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Research article

Additively orthodox semirings with special transversals

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Abstract: A semiring $(S, +, \cdot)$ is called additively orthodox semiring if its additive reduct $(S, +)$ is a orthodox semigroup. In this paper, by introducing some special semiring transversals as the tools, the constructions of additively orthodox semirings with a skew-ring transversal or with a generalized Clifford semiring transversal are established. Meanwhile, it is shown that an additively orthodox semiring with a generalized Clifford semiring transversal is a b-lattice of additively orthodox semirings with skew-ring transversals. Consequently, the corresponding results of Clifford semirings and generalized Clifford semirings in reference (M. K. Sen, S. K. Maity, K. P. Shum, Clifford semirings and generalized Clifford semirings, Taiwan. J. Math., 9 (2005), 433–444) and completely regular semirings in reference (S. K. Maity, M. K. Sen, K. P. Shum, On completely regular semirings, Bull. Cal. Math. Soc., 98 (2006), 319–328) are extended and strengthened.

Keywords: additively orthodox semiring; skew-ring transversal; generalized Clifford semiring transversal
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1. Introduction

A semiring $(S, +, \cdot)$ is an algebraic structure consisting of a non-empty set S together with two binary operations $+$ and $\cdot$ on S such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by distributivity, that is, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. A semiring $(S, +, \cdot)$ is called a b-lattice [22] if its additive reduct $(S, +)$ is a semilattice and its multiplicative reduct $(S, \cdot)$ is a band. Also, a skew-ring [22] $(S, +, \cdot)$ means a semiring whose additive reduct $(S, +)$ is a group, not necessarily an abelian group.

The algebraic theory of semirings has some important applications in automata theory, optimization theory and models of discrete event networks, etc. [11].

As one of the development power of semiring theory, semirings were studied by various researchers using techniques coming from semigroup theory. Many authors extended the concepts and results of semigroups to semirings in recent several decades. For instance, Zeleznikow [25] studied the regular semirings in which both additive and multiplicative semigroups are regular. Sen, Maity and Shum [22] defined the Clifford semiring which is a completely regular and an additively inverse semiring such that the set of its additive idempotents is a distributive sublattice (without assuming that its additive reduct is commutative), and then verified that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of skew-rings. Moreover, they introduced generalized Clifford semirings which are completely regular and inverse semirings such that its additive idempotent set is a k-ideal, proved that a semiring S is a generalized Clifford semiring if and only if S is a strong b-lattice of skew-rings, and if and only if it is an additively inverse semiring and is a subdirect product of a b-lattice and a skew-ring. Maity, Sen and Shum [14] had also extended completely regular semigroups to completely regular semirings. Karvellas [13] introduced the additively inverse semirings whose additive reduct is inverse semigroup. For more literatures, we can see [7, 8, 10, 14, 15, 18, 21–23], etc.

As we know, the ideas of transversals are very important to study the structure of semigroups. Blyth and McFadden [4] introduced the notion of transversals, and they give a complete description of the regular semigroups with multiplicative inverse transversals. From then on, there are many papers devoted to inverse transversal [2, 16, 17, 19, 20], etc. For instance, McAlister and McFadden [16] showed how to construct a regular semigroup containing a quasi-ideal inverse transversal. Saito [20] gave the construction method of a regular semigroup having inverse transversals. For generalizing the concept of inverse transversal, Chen [5] introduced the orthodox transversals for regular semigroups and obtained a structure theorem for the regular semigroups with quasi-ideal orthodox transversals. As an analog of the inverse transversal in classes of abundant semigroups, El-Qallali [6] introduced the concept of adequate transversal for abundant semigroups. Furthermore, he established the structure of abundant semigroups with a multiplicative type A transversal which satisfies the regularity condition. Guo [9] investigated abundant semigroups with a multiplicative adequate transversal and established a structure of this class of abundant semigroups.

In this paper, our aim is to introduce the idea of transversals in additively orthodox semiring and to establish the constructions of additively orthodox semirings with a skew-ring transversal or with a generalized Clifford semiring transversal. In Section 2, we firstly introduce the concept of additively inverse semiring transversals and recall some results of regular semigroups with an inverse transversal. In Section 3, we describe a construction of additively regular semirings with a skew-ring transversal. In Section 4, we show that an additively orthodox semiring with a generalized Clifford semiring transversal is a b-lattice of additively regular semirings with skew-ring transversals and describe a construction of additively orthodox semirings with a generalized Clifford semiring transversal. Consequently, the corresponding results of Clifford semirings and generalized Clifford semirings in [22], completely regular semirings in [14] are extended and strengthened.

For the undefined notion and notations about semigroups, readers can be referred to [1, 11, 12, 20, 22, 24].

2. Preliminaries

A semiring $(S, +, \cdot)$ is called additively regular semiring (additively orthodox semiring, additively inverse semiring, respectively) if its additive reduct $(S, +)$ is a regular (orthodox, inverse, respectively) semigroup. For the sake of brevity, we sometimes just denote the semiring $(S, +, \cdot)$ by S.

Let $(S, +, \cdot)$ be a semiring. We denote the Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$ on additive reduct $(S, +)$ by $\overset{+}{\mathcal{L}}, \overset{+}{\mathcal{R}}, \overset{+}{\mathcal{H}}, \overset{+}{\mathcal{D}}, \overset{+}{\mathcal{J}}$, the set of all additive idempotents of S by $E^+(S)$, and the set of additive inverses by $V^+(x)$ for each $x \in S$.

Firstly we introduce the concept of additively inverse semiring transversal and recall some results of regular semigroups with an inverse transversal.

Definition 2.1. *Let S be an additively regular semiring, and S° an additively inverse subsemiring of S. We call S° is an additively inverse semiring transversal of S if it contains precisely one additive inverse of each $x \in S$. And denote the uniquely additive inverse by x° and $(x^\circ)^\circ$ by $x^{\circ\circ}$.*

Obviously, if an additively regular semiring S has an additively inverse semiring transversal S°, then $(S^\circ, +)$ is an inverse transversal of $(S, +)$. According to some results listed in [1, 3, 20, 24], the following lemmas are true :

Lemma 2.1. *Let S be an additively regular semiring with an additively inverse semiring transversal S°. Then,*

(1) for all $x \in S$, $x^{\circ\circ\circ} = x^\circ$;

(2) for all $x, y \in S$, $(x^\circ + y)^\circ = y^\circ + x^{\circ\circ}$, $(x + y^\circ)^\circ = y^{\circ\circ} + x^\circ$;

(3) $(S, +)$ is an orthodox semigroup if and only if $(x + y)^\circ = y^\circ + x^\circ$ for all $x, y \in S$.

The following important results are also from [1, 3, 20, 24]:

$$L_\circ \triangleq \{x \in S : x = x + x^\circ + x^{\circ\circ}\} = \{x \in S : x^\circ + x = x^\circ + x^{\circ\circ}\} = \cup_{e \in E^+(S^\circ)} \overset{+}{L}_e,$$

$$R_\circ \triangleq \{x \in S : x = x^{\circ\circ} + x^\circ + x\} = \{x \in S : x + x^\circ = x^{\circ\circ} + x^\circ\} = \cup_{e \in E^+(S^\circ)} \overset{+}{R}_e.$$

$(L_\circ, +)$ and $(R_\circ, +)$ are respectively left and right inverse subsemigroups of $(S, +)$ and $L_\circ \cap R_\circ = S^\circ$.

$$I_\circ \triangleq \{x \in S : x = x + x^\circ\} = \{x \in S : x^\circ = x^\circ + x\} = \{x + x^\circ : x \in S\} = E^+(L_\circ),$$

$$\Lambda_\circ \triangleq \{x \in S : x = x^\circ + x\} = \{x \in S : x^\circ = x + x^\circ\} = \{x^\circ + x : x \in S\} = E^+(R_\circ).$$

$(I_\circ, +)$ and $(\Lambda_\circ, +)$ are a left regular band and a right regular band respectively and $I_\circ \cap \Lambda_\circ = E^+(S^\circ)$.

