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Research article

The q-WZ pairs and divisibility properties of certain polynomials

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Abstract: Using the q-WZ (Wilf-Zeilberger) pairs we give divisibility properties of certain polynomials. These results may deemed generalizations of some q-congruences obtained by Guo earlier, or q-analogues of some congruences of Sun. For example, we prove that, for $n \geqslant 1$ and $0 \leqslant j \leqslant n$, the following two polynomials

$$\sum_{k=j}^{n}(-1)^k[3k-2j+1]\begin{bmatrix}2k-2j\\k\end{bmatrix}\frac{(q;q^2)_k(q;q^2)_{k-j}(-q;q)_n^3}{(q;q)_k(q^2;q^2)_{k-j}},$$

$$\sum_{k=j}^{n}(-1)^{n-k}q^{(k-j)^2}[4k+1]\frac{(q;q^2)_k^2(q;q^2)_{k+j}(-q;q)_n^6}{(q^2;q^2)_k^2(q^2;q^2)_{k-j}(q;q^2)_j^2}.$$

are divisible by $(1+q^n)^2[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}$. Here $[m]=1+q+\cdots+q^{m-1}$, $(a;q)_m=(1-a)(1-aq)\cdots(1-aq^{m-1})$, and $\begin{bmatrix}m\\k\end{bmatrix}=(q^{m-k+1};q)_k/(q;q)_k$.

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1. Introduction

It was conjectured by Van Hamme [24] that, for any odd prime p,

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{4k+1}{(-64)^k}\binom{2k}{k}^3 \equiv p(-1)^{\frac{p-1}{2}} \pmod{p^3}. \tag{1.1}$$

The congruence (1.1) was first proved by Mortenson [19] using a technical evaluation of a quotient of Gamma functions, and later reproved by Zudilin [28] via the WZ (Wilf-Zeilberger) method. Using the same WZ-pair as Zudilin, Z.-W. Sun [23] proves the following generalization of (1.1): for any positive integer n,

$$\sum_{k=0}^{n}(4k+1)\binom{2k}{k}^3(-64)^{n-k} \equiv 0 \mod 4(2n+1)\binom{2n}{n}. \tag{1.2}$$

Moreover, Z.-W. Sun [22, Conjecture 5.1(i)] proposed the following conjecture: for $n \geqslant 1$,

$$\sum_{k=0}^{n}(3k+1)\binom{2k}{k}^3(-8)^{n-k} \equiv 0 \pmod{4(2n+1)\binom{2n}{n}}, \tag{1.3}$$

$$\sum_{k=0}^{n}(6k+1)\binom{2k}{k}^3(-512)^{n-k} \equiv 0 \pmod{4(2n+1)\binom{2n}{n}}, \tag{1.4}$$

$$\sum_{k=0}^{n}(6k+1)\binom{2k}{k}^3 256^{n-k} \equiv 0 \pmod{4(2n+1)\binom{2n}{n}}. \tag{1.5}$$

The above three congruences were later proved by He [13, 14] using the WZ method again. Recently, still via the WZ method, Sun [21, Theorem 1.1] gave further generalizations of (1.3)–(1.5), such as: for $n \geqslant 1$ and $0 \leqslant j \leqslant n$,

$$\sum_{k=j}^{n}\frac{(6k-2j+1)\binom{2k}{k}\binom{2k+2j}{k+j}\binom{2k-2j}{k-j}\binom{k+j}{k}}{\binom{2j}{j}2^{8k-8n-2j}} \equiv 0 \pmod{4(2n+1)\binom{2n}{n}}. \tag{1.6}$$

It is easy to see that the $j = 0$ case of (1.6) reduces to (1.5). The reader is referred to [1] for a collection of some other interesting applications of the WZ method in recent years.

