\nu-1} (f \circ h \circ \tau)(x)(\zeta \circ h \circ \tau)\, \tau'(x)\, dx$$

$$= \frac{\left(\frac{1}{u} - \frac{1}{v}\right)^{\nu}}{\Gamma(\nu)} \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left(\frac{\frac{1}{u} - \tau(x)}{\frac{1}{u} - \frac{1}{v}}\right)^{\nu-1}$$

$$\times (f \circ h \circ \tau)(x)(\zeta \circ h \circ \tau)\, \tau'(x) \frac{dx}{\frac{1}{u} - \frac{1}{v}} + \frac{\left(\frac{1}{u} - \frac{1}{v}\right)^{\nu}}{\Gamma(\nu)}$$

$$\times \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left(\frac{\tau(x) - \frac{1}{v}}{\frac{1}{u} - \frac{1}{v}}\right)^{\nu-1} (f \circ h \circ \tau)(x)(\zeta \circ h \circ \tau)\, \tau'(x) \frac{dx}{\frac{1}{u} - \frac{1}{v}}.$$

Setting $t_1 = \frac{\frac{1}{u} - \tau(x)}{\frac{1}{u} - \frac{1}{v}}$and $t_2 = \frac{\tau(x) - \frac{1}{v}}{\frac{1}{u} - \frac{1}{v}}$ and using (2.1)

$$\zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f \circ h \circ \tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right) + \zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f \circ h \circ \tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right)$$

$$= \frac{\left(\frac{1}{u} - \frac{1}{v}\right)^{\nu}}{\Gamma(\nu)} \int_0^1 t_1^{\nu-1} f(\zeta(\theta_{u,v}(t_1)))\, \zeta(\theta_{u,v}(t_1))\, dt_1$$

$$+\frac{\left(\frac{1}{u}-\frac{1}{v}\right)^{\nu}}{\Gamma(\nu)}\int_0^1 t_2^{\nu-1} f\left(\zeta\left(\theta^*_{v,u}(t_2)\right)\right)\zeta\left(\theta^*_{v,u}(t_2)\right)dt_2$$

$$=\frac{\left(\frac{1}{u}-\frac{1}{v}\right)^{\nu}}{\Gamma(\nu)}\int_0^1 t^{\nu-1} f\left(\zeta\left(\theta_{u,v}(t)\right)\right)\zeta\left(\theta_{u,v}(t)\right)dt$$
$$+\frac{\left(\frac{1}{u}-\frac{1}{v}\right)^{\nu}}{\Gamma(\nu)}\int_0^1 t^{\nu-1} f\left(\zeta\left(\theta^*_{v,u}(t)\right)\right)\zeta\left(\theta_{u,v}(t)\right)dt. \quad (2.6)$$

Using (2.5) and (2.6) in (2.4), we obtain

$$f\left(\frac{2uv}{u+v}\right)\left[J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)+J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right]$$
$$\leq\zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)$$
$$+\zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{v}\right)\right). \quad (2.7)$$

Thus the first inequality is proved.
To prove the second inequality we use the convexity of f

$$f\left(\zeta\left(\theta_{u,v}(t)\right)\right)+f\left(\zeta\left(\theta^*_{v,u}(t)\right)\right)\leq f(u)+f(v). \quad (2.8)$$

Multiplying both the sides by $t^{\nu-1}\zeta\left(\theta_{u,v}(t)\right)$ and then integrating the resultant with respect to "t" over $[0,1]$, we obtain

$$\int_0^1 t^{\nu-1} f\left(\zeta\left(\theta_{u,v}(t)\right)\right)\zeta\left(\theta_{u,v}(t)\right)dt+\int_0^1 t^{\nu-1} f\left(\zeta\left(\theta^*_{v,u}(t)\right)\right)\zeta\left(\theta_{u,v}(t)\right)dt$$
$$\leq[f(u)+f(v)]\int_0^1 t^{\nu-1}\zeta\left(\theta_{u,v}(t)\right)dt. \quad (2.9)$$

Then by using (2.1) and (2.5) in (2.8) , we obtain (2.3). □

Remark 1. (more specifically), if we utilize Theorem 1 in Theorem 2.2, then

(1) $\tau(x)=x$, then (2.3) takes the form

$$f\left(\frac{2uv}{u+v}\right)\left[J^{\nu}_{\frac{1}{v}+}(\zeta\circ h)\left(\frac{1}{u}\right)+J^{\nu}_{\frac{1}{u}-}(\zeta\circ h)\left(\frac{1}{v}\right)\right]$$
$$\leq\zeta\left(\frac{1}{u}\right)\left({}_{\zeta}J^{\nu}_{\frac{1}{v}+}(f\circ h)\left(\frac{1}{u}\right)\right)+\zeta\left(\frac{1}{v}\right)\left({}_{\zeta}J^{\nu}_{\frac{1}{u}-}(f\circ h)\left(\frac{1}{v}\right)\right)$$
$$\leq\frac{f(u)+f(v)}{2}\left[J^{\nu}_{\frac{1}{v}+}(\zeta\circ h)\left(\frac{1}{u}\right)+J^{\nu}_{\frac{1}{u}-}(\zeta\circ h)\left(\frac{1}{v}\right)\right], \quad (2.10)$$

where ${}_{\zeta}J^{\nu}_{u+}$ and ${}_{\zeta}J^{\nu}_{v-}$ are the left and right weighted Riemann-Liouville fractional operators of order $\nu>0$, defined by

$$\left({}_{\zeta}J^{\nu}_{u+}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_u^x (x-t)^{\nu-1} f(t)\zeta(t)\,dt$$

and

$$\left({}_{\zeta}J^{\nu}_{v-}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_{x}^{v}(t-x)^{\nu-1}f(t)\zeta(t)\,dt.$$

(2) $\tau(x)=x$ and $\nu=1$, then inequality (2.3) transforms into inequality (1.8).

(3) $\tau(x)=x$ and $\zeta(x)=1$, then inequality (2.3) reduces to the inequality (1.9).

(4) $\tau(x)=x$, $\zeta(x)=1$ and $\nu=1$, then inequality (2.3) becomes the inequality (1.7).