Lemma 2.2. *Let $(S, +, \cdot)$ be an additively regular semiring with an additively inverse semiring transversal S°. Then, $L_\circ$, $R_\circ$, $I_\circ$ and $\Lambda_\circ$ are subsemirings of S.*

Proof. For all $x, y \in L_\circ$, there exist $e, f \in E^+(S^\circ)$, such that $x \overset{+}{\mathcal{L}} e$ and $y \overset{+}{\mathcal{L}} f$. Note that $\overset{+}{\mathcal{L}}$ is multiplicative congruence on $(S, \cdot)$. We have $xy \overset{+}{\mathcal{L}} ef \in E^+(S^\circ)$ and so $xy \in L_\circ$. □

Lemma 2.3. *(Theorem 14.7 [1])* $(\forall x, y \in S)$ *If $x \in L_\circ$(or $y \in R_\circ$) then $(x + y)^\circ = y^\circ + x^\circ$.*

Lemma 2.4. *Let $(S,+,\cdot)$ be an additively regular semiring with an additively inverse semiring transversal S°. Then, for all $x,y\in S$,*

(1) $(x^\circ y)^\circ=(xy^\circ)^\circ=x^\circ y^\circ$ and $x^{\circ\circ}y^\circ=(x^\circ y^\circ)^\circ=x^\circ y^{\circ\circ}$;

(2) If $x,y\in L_\circ$(or $x,y\in R_\circ$), then $(xy)^\circ=x^\circ y^{\circ\circ}(=x^{\circ\circ}y^\circ)$ and $(xy)^{\circ\circ}=x^{\circ\circ}y^{\circ\circ}=x^\circ y^{\circ\circ\circ}=x^{\circ\circ\circ}y^\circ=x^\circ y^\circ$.

Proof. (1) By $x^\circ y+x^\circ y^\circ+x^\circ y=x^\circ(y+y^\circ+y)=x^\circ y$ and $x^\circ y^\circ+x^\circ y+x^\circ y^\circ=x^\circ(y^\circ+y+y^\circ)=x^\circ y^\circ$, we have that $x^\circ y^\circ\in V^+(x^\circ y)\cap S^\circ$, Thus $(x^\circ y)^\circ=x^\circ y^\circ$. Similarly, $(xy^\circ)^\circ=x^\circ y^\circ$ and $x^{\circ\circ}y^\circ=(x^\circ y^\circ)^\circ=x^\circ y^{\circ\circ}$.

(2) By (1) and Lemma 2.3, we have

$$\begin{aligned}
\begin{cases} xy+xy^\circ+xy = xy \\ xy^\circ+xy+xy^\circ = xy^\circ \end{cases}
&\Rightarrow \begin{cases} (xy+xy^\circ+xy)^\circ = (xy)^\circ \\ (xy^\circ+xy+xy^\circ)^\circ = (xy^\circ)^\circ \end{cases}\\
&\Rightarrow \begin{cases} (xy)^\circ+(xy+xy^\circ)^\circ = (xy)^\circ \\ (xy^\circ)^\circ+(xy^\circ+xy)^\circ = x^\circ y^\circ \end{cases}\\
&\Rightarrow \begin{cases} (xy)^\circ+(xy^\circ)^\circ+(xy)^\circ = (xy)^\circ \\ x^\circ y^\circ+(xy)^\circ+(xy^\circ)^\circ = x^\circ y^\circ \end{cases}\\
&\Rightarrow \begin{cases} (xy)^\circ+x^\circ y^\circ+(xy)^\circ = (xy)^\circ \\ x^\circ y^\circ+(xy)^\circ+x^\circ y^\circ = x^\circ y^\circ \end{cases}\\
&\Rightarrow (xy)^\circ=(x^\circ y^\circ)^\circ=x^\circ y^{\circ\circ}(=x^{\circ\circ}y^\circ).
\end{aligned}$$

It is easy to verify the second part. □

It is well known that the inverse transversals of a regular semigroup are mutually isomorphic [3]. Actively, the additively inverse semiring transversals of an additively regular semiring are also mutually isomorphic.

Theorem 2.1. *Let S° and S^* to be the additively inverse semiring transversals of additively regular semiring S. Then $S^\circ\cong S^*$.*

Proof. Let $S^\#=L_\circ\cap R_*$. By Theorem 2 and Theorem 3 in [3], $(S^\#,+)$ is an inverse transversal of $(S,+)$ with $(L_\#,+)=(L_\circ,+)$ and $(R_\#,+)=(R_*,+)$. Moreover, $S^\#$ forms a subsemiring of S, so $S^\#$ is an additively inverse semiring transversal of S with $L_\#=L_\circ$ and $R_\#=R_*$. Define a mapping $\varphi_{\#*}:S^\#\to S^*, x\mapsto x^{**}$. By Theorem 2 in [3], $\varphi_{\#*}$ is an isomorphism from $(S^\#,+)$ onto $(S^*,+)$. Now, for $x,y\in S^*$, then $x,y\in R_*=R_\#$, by Lemma 2.3 and Lemma 2.4, $\varphi_{\#*}(xy)=(xy)^{**}=x^{**}y^{**}=\varphi_{\#*}(x)\varphi_{\#*}(y)$. Hence $S^\#\cong S^*$. Similarly, we can show that $S^\#\cong S^\circ$. Therefore, $S^\circ\cong S^\#\cong S^*$. □

Lemma 2.5. *For $x,y\in S$, if $\overset{+}{H}_x$ and $\overset{+}{H}_y$ are respectively contain an additive idempotent 0_x and 0_y, then $\overset{+}{H}_{xy}$contains an additive idempotent $0_{xy}=x0_y=0_x0_y=0_xy$.*

Proof. Since $\overset{+}{\mathcal{H}}$ is a multiplicative congruence on S, then $x\overset{+}{\mathcal{H}}0_x$ and $y\overset{+}{\mathcal{H}}0_y$ imply that $0_xy, 0_x0_y, x0_y\in\overset{+}{H}_{xy}\cap E^+(S)$, and so $x0_y=0_x0_y=0_{xy}=0_xy$. □

3. Additively orthodox semiring with a skew-ring transversal

In this section, we will consider additively orthodox semirings with a skew-ring transversal. For brevity, throughout this section we always assume that S is an additively orthodox semiring with a skew-ring transversal S°. Hence $(S^\circ, +)$ is a group transversal of $(S, +)$. We denote the identity element in $(S^\circ, +)$ by 0°. $(S, +)$ is an orthodox semigroup, which means that $E^+(S)$ forms a subsemigroup of $(S, +)$. Moreover, $E^+(S) \cdot S \subseteqq E^+(S)$, $S \cdot E^+(S) \subseteqq E^+(S)$. Consequently, $E^+(S)$ is an ideal of S. What is more, S° is a $\overset{+}{\mathcal{H}}$-class of S, and $\overset{+}{\mathcal{D}} = S \times S$ since $x\overset{+}{\mathcal{D}}x^\circ$ for each x in S. Therefore S has only one $\overset{+}{\mathcal{D}}$-class.

