Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 4125–4136.
DOI: 10.3934/math.2022228
Received: 20 August 2021
Revised: 28 November 2021
Accepted: 07 December 2021
Published: 15 December 2021

Research article

New q-supercongruences arising from a summation of basic hypergeometric series

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Abstract: With the help of a summation of basic hypergeometric series, the creative microscoping method recently introduced by Guo and Zudilin, and the Chinese remainder theorem for coprime polynomials, we prove some new q-supercongruences on sums of q-shifted factorials. Especially, we give a q-analogue of a formula due to Liu [14].

Keywords: q-supercongruence; basic hypergeometric series; creative microscoping method; Chinese remainder theorem for coprime polynomials
Mathematics Subject Classification: 11A07, 11B65, 33D15

1. Introduction

For any complex variable x, define the shifted-factorial to be

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{when} \quad n \in \mathbb{Z}^+.$$

Let p be an odd prime and $\mathbb{Z}_p$ denote the ring of all p-adic integers. Define Morita's p-adic Gamma function (cf. [17, Chapter 7]) by

$$\Gamma_p(0) = 1 \quad \text{and} \quad \Gamma_p(n) = (-1)^n \prod_{\substack{1\leqslant k<n \\ p\nmid k}} k, \quad \text{when} \quad n \in \mathbb{Z}^+.$$

Noting $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ related to the p-adic norm $|\cdot|_p$, for each $x \in \mathbb{Z}_p$, the definition of p-adic Gamma function can be extended as

$$\Gamma_p(x) = \lim_{\substack{n\in\mathbb{N} \\ |x-n|_p\to 0}} \Gamma_p(n).$$

Two properties of the p-adic Gamma function in common use can be stated as follows:

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } p \nmid x, \\ -1, & \text{if } p \mid x, \end{cases}$$
$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\langle -x\rangle_p - 1},$$

where $\langle x\rangle_p$ indicates the least nonnegative residue of x modulo p, i.e., $\langle x\rangle_p \equiv x \pmod{p}$ and $\langle x\rangle_p \in \{0, 1, \ldots, p-1\}$. In 2016, Long and Ramakrishna [16, Proposition 25] showed that, for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \frac{(1/3)_k^3}{k!^3} \equiv \begin{cases} \Gamma_p(1/3)^6 \pmod{p^3}, & \text{if } p \equiv 1 \pmod{6}, \\ -\dfrac{p^2}{3}\Gamma_p(1/3)^6 \pmod{p^3}, & \text{if } p \equiv 5 \pmod{6}. \end{cases} \tag{1.1}$$

Similarly, Liu [14, Theorem 1.1] proved that, for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \frac{(-1/3)_k^3}{k!^3} \equiv \begin{cases} -18p^2\Gamma_p(2/3)^6 \pmod{p^3}, & \text{if } p \equiv 1 \pmod{6}, \\ 54\Gamma_p(2/3)^6 \pmod{p^3}, & \text{if } p \equiv 5 \pmod{6}. \end{cases} \tag{1.2}$$

For all complex numbers x and q, define the q-shifted factorial to be

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad \text{when} \quad n \in \mathbb{Z}^+.$$

For simplicity, we also adopt the compact notation:

$$(x_1, x_2, \ldots, x_r; q)_n = (x_1;q)_n(x_2;q)_n\cdots(x_r;q)_n,$$

where $r \in \mathbb{Z}^+$ and $n \in \mathbb{N}$. Following Gasper and Rahman [1], define the basic hypergeometric series ${}_{r+1}\phi_r$ to be

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q,\, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, b_2, \ldots, b_r; q)_k} z^k.$$

Then the q-Saalschütz identity (cf. [1, Appendix (II.12)]) can be expressed as

$${}_3\phi_2\left[\begin{matrix} a,\, b,\, q^{-n} \\ c,\, abq^{1-n}/c \end{matrix}; q,\, q\right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \tag{1.3}$$

Recently, Guo [2] established three q-supercongruences via the creative microscoping method (introduced by Guo and Zudilin [9]), and the Chinese remainder theorem for polynomials. Similarly, Wei, Liu, and Wang [21, Theorems 1.1 and 1.2] provided a q-analogue of (1.1). For more q-analogues of supercongruences, we refer the reader to [3–8, 10–13, 15, 18, 20, 22].