Lemma 2.3. *Let $f: I\subseteq(0,\infty)\to\Re$ be an L^1 function with $f'\in L^1$ for $0<u<v$, $u,v\in I^{\circ}$ and $\zeta:[u,v]\to\Re$ be an integrable, positive and weighted symmetric function with respect to $\frac{2uv}{u+v}$. If τ is an increasing and positive function on $[u,v)$ and $\tau(x)$ is continuous on (u,v), then, we have for $\nu>0$:*

$$\begin{aligned}
&\frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left[\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{t}\tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\right]\\
&\qquad\times(f\circ h\circ\tau)'(t)\,\tau'(t)\,dt\\
&-\frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left[\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)}\tau'(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\right]\\
&\qquad\times(f\circ h\circ\tau)'(t)\,\tau'(t)\,dt\\
&=\left[\frac{f(u)+f(v)}{2}\right]\left[J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right.\\
&\qquad\left.+J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right]\\
&-\left[\zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right)\right.\\
&\qquad\left.+\zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right)\right]
\end{aligned}\tag{2.11}$$

where $h(x)=\frac{1}{x}$, $x\in\left[\frac{1}{v},\frac{1}{u}\right]$.

Proof. Setting

$$\begin{aligned}
&\frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left[\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{t}\tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\right]\\
&\qquad\times(f\circ h\circ\tau)'(t)\,\tau'(t)\,dt\\
&-\frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left[\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)}\tau'(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\right]\\
&\qquad\times(f\circ h\circ\tau)'(t)\,\tau'(t)\,dt=\chi_1+\chi_2.
\end{aligned}\tag{2.12}$$

By integration by parts, making use of Lemma 2.1, and definitions (1.11) and (1.12), we obtain

$$\chi_1=\frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left[\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{t}\tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\right]$$

$$
\begin{aligned}
&\times d\left[(f\circ h\circ\tau)(t)\right]\\
&=\frac{f(u)}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\tau^{'}(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\tau'(x)\,dx\\
&-\frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\tau^{'}(t)\left(\frac{1}{u}-\tau(t)\right)^{\nu-1}(\zeta\circ h\circ\tau)(t)(f\circ h\circ\tau)(t)\,dt\\
&=\frac{f(u)}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\tau^{'}(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\\
&-\zeta\left(\frac{1}{u}\right)\frac{(\zeta\circ\tau)^{-1}\left(\tau^{-1}\left(\frac{1}{u}\right)\right)}{\Gamma(\nu)}\\
&\times\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\tau^{'}(t)\left(\frac{1}{u}-\tau(t)\right)^{\nu-1}(\zeta\circ h\circ\tau)(t)(f\circ h\circ\tau)(t)\,dt\\
&=f(u)\left(J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\\
&-\zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f\circ h\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{u}\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_2&=-\frac{1}{\Gamma(\nu)}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left[\int_t^{\tau^{-1}\left(\frac{1}{u}\right)}\tau^{'}(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\right]\\
&\times d\left[(f\circ h\circ\tau)(t)\right]\\
&=f(v)\left(J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\\
&-\zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f\circ h\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{v}\right)\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\chi_1+\chi_2=f(u)&\left(J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right)\\
&-\zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f\circ h\circ\tau)\right)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\\
&+f(v)\left(J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right)\\
&-\zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right).
\end{aligned}
\tag{2.13}
$$

Since

$$
J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)=J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)
$$

$$= \frac{1}{2}\left[J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)+J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right].$$

Thus, we obtain from (2.13) that

$$\begin{aligned}\chi_1+\chi_2 = \left[\frac{f(u)+f(v)}{2}\right]&\left[J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right.\\ &\left.+\left[J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right]\right.\\ &-\left[\zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right)\right.\\ &\left.+\zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right)\right].\end{aligned} \tag{2.14}$$

Which is the required result. □

Remark 2. Particularly, in Lemma 2.3, if we take:

(1) $\tau(x)=x$, then equality (2.11) becomes

$$\begin{aligned}&\frac{1}{\Gamma(\nu)}\int_{\frac{1}{v}}^{\frac{1}{u}}\left[\int_{\frac{1}{v}}^{t}\left(\frac{1}{u}-x\right)^{\nu-1}(\zeta\circ h)(x)\,dx\right](f\circ h)'(t)\,dt\\ &\quad-\frac{1}{\Gamma(\nu)}\int_{\frac{1}{v}}^{\frac{1}{u}}\left[\int_{t}^{\frac{1}{u}}\left(x-\frac{1}{v}\right)^{\nu-1}(\zeta\circ h)(x)\,dx\right](f\circ h)'(t)\,dt\\ &=\left[\frac{f(u)+f(v)}{2}\right]\left[J^{\nu}_{\frac{1}{v}+}(\zeta\circ h)\left(\frac{1}{u}\right)+J^{\nu}_{\frac{1}{u}+}(\zeta\circ h)\left(\frac{1}{v}\right)\right]\\ &\quad-\left[\zeta\left(\frac{1}{u}\right)\left({}_{\zeta}J^{\nu}_{\frac{1}{v}+}(f\circ h)\left(\frac{1}{u}\right)\right)+\zeta\left(\frac{1}{v}\right)\left({}_{\zeta}J^{\nu}_{\frac{1}{u}-}(f\circ h)\left(\frac{1}{v}\right)\right)\right].\end{aligned} \tag{2.15}$$

where ${}_{\zeta}J^{\nu}_{u+}$ and ${}_{\zeta}J^{\nu}_{v-}$ are defined in Remark 1.