Proposition 3.1. *Let S be an additively orthodox semiring with a skew-ring transversal S°. Then we have the following statements.*

(1) For any $x, y \in S$, $\overset{+}{H}_x \cap E^+(S) \neq \emptyset$, and $(x + y) \in \overset{+}{R}_x \cap \overset{+}{L}_y$;

(2) For any $e \in I_\circ, f \in \Lambda_\circ$, $f + e = 0^\circ$;

(3) For any $e \in I_\circ, f \in \Lambda_\circ$, $fe = 0^\circ e + f0^\circ$ and $ef = e0^\circ + 0^\circ f$.

Proof. (1) For any $x \in S$, $(x + x^\circ) + (x^\circ + x) \in E^+(S)$ since $(S, +)$ is an orthodox semigroup. Moreover, $\overset{+}{L}_{x+x^\circ} \cap \overset{+}{R}_{x^\circ+x}$ contains an additive idempotent 0°, then $(x + x^\circ) + (x^\circ + x) \in \overset{+}{R}_{x+x^\circ} \cap \overset{+}{L}_{x^\circ+x} = \overset{+}{H}_x$.

Hence, $\overset{+}{L}_y \cap \overset{+}{R}_x$ contains an additive idempotent for any $x, y \in S$. Therefore, $(x + y) \in \overset{+}{R}_x \cap \overset{+}{L}_y$ by Proposition 2.3.7 in [12].

(2) For any $e \in I_\circ = E^+(L_\circ) = \overset{+}{L}_{0^\circ}, f \in \Lambda_\circ = E^+(R_\circ) = \overset{+}{R}_{0^\circ}$, $(f + e) \in E^+(S)$ since $(S, +)$ is an orthodox semigroup. Moreover, by (1), $(f + e) \in \overset{+}{R}_f \cap \overset{+}{L}_e = \overset{+}{R}_{0^\circ} \cap \overset{+}{L}_{0^\circ} = \overset{+}{H}_{0^\circ}$. Therefore, $f + e = 0^\circ$.

(3) Note that $\overset{+}{\mathcal{R}}$ and $\overset{+}{\mathcal{L}}$ are the multiplicative congruences on S. Then $0^\circ\overset{+}{\mathcal{R}}f$ implies that $e0^\circ\overset{+}{\mathcal{R}}ef$ and $0^\circ\overset{+}{\mathcal{L}}e$ implies that $0^\circ f\overset{+}{\mathcal{L}}ef$. Hence, by (1), $(e0^\circ + 0^\circ f) \in \overset{+}{R}_{e0^\circ} \cap \overset{+}{L}_{0^\circ f} = \overset{+}{R}_{ef} \cap \overset{+}{L}_{ef} = \overset{+}{H}_{ef}$. Moreover, $ef, e0^\circ + 0^\circ f \in E^+(S)$. Therefore, $ef = e0^\circ + 0^\circ f$. Similarly, we can get that $fe = 0^\circ e + f0^\circ$. □

Theorem 3.1. *Let G be a skew-ring, I be a semiring whose additive reduct is a right zero band with multiplicative idempotents and Λ a left zero band with multiplicative idempotents. Define the addition and multiplication on the set*

$$W = I \times G \times \Lambda$$

by

$$(i, x, \lambda) + (j, y, \mu) = (i, x + y, \mu)$$

and

$$(i, x, \lambda) \cdot (j, y, \mu) = (ij, xy, \lambda\mu).$$

Then W is an additively orthodox semiring with a skew-ring transversal isomorphic to G. Conversely, every additively orthodox semiring with a skew-ring transversal can be constructed in this way.

Proof. Firstly, we prove that W is a semiring. It is clear that $(W,+)$ and $(W,\cdot)$ are semigroups. For any $(i, x, \lambda), (j, y, \mu), (k, z, \nu) \in W$,

$$\begin{aligned}
&(i,x,\lambda)\cdot[(j,y,\mu)+(k,z,\nu)]\\
&=(i,x,\lambda)\cdot(j,y+z,\nu)\\
&=(ij,x(y+z),\lambda\nu)\\
&=(ij,xy+xz,\lambda\nu)\\
&=(ij,xy,\lambda\mu)+(ik,xz,\lambda\nu)\\
&=(i,x,\lambda)\cdot(j,y,\mu)+(i,x,\lambda)\cdot(k,z,\nu).
\end{aligned}$$

Similarly, we get that

$$[(j,y,\mu)+(k,z,\nu)]\cdot(i,x,\lambda)=(j,y,\mu)\cdot(i,x,\lambda)+(k,z,\nu)\cdot(i,x,\lambda).$$

So W is a semiring.

Furthermore, we will show that the additive reduct of W is orthodox semigroup. For each $(i,x,\lambda)\in W$, there exist $(i,x^\circ,\lambda)\in W$ (where x° stands for the additive inverse of x in G), such that $(i,x,\lambda)+(i,x^\circ,\lambda)+(i,x,\lambda)=(i,x+x^\circ+x,\lambda)=(i,x,\lambda)$. So $(W,+)$ is regular semigroup. Moreover, $(i,x,\lambda)\in E^+(W)$ if and only if x is the identity in G (denoted by 0°) of $(W,+)$. Then $(E^+(W),+)$ is closed under addition since $(i,0^\circ,\lambda)+(j,0^\circ,\mu)=(i,0^\circ,\mu)\in E^+(W)$. Hence $(W,+)$ is orthodox semigroup as required.

Choose one multiplicative idempotent $i\in I$ and $\lambda\in\Lambda$ respectively, and then let $W^\circ=\{(0_i,x,0_\lambda)\in W : x\in G\}$. Now, define a mapping $\varphi: W^\circ\to G,(i,x,\lambda)\mapsto x$. It is clear that φ is bijection from W° onto G. For any $(i,x,\lambda),(i,y,\lambda)\in W^\circ$,

$$\varphi((i,x,\lambda)+(i,y,\lambda))=\varphi((i,x+y,\lambda))=x+y=\varphi((i,x,\lambda))+\varphi((i,y,\lambda))$$

and

$$\varphi((i,x,\lambda)\cdot(i,y,\lambda))=\varphi((i\cdot i,x\cdot y,\lambda\cdot\lambda))=\varphi((i,x\cdot y,\lambda))=x\cdot y=\varphi((i,x,\lambda))\cdot\varphi((i,y,\lambda)).$$

Consequently, $W^\circ\cong G$. So W° is a sub-skew-ring of W.

Next, we prove that W° is a sub-skew-ring transversal of W. For any $(j,y,\mu),(i,x,\lambda)\in W$,

$$\begin{aligned}
(i,x,\lambda)\in V^+((j,y,\mu)) &\Leftrightarrow \begin{cases}(j,y,\mu)+(i,x,\lambda)+(j,y,\mu) &= (j,y,\mu)\\ (i,x,\lambda)+(j,y,\mu)+(i,x,\lambda) &= (i,x,\lambda)\end{cases}\\
&\Leftrightarrow \begin{cases}(j,y+x+y,\mu) &= (j,y,\mu)\\ (i,x+y+x,\lambda) &= (i,x,\lambda)\end{cases}\\
&\Leftrightarrow \begin{cases}y+x+y &= y\\ x+y+x &= x\end{cases}\\
&\Leftrightarrow\ x=y^\circ.
\end{aligned}$$

So (j,y,μ) has exactly one additive inverse $(i,y^\circ,\lambda)\in W^\circ$. Hence W° is a sub-skew-ring transversal of W as required.