Let $[n] = (1-q^n)/(1-q)$ be the q-integer and $\Phi_n(q)$ the n-th cyclotomic polynomial in q:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(k,n)=1}} (q - \zeta^k),$$

where ζ is an n-th primitive root of unity. Motivated by the work just mentioned, we shall establish the following two theorems.

Theorem 1.1. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 3$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{(2n+1)/3} \frac{(q^{-1};q^3)_k^3}{(q^3;q^3)_k^3} q^{9k} \equiv q^{(2-2n)/3}(1+q)\frac{(q;q^3)_{(2n+1)/3}^2}{(q^3;q^3)_{(2n+1)/3}^2} \times \left\{3 - [2n]^2\left(\sum_{i=1}^{(2n+1)/3} \frac{3q^{3i-2}}{[3i-2]^2} - \frac{1+5q+3q^2}{1+q}\right)\right\}.$$

Theorem 1.2. *Let n be a positive integer with $n \equiv 2 \pmod 3$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{(n+1)/3} \frac{(q^{-1};q^3)_k^3}{(q^3;q^3)_k^3} q^{9k} \equiv q^{(2-n)/3}(1+q)\frac{(q;q^3)_{(n+1)/3}^2}{(q^3;q^3)_{(n+1)/3}^2} \times \left\{\theta_n(q) + [n]^2\left(\sum_{i=1}^{(n+1)/3} \frac{3q^{3i-2}}{[3i-2]^2} - \frac{1+5q+3q^2}{1+q}\right)\right\},$$

where

$$\theta_n(q) = \frac{(1-q-3q^2)(1-2q^n) + (4-4q-6q^2+3q^3)q^{2n}}{(1+q)(q-q^n)^2}.$$

It is not difficult to understand that Theorems 1.1 and 1.2 give a q-analogue of (1.2). Letting $n = p$ be an prime and taking $q \to 1$ in the above two theorems, we obtain the following conclusions.

Corollary 1.3. *Let p be a prime such that $p \equiv 1 \pmod 6$. Then*

$$\sum_{k=0}^{(2p+1)/3} \frac{(-1/3)_k^3}{k!^3} \equiv \frac{6(1/3)_{(2p+1)/3}^2}{(1)_{(2p+1)/3}^2}\left\{1 + 6p^2 - \sum_{i=1}^{(2p+1)/3} \frac{4p^2}{(3i-2)^2}\right\} \pmod{p^3}.$$

Corollary 1.4. *Let p be a prime such that $p \equiv 5 \pmod 6$. Then*

$$\sum_{k=0}^{(p+1)/3} \frac{(-1/3)_k^3}{k!^3} \equiv \frac{54(1/3)_{(p+1)/3}^2}{(1)_{(p-2)/3}^2}\left\{1 + \frac{p^2}{(p+1)^2}\sum_{i=1}^{(p+1)/3} \frac{1}{(3i-2)^2}\right\} \pmod{p^3}.$$

In order to explain the equivalence of (1.2) and Corollaries 1.3 and 1.4, we need to verify the following relations.

Proposition 1.5. *Let p be a prime such that $p \equiv 1 \pmod 6$. Then*

$$\frac{(1/3)_{(2p+1)/3}^2}{(1)_{(2p+1)/3}^2}\left\{1 + 6p^2 - \sum_{i=1}^{(2p+1)/3} \frac{4p^2}{(3i-2)^2}\right\} \equiv -3p^2\Gamma_p(2/3)^6 \pmod{p^3}.$$

Proposition 1.6. *Let p be a prime such that $p \equiv 5 \pmod 6$. Then*

$$\frac{(1/3)_{(p+1)/3}^2}{(1)_{(p-2)/3}^2}\left\{1 + \frac{p^2}{(p+1)^2}\sum_{i=1}^{(p+1)/3} \frac{1}{(3i-2)^2}\right\} \equiv \Gamma_p(2/3)^6 \pmod{p^3}.$$

The rest of the paper is arranged as follows. The proof of Theorems 1.1 and 1.2 will be given in Section 2. To this end, we first derive a q-supercongruence modulo $(1-aq^{tn})(a-q^{tn})(b-q^{tn})$, where $t \in \{1,2\}$, by using a summation of basic hypergeometric series, the creative microscoping method, and the Chinese remainder theorem for coprime polynomials. Finally, the proof of Propositions 1.5 and 1.6 will be displayed in Section 3.

