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Research article

Certain inequalities in frame of the left-sided fractional integral operators having exponential kernels

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Abstract: By virtue of the left-sided fractional integral operators having exponential kernels, proposed by Ahmad et al. in [J. Comput. Appl. Math. 353:120-129, 2019], we create the left-sided fractional Hermite–Hadamard type inequalities for convex mappings. Moreover, to study certain fractional trapezoid and midpoint type inequalities via the differentiable convex mappings, two fractional integral identities are proven. Also, we show the important connections of the derived outcomes with those classical integrals clearly. Finally, we provide three numerical examples to verify the correctness of the presented inequalities that occur with the variation of the parameter μ.

Keywords: Hermite–Hadamard's inequalities; fractional integrals; convex mappings
Mathematics Subject Classification: 26A33, 26A51, 26D15, 26D10

1. Introduction

The convexity of functions is an impressive tool, which is applicable, particularly in several distinct areas of engineering mathematics and applied analysis. Recently, a large number of researchers, including mathematicians, engineers and scientists, have devoted themselves to studying the inequalities and properties in association with convexity in certain diverse directions. For example, Du et al. [10] proposed some k-fractional extensions of the trapezium inequalities in connection with the generalized semi-(m, h)-preinvexity, Kunt et al. [21] presented the improved version of fractional Hermite–Hadamard type inequalities for the convex functions, and Mehrez et al. [28] gave the new Hermite–Hadamard type integral inequalities with regard to the convex functions and their related applications. For more outcomes with regard to diverse types of the convexity please see [25, 34] and the references cited in them. And the Hermite–Hadamard's inequalities, one of the most distinguished mathematical inequalities considering convex mappings, are also applied diffusely in plenty of other aspects of computational mathematics. Let us recognize them as below.

Suppose that $\phi : \Lambda \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval Λ of real numbers, and $\theta, \xi \in \Lambda$ along with $\theta \neq \xi$. The following inequalities, to be named as the Hermite–Hadamard's inequalities, are frequently put into use in engineering mathematics and applied analysis.

$$\phi\left(\frac{\theta+\xi}{2}\right) \leq \frac{1}{\xi-\theta}\int_{\theta}^{\xi}\phi(\omega)\mathrm{d}\omega \leq \frac{\phi(\theta)+\phi(\xi)}{2}. \tag{1.1}$$

The classical integral inequalities, which have given rise to considerable attention from plenty of authors, provided error bounds for the mean value of a continuous convex mapping $\phi : [\theta, \xi] \to \mathbb{R}$. There have been a large amount of studies, regarding the Hermite–Hadamard type inequalities, on the basis of other various types of convex mappings, such as convex mappings [15], s-convex mappings [24], generalized m-convex mappings [12], (α, m)-convex mappings [37], exponential trigonometric convex mappings [17], h-convex mappings [8], h-preinvex mappings [27], r-preinvex mappings [13], N-quasiconvex mappings [1] and so on. For more findings with regard to this topical subject, the interested readers may refer to [11, 18, 23, 26] and the references therein.

In [9], the authors acquired trapezoid type inequalities, in association with the Hadamard's inequality, for the first-order differentiable convex mappings. They took advantage of the following lemma to deduce their findings.

Lemma 1. *Assume that $\phi : \Lambda^\circ \to \mathbb{R}$ is a differentiable mapping defined on the interval Λ°, in which Λ° is the interior of Λ, $\theta, \xi \in \Lambda^\circ$ along with $\theta < \xi$. If the mapping $\phi' \in L^1([\theta, \xi])$, then the following identity holds true*

$$\frac{\phi(\theta)+\phi(\xi)}{2} - \frac{1}{\xi-\theta}\int_{\theta}^{\xi}\phi(\omega)\mathrm{d}\omega = \frac{\xi-\theta}{2}\int_{0}^{1}(1-2\omega)\phi'(\omega\theta+(1-\omega)\xi)\mathrm{d}\omega. \tag{1.2}$$

To create the midpoint type integral inequalities, Kirmaci demonstrated the following lemma in [20].

Lemma 2. *Under the same prerequisites of Lemma 1, we obtain the following identity*

$$\begin{aligned}&\frac{1}{\xi-\theta}\int_{\theta}^{\xi}\phi(\omega)\mathrm{d}\omega - \phi\left(\frac{\theta+\xi}{2}\right)\\ &= (\xi-\theta)\left[\int_{0}^{1/2}\omega\phi'(\omega\theta+(1-\omega)\xi)\mathrm{d}\omega + \int_{1/2}^{1}(\omega-1)\phi'(\omega\theta+(1-\omega)\xi)\mathrm{d}\omega\right].\end{aligned} \tag{1.3}$$

The next conception, regarding the affine mapping and the related theorem, are of importance to our study.