(2) $\tau(x)=x$ and $\zeta(x)=1$, then equality (2.11) becomes

$$\begin{aligned}&\frac{1}{2}\left(\frac{uv}{v-u}\right)\left[\int_0^1(1-t^{\nu})(f\circ h)'\left(\frac{tu+(1-t)v}{uv}\right)dt\right.\\ &\left.\quad-\int_0^1(1-t^{\nu})(f\circ h)'\left(\frac{tv+(1-t)u}{uv}\right)dt\right]=\frac{f(u)+f(v)}{2}\\ &\quad-\frac{\Gamma(\nu+1)}{2}\left(\frac{uv}{v-u}\right)^{\nu}\left[J^{\nu}_{\frac{1}{v}+}(f\circ h)\left(\frac{1}{u}\right)+J^{\nu}_{\frac{1}{u}-}(f\circ h)\left(\frac{1}{v}\right)\right].\end{aligned} \tag{2.16}$$

(3) $\tau(x)=x$, $\zeta(x)=1$ and $\nu=1$, we obtain

$$\begin{aligned}&\frac{uv(v-u)}{2}\int_0^1\frac{(1-2t)}{(t+u(1-t)v)^2}f\left(\frac{uv}{t+u(1-t)v}\right)dt\\ &\quad=\frac{f(u)+f(v)}{2}-\left(\frac{uv}{v-u}\right)\int_u^v\frac{f(x)}{x^2}dx.\end{aligned} \tag{2.17}$$

We will use the following notations for the rest of this section:

$$
\begin{aligned}
{}_{\zeta}\chi^{\nu;\tau}(u,v) = \left[\frac{f(u)+f(v)}{2}\right]\Bigg[& J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right) \\
& + J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)+}(\zeta\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\Bigg] \\
& - \left[\zeta\left(\frac{1}{u}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{v}\right)+}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{u}\right)\right)\right)\right. \\
& \left. + \zeta\left(\frac{1}{v}\right)\left({}_{\zeta\circ\tau}J^{\nu;\tau}_{\tau^{-1}\left(\frac{1}{u}\right)-}(f\circ h\circ\tau)\left(\tau^{-1}\left(\frac{1}{v}\right)\right)\right)\right]. \quad (2.18)
\end{aligned}
$$

Theorem 2.4. *Let $f : I \subseteq (0,\infty) \to \Re$ be an L^1 function with $f' \in L^1$ for $0 < u < v$, $u, v \in I^\circ$ and $\zeta : [u,v] \to \Re$ be an integrable, positive and weighted symmetric function with respect to $\frac{2uv}{u+v}$. If $\left|(f\circ h)'\right|$ is harmonic convex on $[u,v]$, τ is an increasing and positive function on $[u,v)$, and $\tau(x)$ is continuous on (u,v), then, we have for $\nu > 0$:*

$$
\left|{}_{\zeta}\chi^{\nu;\tau}(u,v)\right| \le \frac{\|\zeta\circ h\circ\tau\|_\infty \left[\begin{array}{c} u_\tau(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right| \\ +v_\tau(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{v}\right)\right| \end{array}\right]}{\Gamma(\nu+1)}, \quad (2.19)
$$

where $h(x) = \frac{1}{x}$, $x \in \left[\frac{1}{v}, \frac{1}{u}\right]$,

$$
\begin{aligned}
u_\tau(\nu,u,v) = & \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}} \frac{u(\tau(t)-v)}{\tau(t)(v-u)} \\
& \times\left[\left(\frac{1}{u}-\tau(t)\right)^\nu - \left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-\tau(t)\right)\right)^\nu\right]\tau'(t)\,dt \\
& + \int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)} \frac{u(\tau(t)-v)}{\tau(t)(v-u)} \\
& \times\left[\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^\nu - \left(\frac{1}{u}-\tau(t)\right)^\nu\right]\tau'(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
v_\tau(\nu,u,v) = & \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}} \frac{v(u-\tau(t))}{\tau(t)(v-u)} \\
& \times\left[\left(\frac{1}{u}-\tau(t)\right)^\nu - \left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^\nu\right]\tau'(t)\,dt \\
& + \int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)} \frac{v(u-\tau(t))}{\tau(t)(v-u)} \\
& \times\left[\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^\nu - \left(\frac{1}{u}-\tau(t)\right)^\nu\right]\tau'(t)\,dt.
\end{aligned}
$$

Proof. According to (2.11) of Lemma 2.3, we obtain

$$\left|{}_{\zeta}\chi^{\nu;\tau}(u,v)\right| \leq \frac{1}{\Gamma(\nu)} \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left| \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} \right.$$
$$\left. \times (\zeta \circ h \circ \tau)(x)\,dx - \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)} \tau'(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx \right|$$
$$\times \left|\left(f' \circ h \circ \tau\right)(t)\right| \tau'(t)\,dt \quad (2.20)$$

We know that ζ is a harmonic symmetric with respect to $\frac{2uv}{u+v}$, we observed that

$$\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)} \tau'(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx$$
$$= \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)} \tau'(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1} (\zeta \circ h \circ \tau)\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-x\right)dx$$
$$= \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx.$$

Hence

$$\left| \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx \right.$$
$$\left. - \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)} \tau'(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx \right|$$
$$= \left| \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx \right|. \quad (2.21)$$

From (2.21) we get

$$\left| \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx \right.$$
$$\left. - \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)} \tau'(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx \right|$$
$$= \left| \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx \right|$$
$$\leq \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta \circ h \circ \tau)(x)\,dx \quad (2.22)$$

for $t \in \left[\tau^{-1}\left(\frac{1}{v}\right), \frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}\right]$ or

$$\left|\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta\circ h\circ\tau)(x)\,dx \right.$$
$$\left. - \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)} \left|\tau'(x)\right| \left(\tau(x)-\frac{1}{v}\right)^{\nu-1} (\zeta\circ h\circ\tau)(x)\,dx\right|$$
$$= \left|\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta\circ h\circ\tau)(x)\,dx\right|$$
$$\leq \int_{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}^{t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta\circ h\circ\tau)(x)\,dx \quad (2.23)$$

for $t \in \left[\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}, \tau^{-1}\left(\frac{1}{u}\right)\right]$.
By applying the harmonic convexity of $\left|f'\right|$ on $[u,v]$ for $t \in \left[\tau^{-1}\left(\frac{1}{v}\right), \tau^{-1}\left(\frac{1}{u}\right)\right]$, we get

$$\left|\left(f'\circ h\circ\tau\right)(t)\right|$$
$$\leq \frac{u(\tau(t)-v)}{\tau(t)(v-u)}\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right| + \frac{v(u-\tau(t))}{\tau(t)(v-u)}\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right|. \quad (2.24)$$