2. Proof of Theorems 1.1 and 1.2

In order to prove Theorems 1.1 and 1.2, we require the following lemma.

Lemma 2.1.

$$
{}_3\phi_2\left[\begin{matrix} a, b, q^{-m} \\ q, abq^{2-m} \end{matrix} ; q, q^3\right] = \frac{(1/a, 1/b; q)_m}{(q, 1/ab; q)_m}
\times\left\{\frac{q^m(1-q^m)(q-abq^2-(1+q-aq-bq)q^m)}{(1-abq)(aq-q^m)(bq-q^m)} - \frac{1-ab-(2-a-b)q^m}{(1-a)(1-b)}\right\}.
$$

Proof. By comparing the k-th summands in the summations, it is easy to see that

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a,\ b,\ xq,\ q^{-m} \\ cq,\ x,\ abq^{1-m}/c \end{matrix} ; q,\ q\right] \\
&= \frac{(1-c)(ab-cxq^m)}{(1-x)(ab-c^2q^m)}{}_3\phi_2\left[\begin{matrix} a,\ b,\ q^{-m} \\ c,\ abq^{1-m}/c \end{matrix} ; q,\ q\right] \\
&\quad + \frac{(c-x)(ab-cq^m)}{(1-x)(ab-c^2q^m)}{}_3\phi_2\left[\begin{matrix} a,\ b,\ q^{-m} \\ cq,\ abq^{-m}/c \end{matrix} ; q,\ q\right].
\end{aligned}
$$

Evaluating the two series on the right-hand side by (1.3), we get

$$
{}_4\phi_3\left[\begin{matrix} a,\ b,\ xq,\ q^{-m} \\ cq,\ x,\ abq^{1-m}/c \end{matrix} ; q,\ q\right] = \Omega_m(q; a, b, c, x), \tag{2.1}
$$

where

$$
\Omega_m(q; a, b, c, x) = \frac{(c/a, c/b; q)_m}{(qc, c/ab; q)_m}
\times\left\{\frac{(1-cq^m)(ab-cxq^m)}{(1-x)(ab-c^2q^m)} + \frac{(c-x)(ab-c)(a-cq^m)(b-cq^m)}{(1-x)(a-c)(b-c)(ab-c^2q^m)}\right\}.
$$

Similarly, it is also routine to confirm the relation

$$
\begin{aligned}
&{}_5\phi_4\left[\begin{matrix} a,\ b,\ xq,\ yq,\ q^{-m} \\ cq^2,\ x,\ y,\ abq^{1-m}/c \end{matrix} ; q,\ q\right] \\
&= \frac{(1-cq)(ab-cyq^m)}{(1-y)(ab-c^2q^{m+1})}{}_4\phi_3\left[\begin{matrix} a,\ b,\ xq,\ q^{-m} \\ cq,\ x,\ abq^{1-m}/c \end{matrix} ; q,\ q\right] \\
&\quad + \frac{(cq-y)(ab-cq^m)}{(1-y)(ab-c^2q^{m+1})}{}_4\phi_3\left[\begin{matrix} a,\ b,\ xq,\ q^{-m} \\ cq^2,\ x,\ abq^{-m}/c \end{matrix} ; q,\ q\right].
\end{aligned}
$$

Calculating the two series on the right-hand side via (2.1), we arrive at

$$
{}_5\phi_4\left[\begin{matrix} a,\ b,\ xq,\ yq,\ q^{-m} \\ cq^2,\ x,\ y,\ abq^{1-m}/c \end{matrix} ; q,\ q\right]
$$

$$= \frac{(1-cq)(ab-cyq^m)}{(1-y)(ab-c^2q^{m+1})}\Omega_m(q;a,b,c,x)$$
$$+ \frac{(cq-y)(ab-cq^m)}{(1-y)(ab-c^2q^{m+1})}\Omega_m(q;a,b,cq,x).$$

Letting $c \to q^{-1}$, $x \to \infty$, $y \to \infty$ in the last equation, we are led to Lemma 2.1. □

Subsequently, we shall deduce the following united parametric extension of Theorems 1.1 and 1.2.