On the other hand, by finding a q-analogue of the WZ pair in Zudilin's proof of (1.1), Guo [3] gave the following q-analogue of (1.2):

$$\sum_{k=0}^{n}(-1)^k q^{k^2}[4k+1]\begin{bmatrix}2k\\k\end{bmatrix}^3(-q^{k+1};q)_{n-k}^6 \equiv 0 \pmod{(1+q^n)^2[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}}. \tag{1.7}$$

Here and in what follows, the *q-shifted factorials* are defined by

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}, \quad \text{where} \quad (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j),$$

the *q-integers* are defined as $[n] = [n]_q = (1-q^n)/(1-q)$, and the *$q$-binomial coefficients* are given by

$$\begin{bmatrix}m\\k\end{bmatrix} = \begin{bmatrix}m\\k\end{bmatrix}_q = \begin{cases}\dfrac{(q;q)_m}{(q;q)_k(q;q)_{m-k}} & \text{if } 0 \leqslant k \leqslant m,\\ 0 & \text{otherwise.}\end{cases}$$

In his subsequent work [4–6], Guo also gave similar q-analogues of (1.3)–(1.5). For more recent progress on q-congruences, see [7–11, 15–18, 20, 25, 26, 29].

In this paper, we shall give q-analogues of Sun's generalizations of (1.3)–(1.5), including a q-analogue of (1.6). We shall also give a further generalization of (1.7). Our main results can be stated as follows.

Theorem 1.1. *Let n be a positive integer and* $0 \leqslant j \leqslant n$. *Then modulo* $(1+q^n)^2[2n+1]\begin{bmatrix}2n\\ n\end{bmatrix}$,

$$\sum_{k=j}^{n}(-1)^k[3k-2j+1]\begin{bmatrix}2k-2j\\ k\end{bmatrix}\frac{(q;q^2)_k(q;q^2)_{k-j}(-q;q)_n^3}{(q;q)_k(q^2;q^2)_{k-j}} \equiv 0, \tag{1.8}$$

$$\sum_{k=j}^{n}(-1)^k[6k-2j+1]\frac{(q;q^2)_{k+j}(q;q^2)_{k-j}^2(-q;q)_n^6(-q^2;q^2)_n^3}{(q^4;q^4)_k^2(q^4;q^4)_{k-j}} \equiv 0, \tag{1.9}$$

$$\frac{1+q^n}{1+q^{2n}}\sum_{k=j}^{n}q^{(k-j)^2}[6k-2j+1]\frac{(q^2;q^4)_k(q;q^2)_{k-j}(q;q^2)_{k+j}(-q;q)_n^4(-q^2;q^2)_n^4}{(q^4;q^4)_k^2(q^4;q^4)_{k-j}(q^2;q^4)_j} \equiv 0. \tag{1.10}$$

It is easy to see that, when $q=1$ the congruence (1.10) reduces to

$$\sum_{k=j}^{n}\frac{2^{8n-2k}(6k-2j+1)(\frac{1}{2})_k(\frac{1}{2})_{k-j}(\frac{1}{2})_{k+j}}{(1)_k^2(1)_{k-j}(\frac{1}{2})_j} \equiv 0 \pmod{4(2n+1)\binom{2n}{n}},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geqslant 1$ and $(a)_0 = 1$. This congruence is exactly an equivalent form of (1.6). Similarly, the congruences (1.8) and (1.9) in the $q=1$ case reduce to the other two results in [21, Theorem 1.1].

Theorem 1.2. *Let n be a positive integer and* $0 \leqslant j \leqslant n$. *Then*

$$\sum_{k=j}^{n}(-1)^k q^{(k-j)^2}[4k+1]\frac{(q;q^2)_k^2(q;q^2)_{k+j}(-q;q)_n^6}{(q^2;q^2)_k^2(q^2;q^2)_{k-j}(q;q^2)_j^2} \equiv 0 \pmod{(1+q^n)^2[2n+1]\begin{bmatrix}2n\\ n\end{bmatrix}}. \tag{1.11}$$

It is clear that the $q=1$ case of (1.11) gives

$$\sum_{k=j}^{n}\frac{(-1)^k(4k+1)\binom{2k}{k}^2\binom{2k+2j}{k+j}\binom{k+j}{k-j}}{\binom{2j}{j}64^{k-n}} \equiv 0 \pmod{4(2n+1)\binom{2n}{n}},$$

which is a generalization of (1.2), and was neglected by Sun [21].