Conversely, let S be an additively orthodox semiring with a skew-ring transversal S°. By Lemma 2.2, $(I_\circ,+)$ is a semiring whose additive reduct is a right zero band with multiplicative idempotents 0° and $(\Lambda_\circ, +)$ a left zero band with multiplicative idempotents 0°. Thus we can construct a semiring $W = I_\circ \times S^\circ \times \Lambda_\circ$ with the addition and multiplication by the rules

$$(0_i, x, 0_\lambda) + (0_j, y, 0_\mu) = (0_i, x + y, 0_\mu)$$

and

$$(0_i, x, 0_\lambda) \cdot (0_j, y, 0_\mu) = (0_i 0_j, xy, 0_\lambda 0_\mu).$$

Now, we define a mapping φ from W to S such that $\varphi((0_i, x, 0_\lambda)) = 0_i + x + 0_\lambda$. For any $x \in S, x + x^\circ + x^{\circ\circ} + x^\circ + x = x$ and so $\varphi((x + x^\circ, x^{\circ\circ}, x^\circ + x)) = x$. Then φ is surjective from W onto S. If $\varphi((0_i, x, 0_\lambda)) = \varphi((0_j, y, 0_\mu))$ for $(0_i, x, 0_\lambda), (0_j, y, 0_\mu) \in W$, then $0_i + x + 0_\lambda = 0_j + y + 0_\mu$. Hence,

$$\begin{aligned}
& (0_i + x + 0_\lambda)^{\circ\circ} = (0_j + y + 0_\mu)^{\circ\circ} \\
\Rightarrow\ & 0_i^{\circ\circ} + x^{\circ\circ} + 0_\lambda^{\circ\circ} = 0_j^{\circ\circ} + y^{\circ\circ} + 0_\mu^{\circ\circ} \quad \text{(by Lemma 2.1 (3))} \\
\Rightarrow\ & 0^\circ + x^{\circ\circ} + 0^\circ = 0^\circ + y^{\circ\circ} + 0^\circ \\
\Rightarrow\ & x^{\circ\circ} = y^{\circ\circ} \Rightarrow x = y.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
0_i &= 0_i + x + 0_\lambda + 0_\lambda^\circ + x^\circ + 0_i^\circ \\
&= (0_i + x + 0_\lambda) + (0_i + x + 0_\lambda)^\circ \\
&= (0_j + y + 0_\mu) + (0_j + y + 0_\mu)^\circ \\
&= (0_j + y + 0_\mu) + (0_\mu^\circ + y^\circ + 0_j^\circ) \\
&= 0_j.
\end{aligned}$$

Similarly, we can get that $0_\lambda = 0_\mu$. It is clear that φ is bijective.

Finally,

$$\begin{aligned}
& \varphi((0_i, x, 0_\lambda) + (0_j, y, 0_\mu)) \\
&= \varphi((0_i, x + y, 0_\mu)) \\
&= 0_i + x + y + 0_\mu \\
&= 0_i + x + 0^\circ + y + 0_\mu \\
&= 0_i + x + 0_\lambda + 0_j + y + 0_\mu \quad \text{(by Proposition 3.1 (2))} \\
&= \varphi((0_i, x, 0_\lambda)) + \varphi((0_j, y, 0_\mu)),
\end{aligned}$$

and by Proposition 3.1,

$$
\begin{aligned}
&\varphi((0_i, x, 0_\lambda) \cdot (0_j, y, 0_\mu)) \\
&= \varphi((0_i0_j, xy, 0_\lambda 0_\mu)) \\
&= 0_i0_j + xy + 0_\lambda 0_\mu \\
&= 0_i(0_j + 0^\circ) + 0^\circ(0_\mu + 0_j) + xy + 0^\circ(0_\mu + 0_j) + 0_\lambda(0^\circ + 0_\mu) \\
&= 0_i0_j + (0_i0^\circ + 0^\circ 0_\mu) + 0^\circ 0_j + xy + 0^\circ 0_\mu + (0^\circ 0_j + 0_\lambda 0^\circ) + 0_\lambda 0_\mu \\
&= 0_i0_j + 0_i0_\mu + 0^\circ 0_j + xy + 0^\circ 0_\mu + 0_\lambda 0_j + 0_\lambda 0_\mu \\
&= 0_i0_j + 0_i(0^\circ + 0_\mu) + 0^\circ 0_j + xy + 0^\circ 0_\mu + 0_\lambda(0_j + 0^\circ) + 0_\lambda 0_\mu \\
&= 0_i0_j + 0_i0^\circ + 0_i0_\mu + 0^\circ 0_j + xy + 0^\circ 0_\mu + 0_\lambda 0_j + 0_\lambda 0^\circ + 0_\lambda 0_\mu \\
&= 0_i0_j + 0_iy + 0_i0_\mu + x0_j + xy + x0_\mu + 0_\lambda 0_j + 0_\lambda y + 0_\lambda 0_\mu \\
&= (0_i + x + 0_\lambda) \cdot (0_j + y + 0_\mu) \\
&= \varphi((0_i, x, 0_\lambda)) \cdot \varphi((0_j, y, 0_\mu)).
\end{aligned}
$$

Therefore, φ is a semiring isomorphism. □

4. Additively orthodox semiring with a generalized Clifford semiring transversal

In this section, we will consider additively orthodox semirings with a generalized Clifford semiring transversal. From now on, S always denotes an additively orthodox semiring with a generalized Clifford semiring transversal S°.

Since S° is a generalized Clifford semiring, then it is a strong b-lattice of skew-rings $S^\circ = < Y, S^\circ_\alpha, \phi_{\alpha,\beta} >$, where S°_α is a sub-skew-ring and a $\overset{+}{\mathcal{H}}$-class of S° for every $\alpha \in Y$([22]). We denote the uniquely additive idempotent in S°_α by 0_α, then $0_\alpha + 0_\beta = 0_{\alpha+\beta}$ and $0_\alpha 0_\beta = 0_{\alpha\beta}$.

Since $\overset{+}{\mathcal{H}}(S^\circ) \subseteq \overset{+}{\mathcal{H}}(S)$, $\overset{+}{\mathcal{H}}_{0_\alpha}(S^\circ) \subseteq \overset{+}{\mathcal{H}}_{0_\alpha}(S)$. On the other hand, $\overset{+}{\mathcal{H}}_{0_\alpha}(S^\circ) \subseteq \overset{+}{\mathcal{L}}_{0_\alpha}(S) \cap \overset{+}{\mathcal{R}}_{0_\alpha}(S) \subseteq \mathcal{L}_\circ(S) \cap \mathcal{R}_\circ(S) = S^\circ$, and so $\overset{+}{\mathcal{H}}_{0_\alpha}(S)$ is a sub-skew-ring of S°. Then $\overset{+}{\mathcal{H}}_{0_\alpha}(S) \subseteq \overset{+}{\mathcal{H}}_{0_\alpha}(S^\circ)$. Therefore, $\overset{+}{\mathcal{H}}_{0_\alpha}(S^\circ) = \overset{+}{\mathcal{H}}_{0_\alpha}(S)$, which means that S_α is sub-skew-ring and $\overset{+}{\mathcal{H}}$-class of S.

Now, we will show that every $\overset{+}{\mathcal{D}}$-class of S contains only one S°_α.