Theorem 2.2. *Let n be a positive integer with $n \equiv 3-t \pmod 3$ and $t \in \{1,2\}$. Then, modulo $(1-aq^{tn})(a-q^{tn})(b-q^{tn})$,*

$$\sum_{k=0}^{(tn+1)/3} \frac{(aq^{-1},q^{-1}/a,q^{-1}/b;q^3)_k}{(q^3;q^3)_k^2(q^3/b;q^3)_k}q^{9k}$$
$$\equiv \frac{(b-q^{tn})(ab-1-a^2+aq^{tn})}{(a-b)(1-ab)}\frac{(bq,q;q^3)_{(tn+1)/3}}{(bq)^{(tn+1)/3}(1/b,q^3;q^3)_{(tn+1)/3}}A_n(q;b,t)$$
$$+ \frac{(1-aq^{tn})(a-q^{tn})}{(a-b)(1-ab)}\frac{(aq,q/a;q^3)_{(tn+1)/3}}{b^{(tn+1)/3}(1/b,1/bq;q^3)_{(tn+1)/3}}B(q;a,b), \tag{2.2}$$

where

$$A_n(q;b,t) = \frac{b(1-q^{tn+1})\{q^{tn+2}/b-q+q^{tn-1}(1+q^3-q^{tn+2}-q^2/b)\}}{(1-q)(1-bq^{tn-1})(1-q^{tn+1}/b)}$$
$$- \frac{1-q^{tn-2}/b-q^{tn+1}(2-q^{tn-1}-q^{-1}/b)}{(1-q^{tn-1})(1-q^{-1}/b)},$$
$$B(q;a,b) = \frac{(1-bq)\{1-q-b(q^{-2}+q-a-1/a)\}}{q(1-q)(1-ab/q)(1-b/aq)}$$
$$- \frac{1-q^{-2}-b(2q-a-1/a)}{bq(1-aq^{-1})(1-q^{-1}/a)}.$$

Proof. When $a = q^{-tn}$ or $a = q^{tn}$, the left-hand side of (2.2) is equal to

$$\sum_{k=0}^{(tn+1)/3} \frac{(q^{-1-tn},q^{-1+tn},q^{-1}/b;q^3)_k}{(q^3;q^3)_k^2(q^3/b;q^3)_k}q^{9k} = {}_3\phi_2\left[\begin{matrix} q^{-1-tn},q^{-1+tn},q^{-1}/b \\ q^3,q^3/b \end{matrix};q^3,\ q^9\right]. \tag{2.3}$$

According to Lemma 2.1, the right-hand side of (2.3) can be written as

$$\frac{(bq,q;q^3)_{(tn+1)/3}}{(bq)^{(tn+1)/3}(1/b,q^3;q^3)_{(tn+1)/3}}A_n(q;b,t).$$

Since $(1-aq^{tn})$ and $(a-q^{tn})$ are relatively prime polynomials, we have the following result: modulo $(1-aq^{tn})(a-q^{tn})$,

$$\sum_{k=0}^{(tn+1)/3} \frac{(aq^{-1},q^{-1}/a,q^{-1}/b;q^3)_k}{(q^3;q^3)_k^2(q^3/b;q^3)_k}q^{9k} \equiv \frac{(bq,q;q^3)_{(tn+1)/3}}{(bq)^{(tn+1)/3}(1/b,q^3;q^3)_{(tn+1)/3}}A_n(q;b,t). \tag{2.4}$$

When $b = q^{tn}$, the left-hand side of (2.2) is equal to

$$\sum_{k=0}^{(tn+1)/3} \frac{(aq^{-1}, q^{-1}/a, q^{-1-tn}; q^3)_k}{(q^3; q^3)_k^2 (q^{3-tn}; q^3)_k} q^{9k} = {}_3\phi_2\left[\begin{matrix} aq^{-1}, q^{-1}/a, q^{-1-tn} \\ q^3, q^{3-tn} \end{matrix}; q^3, q^9\right]. \tag{2.5}$$