The rest of the paper is organized as follows. We first establish a general divisibility results based on the q-WZ machinery in the next section. We shall prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively. Finally, we shall propose some open problems in Section 5.

2. The q-WZ pair and an auxillary result

Let $A = A(n,k)$ be a double-indexed sequence with values in a suitable ground-field containing the rational number field and q. Recall that the sequence A is called q-hypergeometric in both parameters if both quotients

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are rational functions in q^n and q^k over certain field for all n and k whenever the quotients are well-defined. We say that two q-hypergeometric functions $F(n,k)$ and $G(n,k)$ form a q-WZ pair if they satisfy the following relation:

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k). \tag{2.1}$$

Wilf and Zeilberger [27] showed that in this case there exists a rational function $C(n, k)$ in q^n and q^k such that $F(n, k) = C(n, k)G(n, k)$. The function $C(n, k)$ is usually called the certificate of the pair (F, G).

We have the following q-version of [21, Theorem 2.1].

Theorem 2.1. *Let $F(n, k)$ and $G(n, k)$ be a q-WZ pair. Assume that $F(n, k) = G(n, k) = 0$ for $n < k$. Let $A_N(q)$ be a polynomial in q with integer coefficients such that $A_N(q)G(N + 1, k)$ is also a polynomial in q with integer coefficients for all $k \geqslant 1$. If $P_N(q)$ is a polynomial in q with integer coefficients satisfying*

(i) $A_N(q)G(N + 1, k) \equiv 0 \mod P_N(q)$, *for all* $k \geqslant 1$;

(ii) $A_N(q)F(N, N) \equiv 0 \mod P_N(q)$.

Then, for all $m \geqslant 0$,

$$A_N(q) \sum_{n=m}^{N} F(n, m) \equiv 0 \mod P_N(q). \tag{2.2}$$

Proof. Our proof is similar to that of [21, Theorem 2.1]. For the sake of completeness, we provide it here. We proceed by induction on m.

Summing (2.1) over k from 1 to N, we obtain

$$F(n, 0) - F(n, N) = \sum_{k=1}^{N} (G(n + 1, k) - G(n, k)).$$

Multiplying both sides of the above identity by $A_N(q)$ and then summing it over n from 0 to N, we get

$$A_N(q) \sum_{n=0}^{N} F(n, 0) - A_N(q)F(N, N) = A_N(q) \sum_{k=1}^{N} G(N + 1, k), \tag{2.3}$$

where we have used $F(n, N) = 0$ for $n < N$ and $G(0, k) = 0$ for $k \geqslant 1$. From (2.3) and the conditions (i) and (ii), we immediately deduce that

$$A_N(q) \sum_{n=0}^{N} F(n, 0) \equiv 0 \mod P_N(q).$$

Namely, the congruence (2.2) holds for $m = 0$.

We now assume that (2.2) is true for some $m = k$ with $k \geqslant 0$. Similarly as before, multiplying both sides of (2.1) by $A_N(q)$, shifting $k \to k + 1$, and then summing it over n from k to N, we have

$$A_N(q) \sum_{n=k}^{N} F(n, k) - A_N(q) \sum_{n=k}^{N} F(n, k + 1) = A_N(q)G(N + 1, k + 1).$$

Thus, by the condition (i) and the induction hypothesis, we get

$$A_N(q) \sum_{n=k}^{N} F(n, k + 1) \equiv A_N(q) \sum_{n=k+1}^{N} F(n, k + 1) \equiv 0 \mod P_N(q).$$

This completes the inductive step, and therefore (2.2) is true for all $m \geqslant 0$. □

3. Proof of Theorem 1.1

Proof of (1.8). For positive integers n, define

$$(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n} = \frac{(-1)^n a^{-n} q^{n(n+1)/2}}{(q/a;q)_n}.$$

The following functions F and G introduced in [6]:

$$F(n,k) = (-1)^n[3n-2k+1]\begin{bmatrix}2n-2k\\ n\end{bmatrix}\frac{(q;q^2)_n(q;q^2)_{n-k}}{(q;q)_n(q^2;q^2)_{n-k}},$$
$$G(n,k) = (-1)^{n+1}[n]\begin{bmatrix}2n-2k\\ n-1\end{bmatrix}\frac{(q;q^2)_n(q;q^2)_{n-k}q^{n+1-2k}}{(q;q)_n(q^2;q^2)_{n-k}}.$$

satisfy the relation (2.1). Namely, they form a q-WZ pair.

