Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 4137–4152.
DOI: 10.3934/math.2022229
Received: 02 July 2021
Revised: 25 November 2021
Accepted: 10 December 2021
Published: 15 December 2021

Research article

The linear k-arboricity of digraphs

Xiaoling Zhou[1], **Chao Yang**[2] **and Weihua He**[1,*]

[1] School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou 510520, China

[2] School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou 510520, China

* **Correspondence:** Email: hwh12@gdut.edu.cn.

Abstract: A linear k-diforest is a directed forest in which every connected component is a directed path of length at most k. The linear k-arboricity of a digraph D is the minimum number of linear k-diforests needed to partition the arcs of D. In this paper, we study the linear k-arboricity for digraphs, and determine the linear 3-arboricity and linear 2-arboricity for symmetric complete digraphs and symmetric complete bipartite digraphs.

Keywords: linear k-arboricity; digraphs; symmetric complete digraphs; symmetric complete bipartite digraphs
Mathematics Subject Classification: 05C70, 05C38

1. Introduction

In this paper, a digraph is a finite loopless directed graph without parallel arcs (arcs with the same head and the same tail) and an undirected graph is also a finite and simple graph. A linear forest is a forest in which every connected component is a path. The linear arboricity of a graph G, defined by Harary [14], is the minimum number of linear forests that partition the edges of G and is denoted by $la(G)$. Later, Habib and Péroche [13] introduced the linear k-arboricity of a graph G, which is the minimum number of k-linear forests (forests in which every connected component is a path of length at most k) required to partition the edges of G and is denoted by $la_k(G)$. Moreover, Akiyama et al. [1] proposed a conjecture about the value of linear arboricity and Habib and Péroche [13] proposed a conjecture about the value of linear k-arboricity which subsumes Akiyama's conjecture. Aimed at these two conjectures, considerable works have been done over the years (see [2, 3, 6–12, 16, 19–22]).

It is natural to consider similar problems for digraphs. Let $D = (V(D), A(D))$ be a digraph. We denote $\Delta^+(D) = max\{d^+(v)|\ for\ all\ v \in V\}$, $\Delta^-(D) = max\{d^-(v)|\ for\ all\ v \in V\}$ and $\Delta(D) =$

$max\{\Delta^+(D), \Delta^-(D)\}$. The underlying graph $S(D)$ of D is the undirected simple graph with the same vertex set of D by replacing each arc by an edge with the same ends. A linear diforest is a directed forest in which every connected component is a directed path. The linear arboricity of D, defined by Nakayama and Péroche [17], is the minimum number of linear diforests that partition the arcs of D and is denoted by $\overrightarrow{la}(D)$. Nakayama and Péroche [17] also conjectured that $\overrightarrow{la}(D) \leq \Delta(D) + 1$. Since every digraph can be a regular digraph by adding arcs, Nakayama-Péroche conjecture is equivalent to say that the linear arboricity of a d-regular digraph D (i.e. every vertex in D has in-degree d and out-degree d) is $d + 1$. In 2017, He et al. [15] found that the symmetric complete digraphs K_3^* and K_5^* have the linear arboricity $d + 2$ ($d = 2, 4$ respectively), which is contrary to Nakayama-Péroche conjecture. Then they conjectured that the linear arboricity of a d-regular digraph D is $d + 1$ except D is K_3^* or K_5^*.

In this paper, we study the linear k-arboricity for digraphs. The linear k-arboricity of a digraph D is the minimum number of linear k-diforests (diforests in which every connected component is a directed path of length at most k) that partition the arcs of D and is denoted by $\overrightarrow{la}_k(D)$.

This paper is organized as follows: In Section 2, we introduce some notations and obtain the upper bound and the lower bound of the linear k-arboricity for general digraphs. In Sections 3 and 4, we study the linear 3-arboricity and linear 2-arboricity for symmetric complete digraphs respectively. In Sections 5 and 6, we study the linear 3-arboricity and linear 2-arboricity for symmetric complete bipartite digraphs respectively.

2. Preliminaries

For an undirected graph G with n vertices, Habib and Péroche [13] conjectured that $la_k(G) \leq \lceil \frac{\Delta(G)n+1}{2\lfloor kn/(k+1)\rfloor} \rceil$ when $\Delta(G) < n - 1$ and $la_k(G) \leq \lceil \frac{\Delta(G)n}{2\lfloor kn/(k+1)\rfloor} \rceil$ when $\Delta(G) = n - 1$. Based on the linear arboricity conjecture for digraphs in [15] and Habib-Péroche conjecture, we propose the following conjecture for the linear k-arboricity in digraphs.

Conjecture 2.1. *For a digraph D with n vertices, if $k = n - 1$,*

$$\overrightarrow{la}_k(D) \leq \begin{cases} \lceil \dfrac{\Delta(D)n}{\lfloor kn/(k+1)\rfloor} \rceil & \text{when } \Delta(D) = n-1 \text{ and } D \text{ is not } K_3^* \text{ and } K_5^*, \\ \lceil \dfrac{\Delta(D)n+1}{\lfloor kn/(k+1)\rfloor} \rceil & \text{when } \Delta(D) < n-1 \text{ or } D \text{ is } K_3^* \text{ or } K_5^*. \end{cases}$$

If $k < n - 1$,

$$\overrightarrow{la}_k(D) \leq \begin{cases} \lceil \dfrac{\Delta(D)n}{\lfloor kn/(k+1)\rfloor} \rceil & \text{when } \Delta(D) = n-1, \\ \lceil \dfrac{\Delta(D)n+1}{\lfloor kn/(k+1)\rfloor} \rceil & \text{when } \Delta(D) < n-1. \end{cases}$$

It is easy to obtain the following lemmas.

Lemma 2.1. *Let H be a subdigraph of a digraph D. Then $\overrightarrow{la}_k(H) \leq \overrightarrow{la}_k(D)$.*

Lemma 2.2. *For a digraph D with n vertices,*

$$\overrightarrow{la}_1(D) \geq \overrightarrow{la}_2(D) \geq \ldots \geq \overrightarrow{la}_{n-1}(D) = \overrightarrow{la}(D).$$

Lemma 2.3. *For a digraph $D = (V(D), A(D))$ with n vertices,*

$$\overrightarrow{la}_k(D) \geq \max\left\{\Delta(D), \left\lceil \frac{|A(D)|}{\lfloor \frac{kn}{k+1} \rfloor} \right\rceil\right\}.$$

If D is a symmetric digraph, we just give two opposite directions to the linear forests of the minimum linear k-forests partition of $S(D)$ and get the following trivial upper bound for $\overrightarrow{la}_k(D)$.

Lemma 2.4. *Let D be a symmetric digraph. Then* $\overrightarrow{la}_k(D) \leq 2la_k(S(D))$.

In this paper, we mainly study the linear k-arboricity for symmetric complete digraphs and symmetric complete bipartite digraphs. Fu et al. [11, 12, 22] studied linear 2-arboricity and 3-arboricity of complete graphs K_n and complete bipartite graphs $K_{n,n}$.

Theorem 2.1. *[12]*

$$la_3(K_n) = \begin{cases} \left\lceil \dfrac{2n-2}{3} \right\rceil & \text{when } n \equiv 0, 4, 8, 11\ (mod 12), \\ \left\lceil \dfrac{2n}{3} \right\rceil & \text{when } n \equiv 1, 2, 3, 5, 6, 7, 9, 10\ (mod 12). \end{cases}$$

Theorem 2.2. *[12]*

$$la_3(K_{n,n}) = \begin{cases} \left\lceil \dfrac{2n}{3} \right\rceil & \text{when } n \equiv 0, 1, 2, 4, 5\ (mod\ 6), \\ \left\lceil \dfrac{2n+2}{3} \right\rceil & \text{when } n \equiv 3\ (mod\ 6). \end{cases}$$

Theorem 2.3. *[6, 22]*

$$la_2(K_n) = \left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil.$$

Theorem 2.4. *[11]*

$$la_2(K_{n,n}) = \left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil.$$

Let $K^*_{n,n}$ be a symmetric complete bipartite digraph with partite sets $X = \{x_0, x_1, ..., x_{n-1}\}$ and $Y = \{y_0, y_1, ..., y_{n-1}\}$. We define the bipartite difference of the undirected edge x_py_q in $S(K^*_{n,n})$ as the value $q-p(mod\ n)$. Those edges in $S(K^*_{n,n})$ with the same value of the bipartite difference must be a matching. In particular, we denote the set of edges of the bipartite difference i in $S(K^*_{n,n})$ by M_i ($i = 0, 1, ..., n-1$). In $K^*_{n,n}$, for $i = 0, 1, ..., n-1$, we define $\overrightarrow{M}_i = \{x_d y_{d+i(mod\ n)} | d = 0, 1, ..., n-1\}$ and $\overleftarrow{M}_i = \{y_{d+i(mod\ n)} x_d | d = 0, 1, ..., n-1\}$. Thus, we can partition the arcs of $K^*_{n,n}$ into $2n$ pairwise arc-disjoint perfect matchings $\overrightarrow{M}_0, \overrightarrow{M}_1, ..., \overrightarrow{M}_{n-1}, \overleftarrow{M}_0, \overleftarrow{M}_1, ..., \overleftarrow{M}_{n-1}$. Similarly as in [12], we have the following two useful results.

