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*Research article*

# A regularity criterion for liquid crystal flows in terms of the component of velocity and the horizontal derivative components of orientation field

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**Abstract:** In this paper, we establish a regularity criterion for the 3D nematic liquid crystal flows. More precisely, we prove that the local smooth solution $(u, d)$ is regular provided that velocity component $u_3$, vorticity component $\omega_3$ and the horizontal derivative components of the orientation field $\nabla_h d$ satisfy

$$\int_0^T \|u_3\|_{L^p}^{\frac{2p}{p-3}} + \|\omega_3\|_{L^q}^{\frac{2q}{2q-3}} + \|\nabla_h d\|_{L^a}^{\frac{2a}{a-3}} \mathrm{d}t < \infty,$$
$$\text{with } \; 3 < p \le \infty, \; \frac{3}{2} < q \le \infty, \; 3 < a \le \infty.$$

**Keywords:** liquid crystal flow; velocity component; regularity criterion
**Mathematics Subject Classification:** 35B65, 35Q35, 76A15

## 1. Introduction

In this paper, we will consider the following three-dimensional (3D) nematic liquid crystal flows:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \partial_t d + u \cdot \nabla d = \gamma(\Delta d - f(d)), \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), d(x, 0) = d_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ is the velocity field, $d = (d_1, d_2, d_3) \in \mathbb{R}^3$ is the macroscopic average of molecular orientation field and $p$ represents the scalar pressure. The notation $\nabla d \odot \nabla d$ represents the

$3 \times 3$ matrix of which the $(i, j)$ entry can be denoted by

$$\sum_{k=1}^{3} \partial_i d_k \partial_j d_k (1 \le i, j \le 3),$$

and

$$f(d) = \frac{1}{|\eta|^2}(|d|^2 - 1)d.$$

$u_0$ is the initial velocity with $\nabla \cdot u_0 = 0$, $d_0$ is initial orientation vector with $|d_0| \le 1$. Here, $\mu, \lambda, \gamma, \eta$ are all positive constants. And to simplify the presentation, we shall assume that $\mu = \lambda = \gamma = \eta = 1$ in this paper.

The hydrodynamic theory of liquid crystals was established by Ericksen and Leslie during 1960s (see [4, 10]). And the system (1.1) is a simplified version of the Ericksen-Leslie model which still retains most of the essential features of the hydrodynamic equations for nematic liquid crystal (see [8]). One of the most significant studies in this area was made by Lin and Liu [9], where they established the existence of global-in-time weak solutions and local-in-time classical solutions. When the orientation field $d$ equals a constant, the above equations reduce to the incompressible Navier-Stokes equations. For well-known Prodi-Serrin type regularity criterion, people paid much focus on decomposing the integral term about $u \cdot \nabla u$ and got some improving results based on the components of velocity field $u$ and the gradient of the velocity field $\nabla u$, readers can refer to [1–3, 7, 14, 20, 21, 23, 24]. Naturally, these related results were extended to the liquid crystal flows, see [5, 6, 11, 12, 16–19, 22], and references therein. Moreover, these Prodi-Serrin type regularity criteria based on velocity field indicate that the velocity field $u$ plays a more dominate role than the orientation field $d$ does on the regularity of solutions to the system (1.1).

In [13], Qian established the regularity criterion for system (1.1). That is, if

$$\begin{aligned}\int_0^T \|u_3\|_{L^p}^q + \|\omega_3\|_{L^a}^b + \|\partial_3 u_h\|_{L^a}^b \mathrm{d}t < M, \ for\ some\ \ M > 0 \\ and\ \ \frac{3}{p} + \frac{2}{q} = 1,\ \frac{3}{a} + \frac{2}{b} = 2,\ 3 < p \le \infty, \frac{3}{2} < a \le \infty \end{aligned} \tag{1.2}$$

where $u_h = (u_1, u_2)$, $\omega_3 = \partial_1 u_2 - \partial_2 u_1$, then the solution is regular. Later, Qian [15] proved the following regularity criterion:

$$\begin{aligned}\int_0^T \|u_3\|_{L^p}^q + \|\partial_3 u_h\|_{L^a}^b + \|\nabla_h \nabla d\|_{L^a}^b \mathrm{d}t < M, \ for\ some\ \ M > 0 \\ and\ \ \frac{3}{p} + \frac{2}{q} = 1,\ \frac{3}{a} + \frac{2}{b} = 2,\ 3 < p \le \infty, \frac{3}{2} < a \le \infty. \end{aligned} \tag{1.3}$$