Since $[N+1]\begin{bmatrix}2N+2\\ N+1\end{bmatrix}/(1+q^{N+1}) = [2N+1]\begin{bmatrix}2N\\ N\end{bmatrix}$ and $\begin{bmatrix}2N\\ N\end{bmatrix} \equiv 0 \pmod{1+q^N}$, we have

$$\begin{aligned}&(-q;q)_N^3 G(N+1,k)\\ &= (-1)^N[N+1]\begin{bmatrix}2N+2\\ N+1\end{bmatrix}\begin{bmatrix}2N-2k+2\\ N\end{bmatrix}\begin{bmatrix}2N-2k+2\\ N-k+1\end{bmatrix}\frac{(-q;q)_N^2 q^{N-2k+2}}{(1+q^{N+1})(-q;q)_{N-k+1}^2}\\ &\equiv 0 \pmod{(1+q^N)^2[2N+1]\begin{bmatrix}2N\\ N\end{bmatrix}}\end{aligned}$$

for $k=1$ or $k \geqslant 2$. It is clear that $F(N,N)=0$. The proof of (1.8) then follows from Theorem 2.1. □

Proof of (1.9). We again use a q-WZ pair to prove (1.9). The q-WZ pair has already been given in [4]:

$$F(n,k) = (-1)^{n+k}\frac{[6n-2k+1](q;q^2)_{n+k}(q;q^2)_{n-k}^2}{(q^4;q^4)_n^2(q^4;q^4)_{n-k}},$$
$$G(n,k) = \frac{(-1)^{n+k}(q;q^2)_{n+k-1}(q;q^2)_{n-k}^2}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}}.$$

By [4, Lemma 3.2], for $1 \leqslant k \leqslant N$, we have

$$\begin{aligned}(-q;q)_N^6(-q^2;q^2)_N^3 G(N+1,k) &= \frac{(q;q^2)_{N+k}(q;q^2)_{N-k+1}^2(-q;q)_N^6}{(1-q)(q^2;q^2)_N^2(q^2;q^2)_{N-k+1}}\frac{(-q^2;q^2)_N}{(-q^2;q^2)_{N-k+1}}\\ &\equiv 0 \pmod{(1+q^N)^2[2N+1]\begin{bmatrix}2N\\ N\end{bmatrix}},\end{aligned}$$

because $(-q^2;q^2)_N/(-q^2;q^2)_{N-k+1}$ is clearly a polynomial in q with integer coefficients.

It is easy to see that

$$F(N,N) = [4N+1]\frac{(q;q^2)_{2N}}{(q^4;q^4)_N^2} = \frac{[4N+1]}{(-q^2;q^2)_N^2(-q;q)_{2N}(-q;q)_N^2}\begin{bmatrix}4N\\ 2N\end{bmatrix}\begin{bmatrix}2N\\ N\end{bmatrix}.$$

By [5, Lemma 3.1], we have

$$(-q;q)_N^2\begin{bmatrix}4N+1\\2N\end{bmatrix}\equiv 0 \pmod{(1+q^N)(-q;q)_{2N}}, \tag{3.1}$$

and so

$$\begin{aligned}(-q;q)_N^6(-q^2;q^2)_N^3F(N,N) &= (-q;q)_N^4(-q^2;q^2)_N\frac{[2N+1]}{(-q;q)_{2N}}\begin{bmatrix}4N+1\\2N\end{bmatrix}\begin{bmatrix}2N\\N\end{bmatrix}\\ &\equiv 0 \pmod{(1+q^N)^2[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}}.\end{aligned}$$