Definition 1. *[23] A mapping ϕ defined on the interval Λ has a support at $\omega_0 \in \Lambda$, if there exists an affine mapping $M(\omega) = \phi(\omega_0) + m(\omega - \omega_0)$, satisfying that $M(\omega) \leq \phi(\omega)$ for all $\omega \in \Lambda$. The graph of support mapping M is described as a line of support for the mapping ϕ at ω_0.*

Theorem 1. *[23] The mapping $\phi : (\theta, \xi) \to \mathbb{R}$ is convex if and only if there is at minimum one line of support for ϕ at each $\omega_0 \in (\theta, \xi)$.*

Next, let us retrospect certain fractional integrals.

Definition 2. *Assume that the mapping $\phi \in L^1([\theta, \xi])$. The left-sided and right-sided Riemann–Liouville integrals $\mathcal{J}^{\mu}_{\theta^+}\phi$ and $\mathcal{J}^{\mu}_{\xi^-}\phi$ of order $\mu > 0$ are defined as*

$$\mathcal{J}^{\mu}_{\theta^+}\phi(s) = \frac{1}{\Gamma(\mu)} \int_{\theta}^{s} (s-\omega)^{\mu-1}\phi(\omega)\mathrm{d}\omega,$$

and

$$\mathcal{J}^{\mu}_{\xi^-}\phi(s) = \frac{1}{\Gamma(\mu)} \int_{s}^{\xi} (\omega-s)^{\mu-1}\phi(\omega)\mathrm{d}\omega,$$

with $\theta < s < \xi$, respectively, in which $\Gamma(\cdot)$ is the gamma function, defined by $\Gamma(\mu) = \int_0^{\infty} e^{-\omega}\omega^{\mu-1}\mathrm{d}\omega, \mathrm{Re}(\mu) > 0$.

The next two Hermite–Hadamard type integral inequalities, by means of the left-sided and right-sided Riemann–Liouville fractional integrals, respectively, were presented by Kunt et al.

Theorem 2. *[23] Assume that the mapping $\phi : [\theta, \xi] \to \mathbb{R}$ is convex for $\theta, \xi \in \mathbb{R}$ together with $\theta < \xi$. If the mapping $\phi \in L^1([\theta, \xi])$, and $\mu > 0$, then the following inequalities for the left-sided Riemann–Liouville fractional integrals hold true*

$$\phi\left(\frac{\mu\theta + \xi}{\mu + 1}\right) \leq \frac{\Gamma(\mu+1)}{(\xi-\theta)^{\mu}}\mathcal{J}^{\mu}_{\theta^+}\phi(\xi) \leq \frac{\mu\phi(\theta) + \phi(\xi)}{\mu+1}. \tag{1.4}$$

Theorem 3. *[22] With the same assumptions mentioned in Theorem 2, we have the following inequalities for the right-sided Riemann–Liouville fractional integrals*

$$\phi\left(\frac{\theta + \mu\xi}{\mu + 1}\right) \leq \frac{\Gamma(\mu+1)}{(\xi-\theta)^{\mu}}\mathcal{J}^{\mu}_{\xi^-}\phi(\theta) \leq \frac{\phi(\theta) + \mu\phi(\xi)}{\mu+1}. \tag{1.5}$$

In 2019, Ahmad et al. considered the fractional integral operators having exponential kernels as below.

Definition 3. *[3] Assume that the mapping $\phi \in L^1([\theta, \xi])$. The left-side and right-side fractional integrals $\mathcal{I}^{\mu}_{\theta^+}\phi$ and $\mathcal{I}^{\mu}_{\xi^-}\phi$ of order $\mu \in (0,1)$ having exponential kernels are respectively defined by*

$$\mathcal{I}^{\mu}_{\theta^+}\phi(s) = \frac{1}{\mu}\int_{\theta}^{s} \exp\left(-\frac{1-\mu}{\mu}(s-\omega)\right)\phi(\omega)\mathrm{d}\omega, \quad s > \theta,$$

and

$$\mathcal{I}^{\mu}_{\xi^-}\phi(s) = \frac{1}{\mu}\int_{s}^{\xi} \exp\left(-\frac{1-\mu}{\mu}(\omega-s)\right)\phi(\omega)\mathrm{d}\omega, \quad s < \xi.$$

If we consider to take $\mu \to 1$, then we obtain that

$$\lim_{\mu\to 1}\mathcal{I}^{\mu}_{\theta^+}\phi(s) = \int_{\theta}^{s}\phi(\omega)\mathrm{d}\omega, \quad \lim_{\mu\to 1}\mathcal{I}^{\mu}_{\xi^-}\phi(s) = \int_{s}^{\xi}\phi(\omega)\mathrm{d}\omega.$$

Moreover, in view of

$$\lim_{\mu\to 0}\frac{1}{\mu}\exp\left(-\frac{1-\mu}{\mu}(s-\omega)\right)=\delta(s-\omega),$$

we observe that

$$\lim_{\mu\to 0}\mathcal{I}^{\mu}_{\theta^+}\phi(s)=\phi(s),\quad \lim_{\mu\to 0}\mathcal{I}^{\mu}_{\xi^-}\phi(s)=\phi(s).$$

In the same paper, the following Hermite–Hadamard type inequalities, by virtue of fractional integrals having exponential kernels, were proved by Ahmad et al.