Inspired by the above results, we establish the following regularity criterion:

**Theorem 1.1.** *Suppose the initial data $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $d_0 \in H^2(\mathbb{R}^3)$, and let $(u, d)$ be a smooth solution to the system (1.1) on $[0, T)$ for some $0 < T < \infty$. If $(u, d)$ satisfies the following condition*

$$\int_0^T \|u_3\|_{L^p}^{\frac{2p}{p-3}} + \|\omega_3\|_{L^q}^{\frac{2q}{2q-3}} + \|\nabla_h d\|_{L^a}^{\frac{2a}{a-3}} dt < \infty, with\ \ 3 < p \le \infty,\ \frac{3}{2} < q \le \infty,\ 3 < a \le \infty, \tag{1.4}$$

then $(u, d)$ can be extended beyond $T$.

**Remark 1.1.** *In [20], Zhang has decomposed the integral* $\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx$ *into the several integrals containing* $u_3$ *and* $\omega_3$ *for the Navier-Stokes equation, and the corresponding criterion is*

$$\int_0^T \|u_3\|_{L^p}^{\frac{2p}{p-3}} + \|\omega_3\|_{L^q}^{\frac{2q}{2q-3}} dt < \infty,\ \text{with } 3 < p \le \infty,\ \frac{3}{2} < q \le \infty. \tag{1.5}$$

*So the condition on* $\partial_3 u_h$ *in (1.2) can be removed and the condition on* $\partial_3 u_h$ *in (1.3) can be replaced. And, the regularity condition of orientation field d is needed to control the term* $\nabla\cdot(\nabla d\odot\nabla d)$ *in view of (1.3).*

**Remark 1.2.** *Compared with the corresponding results (1.2), we replace the conditions on* $\partial_3 u_h$ *with* $\nabla_h d$ *because we can not control the 2-order higher derivatives term* $\nabla\cdot(\nabla d\odot\nabla d)$ *by only* $u_3$ *and* $\omega_3$*. Compared with (1.3), we reduce 1-order derivative on orientation field d, which improves the result of (1.3).*

Throughout this paper, the letter $C$ means a generic constant which may vary from line to line, and the directional derivatives of a function $\varphi$ are denoted by $\partial_i\varphi = \frac{\partial\varphi}{\partial x_i}$ $(i = 1, 2, 3)$.

## 2. Proof of Theorem 1.1

According to the local well-posedness of smooth solution established by Lin and Liu [9] , we only need to establish the priori estimates. And we have the following standard $L^2$ estimate (for example, see [17, p.2-3] )

$$(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + 2\int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \|d|\nabla d|\|_{L^2}^2 + \frac{1}{2}\|\nabla|d|^2\|_{L^2}^2)\mathrm{d}t \le C(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2). \tag{2.1}$$

By an argument similar to [17, Eq (2.7)], we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 \\
=&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x + \int_{\mathbb{R}^3}\nabla\cdot(\nabla d\odot\nabla d)\cdot\Delta u\mathrm{d}x \\
&-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla d)\cdot\Delta d\mathrm{d}x - \int_{\mathbb{R}^3}\Delta(|d|^2 d - d)\cdot\Delta d\mathrm{d}x \\
:=&I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.2}$$

In the following part, we estimate the terms above one by one. For $I_1$ referring to [20, (2.1)–(2.7)], (or see [11]), $I_1$ can be decomposed as follows:

$$\begin{aligned}
I_1 =& \sum_{i,j,k,l=1}^{3}\int_{\mathbb{R}^3}\alpha_{11ijkl}\partial_1 u_1\partial_i u_j\partial_k u_l \\
&+ \sum_{i,j,k,l=1}^{3}\int_{\mathbb{R}^3}\alpha_{12ijkl}\partial_1 u_2\partial_i u_j\partial_k u_l
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j,k,l=1}^{3}\int_{\mathbb{R}^3}\alpha_{21ijkl}\partial_2 u_1\partial_i u_j\partial_k\\
&+\sum_{i,j,k,l=1}^{3}\int_{\mathbb{R}^3}\alpha_{22ijkl}\partial_2 u_2\partial_i u_j\partial_k\\
=&I_{11}+I_{12}+I_{13}+I_{14},
\end{aligned}
$$

where $\alpha_{mnijkl},\ 1\le m,n\le 2,\ 1\le i,j,k,l\le 3$, are suitable integers. And the purpose is to rewrite $\partial_m u_n$ by $u_3$ and $\omega_3$, $1\le m,n\le 2$.