Proposition 4.1. *For all $a \in S$, $|\overset{+}{D}_a \cap E^+(S^\circ)| = 1$.*

Proof. For any $a \in S$, $(a + a^\circ)^\circ \in \overset{+}{D}_a \cap E^+(S^\circ)$ implies that $\overset{+}{D}_a \cap E^+(S^\circ) \neq \emptyset$. Let $0^\circ_\alpha, 0^\circ_\beta \in \overset{+}{D}_a \cap E^+(S^\circ)$. Clearly, $\overset{+}{L}_{0^\circ_\alpha} \cap \overset{+}{R}_{0^\circ_\beta} \subseteq L_\circ \cap R_\circ = S^\circ$. For $b^\circ \in \overset{+}{L}_{0^\circ_\alpha} \cap \overset{+}{R}_{0^\circ_\beta}$ we have $b^{\circ\circ} \in \overset{+}{L}_{0^\circ_\alpha} \cap \overset{+}{R}_{0^\circ_\beta}$ such that $b^\circ + b^{\circ\circ} = 0^\circ_\alpha$ and $b^{\circ\circ} + b^\circ = 0^\circ_\beta$. Hence, $0^\circ_\alpha = b^\circ + b^{\circ\circ} = b^{\circ\circ} + b^\circ = 0^\circ_\beta$. That is, $|\overset{+}{D}_a \cap E^+(S^\circ)| = 1$ as required. □

Since every $\overset{+}{\mathcal{D}}$-class contains only one S°_α, we can denote the $\overset{+}{\mathcal{D}}$-class which contains S°_α by $\overset{+}{D}_\alpha$.

Proposition 4.2. *(1) $\overset{+}{\mathcal{D}}$ is a congruence on semiring S, and $S/\overset{+}{\mathcal{D}} \cong E^+(S^\circ) \cong Y$;*

(2) $\overset{+}{D}_\alpha$ is a subsemiring of S for every $\alpha \in Y$.

Proof. (1) For $a\overset{+}{\mathcal{D}}0^\circ_\alpha$ and $b\overset{+}{\mathcal{D}}0^\circ_\beta$, $(a+b)\overset{+}{\mathcal{D}}(a+b)^\circ = (b^\circ + a^\circ) \in S^\circ_{\alpha+\beta}$ implies that $(a+b)\overset{+}{\mathcal{D}}0^\circ_{\alpha+\beta} = 0^\circ_\alpha + 0^\circ_\beta$.

(2) For any $a, b \in \overset{+}{D}_\alpha$, $(a+b)\overset{+}{\mathcal{D}}(a+b)^\circ = (b^\circ + a^\circ) \in \overset{+}{H}_{0^\circ_\alpha} \subseteq \overset{+}{\mathcal{D}}_\alpha$. Furthermore, $a\overset{+}{\mathcal{D}}0^\circ_\alpha, b\overset{+}{\mathcal{D}}0^\circ_\alpha$ implies that $ab\overset{+}{\mathcal{D}}0^\circ_\alpha 0^\circ_\alpha = 0^\circ_\alpha$, that is, $ab \in \overset{+}{D}_\alpha$. □

Since S°_α is a sub-skew-ring of $\overset{+}{D}_\alpha$ and contains the unique additive inverse x° for every $x \in \overset{+}{D}_\alpha$, S°_α is a skew-ring transversal of $\overset{+}{D}_\alpha$.

Now, we can get an interesting theorem as follow:

Theorem 4.1. *S is a b-lattice of additively orthodox semirings with skew-ring transversals.*

Theorem 4.1 gives a "gross" structure of S but not its "fine" structure. Now we will study a "fine structure" of S in another way. To achieve our aim, we need the following lemmas.

Lemma 4.1. *For each $0_\alpha \in E^+(S^\circ)$, let $I_\alpha = \{0_i \in I : 0^\circ_i = 0_\alpha\}$ and $\Lambda_\alpha = \{0_\lambda \in \Lambda : 0^\circ_\lambda = 0_\alpha\}$. Then*

(1) I_α [$resp.\Lambda_\alpha$] is an additively right [$resp.left$] zero semiring;

(2) For all $0_i \in I_\alpha, 0_j \in I_\beta$ [$0_\lambda \in \Lambda_\alpha, 0_\mu \in \Lambda_\beta$], $0_i + 0_j \in I_{\alpha+\beta}$, $0_i \cdot 0_j \in I_{\alpha\cdot\beta}$, [$0_\lambda + 0_\mu \in \Lambda_{\alpha+\beta}, 0_\lambda \cdot 0_\mu \in \Lambda_{\alpha\cdot\beta}$];

(3) $I_\circ = \sum\{I_\alpha : 0_\alpha \in E^+(S^\circ)\}$ and $\Lambda_\circ = \sum\{\Lambda_\alpha : 0_\alpha \in E^+(S^\circ)\}$, where $\sum$ denotes disjoint union.

Proof. (1) For all $0_i, 0_j \in I_\alpha$, $0_i + 0_j = 0_i$. And by Lemma 2.4, $(0_i 0_j)^\circ = 0^\circ_i 0^{\circ\circ}_j = 0_\alpha$ implies that $0_i 0_j \in I_\alpha$.

(2) For all $0_i \in I_\alpha, 0_j \in I_\beta$, by Theorem 14.7 in [1], $(0_i + 0_j)^\circ = 0^\circ_j + 0^\circ_i = 0_\beta + 0_\alpha = 0_{\beta+\alpha} = 0_{\alpha+\beta}$. By Lemma 2.4, $(0_i 0_j)^\circ = 0^\circ_i 0^{\circ\circ}_j = 0_\alpha 0_\beta = 0_{\alpha\beta}$.

(3) Clear. □

Since $0_i \in I_\alpha$ if and only if $0_i\overset{+}{\mathcal{L}}0_\alpha$, by Lemma 4.1, $\overset{+}{\mathcal{L}}$ is a semiring congruence on subsemiring $I_\circ$, and $I_\circ/\overset{+}{\mathcal{L}} \cong E^+(S^\circ)$, that is, $I_\circ$ is a b-lattice of additively right zero semirings $\{I_\alpha : 0_\alpha \in E^+(S^\circ)\}$. Similarly, $\Lambda_\circ$ is a b-lattice of additively left zero semirings $\{\Lambda_\alpha : 0_\alpha \in E^+(S^\circ)\}$.

Lemma 4.2. *For each $(x, y) \in S^\circ \times S^\circ$, let $\alpha_{(x,y)} : \Lambda_{x^\circ+x} \times I_{y+y^\circ} \to I_\circ$ and $\beta_{(x,y)} : \Lambda_{x^\circ+x} \times I_{y+y^\circ} \to \Lambda_\circ$ be mappings defined by $(0_\lambda, 0_i)\alpha_{(x,y)} = x + 0_\lambda + 0_i + x^\circ$ and $(0_\lambda, 0_i)\beta_{(x,y)} = y^\circ + 0_\lambda + 0_i + y$. Then,*

(1) $(0_\lambda, 0_i)\alpha_{(x,y)} \in I_{x+y+(x+y)^\circ}$ and $(0_\lambda, 0_i)\beta_{(x,y)} \in \Lambda_{(x+y)^\circ+x+y}$;

(2) for $0_\lambda \in \Lambda_{x^\circ+x}$, $0_j \in I_{y+y^\circ}$, $0_\mu \in \Lambda_{y^\circ+y}$, $0_k \in I_{z+z^\circ}$,

$$(0_\lambda, 0_j)\alpha_{(x,y)} + ((0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu, 0_k)\alpha_{(x+y,z)} = (0_\lambda, 0_j + (0_\mu, 0_k)\alpha_{(y,z)})\alpha_{(x,y+z)}$$

and

$$(0_\lambda, 0_j + (0_\mu, 0_k)\alpha_{(y,z)})\beta_{(x,y+z)} + (0_\mu, 0_k)\beta_{(y,z)} = ((0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu, 0_k)\beta_{(x+y,z)};$$

(3) $(x^\circ + x, y + y^\circ)\alpha_{(x,y)} = x + y + (x + y)^\circ$ and $(x^\circ + x, y + y^\circ)\beta_{(x,y)} = (x + y)^\circ + x + y$.