By Lemma 2.1, the right-hand side of (2.5) can be expressed as

$$\frac{(aq, q/a; q^3)_{(tn+1)/3}}{(q^2, q^3; q^3)_{(tn+1)/3}} \times \left\{\frac{q^{tn}(1-q^{tn+1})\{1-q-q^{tn}(q^{-2}+q-a-1/a)\}}{(1-q)(1-aq^{tn-1})(1-q^{tn-1}/a)} - \frac{1-q^{-2}-q^{tn}(2q-a-1/a)}{(1-aq^{-1})(1-q^{-1}/a)}\right\}.$$

Then we obtain the conclusion: modulo $(b - q^{tn})$,

$$\sum_{k=0}^{(tn+1)/3} \frac{(aq^{-1}, q^{-1}/a, q^{-1}/b; q^3)_k}{(q^3; q^3)_k^2 (q^3/b; q^3)_k} q^{9k} \equiv \frac{(aq, q/a; q^3)_{(tn+1)/3}}{b^{(tn+1)/3}(1/b, 1/bq; q^3)_{(tn+1)/3}} B(q; a, b). \tag{2.6}$$

It is clear that the polynomials $(1 - aq^{tn})(a - q^{tn})$ and $(b - q^{tn})$ are relatively prime. Noting the q-congruences

$$\frac{(b-q^{tn})(ab-1-a^2+aq^{tn})}{(a-b)(1-ab)} \equiv 1 \pmod{(1-aq^{tn})(a-q^{tn})},$$
$$\frac{(1-aq^{tn})(a-q^{tn})}{(a-b)(1-ab)} \equiv 1 \pmod{(b-q^{tn})}$$

and employing the Chinese remainder theorem for coprime polynomials, we get Theorem 2.2 from (2.4) and (2.6). □

Proof of Theorem 1.1. Letting $b \to 1, t = 2$ in Theorem 2.2, we arrive at the formula: modulo $\Phi_n(q)(1-aq^{2n})(a-q^{2n})$,

$$\begin{aligned}
&\sum_{k=0}^{(2n+1)/3} \frac{(aq^{-1}, q^{-1}/a, q^{-1}; q^3)_k}{(q^3; q^3)_k^3} q^{9k} \\
&\equiv \frac{(1-a)^2 + (1-aq^{2n})(a-q^{2n})}{(1-a)^2} \frac{(q; q^3)_{(2n+1)/3}^2}{q^{(2n+1)/3}(q^3; q^3)_{(2n+1)/3}^2} C_n(q) \\
&\quad + \frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2} \frac{(aq, q/a; q^3)_{(2n+1)/3}}{(q^2, q^3; q^3)_{(2n-2)/3}} D(q; a) \\
&\equiv \frac{(q; q^3)_{(2n+1)/3}^2}{q^{(2n+1)/3}(q^3; q^3)_{(2n+1)/3}^2} C_n(q) + \frac{(1-aq^{2n})(a-q^{2n})}{q^{(2n+1)/3}(1-a)^2} \\
&\quad \times \left\{-\frac{(q; q^3)_{(2n+1)/3}^2}{(q^3; q^3)_{(2n+1)/3}^2}(3q+3q^2) + \frac{(aq, q/a; q^3)_{(2n+1)/3}}{(q^3; q^3)_{(2n+1)/3}^2}(1-q)^2 D(q; a)\right\},
\end{aligned} \tag{2.7}$$

where

$$C_n(q)=\frac{q^3+q^{2n}(1+q^{4n})(1-3q+q^3-3q^4)}{q(1-q)^2(1-q^{2n-1})^2}+\frac{q^{4n}(1-3q+6q^2+2q^3-3q^4+3q^5)+q^{8n+3}}{q(1-q)^2(1-q^{2n-1})^2},$$

$$D(q;a)=\frac{(1+a+a^2)(a-3aq+q^3+a^2q^3-3aq^4)+3a^2q^2(2+q^3)}{q(1-q)^2(1-aq)^2(1-a/q)^2}.$$