Denoting by $\Delta_h=\partial_1\partial_1+\partial_2\partial_2$ the horizontal Laplacian, and $\mathfrak{R}_m=\frac{\partial_m}{\sqrt{-\Delta_h}}$ the two-dimension Riesz transformation, it was shown in [20, (2.2)–(2.4)], that

$$
\Delta_h u_1=-\partial_2\omega_3-\partial_1\partial_3 u_3,\ \Delta_h u_2=\partial_1\omega_3-\partial_2\partial_3 u_3.
$$

$$
\partial_m u_1=\frac{\partial_2}{\sqrt{-\Delta_h}}\frac{\partial_m}{\sqrt{-\Delta_h}}\omega_3+\frac{\partial_1}{\sqrt{-\Delta_h}}\frac{\partial_m}{\sqrt{-\Delta_h}}\partial_3 u_3=\mathfrak{R}_2\mathfrak{R}_m\omega_3+\mathfrak{R}_1\mathfrak{R}_m\partial_3 u_3, \tag{2.3}
$$

$$
\partial_m u_2=\mathfrak{R}_1\mathfrak{R}_m\omega_3+\mathfrak{R}_2\mathfrak{R}_m\partial_3 u_3,\ 1\le m,n\le 2. \tag{2.4}
$$

The term $I_{11}$ can be expressed as

$$
\begin{aligned}
I_{11}=&\sum_{i,j,k,l=1}^{3}\int_{\mathbb{R}^3}\alpha_{11ijkl}\partial_1 u_1\partial_i u_j\partial_k u_l\mathrm{d}x\\
=&\sum_{i,j,k,l=1}^{3}\int_{\mathbb{R}^3}\alpha_{11ijkl}(\mathfrak{R}_2\mathfrak{R}_1\omega_3+\mathfrak{R}_1\mathfrak{R}_1\partial_3 u_3)\partial_i u_j\partial_k u_l\mathrm{d}x\\
=&\sum_{i,j,k,l=1}^{3}\int_{\mathbb{R}^3}\alpha_{11ijkl}\mathfrak{R}_2\mathfrak{R}_1\omega_3\partial_i u_j\partial_k u_l\mathrm{d}x\\
&-\sum_{i,j,k,l=1}^{3}\int_{\mathbb{R}^3}\alpha_{11ijkl}\mathfrak{R}_1\mathfrak{R}_1 u_3(\partial_3\partial_i u_j\partial_k u_l+\partial_i u_j\partial_3\partial_k u_l)\mathrm{d}x,
\end{aligned}
$$

by (2.3) and integration by parts. Because the Riesz transformation is bounded from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for $1<p<\infty$, we have

$$
\begin{aligned}
I_{11}\le&C\|u_3\|_{L^p}\|\nabla u\|_{L^{\frac{2p}{p-2}}}\|\nabla^2 u\|_{L^2}+C\|\omega_3\|_{L^q}\|\nabla u\|^2_{L^{\frac{2q}{q-1}}}\\
\le&C\|u_3\|_{L^p}\|\nabla u\|_{L^2}^{\frac{p-3}{p}}\|\nabla^2 u\|_{L^2}^{\frac{p+3}{p}}+C\|\omega_3\|_{L^q}\|\nabla u\|_{L^2}^{\frac{2q-3}{q}}\|\nabla^2 u\|_{L^2}^{\frac{3}{q}}\\
\le&C(\|u_3\|_{L^p}^{\frac{2p}{p-3}}+\|\omega_3\|_{L^q}^{\frac{2q}{2q-3}})\|\nabla u\|_{L^2}^2+\frac{1}{16}\|\Delta u\|_{L^2}^2,
\end{aligned}
$$

where $p>3, q>\frac{3}{2}$.