Lemma 2.5. *If $n \geq 4$ is even and $\alpha \in \{0, 1, ..., n-3\}$, then the arcs in the union $\{\overrightarrow{M}_{\alpha}, \overleftarrow{M}_{\alpha+1}, \overrightarrow{M}_{\alpha+2}\}$ in $K^*_{n,n}$ can form two arc-disjoint linear 3-diforests and $\{\overleftarrow{M}_{\alpha}, \overrightarrow{M}_{\alpha+1}, \overleftarrow{M}_{\alpha+2}\}$ can form another two arc-disjoint linear 3-diforests.*

Lemma 2.6. *If $n \geq 3$ is odd, $\alpha \in \{0, 1, ..., n-3\}$ and e is an arc of $\overleftarrow{M}_{\alpha+1}$, then $\{\overrightarrow{M}_{\alpha}, \overleftarrow{M}_{\alpha+1} - \{e\}, \overrightarrow{M}_{\alpha+2}\}$ in $K^*_{n,n}$ can form two arc-disjoint linear 3-diforests. And if e is an arc of $\overrightarrow{M}_{\alpha+1}$, then $\{\overleftarrow{M}_{\alpha}, \overrightarrow{M}_{\alpha+1} - \{e\}, \overleftarrow{M}_{\alpha+2}\}$ can form another two arc-disjoint linear 3-diforests.*

3. Linear 3-arboricity of symmetric complete digraphs

In this section, we determine the linear 3-arboricity of K^*_n. Firstly, we propose an operation of replacing arcs in $K^*_{n,n}$. Let $X = \{x_0, x_1, ..., x_{n-1}\}$ and $Y = \{y_0, y_1, ..., y_{n-1}\}$ be partite sets of $K^*_{n,n}$. Suppose it exists the following directed 3-path $y_i y_{i+d} x_i x_{i+d}$ by adding arcs $x_i x_{i+d}$ and $y_i y_{i+d}$ in $K^*_{n,n}$ as shown in Figure 1. Then we replace the arc $y_{i+d} x_i \in \overleftarrow{M}_d$ by the arc $x_{i+d} y_i \in \overrightarrow{M}_{n-d}$ and we get another directed 3-path $x_i x_{i+d} y_i y_{i+d}$. We call this operation replacing arc operation. In this operation we can use the arcs in a matching to replace the arcs in another matching contained in some directed paths.

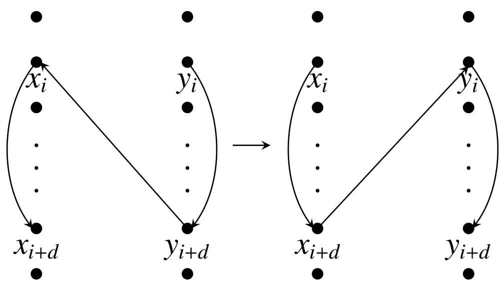

Figure 1. The replacing arc operation.

We need to mention that some of the proof in the following propositions are similar as the proof for the linear 3-arboricity of K_n [12], and we will omit some analogous and tedious proof in this section.

Proposition 3.1. $\overrightarrow{la}_3(K^*_n) \leq 2\lceil \frac{2n-2}{3} \rceil - 1$ *when* $n \equiv 0$ *(mod* 12).

Proof. Let $n = 12t$, $m = \frac{n}{2}$. In [12] it is proved that $S(K^*_n)$ can be decomposed into $m-1$ pairwise edge-disjoint linear 3-forests and couple of matchings. Thus, by giving two opposite directions to edges of those linear 3-forests and matchings of $S(K^*_n)$, in K^*_n, we have
(1) $2(m-1)$ pairwise arc-disjoint linear 3-diforests;
(2) m pairwise arc-disjoint perfect matchings $\overrightarrow{M}_d = \{x_i y_{i+d (mod\ m)} | i = 0, 1, ..., m-1\}$ $(d = 0, 1, 2, ... \frac{m}{2} - 1)$, $\overleftarrow{M}_d = \{y_{i+d (mod\ m)} x_i | i = 0, 1, ..., m-1\}$ $(d = 0, 1, 2, ... \frac{m}{2} - 1)$;
(3) $\overrightarrow{M}_{\frac{m}{2}} = \{x_i y_{i+\frac{m}{2} (mod\ m)} | i = 0, 1, ..., \frac{m}{2} - 1\}$ and $\overleftarrow{M}_{\frac{m}{2}} = \{y_{i+\frac{m}{2} (mod\ m)} x_i | i = 0, 1, ..., \frac{m}{2} - 1\}$.

By Lemma 2.5, we can construct $\frac{2m}{3}$ linear 3-diforests using these matchings in (2).

Then for the directed 3-paths $y_{i+\frac{m}{2}} y_i x_{i+\frac{m}{2}} x_i$, $i \in \{0, 1, ..., \frac{m}{2} - 1\}$ from those $2(m-1)$ linear 3-diforests in (1), by using the arcs in $\overrightarrow{M}_{\frac{m}{2}}$, we apply the replacing arc operation for $y_{i+\frac{m}{2}} y_i x_{i+\frac{m}{2}} x_i$ and get new paths $x_{i+\frac{m}{2}} x_i y_{i+\frac{m}{2}} y_i$. Note that in the whole operation, the arcs in the matching $\overleftarrow{M}'_{\frac{m}{2}} = \{y_i x_{i+\frac{m}{2} (mod\ m)} | i =$

$0, 1, ..., \frac{m}{2} - 1\}$ are removed from the paths. It is not hard to see that the arcs of $\overleftarrow{M}'_{\frac{m}{2}}$ and $\overleftarrow{M}_{\frac{m}{2}}$ can form one linear 3-diforest.

Accordingly, $\overrightarrow{la}_3(K_n^*) \leq 2(m-1) + \frac{2m}{3} + 1 = 2\lceil\frac{2n-2}{3}\rceil - 1$. □

Proposition 3.2. $\overrightarrow{la}_3(K_n^*) \leq 2\lceil\frac{2n}{3}\rceil - 1$ *when* $n \equiv 2$ $(mod\ 12)$.

Proof. Let $n = 12t + 2$, $m = \frac{n-2}{2}$, $t \geq 1$ (when $t = 0$, it is trivial). In K_n^*, we have
(1) $2(m + 1)$ pairwise arc-disjoint linear 3-diforests;
(2) m pairwise arc-disjoint perfect matchings $\overrightarrow{M}_d = \{x_i y_{i+d(mod\ m)} | i = 0, 1, ..., m-1\}$ $(d = 0, 1, 2, ...\frac{m}{2} - 1)$, $\overleftarrow{M}_d = \{y_{i+d(mod\ m)} x_i | i = 0, 1, ..., m - 1\}$ $(d = 0, 1, 2, ...\frac{m}{2} - 1)$;
(3) $\overrightarrow{M}_{\frac{m}{2}} = \{x_i y_{i+\frac{m}{2}(mod\ m)} | i = 0, 1, ..., \frac{m}{2} - 1\}$ and $\overleftarrow{M}_{\frac{m}{2}} = \{y_{i+\frac{m}{2}(mod\ m)} x_i | i = 0, 1, ..., \frac{m}{2} - 1\}$.

By Lemma 2.5, we can construct $\frac{2m}{3}$ linear 3-diforests using these matchings in (2).

Then, by the similar replacing arc operation in Proposition 3.1, we have $\overrightarrow{la}_3(K_n^*) \leq 2(m+1)+\frac{2m}{3}+1 = 2\lceil\frac{2n}{3}\rceil - 1$. □

Proposition 3.3. $\overrightarrow{la}_3(K_n^*) \leq 2\lceil\frac{2n}{3}\rceil - 1$ *when* $n \equiv 5$ $(mod\ 12)$.

Proof. Let $n = 12t + 5$, $m = \frac{n-1}{2}$, $t \geq 0$.

When $t = 0$, let $V(K_5^*) = \{x_0, x_1, x_2, x_3, x_4\}$. Then we can easily find 7 arc-disjoint linear 3-diforests to partition the arcs of K_5^*: $\{x_1x_2x_0x_4\}$, $\{x_0x_3x_1, x_4x_2\}$, $\{x_0x_1, x_4x_3x_2\}$, $\{x_2x_1x_4x_0\}$, $\{x_1x_3x_0, x_2x_4\}$, $\{x_1x_0, x_2x_3x_4\}$, $\{x_0x_2, x_4x_1\}$.

Now we assume that $t \geq 1$. In K_n^*, we have
(1) $2(m + 1)$ pairwise arc-disjoint linear 3-diforests;
(2) $\frac{m}{2} - 1$ pairwise arc-disjoint perfect matchings $\overrightarrow{M}_d = \{x_i y_{i+d(mod\ m)} | i = 0, 1, ..., m-1\}$, $d = 0, 2, ...\frac{m}{2} - 1$;
(3) $\frac{m}{2} - 1$ pairwise arc-disjoint perfect matchings $\overleftarrow{M}_d = \{y_{i+d(mod\ m)} x_i | i = 0, 1, ..., m-1\}$, $d = 0, 2, ...\frac{m}{2} - 1$;
(4) $\overrightarrow{M}_{\frac{m}{2}} = \{x_i y_{i+\frac{m}{2}(mod\ m)|i=0,1,...,\frac{m}{2}-1}\}$ and $\overleftarrow{M}_{\frac{m}{2}} = \{y_{i+\frac{m}{2}(mod\ m)} x_i | i = 0, 1, ..., \frac{m}{2} - 1\}$.