Therefore, by Theorem 2.1, the congruence (1.9) holds. □

Proof of (1.10). The following functions, introduced in [5],

$$F(n,k)=\frac{q^{(n-k)^2}[6n-2k+1](q^2;q^4)_n(q;q^2)_{n-k}(q;q^2)_{n+k}}{(q^4;q^4)_n^2(q^4;q^4)_{n-k}(q^2;q^4)_k},$$

$$G(n,k)=\frac{q^{(n-k)^2}(q^2;q^4)_n(q;q^2)_{n-k}(q;q^2)_{n+k-1}}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}(q^2;q^4)_k},$$

form a q-WZ pair. By [5, Lemma 3.2], for $1\leqslant k\leqslant N$, we have

$$\begin{aligned}&(-q;q)_N^4(-q^2;q^2)_N^4G(N+1,k)\\ &=\frac{q^{(N-k+1)^2}(-q;q)_N^4(-q^2;q^2)_N(q^2;q^4)_{N+1}(q;q^2)_{N-k+1}(q;q^2)_{N+k}}{(1-q)(q^2;q^2)_N^2(q^2;q^2)_{N-k+1}(q^2;q^4)_k}\frac{(-q^2;q^2)_N}{(-q^2;q^2)_{N-k+1}}\\ &\equiv 0 \pmod{(1+q^N)(1+q^{2N})[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}F(N,N)&=[4N+1]\frac{(q;q^2)_{2N}}{(q^4;q^4)_N^2}\\ &=\frac{[4N+1]}{(-q^2;q^2)_N^2(-q;q)_{2N}(-q;q)_N^2}\begin{bmatrix}4N\\2N\end{bmatrix}\begin{bmatrix}2N\\N\end{bmatrix}.\end{aligned}$$

By (3.1), we get

$$\begin{aligned}(-q;q)_N^4(-q^2;q^2)_N^4F(N,N) &= (-q;q)_N^2(-q^2;q^2)_N^2\frac{[2N+1]}{(-q;q)_{2N}}\begin{bmatrix}4N+1\\2N\end{bmatrix}\begin{bmatrix}2N\\N\end{bmatrix}\\ &\equiv 0 \pmod{(1+q^N)(1+q^{2N})[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}}.\end{aligned}$$

Therefore, by Theorem 2.1, we obtain

$$(-q;q)_N^4(-q^2;q^2)_N^2\sum_{n=j}^{N}F(n,j)\equiv 0 \pmod{(1+q^N)^2[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}}.$$

Namely, the congruence (1.10) holds. □

4. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. This time we need the following q-WZ pair in [3]:

$$F(n,k) = (-1)^{n+k} q^{(n-k)^2} \frac{[4n+1](q;q^2)_n^2 (q;q^2)_{n+k}}{(q^2;q^2)_n^2 (q^2;q^2)_{n-k} (q;q^2)_k^2},$$

$$G(n,k) = \frac{(-1)^{n+k} q^{(n-k)^2} (q;q^2)_n^2 (q;q^2)_{n+k-1}}{(1-q)(q^2;q^2)_{n-1}^2 (q^2;q^2)_{n-k} (q;q^2)_k^2}.$$

By [3, Lemma 4.2], for $1 \leqslant k \leqslant N$, we have

$$(-q;q)_N^6 G(N+1,k) \equiv 0 \pmod{(1+q^N)^2[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}}.$$

Moreover, since

$$F(N,N) = [4N+1]\frac{(q;q^2)_{2N}}{(q^2;q^2)_N^2} = \frac{[4N+1]}{(-q;q)_{2N}^2}\begin{bmatrix}4N\\2N\end{bmatrix}\begin{bmatrix}2N\\N\end{bmatrix}_{q^2},$$

we immediately get

$$\begin{aligned}(-q;q)_N^6 F(N,N) &= (-q;q)_N^6 \frac{[2N+1]}{(-q;q)_{2N}^2}\begin{bmatrix}4N+1\\2N\end{bmatrix}\begin{bmatrix}2N\\N\end{bmatrix}_{q^2} \\ &\equiv 0 \pmod{(1+q^N)^2[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}}.\end{aligned}$$

The proof of (1.11) then follows readily from Theorem 2.1.