By the L'Hôspital rule, we have

$$\lim_{a\to1}\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\Big\{-(q;q^3)^2_{(2n+1)/3}(3q+3q^2)+(aq,q/a;q^3)_{(2n+1)/3}(1-q)^2D(q;a)\Big\}$$
$$=-q(1+q)[2n]^2(q;q^3)^2_{(2n+1)/3}\bigg\{\sum_{i=1}^{(2n+1)/3}\frac{3q^{3i-2}}{[3i-2]^2}-\frac{1+5q+3q^2}{1+q}\bigg\}.$$

Letting $a\to1$ in (2.7) and utilizing the above limit, we are led to the q-supercongruence: modulo $\Phi_n(q)^3$,

$$\begin{aligned}&\sum_{k=0}^{(2n+1)/3}\frac{(q^{-1};q^3)_k^3}{(q^3;q^3)_k^3}q^{9k}\\
&\equiv\frac{(q;q^3)^2_{(2n+1)/3}}{q^{(2n+1)/3}(q^3;q^3)^2_{(2n+1)/3}}C_n(q)\\
&\quad-q(1+q)[2n]^2\frac{(q;q^3)^2_{(2n+1)/3}}{q^{(2n+1)/3}(q^3;q^3)^2_{(2n+1)/3}}\bigg\{\sum_{i=1}^{(2n+1)/3}\frac{3q^{3i-2}}{[3i-2]^2}-\frac{1+5q+3q^2}{1+q}\bigg\}\\
&\equiv q^{(2-2n)/3}(1+q)\frac{(q;q^3)^2_{(2n+1)/3}}{(q^3;q^3)^2_{(2n+1)/3}}\\
&\quad\times\bigg\{3-[2n]^2\bigg(\sum_{i=1}^{(2n+1)/3}\frac{3q^{3i-2}}{[3i-2]^2}-\frac{1+5q+3q^2}{1+q}\bigg)\bigg\}.\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Letting $b\to1, t=1$ in Theorem 2.2, we obtain the result: modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,

$$\begin{aligned}&\sum_{k=0}^{(n+1)/3}\frac{(aq^{-1},q^{-1}/a,q^{-1};q^3)_k}{(q^3;q^3)_k^3}q^{9k}\\
&\equiv\frac{(1-a)^2+(1-aq^n)(a-q^n)}{(1-a)^2}\frac{(q;q^3)^2_{(n+1)/3}}{q^{(n+1)/3}(q^3;q^3)^2_{(n+1)/3}}C_{n/2}(q)\end{aligned}$$

$$
\begin{aligned}
&+\ \frac{(1-aq^n)(a-q^n)}{(1-a)^2}\frac{(aq,q/a;q^3)_{(n+1)/3}}{(q^2,q^3;q^3)_{(n-2)/3}}D(q;a)\\
\equiv\ &\frac{(q;q^3)^2_{(n+1)/3}}{q^{(n+1)/3}(q^3;q^3)^2_{(n+1)/3}}C_{n/2}(q)+\frac{(1-aq^n)(a-q^n)}{q^{(n+1)/3}(1-a)^2}\\
&\times\left\{\frac{(q;q^3)^2_{(n+1)/3}}{(q^3;q^3)^2_{(n+1)/3}}(3q+3q^2)-\frac{(aq,q/a;q^3)_{(n+1)/3}}{(q^3;q^3)^2_{(n+1)/3}}(1-q)^2D(q;a)\right\}.
\end{aligned}
\tag{2.8}
$$

By the L'Hôspital rule, we have

$$
\begin{aligned}
&\lim_{a\to1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left\{(q;q^3)^2_{(n+1)/3}(3q+3q^2)-(aq,q/a;q^3)_{(n+1)/3}(1-q)^2D(q;a)\right\}\\
&=q(1+q)[n]^2(q;q^3)^2_{(n+1)/3}\left\{\sum_{i=1}^{(n+1)/3}\frac{3q^{3i-2}}{[3i-2]^2}-\frac{1+5q+3q^2}{1+q}\right\}.
\end{aligned}
$$