Proof. It is from Lemma 2.4 in [20]. □

Let M and N be two sets. A partial mapping from M to N is a mapping from a subset C of M into N. The set of all partial mappings from M to N is denoted by $PT(M, N)$. Then, by Lemma 4.2, $\alpha_{(x,y)} \in PT(\Lambda_\circ \times I_\circ, I_\circ)$ and $\beta_{(x,y)} \in PT(\Lambda_\circ \times I_\circ, \Lambda_\circ)$ with $dom(\alpha_{(x,y)}) = dom(\beta_{(x,y)}) = \Lambda_{x^\circ+x} \times I_{y+y^\circ}$.

By Lemma 4.2, it is not difficult to prove the following lemma.

Lemma 4.3. *For any* $x, y \in S^\circ$, *if* $0_i \in I_{x+x^\circ}, 0_\lambda \in \Lambda_{x^\circ+x}, 0_j \in I_{y+y^\circ}$ *and* $0_\mu \in \Lambda_{y^\circ+y}$, *then*

$$(0_i + x + 0_\lambda) + (0_j + y + 0_\mu) = 0_i + (0_\lambda, 0_j)\alpha_{(x,y)} + x + y + (0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu$$

with $0_i + (0_\lambda, 0_j)\alpha_{(x,y)} \in I_{x+y+(x+y)^\circ}$ *and* $(0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu \in \Lambda_{(x+y)^\circ+x+y}$.

Lemma 4.4. *For any* $0_i \in I_\circ, 0_\lambda \in \Lambda_\circ$, $0_i0_\lambda = 0_i0_\lambda^\circ + 0_i^\circ 0_\lambda$ *and* $0_\lambda 0_i = 0_\lambda^\circ 0_i + 0_\lambda 0_i^\circ$.

Proof. Since $\overset{+}{\mathcal{R}}$ and $\overset{+}{\mathcal{L}}$ are the multiplicative congruences on S, $0_\lambda^\circ \overset{+}{\mathcal{R}} 0_\lambda$ implies that $0_i0_\lambda^\circ \overset{+}{\mathcal{R}} 0_i0_\lambda$ and $0_i^\circ \overset{+}{\mathcal{L}} 0_i$ implies that $0_i^\circ 0_\lambda \overset{+}{\mathcal{L}} 0_i0_\lambda$. By $0_i^\circ 0_\lambda^\circ \in \overset{+}{L}_{0_i^\circ 0_\lambda} \cap \overset{+}{R}_{0_i0_\lambda^\circ}$, we have that $(0_i0_\lambda^\circ + 0_i^\circ 0_\lambda) \in \overset{+}{R}_{0_i^\circ 0_\lambda} \cap \overset{+}{L}_{0_i0_\lambda^\circ} = \overset{+}{H}_{0_i0_\lambda}$, and so $0_i0_\lambda = 0_i0_\lambda^\circ + 0_i^\circ 0_\lambda$. Similarly, we can prove that $0_\lambda 0_i = 0_\lambda^\circ 0_i + 0_\lambda 0_i^\circ$. □

Lemma 4.5. *For any* $x, y \in S^\circ$, *if* $0_i \in I_{x+x^\circ}, 0_\lambda \in \Lambda_{x^\circ+x}, 0_j \in I_{y+y^\circ}$ *and* $0_\mu \in \Lambda_{y^\circ+y}$, *then* $(0_i + x + 0_\lambda)(0_j + y + 0_\mu) = 0_i0_j + xy + 0_\lambda 0_\mu$ *with* $0_i0_j \in I_{xy+(xy)^\circ}$ *and* $0_\lambda 0_\mu \in I_{(xy)^\circ+xy}$.

Proof. For any $x, y \in S^\circ$, by Theorem 2.1 in [22],$x + x^\circ = x^\circ + x$, and $y + y^\circ = y^\circ + y$, then $x \in \overset{+}{R}_{x+x^\circ} \cap \overset{+}{L}_{x^\circ+x} = \overset{+}{H}_{x^\circ+x}$ and $y \in \overset{+}{R}_{y+y^\circ} \cap \overset{+}{L}_{y^\circ+y} = \overset{+}{H}_{y^\circ+y}$.

$$\begin{aligned}
&(0_i + x + 0_\lambda)(0_j + y + 0_\mu)\\
&= 0_i0_j + 0_iy + 0_i0_\mu + x0_j + xy + x0_\mu + 0_\lambda 0_j + 0_\lambda y + 0_\lambda 0_\mu\\
&= 0_i0_j + 0_i0_\mu^\circ + 0_i0_\mu + 0_i^\circ 0_j + xy + 0_\lambda^\circ 0_\mu + 0_\lambda 0_j + 0_\lambda 0_j^\circ + 0_\lambda 0_\mu \text{ (by Lemma 2.5)}\\
&= 0_i0_j + 0_i0_\mu + 0_i^\circ 0_j + xy + 0_\lambda^\circ 0_\mu + 0_\lambda 0_j + 0_\lambda 0_\mu \text{ (by Lemma 4.1)}\\
&= 0_i0_j + (0_i0_\mu^\circ + 0_i^\circ 0_\mu) + 0_i^\circ 0_j + xy + 0_\lambda^\circ 0_\mu + (0_\lambda^\circ 0_j + 0_\lambda 0_j^\circ) + 0_\lambda 0_\mu \text{ (by Lemma 4.4)}\\
&= 0_i0_j + 0_i^\circ 0_j^\circ + xy + 0_\lambda^\circ 0_\mu^\circ + 0_\lambda 0_\mu \text{ (by Lemma 4.1)}\\
&= 0_i0_j + xy + 0_\lambda 0_\mu.
\end{aligned}$$

□

Lemma 4.6. *For any* $x \in S^\circ$, *if* $0_i \in I_{x+x^\circ}$ *and* $0_\lambda \in \Lambda_{x^\circ+x}$, *then* $0_i = 0_i + x + 0_\lambda + x^\circ$ *and* $0_\lambda = x^\circ + 0_i + x + 0_\lambda$.

Proof. Since $0_i \overset{+}{\mathcal{L}} (x + x^\circ)$ and $0_\lambda \overset{+}{\mathcal{R}} (x^\circ + x)$, we have $0_i = 0_i + x + x^\circ = 0_i + x + 0_\lambda + x^\circ$. Similarly, $0_\lambda = x^\circ + 0_i + x + 0_\lambda$. □

Lemma 4.7. *If* $0_i \in I_{x^\circ+x}$, $0_\lambda \in \Lambda_{x^\circ+x}$, $0_j \in I_{y+y^\circ}$, $0_\mu \in \Lambda_{y+y^\circ}$, $0_k \in I_{z+z^\circ}$, $0_\nu \in \Lambda_{z^\circ+z}$, *then*

$$0_i0_k + (0_\lambda, 0_j)\alpha_{(x,y)}0_k = 0_i0_k + (0_\lambda 0_\nu, 0_j0_k)\alpha_{(xz,yz)},$$

$$(0_\lambda, 0_j)\beta_{(x,y)}0_k + 0_\mu 0_\nu = (0_\lambda 0_\nu, 0_j0_k)\beta_{(xz,yz)} + 0_\mu 0_\nu,$$

$$0_k0_i + 0_k(0_\lambda, 0_j)\alpha_{(x,y)} = 0_k0_i + (0_\nu 0_\lambda, 0_k0_j)\alpha_{(zx,zy)}$$

and

$$0_k(0_\lambda, 0_j)\beta_{(x,y)} + 0_\nu 0_\mu = (0_\nu 0_\lambda, 0_k0_j)\beta_{(zx,zy)} + 0_\nu 0_\mu.$$

Proof. Let $0_i \in I_{x+x^\circ}$, $0_\lambda \in \Lambda_{x^\circ+x}$, $0_j \in I_{y+y^\circ}$, $0_\mu \in \Lambda_{y+y^\circ}$, $0_k \in I_{z+z^\circ}$, $0_\nu \in \Lambda_{z^\circ+z}$.