By Lemma 2.5, we can construct $\frac{2}{3}(m - 8) = \frac{2m-16}{3}$ linear 3-diforests by the matchings in (2) and (3) except the matchings $\overrightarrow{M}_0$, $\overleftarrow{M}_0$, $\overrightarrow{M}_{\frac{m}{2}-2}$, $\overleftarrow{M}_{\frac{m}{2}-2}$,$\overrightarrow{M}_{\frac{m}{2}-1}$, $\overleftarrow{M}_{\frac{m}{2}-1}$.

When t is even and $t \geq 2$, $\overrightarrow{M}_0$, $\overleftarrow{M}_{\frac{m}{2}-2}$, $\overrightarrow{M}_{\frac{m}{2}-1}$ can form two linear 3-diforests as

$$\{x_{6t+1+i(mod\ 6t+2)} y_{3t-1+i(mod\ 6t+2)} x_i y_i | i = 0, 2, 4, ..., 6t\}$$

$$and\ \ \{x_{6t+1+i(mod\ 6t+2)} y_{3t-1+i(mod\ 6t+2)} x_i y_i | i = 1, 3, 5, ..., 6t + 1\}.$$

Similarly, $\overleftarrow{M}_0$, $\overrightarrow{M}_{\frac{m}{2}-2}$, $\overleftarrow{M}_{\frac{m}{2}-1}$ can form another two arc-disjoint linear 3-diforests.

When t is odd, $\overleftarrow{M}_0$, $\overrightarrow{M}_{\frac{m}{2}-1}$, $\overleftarrow{M}_{\frac{m}{2}-2}$ can form two linear 3-diforests as

$$\{y_i x_i y_{3t+i} x_{6t+3+i} | i = 0, 2, 4, ..., 6t\}$$

$$and\ \ \{y_i x_i y_{3t+i} x_{6t+3+i} | i = 1, 3, 5, ..., 6t + 1\}.$$

Similarly, $\overrightarrow{M}_0$, $\overleftarrow{M}_{\frac{m}{2}-1}$, $\overrightarrow{M}_{\frac{m}{2}-2}$ can form another two arc-disjoint linear 3-diforests.

In addition, we apply the replacing arc operation for the arcs of the matching $\overrightarrow{M}_{\frac{m}{2}}$ in some linear 3-diforests of (1) and obtain a new matching $\overleftarrow{M}'_{\frac{m}{2}} = \{y_i x_{i+\frac{m}{2}(mod\ m)} | i = 0, 1, ..., \frac{m}{2} - 1\}$. The arcs of $\overleftarrow{M}'_{\frac{m}{2}}$ and $\overleftarrow{M}_{\frac{m}{2}}$ also can form one linear 3-diforest.

Accordingly, $\overrightarrow{la}_3(K_n^*) \le 2(m + 1) + \frac{2m-16}{3} + 4 + 1 = 2\lceil \frac{2n}{3} \rceil - 1$. □

Proposition 3.4. $\overrightarrow{la}_3(K_n^*) \le 2\lceil \frac{2n}{3} \rceil - 1$ *when* $n \equiv 7$ $(mod\ 12)$.

Proof. Let $n = 12t + 7$, $m = \frac{n-1}{2}$, $t \ge 0$.

When $t = 0$, let $V(K_7^*) = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$. We can find 9 arc-disjoint linear 3-diforests to partition the arcs of K_7^*: $\{x_4x_1x_0, x_3x_2x_5x_6\}$, $\{x_5x_0x_3x_1, x_6x_4x_2\}$, $\{x_0x_6x_1x_2, x_4x_3x_5\}$, $\{x_0x_1x_4, x_6x_5x_2x_3\}$, $\{x_1x_3x_0x_5, x_2x_4x_6\}$, $\{x_2x_1x_6x_0, x_3x_4x_5\}$, $\{x_3x_6x_2x_0, x_1x_5x_4\}$, $\{x_0x_4, x_5x_1, x_2x_6x_3\}$, $\{x_4x_0x_2, x_5x_3\}$.

In K_n^*, we have

(1) $2m$ pairwise arc-disjoint linear 3-diforests;

(2) $\frac{m-3}{2}$ pairwise arc-disjoint perfect matchings $\overrightarrow{M}_d = \{x_i y_{i+d(mod\ m)} | i = 0, 1, ..., m - 1\}, d = 0, 1, 2, ... \frac{m-5}{2}$;

(3) $\frac{m-3}{2}$ pairwise arc-disjoint perfect matchings $\overleftarrow{M}_d = \{y_{i+d(mod\ m)} x_i | i = 0, 1, ..., m - 1\}, d = 0, 1, 2, ... \frac{m-5}{2}$;

(4) $\overrightarrow{M}_{\frac{m-3}{2}} = \{x_i y_{i+d(mod\ m)} | i = 0, 1, ..., m - 1\}$, $\overleftarrow{M}_{\frac{m-1}{2}} = \{y_{i+d(mod\ m)} x_i | i = 0, 1, ..., m - 1\}$, and $\overleftarrow{M}_{\frac{m-3}{2}} = \{y_{i+d(mod\ m)} x_i | i = 0, 1, ..., m - 1\}$, $\overrightarrow{M}_{\frac{m-1}{2}} = \{x_i y_{i+d(mod\ m)} | i = 0, 1, ..., m - 1\}$.

Similarly to the proof of Proposition 3.7 in [12], we can construct $2m + \frac{2}{3}(m - 3)$ arc-disjoint linear 3-diforests using the linear 3-diforests in (1) and the matchings in (2) and (3).

Next, we apply the replacing arc operation for the arcs of $\overleftarrow{M}_{\frac{m-1}{2}}$ in some linear 3-diforests of (1) and obtain a new matching $\overrightarrow{M}_{\frac{m+1}{2}}$. Also, we obtain $\overleftarrow{M}_{\frac{m+3}{2}}$ by replacing the arcs of $\overrightarrow{M}_{\frac{m-3}{2}}$ in some linear 3-diforests of (1).

Now there are only four matchings left: $\overleftarrow{M}_{\frac{m-3}{2}}$, $\overrightarrow{M}_{\frac{m-1}{2}}$, $\overrightarrow{M}_{\frac{m+1}{2}}$, $\overleftarrow{M}_{\frac{m+3}{2}}$. In the following, we prove that these four matchings can form three arc-disjoint linear 3-diforests. For convenience, we denote these four matchings by $\overleftarrow{M}_{3t}$, $\overrightarrow{M}_{3t+1}$, $\overrightarrow{M}_{3t+2}$, $\overleftarrow{M}_{3t+3}$.

We partition $\overrightarrow{M}_{3t+1}$ into three pairwise arc-disjoint matchings $\overrightarrow{W}_0 = \{x_{4t+3} y_{t+1}\}$, $\overrightarrow{W}_1 = \{x_i y_{i+3t+1(mod\ 6t+3)} | i = 0, 2, 4, ..., 4t + 2, 4t + 5, 4t + 7, ..., 6t + 1\}$ and $\overrightarrow{W}_2 = \{x_i y_{i+3t+1(mod\ 6t+3)} | i = 1, 3, 5, ..., 4t + 1, 4t + 4, 4t + 6, ..., 6t + 2\}$. Also, we partition $\overleftarrow{M}_{3t+3}$ into two arc-disjoint matchings $\overleftarrow{W}'_1 = \{y_{i+3t+3(mod\ 6t+3)} x_i | i = 0, 2, 4, ..., 4t + 2, 4t + 5, 4t + 7, ..., 6t + 1\}$ and $\overleftarrow{W}'_2 = \{y_{i+3t+3(mod\ 6t+3)} x_i | i = 1, 3, 5, ..., 4t + 1, 4t + 3, 4t + 4, 4t + 6, ..., 6t + 2\}$. Then the arcs in $\overleftarrow{M}_{3t} \cup \overrightarrow{W}_2$, $\overrightarrow{M}_{3t+2} \cup \overrightarrow{W}'_1$, $\overrightarrow{W}_1 \cup \overleftarrow{W}'_2$ can form three linear 3-diforests, which are denoted by L_1, L_2 and L_3 respectively. We move the arc $y_{t+1} x_{4t+4}$ of L_1 into L_3, add $\overrightarrow{W}_0$ into L_1 and finally obtain three arc-disjoint linear 3-diforests by using $\overleftarrow{M}_{3t}$, $\overrightarrow{M}_{3t+1}$, $\overrightarrow{M}_{3t+2}$, $\overleftarrow{M}_{3t+3}$.

Accordingly, $\overrightarrow{la}_3(K_n^*) \le 2m + \frac{2m-6}{3} + 3 = 2\lceil \frac{2n}{3} \rceil - 1$. □

Proposition 3.5. $\overrightarrow{la}_3(K_n^*) \le 2\lceil \frac{2n}{3} \rceil - 1$ *when* $n \equiv 10$ $(mod\ 12)$.