5. Some consequences and open problems

It is easy to see that

$$\frac{(q;q^2)_k}{(q^2;q^2)_k} = \frac{1}{(-q;q)_k^2}\begin{bmatrix}2k\\k\end{bmatrix}.$$

Letting $j=1$ in Theorems 1.1 and 1.2, we obtain the following results.

Corollary 5.1. *Let n be a positive integer. Then, modulo* $(1+q^n)(1+q^{2n})[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}$,

$$\sum_{k=1}^{n} q^{(k-1)^2} \frac{[4k][2k+1][6k-1](-q;q)_n^4(-q^2;q^2)_n^4}{(1+q)[2k-1](-q;q)_k^4(-q^2;q^2)_k^4}\begin{bmatrix}2k\\k\end{bmatrix}^2\begin{bmatrix}2k\\k\end{bmatrix}_{q^2} \equiv 0.$$

Corollary 5.2. *Let n be a positive integer. Then, modulo* $(1+q^n)^2[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}$,

$$\sum_{k=1}^{n} \frac{(-1)^k[k][k-1][3k-1](-q;q)_n^3}{[2k-1]^2(-q;q)_k^3}\begin{bmatrix}2k\\k\end{bmatrix}^3 \equiv 0,$$

$$\sum_{k=1}^{n}\frac{(-1)^k[4k][2k+1][6k-1](-q;q)_n^6(-q^2;q^2)_n^3}{[2k-1]^2(-q;q)_k^6(-q^2;q^2)_k^3}\begin{bmatrix}2k\\k\end{bmatrix}^3\equiv 0,$$

$$\sum_{k=1}^{n}(-1)^k q^{(k-1)^2}[2k][2k+1][4k+1]\begin{bmatrix}2k\\k\end{bmatrix}^3\frac{(-q;q)_n^6}{(-q;q)_k^6}\equiv 0.$$

It is worth mentioning that Guo and the author [12, Theorem 1.4] proved that, for $n\geqslant 1$,

$$\sum_{k=0}^{n}[4k+1]\begin{bmatrix}2k\\k\end{bmatrix}^4(-q^{k+1};q)_{n-k}^8\equiv 0 \pmod{(1+q^n)^3[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}}. \tag{5.1}$$

However, we are unable to prove the following generalization of (5.1):

$$\sum_{k=j}^{n}q^{j(j-2k-1)}[4k+1]\frac{(q;q^2)_k^3(q;q^2)_{k+j}(-q;q)_n^8}{(q^2;q^2)_k^3(q^2;q^2)_{k-j}(q;q^2)_j^2}\equiv 0 \pmod{(1+q^n)^3[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}}.$$

Motivated by (1.11) and the above conjectural generalization of (5.1), we propose the following generalization of [20, Theorem 5.1].

Conjecture 5.3. *Let n be a positive integer and $r\geqslant 2$. Then, for $0\leqslant j\leqslant n$, modulo $(1+q^n)^{2r-2}[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}$,*

$$\sum_{k=j}^{n}(-1)^k q^{(k-j)^2+(r-2)(k-j)}[4k+1]\frac{(q;q^2)_k^{2r-2}(q;q^2)_{k+j}(-q;q)_n^{4r-2}}{(q^2;q^2)_k^{2r-2}(q^2;q^2)_{k-j}(q;q^2)_j^2}\equiv 0;$$

and modulo $(1+q^n)^{2r-1}[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}$,

$$\sum_{k=j}^{n}q^{j(j-2k-1)+(r-2)(k-j)}[4k+1]\frac{(q;q^2)_k^{2r-1}(q;q^2)_{k+j}(-q;q)_n^{4r}}{(q^2;q^2)_k^{2r-1}(q^2;q^2)_{k-j}(q;q^2)_j^2}\equiv 0.$$

6. Conclusions

In Sections 3 and 4, we give proofs of some divisibility properties of certain polynomials by using the q-WZ pairs. Note that the q-WZ pairs are difficult to find, but once a q-WZ pair is given it may play a key role in the proof of a congruence. The Section 5 provides a conjectural generalization of [20, Theorem 5.1].

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Conflict of interest

The author declares that there is no conflict of interest in this paper.

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