On the one hand,

$$\begin{aligned}
&[(0_i + x + 0_\lambda) + (0_j + y + 0_\mu)](0_k + z + 0_\nu)\\
&= (0_i + (0_\lambda, 0_j)\alpha_{(x,y)} + (x + y) + (0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu)(0_k + z + 0_\nu) \text{ (by Lemma 4.3)}\\
&= (0_i0_k + (0_\lambda, 0_j)\alpha_{(x,y)}0_k) + (xz + yz) + ((0_\lambda, 0_j)\beta_{(x,y)}0_\nu + 0_\mu0_\nu) \text{ (by Lemma 4.5)},
\end{aligned}$$

where $(0_i0_k + (0_\lambda, 0_j)\alpha_{(x,y)}0_k) \in I_{(xz+yz)+(xz+yz)^\circ}$, and $((0_\lambda, 0_j)\beta_{(x,y)}0_\nu + 0_\mu0_\nu) \in \Lambda_{(xz+yz)^\circ+(xz+yz)}$.

On the other hand,

$$\begin{aligned}
&[(0_i + x + 0_\lambda) + (0_j + y + 0_\mu)](0_k + z + 0_\nu)\\
&= (0_i0_k + xz + 0_\lambda0_\nu) + (0_j0_k + yz + 0_\mu0_\nu) \text{ (by Lemma 4.5)}\\
&= 0_i0_k + (0_\lambda0_\nu, 0_j0_k)\alpha_{(xz,yz)} + (xz + yz) + (0_\lambda0_\nu, 0_j0_k)\beta_{(xz,yz)} + 0_\mu0_\nu \text{ (by Lemma 4.3)},
\end{aligned}$$

where $(0_i0_k + (0_\lambda0_\nu, 0_j0_k)\alpha_{(xz,yz)}) \in I_{xz+yz+(xz+yz)^\circ}$ and $((0_\lambda0_\nu, 0_j0_k)\beta_{(xz,yz)} + 0_\mu0_\nu) \in \Lambda_{(xz+yz)^\circ+xz+yz}$. Thus,

$$\begin{aligned}
&0_i0_k + (0_\lambda0_\nu, 0_j0_k)\alpha_{(xz,yz)}\\
&= 0_i0_k + (0_\lambda0_\nu, 0_j0_k)\alpha_{(xz,yz)} + (xz + yz) + (0_\lambda0_\nu, 0_j0_k)\beta_{(xz,yz)} + 0_\mu0_\nu + (xz + yz)^\circ \text{ (by Lemma 4.6)}\\
&= [(0_i + x + 0_\lambda) + (0_j + y + 0_\mu)](0_k + z + 0_\nu) + (xz + yz)^\circ\\
&= (0_i0_k + (0_\lambda, 0_j)\alpha_{(x,y)}0_k) + (xz + yz) + ((0_\lambda, 0_j)\beta_{(x,y)}0_\nu + 0_\mu0_\nu) + (xz + yz)^\circ\\
&= 0_i0_k + (0_\lambda, 0_j)\alpha_{(x,y)}0_k \text{ (by Lemma 4.6)}
\end{aligned}$$

and

$$\begin{aligned}
&(0_\lambda0_\nu, 0_j0_k)\beta_{(xz,yz)} + 0_\mu0_\nu\\
&= (xz + yz)^\circ + 0_i0_k + (0_\lambda0_\nu, 0_j0_k)\alpha_{(xz,yz)} + (xz + yz) + (0_\lambda0_\nu, 0_j0_k)\beta_{(xz,yz)} + 0_\mu0_\nu \text{ (by Lemma 4.6)}\\
&= (xz + yz)^\circ + [(0_i + x + 0_\lambda) + (0_j + y + 0_\mu)](0_k + z + 0_\nu)\\
&= (xz + yz)^\circ + (0_i0_k + (0_\lambda, 0_j)\alpha_{(x,y)}0_k) + (xz + yz) + ((0_\lambda, 0_j)\beta_{(x,y)}0_\nu + 0_\mu0_\nu)\\
&= (0_\lambda, 0_j)\alpha_{(x,y)}0_\nu + 0_\mu0_\nu \text{ (by Lemma 4.6)}.
\end{aligned}$$

Similarly, we can prove that

$$0_k0_i + 0_k(0_\lambda, 0_j)\alpha_{(x,y)} = 0_k0_i + (0_\nu0_\lambda, 0_k0_j)\alpha_{(zx,zy)}$$

and

$$0_k(0_\lambda, 0_j)\beta_{(x,y)} + 0_\nu0_\mu = (0_\nu0_\lambda, 0_k0_j)\beta_{(zx,zy)} + 0_\nu0_\mu.$$

□

Theorem 4.2. *Let S° be a generalized Clifford semiring with the b-lattice $E^+(S^\circ)$ as a k-ideal of S° and let I be a b-lattice of additively right zero semirings $\{I_{0_\alpha} : 0_\alpha \in E^+(S^\circ)\}$ and Λ a b-lattice of additively left zero semirings $\{\Lambda_{0_\alpha} : 0_\alpha \in E^+(S^\circ)\}$. Suppose that I and Λ have a common b-lattice transversal $E^+(S^\circ)$. And, for each $(x, y) \in S^\circ \times S^\circ$, there exit $\alpha_{(x,y)} \in PT(\Lambda \times I, I)$ and $\beta_{(x,y)} \in PT(\Lambda \times I, \Lambda)$ satisfying:*

(1) $dom(\alpha_{(x,y)}) = dom(\beta_{(x,y)}) = \Lambda_{x^\circ+x} \times I_{y+y^\circ}$, $(0_\lambda, 0_i)\alpha_{(x,y)} \in I_{x+y+(x+y)^\circ}$ and $(0_\lambda, 0_i)\beta_{(x,y)} \in \Lambda_{(x+y)^\circ+x+y}$;

(2) If $0_\lambda \in \lambda_{x^\circ+x}$, $0_j \in I_{y+y^\circ}$,$0_\mu \in \Lambda_{y^\circ+y}$,$0_k \in I_{z+z^\circ}$, *then*

$$(0_\lambda, 0_j)\alpha_{(x,y)} + ((0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu, 0_k)\alpha_{(x+y,z)} = (0_\lambda, 0_j + (0_\mu, 0_k)\alpha_{(y,z)})\alpha_{(x,y+z)}$$

and

$$(0_\lambda, 0_j + (0_\mu, 0_k)\alpha_{(y,z)})\beta_{(x,y+z)} + (0_\mu, 0_k)\beta_{(y,z)} = ((0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu, 0_k)\beta_{(x+y,z)};$$