Proof. Let $n = 12t + 10$, $m = \frac{n}{2} = 6t + 5$, $t \ge 0$. In K_n^*, we have

(1) $2m$ pairwise arc-disjoint linear 3-diforests;

(2) $\frac{m-1}{2}$ pairwise arc-disjoint matchings $\overrightarrow{M}_d = \{x_i y_{i+d(mod\ m)} | i = 0, 1, ..., m-1\}$, $d = 1, 2, ... \frac{m-1}{2}$;
(3) $\frac{m-1}{2}$ pairwise arc-disjoint matchings $\overleftarrow{M}_d = \{y_{i+d(mod\ m)} x_i | i = 0, 1, ..., m-1\}$, $d = 1, 2, ... \frac{m-1}{2}$.

We assume that t is even. We partition the matchings in (2) and (3) into two groups $\mathcal{M}_1 = \{\overrightarrow{M}_1, \overleftarrow{M}_2, \overrightarrow{M}_3, ..., \overrightarrow{M}_{3t+1}, \overleftarrow{M}_{3t+2}\}$ and $\mathcal{M}_2 = \{\overleftarrow{M}_1, \overrightarrow{M}_2, \overleftarrow{M}_3, ..., \overleftarrow{M}_{3t+1}, \overrightarrow{M}_{3t+2}\}$. Then we apply the replacing arc operation for the arcs of the matchings of $\mathcal{M}_2$ in some linear 3-diforests of (1) and obtain some new matchings which are put in a new group $\mathcal{M}_3 = \{\overrightarrow{M}_{6t+4}, \overleftarrow{M}_{6t+3}, \overrightarrow{M}_{6t+2}, ..., \overrightarrow{M}_{3t+4}, \overleftarrow{M}_{3t+3}\}$. Now the arcs not covered by linear 3-diforests are either in the matchings of $\mathcal{M}_1$ or in the matchings of $\mathcal{M}_3$.

We claim that $\overrightarrow{M}_1$, $\overleftarrow{M}_2$, $\overleftarrow{M}_{6t+3}$, $\overrightarrow{M}_{6t+4}$ can form three pairwise arc-disjoint 3-diforests. First, we partition $\overrightarrow{M}_1$ into two arc-disjoint matchings $\overrightarrow{W}_1 = \{x_i y_{i+1} | i = 0, 2, 4, ..., 6t+4\}$ and $\overrightarrow{W}_2 = \{x_i y_{i+1} | i = 1, 3, 5, ..., 6t+3\}$; we partition $\overleftarrow{M}_{6t+3}$ into two arc-disjoint matchings $\overleftarrow{W}'_1 = \{y_{i+6t+3} x_i | i = 0, 2, 4, ..., 6t+2\}$ and $\overleftarrow{W}'_2 = \{y_{i+6t+3} x_i \cup y_{6t+2} x_{6t+4} | i = 1, 3, 5, ..., 6t+3\}$. Then the arcs in $\overrightarrow{W}_2 \cup \overleftarrow{M}_2$, $\overleftarrow{W}'_1 \cup \overrightarrow{M}_{6t+4}$, $\overrightarrow{W}_1 \cup \overleftarrow{W}'_2$ can form three linear 3-diforests L_1, L_2 and L respectively, where $L = \{x_i y_{i+1} x_{i+3} \cup x_{6t+2} y_{6t+3} \cup y_{6t+4} x_1 \cup y_{6t+2} x_{6t+4} y_0 | i = 0, 2, 4, ..., 6t\}$. Thus, we have proved our claim and it is easy to observe that $y_i (i \in \{2, 4, ..., 6t\})$ are not incident to any arcs in L.

Now we only need to construct linear 3-diforests to cover the remain matchings: $\overrightarrow{M}_3, \overleftarrow{M}_4, ..., \overrightarrow{M}_{3t+1}, \overleftarrow{M}_{3t+2}$ and $\overleftarrow{M}_{3t+3}, \overrightarrow{M}_{3t+4}, ..., \overleftarrow{M}_{6t+1}, \overrightarrow{M}_{6t+2}$. Lemma 2.6 states that we can take away one arc from each $\overleftarrow{M}_{4+6i}$, $\overrightarrow{M}_{7+6i}$, $\overrightarrow{M}_{3t+4+6i}$, $\overleftarrow{M}_{3t+7+6i}$ $(i = 0, 1, ..., \frac{t}{2} - 1)$ when t is even and the remaining arcs can form $4t$ linear 3-diforests. And those arcs that we took away are adjacent to some $y_i (i \in \{2, 4, ..., 6t\})$, so they can be moved into L to form a new linear 3-diforest.

Then we show how we select those arcs $\{e_j, j = 0, 1, ..., 2t-1\}$ of each $\overleftarrow{M}_{4+6i}$, $\overrightarrow{M}_{7+6i}$, $\overrightarrow{M}_{3t+4+6i}$, $\overleftarrow{M}_{3t+7+6i}$ $(i = 0, 1, ..., \frac{t}{2} - 1)$ when t is even.

Case 1.1. $t \neq 10k+2, 10k+6$ and $10k$, $k \geq 0$.

Let $e_i = y_{t+6+10i} x_{t+2+4i} \in \overleftarrow{M}_{4+6i}$, $e_{\frac{t}{2}+i} = x_{t+3+4i} y_{t+10+10i} \in \overrightarrow{M}_{7+6i}$,

$e_{t+i} = x_{3t+3+4i} y_{2+10i} \in \overrightarrow{M}_{3t+4+6i}$, $e_{\frac{3t}{2}+i} = y_{6+10i} x_{3t+4+4i} \in \overleftarrow{M}_{3t+7+6i}$, for all $i \in \{0, 1, 2, ..., \frac{t}{2} - 1\}$.

Case 1.2. $t = 10k+2, k \geq 0$.

When $k \geq 1$, let $e_i = y_{t+4+10i} x_{t+4i} \in \overleftarrow{M}_{4+6i}$, $e_{\frac{t}{2}+i} = x_{t+1+4i} y_{t+8+10i} \in \overrightarrow{M}_{7+6i}$,

$e_{t+i} = x_{3t+9+4i} y_{8+10i} \in \overrightarrow{M}_{3t+4+6i}$, $e_{\frac{3t}{2}+i} = y_{12+10i} x_{3t+10+4i} \in \overleftarrow{M}_{3t+7+6i}$, for all $i \in \{0, 1, 2, ..., \frac{t}{2} - 1\}$.

When $k = 0$, let $e_0 = y_8 x_4 \in \overleftarrow{M}_4$, $e_1 = x_5 y_{12} \in \overrightarrow{M}_7$, $e_2 = x_9 y_2 \in \overrightarrow{M}_{10}$, $e_3 = y_6 x_{10} \in \overleftarrow{M}_{13}$.

Case 1.3. $t = 10k, 10k+6, k \geq 0$.

When $t = 0$, $\overrightarrow{M}_1$, $\overleftarrow{M}_2$, $\overrightarrow{M}_3$, $\overleftarrow{M}_4$ can form three arc-disjoint linear 3-diforests.

When $t \neq 0$, let $e_i = y_{t+4+10i} x_{t+4i} \in \overleftarrow{M}_{4+6i}$, $e_{\frac{t}{2}+i} = x_{t+1+4i} y_{t+8+10i} \in \overrightarrow{M}_{7+6i}$,

$e_{t+i} = x_{3t+3+4i} y_{2+10i} \in \overrightarrow{M}_{3t+4+6i}$, $e_{\frac{3t}{2}+i} = y_{6+10i} x_{3t+4+4i} \in \overleftarrow{M}_{3t+7+6i}$, for all $i \in \{0, 1, 2, ..., \frac{t}{2} - 1\}$.

Now we assume that t is odd and partition the matchings in (2) and (3) into two groups $\mathcal{M}_1 = \{\overrightarrow{M}_1, \overleftarrow{M}_2, \overrightarrow{M}_3, ..., \overleftarrow{M}_{3t+1}, \overrightarrow{M}_{3t+2}\}$ and $\mathcal{M}_2 = \{\overleftarrow{M}_1, \overrightarrow{M}_2, \overleftarrow{M}_3, ..., \overrightarrow{M}_{3t+1}, \overleftarrow{M}_{3t+2}\}$. Similarly to the proof above, we need to select one arc from each $\overleftarrow{M}_{4+6i}$, $\overrightarrow{M}_{7+6i}$, $\overleftarrow{M}_{3t+4+6i}$, $\overrightarrow{M}_{3t+7+6i}$, $i \in \{0, 1, 2, ..., \frac{t-1}{2} - 1\}$.

Case 2.1. $t \neq 10k+1, 10k+3$ and $10k+7$, $k \geq 0$.

Let $e_i = y_{t+5+10i} x_{t+1+4i} \in \overleftarrow{M}_{4+6i}$, $e_{\frac{t-1}{2}+i} = x_{t+2+4i} y_{t+9+10i} \in \overrightarrow{M}_{7+6i}$,

$e_{t-1+i} = y_{2+10i} x_{3t+3+4i} \in \overleftarrow{M}_{3t+4+6i}$, $e_{\frac{3(t-1)}{2}+i} = x_{3t+4+4i} y_{6+10i} \in \overrightarrow{M}_{3t+7+6i}$, for all $i \in \{0, 1, ..., \frac{t-1}{2} - 1\}$.

$e_{2t-2} = y_{6t}x_{3t-1} \in \overleftarrow{M}_{3t+1}$, $e_{2t-1} = y_{6t-2}x_{6t+2} \in \overleftarrow{M}_{6t+1}$.

Case 2.2. $t = 10k + 3, 10k + 7, k \geq 0$.