Applying (2.22)–(2.24) in (2.20), we obtain

$$\left|{}_{\zeta}\chi^{\nu;\tau}(u,v)\right| \leq \frac{\|\zeta\circ h\circ\tau\|_{\infty}}{\Gamma(\nu)}$$
$$\times \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}} \left(\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} dx\right)$$
$$\times \left(\frac{u(\tau(t)-v)}{\tau(t)(v-u)}\left|(f\circ h)'(u)\right| + \frac{v(u-\tau(t))}{\tau(t)(v-u)}\left|(f\circ h)'(v)\right|\right)\tau'(t)\,dt$$
$$+ \frac{\|\zeta\circ h\circ\tau\|_{\infty}}{\Gamma(\nu)} \int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)} \left(\int_{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}^{t} \tau'(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1} dx\right)$$
$$\times \left(\frac{u(\tau(t)-v)}{\tau(t)(v-u)}\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right| + \frac{v(u-\tau(t))}{\tau(t)(v-u)}\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right|\right)\tau'(t)\,dt. \quad (2.25)$$

Let us evaluate the first integral in (2.25)

$$\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}} \left(\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \left(\frac{1}{u}-\tau(x)\right)^{\nu-1} d(\tau(x))\right)$$
$$\times \left(\frac{u(\tau(t)-v)}{\tau(t)(v-u)}\left|(f\circ h)'(u)\right| + \frac{v(u-\tau(t))}{\tau(t)(v-u)}\left|(f\circ h)'(v)\right|\right)\tau'(t)\,dt$$
$$= \frac{1}{\nu}\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}} \left[\left(\frac{1}{u}-\tau(t)\right)^{\nu} - \left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}\right]$$

$$\times\left(\frac{u\left(\tau\left(t\right)-v\right)}{\tau\left(t\right)\left(v-u\right)}\left|\left(f^{'}\circ h\right)\left(\frac{1}{u}\right)\right|+\frac{v\left(u-\tau\left(t\right)\right)}{\tau\left(t\right)\left(v-u\right)}\left|\left(f^{'}\circ h\right)\left(\frac{1}{u}\right)\right|\right)\tau^{'}\left(t\right)dt \quad (2.26)$$

and the value of the second integral in (2.25) is given below

$$\begin{aligned}
&\int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\int_{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}^{t}\tau^{'}\left(x\right)\left(\frac{1}{u}-\tau\left(x\right)\right)^{\nu-1}dx\right)\\
&\qquad\times\left(\frac{u\left(\tau\left(t\right)-v\right)}{\tau\left(t\right)\left(v-u\right)}\left|\left(f\circ h\right)'\left(u\right)\right|+\frac{v\left(u-\tau\left(t\right)\right)}{\tau\left(t\right)\left(v-u\right)}\left|\left(f\circ h\right)'\left(v\right)\right|\right)\tau^{'}\left(t\right)dt\\
&\qquad=\frac{1}{\nu}\int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)}\left[\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}-\left(\frac{1}{u}-\tau\left(t\right)\right)^{\nu}\right]\\
&\qquad\times\left(\frac{u\left(\tau\left(t\right)-v\right)}{\tau\left(t\right)\left(v-u\right)}\left|\left(f^{'}\circ h\right)\left(\frac{1}{u}\right)\right|+\frac{v\left(u-\tau\left(t\right)\right)}{\tau\left(t\right)\left(v-u\right)}\left|\left(f^{'}\circ h\right)\left(\frac{1}{u}\right)\right|\right)\tau^{'}\left(t\right)dt. \quad (2.27)
\end{aligned}$$

Applying (2.21)–(2.27) in (2.20) to obtain the desired inequality (2.19). □

Remark 3. Particularly, in Theorem 2.4, if we take

(1) $\tau\left(x\right)=x$, we have

$$\begin{aligned}
\left|{}_{\zeta}\chi^{\nu}\left(u,v\right)\right|=&\left\|\left[\frac{f\left(u\right)+f\left(v\right)}{2}\right]\left[J_{\frac{1}{v}+}^{\nu}\left(\zeta\circ h\right)\left(\frac{1}{u}\right)+J_{\frac{1}{u}+}^{\nu}\left(\zeta\circ h\right)\left(\frac{1}{v}\right)\right]\right.\\
&\left.-\left[\zeta\left(\frac{1}{u}\right)\left({}_{\zeta}J_{\frac{1}{v}+}^{\nu}\left(f\circ h\right)\left(\frac{1}{u}\right)\right)+\zeta\left(\frac{1}{v}\right)\left({}_{\zeta}J_{\frac{1}{u}-}^{\nu}\left(f\circ h\right)\left(\frac{1}{v}\right)\right)\right]\right\|\\
&\leq\frac{\left\|\zeta\circ h\right\|_{\infty}\left[u\left(\nu,u,v\right)\left|\left(f^{'}\circ h\right)\left(\frac{1}{u}\right)\right|+v\left(\nu,u,v\right)\left|\left(f^{'}\circ h\right)\left(\frac{1}{v}\right)\right|\right]}{\Gamma\left(\nu+1\right)}, \quad (2.28)
\end{aligned}$$

where

$$\begin{aligned}
u\left(\nu,u,v\right)=&\int_{\frac{1}{v}}^{\frac{u+v}{2uv}}\frac{u\left(t-v\right)}{t\left(v-u\right)}\left[\left(\frac{1}{u}-t\right)^{\nu}-\left(t-\frac{1}{v}\right)^{\nu}\right]dt\\
&+\int_{\frac{u+v}{2uv}}^{\frac{1}{u}}\frac{u\left(t-v\right)}{t\left(v-u\right)}\left[\left(t-\frac{1}{v}\right)^{\nu}-\left(\frac{1}{u}-t\right)^{\nu}\right]dt
\end{aligned}$$

and

$$\begin{aligned}
v\left(\nu,u,v\right)=&\int_{\frac{1}{v}}^{\frac{u+v}{2uv}}\frac{v\left(u-t\right)}{t\left(v-u\right)}\left[\left(\frac{1}{u}-t\right)^{\nu}-\left(t-\frac{1}{v}\right)^{\nu}\right]dt\\
&+\int_{\frac{u+v}{2uv}}^{\frac{1}{u}}\frac{v\left(u-t\right)}{t\left(v-u\right)}\left[\left(t-\frac{1}{v}\right)^{\nu}-\left(\frac{1}{u}-\tau\left(t\right)\right)^{\nu}\right]dt.
\end{aligned}$$