The similar argument as $I_{11}$ can be used to terms $I_{12}, I_{13}, I_{14}$, therefore it can be deduced that

$$I_1 \leq \quad C(\|u_3\|_{L^p}^{\frac{2p}{p-3}} + \|\omega_3\|_{L^q}^{\frac{2q}{2q-3}})\|\nabla u\|_{L^2}^2 + \tfrac{1}{4}\|\Delta u\|_{L^2}^2. \tag{2.5}$$

For $I_2$ and $I_3$, by using the fact $\nabla \cdot u = 0$ and integrating by parts several times, we can rewrite it as follows

$$\begin{aligned} I_2 + I_3 =& \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} [(\partial_i\partial_j d_k \partial_j d_k + \partial_i d_k \partial_j\partial_j d_k)\Delta u_i \\ &- (\Delta u_i \partial_i d_k \Delta d_k + 2\nabla u_i \partial_i \nabla d_k \Delta d_k + u_i \partial_i \Delta d_k \Delta d_k)]\mathrm{d}x \\ =& \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} -2\nabla u_i \partial_i \nabla d_k \Delta d_k \mathrm{d}x \\ =& \int_{\mathbb{R}^3} -2\sum_{j,k=1}^{3}\sum_{i=1}^{2} \partial_j u_i \partial_i \partial_j d_k \Delta d_k \mathrm{d}x - \int_{\mathbb{R}^3} 2\sum_{j,k=1}^{3} \partial_j u_3 \partial_3 \partial_j d_k \Delta d_k \mathrm{d}x \\ =& I_{21} + I_{22}. \end{aligned}$$

$$I_{21} = \int_{\mathbb{R}^3} 2\sum_{j,k=1}^{3}\sum_{i=1}^{2} \partial_j u_i \partial_i d_k \partial_j \Delta d_k \mathrm{d}x + \int_{\mathbb{R}^3} 2\sum_{j,k=1}^{3}\sum_{i=1}^{2} \partial_j\partial_j u_i \partial_i d_k \Delta d_k \mathrm{d}x = I_{211} + I_{212}.$$

Next, employing the Hölder inequality, interpolation inequality and Young's inequality, we have

$$\begin{aligned} I_{211} \leq& C\|\nabla_h d\|_{L^a}\|\nabla u\|_{L^{\frac{2a}{a-2}}}\|\nabla\Delta d\|_{L^2} \\ \leq& C\|\nabla_h d\|_{L^a}\|\nabla u\|_{L^2}^{\frac{a-3}{a}}\|\Delta u\|_{L^2}^{\frac{3}{a}}\|\nabla\Delta d\|_{L^2} \\ \leq& C\|\nabla_h d\|_{L^a}^{\frac{2a}{a-3}}\|\nabla u\|_{L^2}^2 + \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2, \end{aligned} \tag{2.6}$$

$$\begin{aligned} I_{212} \leq& C\|\nabla_h d\|_{L^a}\|\Delta d\|_{L^{\frac{2a}{a-2}}}\|\Delta u\|_{L^2} \\ \leq& C\|\nabla_h d\|_{L^a}\|\Delta d\|_{L^2}^{\frac{a-3}{a}}\|\nabla\Delta d\|_{L^2}^{\frac{3}{a}}\|\Delta u\|_{L^2} \\ \leq& C\|\nabla_h d\|_{L^a}^{\frac{2a}{a-3}}\|\Delta d\|_{L^2}^2 + \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2. \end{aligned} \tag{2.7}$$

In the same way, the term $I_{22}$ can be bounded as follows

$$\begin{aligned} I_{22} =& \int_{\mathbb{R}^3} 2\sum_{j,k=1}^{3}(u_3\partial_3\partial_j\partial_j d_k \Delta d_k + u_3 \partial_3 \partial_j d_k \partial_j \Delta d_k \mathrm{d}x)\mathrm{d}x \\ \leq& C\|u_3\|_{L^p}\|\Delta d\|_{L^{\frac{2p}{p-2}}}\|\nabla\Delta d\|_{L^2} \\ \leq& C\|u_3\|_{L^p}\|\Delta d\|_{L^2}^{\frac{p-3}{p}}\|\nabla\Delta d\|_{L^2}^{\frac{3}{p}}\|\nabla\Delta d\|_{L^2} \\ \leq& C\|u_3\|_{L^p}^{\frac{2p}{p-3}}\|\Delta d\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2. \end{aligned} \tag{2.8}$$