Let $e_i = y_{t+1+10i}x_{t-3+4i} \in \overleftarrow{M}_{4+6i}$, $e_{\frac{t-1}{2}+i} = x_{t-2+4i}y_{t+5+10i} \in \overrightarrow{M}_{7+6i}$,

$e_{t-1+i} = y_{6+10i}x_{3t+7+4i} \in \overleftarrow{M}_{3t+4+6i}$, $e_{\frac{3(t-1)}{2}+i} = x_{3t+8+4i}y_{10+10i} \in \overrightarrow{M}_{3t+7+6i}$, for all $i \in \{0, 1, ..., \frac{t-1}{2} - 1\}$.

$e_{2t-2} = y_{6t}x_{3t-1} \in \overleftarrow{M}_{3t+1}$, $e_{2t-1} = y_{6t-2}x_{6t+2} \in \overleftarrow{M}_{6t+1}$.

Case 2.3. $t = 10k + 1, k \geq 0$.

When $k \geq 1$, let $e_i = y_{t+1+10i}x_{t-3+4i} \in \overleftarrow{M}_{4+6i}$, $e_{\frac{t-1}{2}+i} = x_{t-2+4i}y_{t+5+10i} \in \overrightarrow{M}_{7+6i}$,

$e_{t-1+i} = y_{4+10i}x_{3t+5+4i} \in \overleftarrow{M}_{3t+4+6i}$, $e_{\frac{3(t-1)}{2}+i} = x_{3t+6+4i}y_{8+10i} \in \overrightarrow{M}_{3t+7+6i}$, for all $i \in \{0, 1, ..., \frac{t-1}{2} - 1\}$.

$e_{2t-2} = y_{6t}x_{3t-1} \in \overleftarrow{M}_{3t+1}$, $e_{2t-1} = y_{6t-2}x_{6t+2} \in \overleftarrow{M}_{6t+1}$.

When $k = 0$, let $e_0 = y_4x_0 \in \overleftarrow{M}_4$ and $e_1 = y_2x_6 \in \overleftarrow{M}_7$.

We have finished all the cases discussion and the arcs $\{e_i, i = 0, 1, ..., 2t - 1\}$ are what we need. Accordingly, $\overrightarrow{la}_3(K_n^*) \leq 2(6t + 5) + 4t + 3 = 2\lceil \frac{2n}{3} \rceil - 1$. □

Now we conclude the following result for the linear 3-arboricity of K_n^*, which verifies Conjecture 2.1.

Theorem 3.1.

$$\overrightarrow{la}_3(K_n^*) = \begin{cases} 2\left\lceil \dfrac{2n-2}{3} \right\rceil & \text{when } n \equiv 4, 8, 11 \ (mod 12), \\ 2\left\lceil \dfrac{2n}{3} \right\rceil & \text{when } n \equiv 1, 3, 6, 9 \ (mod 12), \\ 2\left\lceil \dfrac{2n-2}{3} \right\rceil - 1 & \text{when } n \equiv 0 \ (mod 12), \\ 2\left\lceil \dfrac{2n}{3} \right\rceil - 1 & \text{when } n \equiv 2, 5, 7, 10 \ (mod 12). \end{cases}$$

Proof. By Lemmas 2.3, 2.4 and Theorem 2.1, $\left\lceil \frac{n(n-1)}{\lfloor \frac{3n}{4} \rfloor} \right\rceil \leq \overrightarrow{la}_3(K_n^*) \leq 2la_3(K_n)$. In addition, with the above five propositions, we have the result. □

4. Linear 2-arboricity of symmetric complete digraphs

In this section, we study the linear 2-arboricity of K_n^*. We first introduce K_3-factorization $\boldsymbol{F} = \{F_1, F_2, ..., F_t\}$ of $K_n (n \geq 3)$: (1) F_i is a spanning subgraph of K_n and each component of F_i is isomorphic to K_3; (2) each edge is in only one F_i $(1 \leq i \leq t)$. And we call each F_i is a K_3-factor of K_n. Similarly, we can define the $\overrightarrow{K}_3$-factorization of $K_n^* (n \geq 3)$ and each component of the $\overrightarrow{K}_3$-factor is a directed K_3.

Lemma 4.1. *Let C_n^* be a symmetric directed cycle with n vertices. If $n \equiv 0$ (mod 6), then* $\overrightarrow{la}_2(C_n^*) = 3$.

Proof. Let $n = 6t$ and $C_n^* = (x_0, x_1, ..., x_{6t-1}, x_0)$. The arcs of C_n^* can be decomposed into three linear 2-diforests: $\{x_ix_{i+1}x_{i+2} | i = 0, 6, ..., 6t - 6\} \cup \{x_{i+2}x_{i+1}x_i | i = 3, 9, ..., 6t - 3\}$, $\{x_ix_{i+1}x_{i+2} | i = 2, 8, ..., 6t - 4\} \cup \{x_{i+2(mod\ 6t)}x_{i+1(mod\ 6t)}x_i | i = 5, 11, ..., 6t - 1\}$ and $\{x_ix_{i+1}x_{i+2} | i = 4, 10, ..., 6t - 2\} \cup \{x_{i+2}x_{i+1}x_i | i = 1, 7, ..., 6t - 5\}$. □

Proposition 4.1. $\overrightarrow{la}_2(K_n^*) \le 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 0\ (mod\ 12)$.

Proof. Let $n = 12t$.

When $t = 1$, we know that $K_{12}^* = K_{6,6}^* \cup 2K_6^*$. Then $\overrightarrow{la}_2(K_{12}^*) \le \overrightarrow{la}_2(K_{6,6}^*) + \overrightarrow{la}_2(K_6^*)$. Since $K_{6,6}^*$ can be decomposed into three arc-disjoint symmetric directed cycles and each such cycle can form three linear 2-forests by Lemma 4.1, $\overrightarrow{la}_2(K_{6,6}^*) \le 9$. Let $V(K_6^*) = \{x_0, x_1, x_2, y_0, y_1, y_2\}$. We decompose $K_6^* = 2K_3^* \cup M_0^* \cup M_1^* \cup M_2^*$, where $M_d^* = \{x_i y_{i+d(mod\ 3)}, y_{i+d(mod\ 3)} x_i | i = 0, 1, 2\}$ $(d = 0, 1, 2)$. $M_0^* \cup M_1^*$ can form a symmetric directed cycle and thus form three linear 2-diforests by Lemma 4.1. $2K_3^* \cup M_2^*$ contains a symmetric directed cycle $x_1x_0x_2y_1y_2y_0x_1$ and still can form three linear 2-diforests by Lemma 4.1. In addition, x_1x_2, x_0y_2, y_0y_1 and x_2x_1, y_2x_0, y_1y_0 form two linear 2-diforests. Thus, $\overrightarrow{la}_2(K_{12}^*) \le \overrightarrow{la}_2(K_{6,6}^*) + \overrightarrow{la}_2(K_6^*) \le 9 + 3 + 3 + 2 = 17$.

Now we assume $t \ge 2$. Baker and Wilson [5] proved that if F is a perfect matching of K_n, $K_n - F$ can be decomposed into $6t - 1$ K_3-factors if and only if $n = 0(mod\ 12)$ and $t \ge 2$. So for two perfect matchings F and F' in K_n^*, which are with opposite directions, we obtain $6t - 1$ $\overrightarrow{K_3}$-factors $F_1, F_2, ..., F_{6t-1}$ and $6t - 1$ $\overrightarrow{K_3}$-factors $F'_1, F'_2, ..., F'_{6t-1}$ with opposite directions in $K_n^* - \{F, F'\}$.

For the union of any two $\overrightarrow{K_3}$-factors, the directed triangles with a common vertex have two possibilities as in Figure 2. It is easy to check that both circumstances can be decomposed into three linear 2-diforests. $F_1, F_2, ..., F_{6t-1}$ and $F'_1, F'_2, ..., F'_{6t-1}$ can be partitioned into pairs of directed triangles with a common vertex, and then can form $3(6t - 1)$ linear 2-diforests. In addition, F and F' also form two linear 2-diforests in a trivial way.

So $\overrightarrow{la}_2(K_n^*) \le 3(6t - 1) + 2 = 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1.$ □

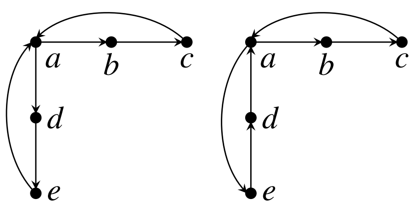

Figure 2. Two directed triangles with a common vertex.

Proposition 4.2. $\overrightarrow{la}_2(K_n^*) \le 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 3\ (mod\ 12)$ *and* $n > 3$.

Proof. Let $n = 12t + 3$. Ray-Chauduri and Wilson [18] proved that K_n can be decomposed into $6t + 1$ K_3 factors if and only if $n = 3\ (mod\ 6)$. Thus, as in Proposition 4.1, we obtain $6t + 1$ $\overrightarrow{K_3}$-factors $F_1, F_2, ..., F_{6t+1}$ and $6t + 1$ $\overrightarrow{K_3}$-factors $F'_1, F'_2, ..., F'_{6t+1}$ with opposite directions in K_n^*. $F_1, F_3, ..., F_{6t+1}$ and $F'_1, F'_2, ..., F'_{6t+1}$ can form $3(6t + 1)$ linear 2-diforests.