(2) $\tau\left(x\right)=x$ and $\zeta(x)=1$, we get

$$\begin{aligned}
\left|\chi^{\nu}(u,v)\right| &= \Bigg|\frac{f(u)+f(v)}{2} - \frac{\Gamma(\nu+1)}{2}\left(\frac{uv}{v-u}\right)^{\nu} \\
&\quad \times \left[J^{\nu}_{\frac{1}{v}+}(f\circ h)\left(\frac{1}{u}\right) + J^{\nu}_{\frac{1}{u}-}(f\circ h)\left(\frac{1}{v}\right)\right]\Bigg| \\
&\leq \left(\frac{uv}{v-u}\right)^{\nu} \frac{\left[u(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right| + v(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{v}\right)\right|\right]}{2}, \quad (2.29)
\end{aligned}$$

where $u(\nu,u,v)$ and $v(\nu,u,v)$ are defined as above.

(3) $\tau(x)=x$, $\zeta(x)=1$ and $\nu=1$, we obtain

$$\begin{aligned}
\left|\chi(u,v)\right| &= \left|\frac{f(u)+f(v)}{2} - \left(\frac{uv}{v-u}\right)\int_{\frac{1}{v}}^{\frac{1}{u}} f(x)\,dx\right| \\
&\leq \left(\frac{uv}{v-u}\right)\frac{\left[u(u,v)\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right| + v(u,v)\left|\left(f'\circ h\right)\left(\frac{1}{v}\right)\right|\right]}{2}, \quad (2.30)
\end{aligned}$$

$$u(u,v) = \frac{v-u}{2uv^2} + \log\left[\frac{4uv}{(u+v)^2}\right]$$

and

$$v(u,v) = \frac{u-v}{2u^2v} + \log\left[\frac{4uv}{(u+v)^2}\right].$$

Theorem 2.5. *Let $f: I\subseteq(0,\infty)\to\Re$ be an L^1 function with $f'\in L^1$ for $0<u<v$, $u,v\in I^{\circ}$ and $\zeta:[u,v]\to\Re$ be an integrable, positive and weighted symmetric function with respect to $\frac{u+v}{2}$. If $\left|f'\right|^q$ is harmonic convex on $[u,v]$ for $q\geq 1$, τ is an increasing and positive function on $[u,v)$, and $\tau(x)$ is continuous on (u,v), then, we have for $\nu>0$:*

$$\left|{}_{\zeta}\chi^{\nu;\tau}(u,v)\right| \leq \frac{\|\zeta\circ h\circ\tau\|_{\infty}}{\Gamma(\nu+1)}\left(C_{\tau}(\nu,u,v)\right)^{1-\frac{1}{q}}\left[\begin{array}{c} u_{\tau}(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right|^q \\ +v_{\tau}(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{v}\right)\right|^q \end{array}\right]^{\frac{1}{q}}, \quad (2.31)$$

where $h(x)=\frac{1}{x}$, $x\in\left[\frac{1}{v},\frac{1}{u}\right]$,

$$\begin{aligned}
C_{\tau}(\nu,u,v) &= \frac{2}{\nu+1}\left[\left(\frac{1}{u}-\frac{1}{v}\right)^{\nu+1} - \left(\frac{1}{u}-\tau\left(\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}\right)\right)^{\nu+1}\right] \\
&\quad - \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}\tau'(t)\,dt \\
&\quad + \int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}\tau'(t)\,dt
\end{aligned}$$

and $u_{\tau}(\nu,u,v)$, $v_{\tau}(\nu,u,v)$ are defined as in Theorem 2.4.

Proof. Applying power-mean inequality to(2.20) and then using (2.22)–(2.24), we get

$$\begin{aligned}
\left|{}_{\zeta}\chi^{\nu;\tau}(u,v)\right| &\leq \frac{1}{\Gamma(\nu)}\left(\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left|\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{t}\tau^{'}(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}\right)\right.\\
&\quad\times(\zeta\circ h\circ\tau)(x)\,dx-\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)}\tau^{'}(x)\left(\tau(x)-\frac{1}{v}\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\Bigg|\\
&\quad\times\left|\left(f^{'}\circ h\circ\tau\right)(t)\right|\tau^{'}(t)\,dt\\
&\leq\frac{1}{\Gamma(\nu)}\left(\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left|\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}\tau^{'}(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\right|\tau^{'}(t)\,dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left|\begin{array}{c}\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}\tau^{'}(x)\\ \left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\end{array}\right|\left|\left(f^{'}\circ h\circ\tau\right)(t)\right|^{q}\tau^{'}(t)\,dt\right)^{\frac{1}{q}}. \qquad (2.32)
\end{aligned}$$

Since $|(\zeta\circ h\circ\tau)(x)|\leq\|\zeta\circ h\circ\tau\|_{\infty}$, hence it is easy to observe that

$$\begin{aligned}
&\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left|\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}\tau^{'}(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}(\zeta\circ h\circ\tau)(x)\,dx\right|\tau^{'}(t)\,dt\\
&\leq\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}\tau^{'}(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}|(\zeta\circ h\circ\tau)(x)|\,dx\right)\tau^{'}(t)\,dt\\
&\leq\|\zeta\circ h\circ\tau\|_{\infty}\left[\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}\left(\int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}\tau^{'}(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}dx\right)\tau^{'}(t)\,dt\right.\\
&\quad\left.+\int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\int_{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t}^{t}\tau^{'}(x)\left(\frac{1}{u}-\tau(x)\right)^{\nu-1}dx\right)\tau^{'}(t)\,dt\right]\\
&=\frac{\|\zeta\circ h\circ\tau\|_{\infty}}{\nu}\left\{\frac{2}{\nu+1}\left[\left(\frac{1}{u}-\frac{1}{v}\right)^{\nu+1}-\left(\frac{1}{u}-\tau\left(\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}\right)\right)^{\nu+1}\right]\right.\\
&\quad-\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}\tau^{'}(t)\,dt\\
&\quad\left.+\int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)}\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}\tau^{'}(t)\,dt\right\}. \qquad (2.33)
\end{aligned}$$