Adding the above inequalities (2.6)–(2.8) together, one obtains

$$I_2 + I_3 \leq C(\|u_3\|_{L^p}^{\frac{2p}{p-3}} + \|\nabla_h d\|_{L^a}^{\frac{2a}{a-3}})(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \frac{1}{4}\|\Delta u\|_{L^2}^2 + \frac{3}{8}\|\nabla\Delta d\|_{L^2}^2. \tag{2.9}$$

For $I_4$, we have

$$\begin{aligned} I_4 \leq& \int_{\mathbb{R}^3} |\Delta d|^2 + \Delta(|d|^2 d)\cdot\Delta d \mathrm{d}x \\ \leq& \|\Delta d\|_{L^2}^2 + C(\|\Delta|d|^2\|_{L^2}\|d\|_{L^6}\|\Delta d\|_{L^3} + \|\Delta d\|_{L^3}\|d\|_{L^6}^2\|\Delta d\|_{L^3}) \\ \leq& \|\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^3}\|d\|_{L^6}^2\|\Delta d\|_{L^3} \\ \leq& \|\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^2} \\ \leq& C\|\Delta d\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2. \end{aligned} \tag{2.10}$$

Hence, inserting (2.5), (2.9) and (2.10) into (2.2) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 \\ \leq& C(1 + \|u_3\|_{L^p}^{\frac{2p}{p-3}} + \|\omega_3\|_{L^q}^{\frac{2q}{2q-3}} + \|\nabla_h d\|_{L^a}^{\frac{2a}{a-3}})(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2), \end{aligned}$$

and it could be derived by Gronwall inequality that

$$\begin{aligned} &\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \int_0^T (\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2)\mathrm{d}t \\ \leq& (\|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2)\exp\left\{\int_0^T C(1 + \|u_3\|_{L^p}^{\frac{2p}{p-3}} + \|\omega_3\|_{L^q}^{\frac{2q}{2q-3}} + \|\nabla_h d\|_{L^a}^{\frac{2a}{a-3}})\mathrm{d}t\right\}. \end{aligned}$$

Then the proof of Theorem 1.1 is completed.

## 3. Conclusions

In this paper, we prove a regular criterion of solution for the 3D nematic liquid crystal flows via velocity component $u_3$, vorticity component $\omega_3$ and the horizontal derivative components of the orientation field $\nabla_h d$, and we hope that the condition on $\nabla_h d$ will be removed in future study.

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## Conflict of interest

All authors declare no conflict of interest in this paper.