Letting $a\to1$ in (2.8) and employing the upper limit, we get the q-supercongruence: modulo $\Phi_n(q)^3$,

$$
\begin{aligned}
&\sum_{k=0}^{(n+1)/3}\frac{(q^{-1};q^3)_k^3}{(q^3;q^3)_k^3}q^{9k}\\
&\equiv\ \frac{(q;q^3)^2_{(n+1)/3}}{q^{(n+1)/3}(q^3;q^3)^2_{(n+1)/3}}C_{n/2}(q)\\
&\quad+q(1+q)[n]^2\frac{(q;q^3)^2_{(n+1)/3}}{q^{(n+1)/3}(q^3;q^3)^2_{(n+1)/3}}\left\{\sum_{i=1}^{(n+1)/3}\frac{3q^{3i-2}}{[3i-2]^2}-\frac{1+5q+3q^2}{1+q}\right\}\\
&\equiv q^{(2-n)/3}(1+q)\frac{(q;q^3)^2_{(n+1)/3}}{(q^3;q^3)^2_{(n+1)/3}}\\
&\quad\times\left\{\theta_n(q)+[n]^2\left(\sum_{i=1}^{(n+1)/3}\frac{3q^{3i-2}}{[3i-2]^2}-\frac{1+5q+3q^2}{1+q}\right)\right\}.
\end{aligned}
$$

Thus we finish the proof of Theorem 1.2. □

3. Proof of Propositions 1.5 and 1.6

Let $\Gamma_p'(x)$ and $\Gamma_p''(x)$ be the first derivative and second derivative of $\Gamma_p(x)$ respectively.

Proof of Proposition 1.5. By means of the properties of the p-adic Gamma function, we arrive at

$$
\begin{aligned}
\frac{(1/3)^2_{(2p+1)/3}}{(1)^2_{(2p+1)/3}}&=\frac{p^2}{(2p+1)^2}\left\{\frac{\Gamma_p((2+2p)/3)\Gamma_p(1)}{\Gamma_p(1/3)\Gamma_p((1+2p)/3)}\right\}^2\\
&=\frac{p^2}{(2p+1)^2}\{\Gamma_p(2/3)\Gamma_p((2+2p)/3)\Gamma_p((2-2p)/3)\}^2.
\end{aligned}
$$

Moreover, it is not difficult to understand that

$$1+6p^2-\sum_{i=1}^{(2p+1)/3}\frac{4p^2}{(3i-2)^2}$$
$$=-3+6p^2-\sum_{i=1}^{(p-1)/3}\frac{4p^2}{(3i-2)^2}-\sum_{i=(p+5)/3}^{(2p+1)/3}\frac{4p^2}{(3i-2)^2}.$$

Then we can proceed as follows:

$$\begin{aligned}&\frac{(1/3)^2_{(2p+1)/3}}{(1)^2_{(2p+1)/3}}\left\{1+6p^2-\sum_{i=1}^{(2p+1)/3}\frac{4p^2}{(3i-2)^2}\right\}\\&=\frac{p^2}{(2p+1)^2}\{\Gamma_p(2/3)\Gamma_p((2+2p)/3)\Gamma_p((2-2p)/3)\}^2\\&\quad\times\left\{-3+6p^2-\sum_{i=1}^{(p-1)/3}\frac{4p^2}{(3i-2)^2}-\sum_{i=(p+5)/3}^{(2p+1)/3}\frac{4p^2}{(3i-2)^2}\right\}\\&\equiv\frac{-3p^2}{(2p+1)^2}\Gamma_p(2/3)^6\\&\equiv-3p^2\Gamma_p(2/3)^6\pmod{p^3}.\end{aligned}$$