Since for $q\geq 1$ and $t\in\left[\tau^{-1}\left(\frac{1}{u}\right),\tau^{-1}\left(\frac{1}{v}\right)\right]$, $\left|f^{'}\right|^{q}$ is convex on $[u,v]$, we get

$$\left|\left(f^{'}\circ h\circ\tau\right)(t)\right|\leq\frac{u(\tau(t)-v)}{\tau(t)(v-u)}\left|\left(f^{'}\circ h\right)\left(\frac{1}{u}\right)\right|^{q}+\frac{v(u-\tau(t))}{\tau(t)(v-u)}\left|\left(f^{'}\circ h\right)\left(\frac{1}{v}\right)\right|^{q}.$$

Thus, now we are able to evaluate the second integral in (2.32)

$$
\begin{aligned}
&\int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left| \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \tau'(x)\left(\tfrac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta\circ h\circ\tau)(x)\,dx \right| \\
&\qquad \times \left|\left(f'\circ h\circ\tau\right)(t)\right|^{q} \tau'(t)\,dt \\
&\leq \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\tau^{-1}\left(\frac{1}{u}\right)} \left| \int_{t}^{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t} \tau'(x)\left(\tfrac{1}{u}-\tau(x)\right)^{\nu-1} (\zeta\circ h\circ\tau)(x)\,dx \right| \\
&\qquad \times \left(\tfrac{u(\tau(t)-v)}{\tau(t)(v-u)} \left|\left(f'\circ h\right)\left(\tfrac{1}{u}\right)\right|^{q} + \tfrac{v(u-\tau(t))}{\tau(t)(v-u)} \left|\left(f'\circ h\right)\left(\tfrac{1}{v}\right)\right|^{q}\right)\tau'(t)\,dt \\
&\leq \frac{\|\zeta\circ h\circ\tau\|_{\infty}}{\nu} \left[\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right|^{q} \left\{ \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{u}\right)}{2}} \frac{u(\tau(t)-v)}{\tau(t)(v-u)} \right.\right. \\
&\qquad \times \left[\left(\frac{1}{u}-\tau(t)\right)^{\nu} - \left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}\right]\tau'(t)\,dt \\
&\qquad + \int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{u}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)} \frac{u(\tau(t)-v)}{\tau(t)(v-u)} \left[\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}\right. \\
&\left.\left. -\left(\frac{1}{u}-\tau(t)\right)^{\nu}\right]\tau'(t)\,dt \right\} + \left|\left(f'\circ h\right)\left(\frac{1}{v}\right)\right|^{q} \left\{ \int_{\tau^{-1}\left(\frac{1}{v}\right)}^{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{u}\right)}{2}} \frac{v(u-\tau(t))}{\tau(t)(v-u)} \right. \\
&\qquad \times \left[\left(\frac{1}{u}-\tau(t)\right)^{\nu} - \left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu}\right]\tau'(t)\,dt \\
&\qquad + \int_{\frac{\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{u}\right)}{2}}^{\tau^{-1}\left(\frac{1}{u}\right)} \frac{v(u-\tau(t))}{\tau(t)(v-u)} \\
&\qquad \times \left.\left.\left[\left(\frac{1}{u}-\tau\left(\tau^{-1}\left(\frac{1}{u}\right)+\tau^{-1}\left(\frac{1}{v}\right)-t\right)\right)^{\nu} - \left(\frac{1}{u}-\tau(t)\right)^{\nu}\right]\tau'(t)\,dt \right\}\right]. \qquad (2.34)
\end{aligned}
$$

Applying (2.33), (2.34) in (2.32), we get (2.31). □

Remark 4. In Theorem 2.5, especially when we take

(1) $\tau(x) = x$, we have

$$
\begin{aligned}
\left|{}_{\zeta}\chi^{\nu}(u,v)\right| &= \left| \left[\frac{f(u)+f(v)}{2}\right] \left[J^{\nu}_{\frac{1}{v}+}(\zeta\circ h)\left(\frac{1}{u}\right) + J^{\nu}_{\frac{1}{u}+}(\zeta\circ h)\left(\frac{1}{v}\right) \right] \right. \\
&\quad \left. - \left[\zeta\left(\frac{1}{u}\right)\left({}_{\zeta}J^{\nu}_{\frac{1}{v}+}(f\circ h)\left(\frac{1}{u}\right)\right) + \zeta\left(\frac{1}{v}\right)\left({}_{\zeta}J^{\nu}_{\frac{1}{u}-}(f\circ h)\left(\frac{1}{v}\right)\right) \right] \right| \\
&\leq \frac{\|\zeta\circ h\|_{\infty}}{\Gamma(\nu+1)} \left(C(\nu,u,v)\right)^{1-\frac{1}{q}} \left[\begin{array}{c} u(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right|^{q} \\ +v(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{v}\right)\right|^{q} \end{array} \right]^{\frac{1}{q}}, \qquad (2.35)
\end{aligned}
$$

where $h(x) = \frac{1}{x}$, $x \in \left[\frac{1}{v}, \frac{1}{u}\right]$,

$$
C(\nu,u,v) = \frac{(2^{\nu}-1)}{2^{\nu}(\nu+1)}\left(\frac{v-u}{uv}\right)^{\nu+1}
$$

$u(\nu,u,v)$ and $v(\nu,u,v)$ are defined in (1) of Remark 3.