Accordingly, $\overrightarrow{la}_2(K_n^*) \leq 3(6t+1) = 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1.$ □

Proposition 4.3. $\overrightarrow{la}_2(K_n^*) \leq 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 2, 10 \ (mod\ 12)$.

Proof. Since Alspach el al. [4] proved that K_n has a Hamiltonian path decomposition when n is even, K_n^* can be decomposed into $\frac{n}{2}$ arc-disjoint symmetric directed n-paths. Each symmetric directed path can be decomposed into three linear 2-forests. Thus, K_n^* can form $\frac{3n}{2}$ arc-disjoint linear 2-diforests.

Accordingly, $\overrightarrow{la}_2(K_n^*) \leq \frac{3n}{2} = 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1.$ □

Proposition 4.4. $\overrightarrow{la}_2(K_n^*) \leq 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 7 \ (mod\ 12)$.

Proof. Let $n = 12t+7$ and $\{v_0, v_1, v_2, ..., v_{12t+6}\}$ be the vertex set of K_n^*. Since Alspach el al. [4] proved that K_n has a Hamiltonian cycle decomposition when n is odd, K_n^* can be decomposed into $6t+3$ symmetric directed Hamiltonian cycles

$C_i^* = v_{12t+6}v_iv_{12t+5+i(mod\ 12t+6)}v_{i+1}v_{12t+4+i(mod\ 12t+6)}...v_{6t+2+i}v_{6t+3+i}v_{12t+6}$ $(0 \leq i \leq 6t+2)$.

Next, we construct symmetric directed paths from C_i^* by removing two kinds of symmetric arcs $v_{3t+1+i}v_{9t+4+i(mod\ 12t+6)}, v_{9t+4+i(mod\ 12t+6)}v_{3t+1+i}$ $(0 \leq i \leq 6t+2)$. Those removed arcs form two matchings and thus form two arc-disjoint linear 2-diforests. In addition, each symmetric directed paths can form three arc-disjoint linear 2-diforests.

Accordingly, $\overrightarrow{la}_2(K_n^*) \leq 3(6t+3)+2 = 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1.$ □

Proposition 4.5. $\overrightarrow{la}_2(K_n^*) \leq 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 5 \ (mod\ 12)$.

Proof. Let $n = 12t+5$. K_n^* can be decomposed into $6t+2$ symmetric directed Hamiltonian cycles

$C_i^* = v_{12t+4}v_iv_{12t+3+i(mod\ 12t+4)}v_{i+1}v_{12t+2+i(mod\ 12t+4)}...v_{6t+1+i}v_{6t+2+i}v_{12t+4}$ $(0 \leq i \leq 6t+1)$.

We obtain symmetric directed paths from C_i^* by removing the symmetric arcs $v_{3t+1+i}v_{9t+3+i(mod\ 12t+4)}$, $v_{9t+3+i(mod\ 12t+4)}v_{3t+1+i}$ $(0 \leq i \leq 6t+1)$. Next, for each such path, we relabel these vertices as $x_0, x_1, x_2, ..., x_{12t+4}$ along the direction: x_0 on behalf of the vertex $v_{9t+3+i(mod\ 12t+4)}$; x_{12t+4} on behalf of the vertex v_{3t+1+i}. Then for each such path, we decompose it into three linear 2-diforests F_1, F_2, F_3 as follows:

$F_1 = \{x_ix_{i+1}x_{i+2} | i = 0, 6, ..., 12t\} \cup \{x_{i+5}x_{i+4}x_{i+3} | i = 0, 6, ..., 12t-6\} \cup \{x_{12t+3}x_{12t+4}\}$;

$F_2 = \{x_{i+3}x_{i+2}x_{i+1}|i = 0, 6, ..., 12t\} \cup \{x_{i+4}x_{i+5}x_{i+6}|i = 0, 6, ..., 12t-6\}$;

$F_3 = \{x_{i+2}x_{i+3}x_{i+4}|i = 0, 6, ..., 12t\} \cup \{x_{i+7}x_{i+6}x_{i+5}|i = 0, 6, ..., 12t-6\} \cup \{x_1x_0\}$.

And we move the arcs $x_{12t+4}x_0 = v_{3t+1+i}v_{9t+3+i(mod\ 12t+4)}$ into F_2 to form a new linear 2-diforest. In addition, the arcs $\{v_{9t+3+i(mod\ 12t+4)}v_{3t+1+i}|i = 0, 1, ..., 6t+1\}$ form a matching and then also a trivial linear 2-diforest.

Accordingly, $\overrightarrow{la}_2(K_n^*) \le 3(6t+2)+1 = 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1.$ □

Now we have the following result for the linear 2-arboricity of K_n^*, which verifies Conjecture 2.1.

Theorem 4.1. *For $K_n^*(n > 3)$,*

$$\overrightarrow{la}_2(K_n^*) = \begin{cases} 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil & when\ n \equiv 1, 4, 6, 8, 9, 11\ (mod 12), \\ 2\left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil - 1 & when\ n \equiv 0, 2, 3, 5, 7, 10\ (mod 12). \end{cases}$$

Proof. By Lemmas 2.3, 2.4 and Theorem 2.3, we know that $\left\lceil \frac{n(n-1)}{\lfloor \frac{2n}{3} \rfloor} \right\rceil \le \overrightarrow{la}_2(K_n^*) \le 2la_2(K_n)$. With all the propositions in this section, we have the result. □

5. Linear 3-arboricity of symmetric complete bipartite digraphs

Let $K_{n,n}^*$ be a symmetric complete bipartite digraph with partite sets $X = \{x_0, x_1, ..., x_n\}$ and $Y = \{y_0, y_1, ..., y_n\}$. We decompose the arc set of $K_{n,n}^*$ into $2n$ pairwise disjoint perfect matchings $\overrightarrow{M}_d = \{x_iy_{i+d(mod\ n)}|i = 0, 1, ..., n-1\}$ and $\overleftarrow{M}_d = \{y_{i+d(mod\ n)}x_i|i = 0, 1, ..., n-1\}$ $(d = 0, 1, 2, ...n-1)$.

Proposition 5.1. $\overrightarrow{la}_3(K_{n,n}^*) \le 2\lceil \frac{2n}{3} \rceil - 1$ *when $n \equiv 2\ (mod\ 6)$.*

Proof. Let $n = 6t+2$. We partition the $2n$ pairwise arc-disjoint perfect matchings of $K_{n,n}^*$ into the following three groups:

(1) $\overrightarrow{M}_2, \overleftarrow{M}_3, \overrightarrow{M}_4, ..., \overrightarrow{M}_{6t}, \overleftarrow{M}_{6t+1}$;

(2) $\overleftarrow{M}_0, \overrightarrow{M}_1, \overleftarrow{M}_2, ..., \overleftarrow{M}_{6t-2}, \overrightarrow{M}_{6t-1}$;

(3) $\overrightarrow{M}_0, \overleftarrow{M}_1, \overleftarrow{M}_{6t}, \overrightarrow{M}_{6t+1}$.

By Lemma 2.5, the perfect matchings in (1) and (2) can form $8t$ arc-disjoint linear 3-diforests.

In addition, we claim that the remaining matchings $\overrightarrow{M}_0, \overleftarrow{M}_1, \overleftarrow{M}_{6t}, \overrightarrow{M}_{6t+1}$ can form three arc-disjoint linear 3-diforests. We partition $\overleftarrow{M}_1$ into two matchings $W_1 = \{y_{i+1}x_i|i = 0, 2, ..., 6t\}$ and $W_2 = \{y_{i+1}x_i|i = 1, 3, ..., 6t+1\}$. And we partition $\overrightarrow{M}_{6t+1}$ into two matchings $W_1' = \{x_iy_{6t+1+i(mod\ 6t+2)}|i = 0, 2, ..., 6t\}$ and $W_2' = \{x_iy_{6t+1+i(mod\ 6t+2)}|i = 1, 3, ..., 6t+1\}$. Then $W_1 \cup W_2'$, $W_2 \cup \overrightarrow{M}_0$, $W_1' \cup \overleftarrow{M}_{6t}$ form three arc-disjoint linear 3-diforests.

Accordingly, $\overrightarrow{la}_3(K_{n,n}^*) \le 8t+3 = 2\lceil \frac{2n}{3} \rceil - 1$. □

Proposition 5.2. $\overrightarrow{la}_3(K^*_{n,n}) \leq 2\lceil\frac{2n+2}{3}\rceil - 1$ *when* $n \equiv 3$ $(mod\ 6)$.

Proof. Let $n = 6t + 3$. We partition the $2n$ pairwise arc-disjoint perfect matchings of $K^*_{n,n}$ into the following two groups:
(1) $\overrightarrow{M}_0, \overleftarrow{M}_1, \overrightarrow{M}_2, ..., \overleftarrow{M}_{6t+1}, \overrightarrow{M}_{6t+2}$;
(2) $\overleftarrow{M}_0, \overrightarrow{M}_1, \overleftarrow{M}_2, ..., \overrightarrow{M}_{6t+1}, \overleftarrow{M}_{6t+2}$.