This verifies the correctness of Proposition 1.5. □

Proof of Proposition 1.6. Through the properties of the p-adic Gamma function, we have

$$\begin{aligned}\frac{(1/3)^2_{(p+1)/3}}{(1)^2_{(p-2)/3}}&=\left\{\frac{\Gamma_p((2+p)/3)\Gamma_p(1)}{\Gamma_p(1/3)\Gamma_p((1+p)/3)}\right\}^2\\&=\{\Gamma_p(2/3)\Gamma_p((2+p)/3)\Gamma_p((2-p)/3)\}^2\\&\equiv\Gamma_p(2/3)^2\left\{\Gamma_p(2/3)+\Gamma_p'(2/3)\frac{p}{3}+\Gamma_p''(2/3)\frac{p^2}{18}\right\}^2\\&\quad\times\left\{\Gamma_p(2/3)-\Gamma_p'(2/3)\frac{p}{3}+\Gamma_p''(2/3)\frac{p^2}{18}\right\}^2\\&\equiv\Gamma_p(2/3)^6\left\{1-\frac{2p^2}{9}G_1(2/3)^2+\frac{2p^2}{9}G_2(2/3)\right\}\pmod{p^3},\end{aligned}\tag{3.1}$$

where $G_1(x)=\Gamma_p'(x)/\Gamma_p(x)$ and $G_2(x)=\Gamma_p''(x)/\Gamma_p(x)$.

Let

$$H_m=\sum_{k=1}^{m}\frac{1}{k},\quad H_m^{(2)}=\sum_{k=1}^{m}\frac{1}{k^2}.$$

In light of the three relations from Wang and Pan [19, Lemmas 2.3 and 2.4]:

$$G_2(0) = G_1(0)^2,$$
$$G_1(2/3) \equiv G_1(0) + H_{(2p-1)/3} \pmod p,$$
$$G_2(2/3) \equiv G_2(0) + 2G_1(0)H_{(2p-1)/3} + H^2_{(2p-1)/3} - H^{(2)}_{(2p-1)/3} \pmod p,$$

we get

$$G_2(2/3) - G_1(2/3)^2 \equiv -H^{(2)}_{(2p-1)/3} \pmod p. \tag{3.2}$$

In view of (3.1) and (3.2), we are led to

$$\frac{(1/3)^2_{(p+1)/3}}{(1)^2_{(p-2)/3}}\left\{1 + \frac{p^2}{(p+1)^2}\sum_{i=1}^{(p+1)/3}\frac{1}{(3i-2)^2}\right\} \tag{3.3}$$
$$\equiv \Gamma_p(2/3)^6\left\{1 - \frac{2p^2}{9}H^{(2)}_{(2p-1)/3}\right\}\left\{1 + \frac{p^2}{(p+1)^2}\sum_{i=1}^{(p+1)/3}\frac{1}{(3i-2)^2}\right\}$$
$$\equiv \Gamma_p(2/3)^6\left\{1 - \frac{2p^2}{9}H^{(2)}_{(2p-1)/3} + \frac{p^2}{(p+1)^2}\sum_{i=1}^{(p+1)/3}\frac{1}{(3i-2)^2}\right\} \pmod{p^3}. \tag{3.4}$$

It is easy to see that

$$\begin{aligned}\sum_{i=1}^{(p+1)/3}\frac{1}{(3i-2)^2} &= H^{(2)}_{p-1} - \frac{1}{9}H^{(2)}_{(p-2)/3} - \sum_{i=1}^{(p-2)/3}\frac{1}{(3i-1)^2}\\ &\equiv -\frac{1}{9}H^{(2)}_{(p-2)/3} - \sum_{i=1}^{(p-2)/3}\frac{1}{(3i-1)^2}\\ &= -\frac{1}{9}H^{(2)}_{(p-2)/3} - \sum_{i=1}^{(p-2)/3}\frac{1}{(p-3i)^2}\\ &\equiv -\frac{2}{9}H^{(2)}_{(p-2)/3}\\ &= -\frac{2}{9}\sum_{i=(2p+2)/3}^{p-1}\frac{1}{(p-i)^2}\\ &\equiv -\frac{2}{9}\sum_{i=(2p+2)/3}^{p-1}\frac{1}{i^2}\\ &\equiv \frac{2}{9}H^{(2)}_{(2p-1)/3} \pmod p.\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4), we confirm the validity of Proposition 1.6. □

4. Conclusions

The main results of this paper are two theorems. They give a q-analogue of (1.2). We hope that more conclusions can be derived form the creative microscoping method and the Chinese remainder theorem for coprime polynomials.

Acknowledgments

The work is supported by the Natural Science Foundation of Hainan Province (No. 2019RC184) and the National Natural Science Foundation of China (No. 12071103).

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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