(2) $\tau(x) = x$ and $\zeta(x) = 1$, we get

$$
\begin{aligned}
|\chi^{\nu}(u,v)| &= \left| \frac{f(u)+f(v)}{2} - \frac{\Gamma(\nu+1)}{2}\left(\frac{uv}{v-u}\right)^{\nu} \right. \\
&\qquad \left. \times \left[J^{\nu}_{\frac{1}{v}+}(f\circ h)\left(\frac{1}{u}\right) + J^{\nu}_{\frac{1}{u}-}(f\circ h)\left(\frac{1}{v}\right) \right] \right| \\
&\le \frac{1}{\Gamma(\nu+1)} \left(C(\nu,u,v)\right)^{1-\frac{1}{q}} \left[\begin{array}{c} u(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right|^{q} \\ +v(\nu,u,v)\left|\left(f'\circ h\right)\left(\frac{1}{v}\right)\right|^{q} \end{array} \right]^{\frac{1}{q}}, \quad (2.36)
\end{aligned}
$$

where $h(x) = \frac{1}{x}$, $x \in \left[\frac{1}{v}, \frac{1}{u}\right]$, $u(\nu,u,v)$, $v(\nu,u,v)$ are defined in (1) of Remark 3 and $C(\nu,u,v)$ is as defined above.

(3) $\tau(x) = x$, $\zeta(x) = 1$ and $\nu = 1$, we obtain

$$
\begin{aligned}
|\chi(u,v)| &= \left| \frac{f(u)+f(v)}{2} - \left(\frac{uv}{v-u}\right) \int_{\frac{1}{v}}^{\frac{1}{u}} f(x)\,dx \right| \\
&\le \frac{1}{\Gamma(\nu+1)} \left(C(u,v)\right)^{1-\frac{1}{q}} \left[\begin{array}{c} u(u,v)\left|\left(f'\circ h\right)\left(\frac{1}{u}\right)\right|^{q} \\ +v(u,v)\left|\left(f'\circ h\right)\left(\frac{1}{v}\right)\right|^{q} \end{array} \right]^{\frac{1}{q}}, \quad (2.37)
\end{aligned}
$$

where $h(x) = \frac{1}{x}$, $x \in \left[\frac{1}{v}, \frac{1}{u}\right]$, $u(u,v)$, $v(u,v)$ are defined in (3) of Remark 3 and $C(u,v) = \left(\frac{v-u}{2uv}\right)^2$.

3. Conclusions

In this study, we proved very important and interesting inequalities of Fejér type for a very fascinating generalized class of functions, namely, harmonic convex functions by using general weighted fractional integral operator which depends upon an increasing function. The results of our study not only generalize a number of findings obtained in [17–19] but one can obtain a number of new results by choosing a increasing function involved. The results can also be an inspiration for young researchers as well as researcher already working in the field of fractional integral inequalities and can further open up new directions of research in mathematical sciences.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

References

1. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.,* **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006
2. T. Abdeljawad, P. O. Mohammed, A. Kashuri, New modified conformable fractional integral inequalities of Hermite-Hadamard type with applications, *J. Funct. Space.*, **2020** (2020), 4352357. https://doi.org/10.1155/2020/4352357
3. A. O. Akdemir, S. I. Butt, M. Nadeem, M. A. Ragusa, New general variants of Chebyshev type inequalities via generalized fractional integral operators, *Mathematics*, **9** (2021), 122. https://doi.org/10.3390/math9020122
4. D. Baleanu, P. O. Mohammed, M. Vivas-Cortez, Y. Rangel-Oliveros, Some modifications in conformable fractional integral inequalities, *Adv. Differ. Equ.,* **2020** (2020), 374. https://doi.org/10.1186/s13662-020-02837-0
5. C. Bardaro, P. L. Butzer, I. Mantellini, The foundations of fractional calculus in the Mellin transform setting with applications, *J. Fourier Anal. Appl.,* **21** (2015), 961–1017. https://doi.org/10.1007/s00041-015-9392-3
6. D. Baleanu, P. O. Mohammed, S. Zeng, Inequalities of trapezoidal type involving generalized fractional integrals, *Alex. Eng. J.,* **59** (2020), 2975–2984. https://doi.org/10.1016/j.aej.2020.03.039
7. F. Chen, S. Wu, Fejér and Hermite-Hadamard type inqequalities for harmonically convex functions, *J. Appl. Math.,* **2014** (2014). https://doi.org/10.1155/2014/386806
8. S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University: Footscray, Australia, 2000.
9. S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.,* **11** (1998) 91–95. https://doi.org/10.1016/S0893-9659(98)00086-X
10. M. R. Delavar, M. Aslani, M. De La Sen, Hermite-Hadamard-Fejér inequality related to generalized convex functions via fractional integrals, *J. Math.*, **2018** (2018), 5864091. https://doi.org/10.1155/2018/5864091
11. L. Fejér, Über die fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss,* **24** (1906), 369–390. https://doi.org/10.1086/141409
12. A. Fernandez, P. O. Mohammed, Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels, *Math. Method. Appl. Sci.,* **44** (2020), 1–18. https://doi.org/10.1002/mma.6188
13. B. Gavrea, I. Gavrea, On some Ostrowski type inequalities, *Gen. Math.,* **18** (2010), 33–44. https://doi.org/10.1016/j.mcm.2007.12.004
14. H. Gunawan, Eridani, Fractional integrals and generalized Olsen inequalities, *Kyungpook Math. J.,* **49** (2009), 31–39. https://doi.org/10.5666/KMJ.2009.49.1.031
15. J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann, *J. Math. Pure. Appl.,* **58** (1893), 171–215.

16. J. Han, P. O. Mohammed, H. Zeng, Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function, *Open Math.,* **18** (2020), 794–806. https://doi.org/10.1515/math-2020-0038

17. İ. İşcan, Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals, *Stud. U. Babes-Bol. Mat.,* **60** (2015), 355–366.

18. İ. İşcan, S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, *Appl. Math. Comput.,* **238** (2014), 237–244. https://doi.org/10.1016/j.amc.2014.04.020

19. İ. İşcan, M. Kunt, N. Yazici, Hermite-Hadamard-Fejér type inequalities for harmonically convex functions via fractional integrals, *New Tre. Math. Sci.,* **4** (2016), 239–253. http://dx.doi.org/10.20852/ntmsci.2016320378

20. F. Jarad, T. Abdeljawad, K. Shah, On the weighted fractional operators of a function with respect to another function, *Fractals*, **28** (2020). https://doi.org/10.1142/S0218348X20400113

21. S. Kaijser, L. Nikolova, L. E. Persson, A. Wedestig, A Hardy type inequalities via convexity, *Math. Inequal. Appl.,* **8** (2005), 403–417.

22. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations,* Elsevier B. V., **204** (2006).

23. M. Kunt, İ. İşcan, On new Hermite-Hadamard-Fejér type inequalities for *p*-convex functions via fractional integrals, *CMMA*, **2** (2017), 1–15.

24. A. Kashuri, T. M. Rassias, New Hermite-Hadamard-Fejer inequalities via k-fractional integrals for di erentiable generalized nonconvex functions, *Filomat,* **34** (2020), 2549–2558. https://doi.org/10.2298/FIL2008549K

25. P. O. Mohammed, I. Brevik, A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals, *Symmetry*, **12** (2020), 610. https://doi.org/10.3390/sym12040610

26. P. O. Mohammed, T. Abdeljawad, Opial integral inequalities for generalized fractional operators with nonsingular kernel, *J. Inequal. Appl.,* **2020** (2020), 148. https://doi.org/10.1186/s13660-020-02419-4

27. H. Kalsoom, M. Vivas-Cortez, M. Amer Latif, H. Ahmad, Weighted midpoint Hermite-Hadamard-Fejér type inequalities in fractional calculus for harmonically convex functions, *Fractal Fract.,* **5** (2021), 252. https://doi.org/10.3390/fractalfract5040252

28. H. Kalsoom, H. Budak, H. Kara, M. A. Ali, Some new parameterized inequalities for co-ordinated convex functions involving generalized fractional integrals, *Open Math.,* **19** (2021), 1153–1186. https://doi.org/10.1515/math-2021-0072

29. P. O. Mohammed, T. Abdeljawad, S. Zeng, A. Kashuri, Fractional Hermite-Hadamard integral inequalities for a new class of convex functions, *Symmetry*, **12** (2020), 1485. https://doi.org/10.3390/sym12091485

30. P. O. Mohammed, T. Abdeljawad, Modification of certain fractional integral inequalities for convex functions, *Adv. Differ. Equ.,* **2020** (2020), 69. https://doi.org/10.1186/s13662-020-2541-2

31. S. Z. Ullah, M. A. Khan, Z. A. Khan, Y. M. Chu, Coordinate strongly s-convex functions and related results, *J. Math. Inequal.,* **14** (2020), 829–843. https://doi.org/10.17719/jisr.11662

32. Y. Khurshid, M. A. Khan, Y. M. Chu, Z. A. Khan, Hermite-Hadamard Fejér inequalities for conformal fractional integrals via preinvex functions, *J. Funct. Space.,* **2019** (2019), 1–10. https://doi.org/10.1155/2019/4976351
33. Z. A. Khan, R. Gul, K. Shah, On impulsive boundary value problem with Riemann-Liouville fractional order derivative, *J. Funct. Space.,* **2021** (2021), 1–11. https://doi.org/10.1155/2021/8331731
34. P. O. Mohammed, Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function, *Math. Method. Appl. Sci.,* **44** (2019), 2314–2324. https://doi.org/10.1002/mma.5784
35. P. O. Mohammed, M. Z. Sarikaya, D. Baleanu, On the generalized Hermite-Hadamard inequalities via the tempered fractional integrals, *Symmetry*, **12** (2020), 595. https://doi.org/10.1039/D0AY90014A
36. I. G. Macdonald, *Symmetric functions and orthogonal polynomials,* American Mathematical Society, New York, 1997.
37. S. Mehmood, F. Zafar, N. Asmin, New Hermite-Hadamard-Fejér type inequalities for (h_1,h_2)-convex functions via fractional calculus, *ScienceAsia*, **46** (2020), 102–108. https://doi.org/10.2306/scienceasia1513-1874.2020.S015
38. T. J. Osler, The fractional derivative of a composite function, *SIAM J. Math. Anal.,* **1** (1970), 288–293. https://doi.org/10.1137/0501026
39. F. Qi, O. P. Mohammed, J. C. Yao, Y. H. Yao, Generalized fractional integral inequalities of Hermite-Hadamard type for (v,m)-convex functions, *J. Inequal. Appl.,* **2019** (2019), 135. https://doi.org/10.1186/s13660-019-2079-6
40. M. Z. Sarikaya, H. Yaldiz, On generalization integral inequalities for fractional integrals, *Nihonkai Math. J.,* **25** (2014), 93–104.
41. M. Z. Sarikaya, C. C. Bilisik, P. O. Mohammed, Some generalizations of Opial type inequalities, *Appl. Math. Inf. Sci.,* **14** (2020), 809–816. https://doi.org/10.18576/amis/140508
42. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.,* **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048
43. Y. Sawano, H. Wadade, On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space, *J. Fourier Anal. Appl.,* **19** (2013), 20–47. https://doi.org/10.1007/s00041-012-9223-8
44. D. P. Shi, B. Y. Xi, F. Qi, Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals of (v,m)-convex functions, *Fract. Differ. Calc.,* **4** (2014), 31–43.
45. M. Vivas-Cortez, T. Abdeljawad, P. O. Mohammed, Y. Rangel-Oliveros, Simpson's integral inequalities for twice differentiable convex functions, *Math. Probl. Eng.,* **2020** (2020), 1936461. https://doi.org/10.1155/2020/1936461
46. C. J. Zhao, W. S. Cheung, On improvements of the Rozanova's inequality, *J. Inequal. Appl.,* **2011** (2011), 33. https://doi.org/10.1186/1029-242X-2011-33
47. A. Scapellato, Riesz potential, Marcinkiewicz integral and their commutators on mixed Morrey spaces, *Filomat,* **34** (2020), 931–944. https://doi.org/10.2298/FIL2003931S

48. A. Abdalmonem, A. Scapellato, Fractional operators with homogeneous kernels in weighted Herz spaces with variable exponent, *Appl. Anal.,* 2020. https://doi.org/10.1080/00036811.2020.1789602

49. T. Y. Zhang, A. P. Ji, F. Qi, On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions, *Abstr. Appl. Anal.,* **2012** (2012), 560586. https://doi.org/10.1155/2012/560586

50. T. Y. Zhang, A. P. Ji, F. Qi, Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means, *Le Mat.,* **68** (2013), 229–239. https://doi.org/10.4418/2013.68.1.17

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