(3) If $0_i \in I_{x^\circ+x}$,$0_\lambda \in \Lambda_{x^\circ+x}$,$0_j \in I_{y+y^\circ}$,$0_\mu \in \Lambda_{y+y^\circ}$,$0_k \in I_{z+z^\circ}$, $0_\nu \in \Lambda_{z^\circ+z}$,*then*

$$0_i0_k + (0_\lambda, 0_j)\alpha_{(x,y)}0_k = 0_i0_k + (0_\lambda0_\nu, 0_j0_k)\alpha_{(xz,yz)},$$

$$(0_\lambda, 0_j)\beta_{(x,y)}0_k + 0_\mu0_\nu = (0_\lambda0_\nu, 0_j0_k)\beta_{(xz,yz)} + 0_\mu0_\nu,$$

$$0_k0_i + 0_k(0_\lambda, 0_j)\alpha_{(x,y)} = 0_k0_i + (0_\nu0_\lambda, 0_k0_j)\alpha_{(zx,zy)}$$

and

$$0_k(0_\lambda, 0_j)\beta_{(x,y)} + 0_\nu0_\mu = (0_\nu0_\lambda, 0_k0_j)\beta_{(zx,zy)} + 0_\nu0_\mu;$$

(4) $(x^\circ + x, y + y^\circ)\alpha_{(x,y)} = x + y + (x + y)^\circ$ *and* $(x^\circ + x, y + y^\circ)\beta_{(x,y)} = (x + y)^\circ + x + y$.

Define addition and multiplication on the set

$$W = \{(0_i, x, 0_\lambda) \in I \times S^\circ \times \Lambda : 0_i \in I_{x+x^\circ}, 0_\lambda \in \Lambda_{x^\circ+x}\}$$

by

$$(0_i, x, 0_\lambda) + (0_j, y, 0_\mu) = (0_i + (0_\lambda, 0_j)\alpha_{(x,y)}, x + y, (0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu)$$

and

$$(0_i, x, 0_\lambda) \cdot (0_j, y, 0_\mu) = (0_i0_j, xy, 0_\lambda0_\mu).$$

Then W *is an additively orthodox semiring with a generalized Clifford semiring transversal isomorphic to* S°*. Conversely, every additively orthodox semiring with a generalized Clifford semiring transversal can be constructed in this way.*

Proof. The associativity of the multiplication is clear. By Theorem 3.6 in [20], $(W, +)$ is an orthodox semigroup with an inverse transversal $(W^\circ, +)$ isomorphic to $(S^\circ, +)$, where $W^\circ = \{(0_i, x, 0_\lambda) \in W : 0_i, 0_\lambda \in E^+(S^\circ), x \in S^\circ\}$ and $(0_\lambda^\circ, x^\circ, 0_i^\circ)$ is the uniquely inverse of $(0_i, x, 0_\lambda)$ in $(W^\circ, +)$. We need to prove the distributive laws of the semiring W. For any $(0_i, x, 0_\lambda), (0_j, y, 0_\mu), (0_k, z, 0_\nu) \in W$, by (3),

$$\begin{aligned}
&[(0_i, x, 0_\lambda) + (0_j, y, 0_\mu)](0_k, z, 0_\nu)\\
&= (0_i + (0_\lambda, 0_j)\alpha_{(x,y)}, x + y, (0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu)(0_k, z, 0_\nu)\\
&= ((0_i + (0_\lambda, 0_j)\alpha_{(x,y)})0_k, (x + y)z, ((0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu)0_\nu)\\
&= (0_i0_k + (0_\lambda, 0_j)\alpha_{(x,y)}0_k, xz + yz, (0_\lambda, 0_j)\beta_{(x,y)}0_\nu + 0_\mu0_\nu)\\
&= (0_i0_k + (0_\lambda0_\nu, 0_j0_k)\alpha_{(xz,yz)}, xz + yz, (0_\lambda0_\nu, 0_j0_k)\beta_{(xz,yz)} + 0_\mu0_\nu)\\
&= (0_i0_k, xz, 0_\lambda0_\nu) + (0_j0_k, yz, 0_\mu0_\nu)\\
&= (0_i, x, 0_\lambda)(0_k, z, 0_\nu) + (0_j, y, 0_\mu)(0_k, z, 0_\nu).
\end{aligned}$$

Thus the distributivity on right is hold. And the distributivity on left can be proved similarly. Hence W is semiring as required. Moreover, $(0_i, x, 0_\lambda) \in W^\circ$ if and only if $0_i^\circ = x + x^\circ$ and $0_\lambda^\circ = x^\circ + x$. By (4), it is not difficult to prove that $W^\circ \cong S^\circ$, so W° is a generalized Clifford subsemiring of W.

Conversely, let S be an additively orthodox semiring with a generalized Clifford semiring transversal S°. By Lemma 4.1, $I_\circ$ and $\Lambda_\circ$ are a b-lattice of additively right zero semirings $\{I_a : a \in E^+(S^\circ)\}$ and a b-lattice of additively left zero semirings $\{\Lambda_a : a \in E^+(S^\circ)\}$, respectively. For each $(x, y) \in S^\circ \times S^\circ$, put $(0_\lambda, 0_i)\alpha_{(x,y)} = x + 0_\lambda + 0_i + x^\circ$ and $(0_\lambda, 0_i)\beta_{(x,y)} = y^\circ + 0_\lambda + 0_i + y$. Then $\alpha_{(x,y)} \in PT(\Lambda_\circ \times I_\circ, I_\circ)$ and $\beta_{(x,y)} \in PT(\Lambda_\circ \times I_\circ, \Lambda_\circ)$, and by Lemma 4.2 and Lemma 4.7, they satisfy the conditions (1)–(4). Put $W = \{(0_i, x, 0_\lambda) \in I_\circ \times S^\circ \times \Lambda_\circ : 0_i \in I_{x+x^\circ}, 0_\lambda \in \Lambda_{x^\circ+x}\}$. Thus we can define the addition and multiplication: For $(0_i, x, 0_\lambda), (0_j, y, 0_\mu) \in W$

$$(0_i, x, 0_\lambda) + (0_j, y, 0_\mu) = (0_i + (0_\lambda, 0_j)\alpha_{(x,y)}, x + y, (0_\lambda, 0_j)\beta_{(x,y)} + 0_\mu)$$

and

$$(0_i, x, 0_\lambda) \cdot (0_j, y, 0_\mu) = (0_i 0_j, xy, 0_\lambda 0_\mu).$$

It is not difficult to verify W is a semiring with an additively inverse semiring transversal $W^\circ = \{(0_i, x, 0_\lambda) \in W : 0_i, 0_\lambda \in E^+(S^\circ), x \in S^\circ\}$. Now, we define a mapping φ from W to S such that $\varphi((0_i, x, 0_\lambda)) = 0_i + x + 0_\lambda$. According to Lemma 4.3 and Lemma 4.5, we know that φ is a semiring isomorphism and the transversal W° of W is isomorphic to S°. □

Remark 1. *From Theorem 4.1 and Theorem 4.2, we can see that the class of additively orthodox semirings with generalized Clifford semiring transversals are actually not only a general extension of the class of Clifford semirings and generalized Clifford semirings studied in [22], but also a general extension of the class of completely regular semirings in [14].*

5. Conclusions

In this paper, the authors introduce some special semiring transversals as the tools, and establish the constructions of additively orthodox semirings with a skew-ring transversal or with a generalized Clifford semiring transversal. The authors also show that an additively orthodox semiring with a generalized Clifford semiring transversal is a b-lattice of additively orthodox semirings with skew-ring transversals. Consequently, the corresponding results of Clifford semirings and generalized Clifford semirings in [22] and completely regular semirings in [14] are extended and strengthened.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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