Let $e_i = y_{4i}x_{6t+2-2i} \in \overleftarrow{M}_{1+6i} (i = 0, 1, ..., t)$,
$e_{t+1+i} = x_{6t+1-2i}y_{4i+2} \in \overrightarrow{M}_{4+6i} (i = 0, 1, ..., t-1)$,
$e_{2t+1+i} = x_{4t+1-2i}y_{4t+2+4i(mod\ 6t+3)} \in \overrightarrow{M}_{1+6i} (i = 0, 1, ..., t)$,
$e_{3t+2+i} = y_{4t+4+4i(mod\ 6t+3)}x_{4t-2i} \in \overleftarrow{M}_{4+6i} (i = 0, 1, ..., t-1)$.

Then we remove the arcs $\{e_j | j = 0, 1, ..., 4t+1\}$ from those perfect matchings. By Lemma 2.6, the perfect matchings of (1) and (2) other than the removed arcs can form $8t + 4$ arc-disjoint linear 3-diforests. In addition, the removed arcs can form another one linear 3-diforests.

Accordingly, $\overrightarrow{la}_3(K^*_{n,n}) \leq 8t + 5 = 2\lceil\frac{2n+2}{3}\rceil - 1$. □

We have the following result for the linear 3-arboricity of $K^*_{n,n}$, which verifies Conjecture 2.1.

Theorem 5.1.

$$\overrightarrow{la}_3(K^*_{n,n}) = \begin{cases} 2\left\lceil\frac{2n}{3}\right\rceil & when\ n \equiv 0, 1, 4, 5\ (mod\ 6), \\ 2\left\lceil\frac{2n}{3}\right\rceil - 1 & when\ n \equiv 2\ (mod\ 6), \\ 2\left\lceil\frac{2n+2}{3}\right\rceil - 1 & when\ n \equiv 3\ (mod\ 6). \end{cases}$$

Proof. By Lemmas 2.3, 2.4 and Theorem 2.2, $\left\lceil\frac{2n^2}{\lfloor\frac{6n}{4}\rfloor}\right\rceil \leq \overrightarrow{la}_3(K^*_{n,n}) \leq 2la_3(K_{n,n})$. With all the propositions in this section, we have the result. □

6. Linear 2-arboricity of symmetric complete bipartite digraphs

Proposition 6.1. $\overrightarrow{la}_2(K^*_{n,n}) \leq 2\left\lceil\frac{n^2}{\lfloor\frac{4n}{3}\rfloor}\right\rceil - 1$ *when* $n \equiv 3$ $(mod\ 12)$.

Proof. Let $n = 12t + 3$. We partition the $2n$ pairwise arc-disjoint perfect matchings of $K^*_{n,n}$ into two groups:
(1) $\overrightarrow{M}_i, \overleftarrow{M}_i, \overrightarrow{M}_{i+1}, \overleftarrow{M}_{i+1}$, $i = 0, 2, 4, ..., 12t$;
(2) $\overleftarrow{M}_{12t+2}, \overrightarrow{M}_{12t+2}$.

For $i \in \{0, 2, 4, ..., 12t\}$, $\overrightarrow{M}_i, \overleftarrow{M}_i, \overrightarrow{M}_{i+1}, \overleftarrow{M}_{i+1}$ can form a symmetric directed cycle and such cycle can be decomposed into three linear 2-diforests by Lemma 4.1. In addition, $\overleftarrow{M}_{12t+2}$ and $\overrightarrow{M}_{12t+2}$ form another two linear 2-diforests.

Accordingly, $\overrightarrow{la}_2(K^*_{n,n}) \leq 3(6t + 1) + 2 = 2\left\lceil\frac{n^2}{\lfloor\frac{4n}{3}\rfloor}\right\rceil - 1$. □

Proposition 6.2. $\overrightarrow{la}_2(K^*_{n,n}) \le 2\left\lceil \dfrac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 4$ $(mod\ 12)$.

Proof. Let $n = 12t + 4$. $K^*_{n,n}$ can be decomposed into $2n$ pairwise arc-disjoint perfect matchings and every four matchings $\overrightarrow{M}_i$, $\overleftarrow{M}_i$, $\overrightarrow{M}_{i+1}$, $\overleftarrow{M}_{i+1}$ $(i = 0, 2, ..., 12t + 2)$ form a symmetric directed cycle $C^*_j(j = \frac{i}{2})$.

We claim that if C is a symmetric directed cycle with $V(C) = \{x_0, x_1, ..., x_{24t+7}\}$ and e is an arc of C, then $C - \{e\}$ can form three arc-disjoint linear 2-diforests. Without loss of generality, we assume that $e = x_0x_{24t+7}$. The three linear 2-diforests F_1, F_2, F_3 are $F_1 = \{x_ix_{i+1}x_{i+2}|i = 0, 6, ..., 24t\} \cup \{x_{i+2}x_{i+1}x_i|i = 3, 9, ..., 24t + 3\} \cup \{x_{24t+6}x_{24t+7}\}$, $F_2 = \{x_{i+2}x_{i+1}x_i|i = 1, 7, ..., 24t + 1\} \cup \{x_ix_{i+1}x_{i+2}|i = 4, 10, ..., 24t + 4\} \cup \{x_{24t+7}x_0\}$ and $F_3 = \{x_ix_{i+1}x_{i+2}|i = 2, 8, ..., 24t + 2\} \cup \{x_{i+2}x_{i+1}x_i|i = 5, 11, ..., 24t + 5\} \cup \{x_1x_0\}$.

Let $e_j = x_{n-1-j}y_j \in C^*_j$ $(j = 0, 1, ...6t + 1)$. By the claim above, $C^*_j - \{e_j\}$ can form three linear 2-diforests. Furthermore, $\{e_j|i = 0, 1, ..., 6t + 1\}$ is a matching and thus form one linear 2-diforest.

Accordingly, $\overrightarrow{la}_2(K^*_{n,n}) \le 3(6t + 2) + 1 = 2\left\lceil \dfrac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1.$ □

Proposition 6.3. $\overrightarrow{la}_2(K^*_{n,n}) \le 2\left\lceil \dfrac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 5$ $(mod\ 12)$.

Proof. Let $n = 12t+5$. We partition the $2n$ pairwise disjoint perfect matchings of $K^*_{n,n}$ into two groups:
(1) $\overrightarrow{M}_i$, $\overleftarrow{M}_i$, $\overrightarrow{M}_{i+1}$, $\overleftarrow{M}_{i+1}$, $(i = 0, 2, ..., 12t + 2)$;
(2) $\overleftarrow{M}_{12t+4}$, $\overrightarrow{M}_{12t+4}$.

Every four matchings $\overrightarrow{M}_i$, $\overleftarrow{M}_i$, $\overrightarrow{M}_{i+1}$, $\overleftarrow{M}_{i+1}$ $(i \in \{0, 2, ..., 12t + 2\})$ form a symmetric directed cycle $C^*_j(j = \frac{i}{2})$.

We claim that if C is a symmetric directed cycle with $V(C) = \{x_0, x_1, ..., x_{24t+9}\}$, and $e = x_0x_{24t+9}$, $e' = x_3x_2$ are two arcs of C, then $C - \{e, e'\}$ can form three arc-disjoint linear 2-diforests, which are $F_1 = \{x_ix_{i+1}x_{i+2}|i = 3, 9, ..., 24t + 3\} \cup \{x_{i+2}x_{i+1}x_i|i = 6, 12, ..., 24t + 6\} \cup \{x_{24t+9}x_0x_1\}$, $F_2 = \{x_ix_{i+1}x_{i+2}|i = 1, 7, ..., 24t+7\} \cup \{x_{i+2}x_{i+1}x_i|i = 4, 10, ..., 24t+4\}$ and $F_3 = \{x_ix_{i+1}x_{i+2}|i = 5, 11, ..., 24t+5\} \cup \{x_{i+2}x_{i+1}x_i|i = 8, 14, ..., 24t + 2, t \ge 1\} \cup \{x_2x_1x_0\} \cup \{x_4x_3\} \cup \{x_{24t+9}x_{24t+8}\}$.

Let $e_j = x_{2j}y_{4j(mod\ n)}$, $e'_j = y_{4j+2(mod\ n)}x_{2j+1} \in C^*_j$, $(j = 0, 1, ...6t+1)$. From our claim above $C^*_j - \{e_j, e'_j\}$ form three linear 2-diforests. Furthermore, $\{e_j|j = 0, 1, ..., 6t + 1\} \cup \{e'_j|j = 0, 1, ..., 6t + 1\}$ is a matching and thus form a linear 2-diforest.

In addition, the remaining matchings $\overleftarrow{M}_{12t+4}$, $\overrightarrow{M}_{12t+4}$ also form two linear 2-diforests.

Accordingly, $\overrightarrow{la}_2(K^*_{n,n}) \le 3(6t + 2) + 1 + 2 = 2\left\lceil \dfrac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1.$ □

Proposition 6.4. $\overrightarrow{la}_2(K^*_{n,n}) \le 2\left\lceil \dfrac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 6$ $(mod\ 12)$.

Proof. Let $n = 12t+6$. For $i \in \{0, 2, ..., 12t+4\}$, the matchings $\overrightarrow{M}_i$, $\overleftarrow{M}_i$, $\overrightarrow{M}_{i+1}$, $\overleftarrow{M}_{i+1}$ can form a symmetric directed cycle and such cycle can decomposed into three linear 2-diforests by Lemma 4.1.