## References


1. C. S. Cao, Sufficient conditions for the regularity to the 3D Navier-Stokes equations, *Discrete Contin. Dyn. Syst.*, **26** (2010), 1141–1151. http://dx.doi.org/10.3934/dcds.2010.26.1141
2. C. S. Cao, E. S. Titi, Regularity criteria for the three-dimensional Navier-Stokes equations, *Indiana Univ. Math. J.*, **57** (2008), 2643–2661.
3. B. Q. Dong, Z. F. Zhang, The BKM criterion for the 3D Navier-Stokes equations via two velocity components, *Nonlinear Anal.: Real World Appl.*, **11** (2010), 2415–2421. https://doi.org/10.1016/j.nonrwa.2009.07.013
4. J. L. Ericksen, Hydrostatic theory of liquid crystals, *Arch. Rational Mech. Anal.*, **9** (1962), 371–378. https://doi.org/10.1007/BF00253358
5. J. S. Fan, B. L. Guo, Regularity criterion to some liquid crystal models and the Landau-Lifshitz equations in $\mathbb{R}^3$, *Sci. China Ser. A-Math.*, **51** (2008), 1787–1797. https://doi.org/10.1007/s11425-008-0013-3
6. W. J. Gu, B. Samet, Y. Zhou, A regularity criterion for a simplified non-isothermal model for nematic liquid crystals, *Funkcial. Ekvac.*, **63** (2020), 247–258.
7. X. J. Jia, Y. Zhou, Remarks on regularity criteria for the Navier-Stokes equations via one velocity component, *Nonlinear Anal.: Real World Appl.*, **15** (2014), 239–245. https://doi.org/10.1016/j.nonrwa.2013.08.002
8. F. H. Lin, Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena, *Commun. Pure Appl. Math.*, **42** (1989), 789–814. https://doi.org/10.1002/cpa.3160420605
9. F. H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, *Commun. Pure Appl. Math.*, **48** (1995), 501–537. https://doi.org/10.1002/cpa.3160480503
10. F. M. Leslie, Some constitutive equations for liquid crystals, *Arch. Ration. Mech. Anal.*, **28** (1968), 265–283. https://doi.org/10.1007/BF00251810
11. Q. Li, B. Q. Yuan, Blow-up criterion for the 3D nematic liquid crystal flows via one velocity and vorticity components and molecular orientations, *AIMS Math.*, **5** (2020), 619–628. https://doi.org/10.3934/math.2020041
12. Q. Liu, J. H. Zhao, S. B. Cui, A regularity criterion for the three-dimensional nematic liquid crystal flow in terms of one directional derivative of the velocity, *J. Math. Phys.*, **52** (2011), 033102. https://doi.org/10.1063/1.3567170
13. C. Y. Qian, Remarks on the regularity criterion for the nematic liquid crystal flows in $\mathbb{R}^3$, *Appl. Math. Lett.*, **274** (2016), 679–689. https://doi.org/10.1016/j.amc.2015.11.007
14. C. Y. Qian, A remark on the global regularity for the 3D Navier-Stokes equations, *Appl. Math. Comput.*, **57** (2016), 126–131. https://doi.org/10.1016/j.aml.2016.01.016
15. C. Y. Qian, A further note on the regularity criterion for the 3D nematic liquid crystal flows, *Appl. Math. Comput.*, **290** (2016), 258–266. https://doi.org/10.1016/j.amc.2016.06.011
16. R. Y. Wei, Y. Li, Z. A. Yao, Two new regularity criteria for nematic liquid crystal flows, *J. Math. Anal. Appl.*, **424** (2015), 636–650. https://doi.org/10.1016/j.jmaa.2014.10.089

17. B. Q. Yuan, Q. Li, Note on global regular solution to the 3D liquid crystal equations, *Appl. Math. Lett.*, **109** (2020), 106491. https://doi.org/10.1016/j.aml.2020.106491

18. B. Q. Yuan, C. Z. Wei, BKM's criterion for the 3D nematic liquid crystal flows in Besov spaces of negative regular index, *J. Nonlinear Sci. Appl.*, **10** (2017), 3030–3037. http://dx.doi.org/10.22436/jnsa.010.06.17

19. B. Q. Yuan, C. Z. Wei, Global regularity of the generalized liquid crystal model with fractional diffusion, *J. Math. Anal. Appl.*, **467** (2018), 948–958. https://doi.org/10.1016/j.jmaa.2018.07.047

20. Z. J. Zhang, Serrin-type regularity criterion for the Navier-Stokes equations involving one velocity and one vorticity component, *Czech. Math. J.*, **68** (2018), 219–225. https://doi.org/10.21136/CMJ.2017.0419-16

21. Z. J. Zhang, F. Alzahrani, T. Hayat, Y. Zhou, Two new regularity criteria for the Navier-Stokes equations via two entries of the velocity Hessian tensor, *Appl. Math. Lett.*, **37** (2014), 124–130. https://doi.org/10.1016/j.aml.2014.06.011

22. L. L. Zhao, F. Q. Li, On the regularity criteria for 3-D liquid crystal flows in terms of the horizontal derivative components of the pressure, *J. Math. Rese. Appl.*, **40** (2020), 165–168.

23. Y. Zhou, M. Pokorný, On the regularity to the solutions of the Navier-Stokes equations via one velocity component, *Nonlinearity*, **23** (2010), 1097–1107.

24. Z. J. Zhang, W. J. Yuan, Y. Zhou, Some remarks on the Navier-Stokes equations with regularity in one direction, *Appl. Math.*, **64** (2019), 301–308. https://doi.org/10.21136/AM.2019.0264-18

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