Accordingly, $\overrightarrow{la}_2(K^*_{n,n}) \leq 3(6t+3) = 2\left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1$. □

Proposition 6.5. $\overrightarrow{la}_2(K^*_{n,n}) \leq 2\left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1$ *when* $n \equiv 8$ (*mod* 12).

Proof. Let $n = 12t+8$. For $i \in \{0, 2, ..., 12t+6\}$, every four matchings $\overrightarrow{M}_i$, $\overleftarrow{M}_i$, $\overrightarrow{M}_{i+1}$, $\overleftarrow{M}_{i+1}$ form a symmetric directed cycle $C^*_j (j = \frac{i}{2})$.

We claim that if C is a symmetric directed cycle with $V(C) = \{x_0, x_1, ..., x_{24t+15}\}$, and $e = x_0x_{24t+15}$, $e' = x_3x_2$ are two arcs of C, then $C - \{e, e'\}$ can form three arc-disjoint linear 2-diforests, which are $F_1 = \{x_ix_{i+1}x_{i+2} | i = 3, 9, ..., 24t+9\} \cup \{x_{i+2}x_{i+1}x_i | i = 6, 12, ..., 24t+12\} \cup \{x_{24t+15}x_0x_1\}$, $F_2 = \{x_ix_{i+1}x_{i+2} | i = 1, 7, ..., 24t+13\} \cup \{x_{i+2}x_{i+1}x_i | i = 4, 10, ..., 24t+10\}$ and $F_3 = \{x_ix_{i+1}x_{i+2} | i = 5, 11, ..., 24t+11\} \cup \{x_{i+2}x_{i+1}x_i | i = 8, 14, ..., 24t+8\} \cup \{x_2x_1x_0\} \cup \{x_4x_3\} \cup \{x_{24t+15}x_{24t+14}\}$. And if we choose $e = x_{24t+15}x_0$, $e' = x_2x_3$, we have the same claim.

Let $e_j = x_{2j}y_{4j (mod\ n)}$, $e'_j = y_{4j+2 (mod\ n)}x_{2j+1} \in C^*_j$, $(j = 0, 1, ..., 3t+1)$; $e_j = y_{4j (mod\ n)}x_{2j}$, $e'_j = x_{2j+1}y_{4j+2 (mod\ n)} \in C^*_j$, $j = 3t+2, 3t+3, ..., 6t+3$. By the claim above, $C^*_j - \{e_j, e'_j\}$ form three linear 2-diforests. Furthermore, the arcs $\{e_j | j = 0, 1, ..., 6t+3\} \cup \{e'_j | j = 0, 1, ..., 6t+3\}$ form a linear 2-diforest $\{x_{2j}y_{4j}x_{6t+4+2j} | j = 0, 1, ..., 3t+1\} \cup \{x_{6t+5+2j}y_{4j+2}x_{2j+1} | j = 0, 1, ..., 3t+1\}$.

Accordingly, $\overrightarrow{la}_2(K^*_{n,n}) \leq 3(6t+4) + 1 = 2\left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1$. □

We have the following result for the linear 2-arboricity of $K^*_{n,n}$, which verifies Conjecture 2.1.

Theorem 6.1.

$$\overrightarrow{la}_2(K^*_{n,n}) = \begin{cases} 2\left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil - 1 & \text{when } n \equiv 3, 4, 5, 6, 8 \ (mod\ 12), \\ 2\left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil & \text{when } n \equiv 0, 1, 2, 9, 10, 11 \ (mod\ 12). \end{cases}$$

Proof. By Lemmas 2.3, 2.4 and Theorem 2.4, $\left\lceil \frac{2n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil \leq \overrightarrow{la}_2(K^*_{n,n}) \leq 2la_2(K_{n,n})$. With all Propositions in this section, we have the result. □

Note that, in Theorem 6.1, we missed one case when $n \equiv 7$ (mod 12). We believe that $\overrightarrow{la}_2(K^*_{n,n}) = 2\left\lceil \frac{n^2}{\lfloor\frac{4n}{3}\rfloor} \right\rceil - 1$ in this case, but we can only prove that $\overrightarrow{la}_2(K^*_{n,n}) \leq 2\left\lceil \frac{n^2}{\lfloor\frac{4n}{3}\rfloor} \right\rceil$ by Lemma 2.4 and Theorem 2.4.

7. Conclusions

In this paper, we determine the linear 3-arboricity for symmetric complete digraphs and symmetric complete bipartite digraphs, and also determine the linear 2-arboricity for symmetric complete digraphs. All these results verify Conjecture 2.1.

Acknowledgments

The authors would like to thank anonymous referees for helpful comments and suggestions which improved the original version of the paper. This work was supported by Guangdong Basic and Applied Basic Research Foundation (No. 2021A1515012047) and Science and Technology Program of Guangzhou (No. 202002030401).

Conflict of interest

The authors declare that they have no competing interests.

References

1. J. Akiyama, G. Exoo, F. Harary, Covering and packing in graphs III: Cyclic and acyclic invariants, *Math. Slovaca*, **30** (1980), 405–417.
2. N. Alon, Probabilistic methods in coloring and decomposition problems, *Discrete Math.*, **127** (1994), 31–46. https://doi.org/10.1016/0012-365X(92)00465-4
3. N. Alon, V. J. Teague, N. C. Wormald, Linear arboricity and linear k-arboricity of regular graphs, *Graph. Combinator.*, **17** (2001), 11–16. https://doi.org/10.1007/PL00007233
4. B. Alspach, J. C. Bermond, D. Sotteau, Decomposition into cycles I: Hamilton decompositions, In: G. Hahn, G. Sabidussi, R. E. Woodrow, *Cycles and rays*, Dordrecht: Springer, 1990, 9–18. https://doi.org/10.1007/978-94-009-0517-7
5. R. D. Baker, R. M. Wilson, Nearly Kirkman triple systems, *Utilitas Math.*, **11** (1977), 289–296.
6. J. C. Bermond, J. L. Fouquet, M. Habib, B. Peroche, On linear k-arboricity, *Discrete Math.*, **52** (1984), 123–132. https://doi.org/10.1016/0012-365X(84)90075-X
7. G. J. Chang, Algorithmic aspects of linear k-arboricity, *Taiwanese J. Math.*, **3** (1999), 71–81. https://doi.org/10.11650/twjm/1500407055
8. G. J. Chang, B. L. Chen, H. L. Fu, K. C. Huang, Linear k-arboricities on trees, *Discrete Appl. Math.*, **103** (2000), 281–287. https://doi.org/10.1016/S0166-218X(99)00247-4

9. B. L. Chen, K. C. Huang, On the linear k-arboricity of K_n and $K_{n,n}$, *Discrete Math.*, **254** (2002), 51–61. https://doi.org/10.1016/S0012-365X(01)00365-X

10. A. Ferber, J. Fox, V. Jain, Towards the linear arboricity conjecture, *J. Comb. Theory B*, **142** (2020), 56–79. https://doi.org/10.1016/j.jctb.2019.08.009

11. H. L. Fu , K. C. Huang, The linear 2-arboricity of complete bipartite graphs, *Ars Combin.*, **38** (1994), 309–318.

12. H. L. Fu, K. C. Huang, C. H. Yen, The linear 3-arboricity of $K_{n,n}$ and K_n, *Discrete Math.*, **308** (2008), 3816–3823. https://doi.org/10.1016/j.disc.2007.07.067

13. M. Habib, B. Peroche, Some problems about linear arboricity, *Discrete Math.*, **41** (1982), 219–220. https://doi.org/10.1016/0012-365X(82)90209-6

14. F. Harary, Coverings and packing in graphs, I, *Ann. New York Acad. Sci.*, **175** (1970), 198–205. https://doi.org/10.1111/j.1749-6632.1970.tb56470.x

15. W. H. He, H. Li, Y. D. Bai, Q. Sun, Linear arboricity of regular digraphs, *Acta Math. Sin. (Engl. Ser.)*, **33** (2017), 501–508. https://doi.org/10.1007/s10114-016-5071-9

16. B. Jackson, N. C. Wormald, On the linear k-arboricity of cubic graphs, *Discrete Math.*, **162** (1996), 293–297. https://doi.org/10.1016/0012-365X(95)00293-6

17. A. Nakayama, B. Peroche, Linear arboricity of digraphs, *Networks*, **17** (1987), 39–53. https://doi.org/10.1002/net.3230170104

18. D. K. Ray-Chauduri, R. M. Wilson, Solution of Kirkman's schoolgirl problem, *Proc. Symp. Pure Math.*, **19** (1971), 187–203.

19. C. Thomassen, Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5, *J. Comb. Theory B*, **75** (1999), 100–109. https://doi.org/10.1006/jctb.1998.1868

20. J. L. Wu, On the linear arboricity of planar graphs, *J. Graph Theor.*, **31** (1999), 129–134. https://doi.org/10.1002/(SICI)1097-0118(199906)31:2<129::AID-JGT5>3.0.CO;2-A

21. B. Xue, L. C. Zuo, On the linear $(n-1)$-arboricity of $K_{n(m)}$, *Discrete Appl. Math.*, **158** (2010), 1546–1550. https://doi.org/10.1016/j.dam.2010.04.013

22. C. H. Yen, H. L. Fu, Linear 2-arboricity of the complete graph, *Taiwanese J. Math.*, **14** (2010), 273–286. https://doi.org/10.11650/twjm/1500405741