Theorem 4. *[3] Suppose that the mapping $\phi:[\theta,\xi]\to\mathbb{R}$ is positive together with $0\le\theta<\xi$ and the mapping $\phi\in L^1([\theta,\xi])$. If ϕ is a convex mapping defined on the interval $[\theta,\xi]$, then we obtain the following inequalities for fractional integrals*

$$\phi\left(\frac{\theta+\xi}{2}\right)\le\frac{1-\mu}{2(1-e^{-\rho})}\left[\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)+\mathcal{I}^{\mu}_{\xi^-}\phi(\theta)\right]\le\frac{\phi(\theta)+\phi(\xi)}{2},\tag{1.6}$$

where

$$\kappa=\frac{1-\mu}{\mu}(\xi-\theta).$$

Fractional calculus, as a forceful tool, has proven to be an vital cornerstone in engineering mathematics and applied sciences. This academic realm has absorbed quite a few mathematicians to take into account this issue. In consequence, certain extraordinary integral inequalities, in light of a fruitful interaction of various approaches of fractional calculus, were brought into force by mass learned men, containing Chen [7] and Mohammed [29] in the research of the Hermite–Hadamard inequalities, Baleanu et al. [4] in the trapezoidal type inequalities involving generalized fractional integrals, and Set et al. [35] in the Simpson type integral inequalities for Riemann–Liouville fractional integral operators, Wang et al. [38] in the Ostrowski type inequalities via Hadamard fractional integral operators, Chen and Katugampola [6] in the Fejér–Hermite–Hadamard type inequalities by means of Katugampola fractional integral operators, Butt et al. [5] in the generalized Hermite–Hadamard type inequalities via ABK-fractional integrals, Agarwal [2] provided some inequalities in association with Hadamard-type k-fractional integral operators, Set et al. [36] and Khan et al. [19] gave certain Hermite–Hadamard type inequalities with regard to the generalized fractional integral operators and the conformable fractional integral operators, respectively. With regard to further momentous findings in connection with the fractional integral operators, we recommend the minded readers to [14, 16, 31] and the bibliographies quoted in them.

Enlightened by the outcomes mentioned above, in particular those created in [22, 23], and as much as we know, there are few articles with regard to the Hermite–Hadamard type inequalities by using only the left-sided fractional integrals or the right-sided fractional integrals. The current paper is designed to investigate the left-sided fractional integral inequalities for convex mappings, which are relevant to the distinguished Hermite–Hadamard's inequalities. To achieve this objective, exploiting only the left-sided fractional integral operators having exponential kernels, we prove the left-sided fractional Hermite–Hadamard type inequalities for convex mappings. Moreover, we construct two integral identities. Under the assistance of these identities, we acquire the fractional trapezoid and midpoint type integral inequalities for the differentiable convex mappings.

2. The left-sided fractional Hermite–Hadamard's inequalities

By virtue of the left-sided fractional integrals having exponential kernels, we derive the following Hermite–Hadamard's inequalities for convex mappings.

Theorem 5. *Suppose that the mapping $\phi : [\theta, \xi] \to \mathbb{R}$ is convex for $\theta, \xi \in \mathbb{R}$ with $\theta < \xi$. If the mapping $\phi \in L^1([\theta, \xi])$, then we deduce the following inequalities for the left-sided fractional integrals having exponential kernels*

$$\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\le\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\le\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\right)\phi(\theta)+\left(\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\right)\phi(\xi), \tag{2.1}$$

where $\kappa = \frac{1-\mu}{\mu}(\xi - \theta)$ and $\mu \in (0, 1)$.

Proof. On account of the convexity of ϕ defined on the interval $[\theta, \xi]$, and employing Theorem 1.1, there exists at minimum one line of support

$$M(\omega)=\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)+m\left[\omega-\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right]\le\phi(\omega), \tag{2.2}$$

for all $\omega \in [\theta, \xi]$ and $m \in \left[\phi'_-\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right), \phi'_+\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right]$.
If we put $\omega = t\theta + (1-t)\xi$, then we deduce that

$$\begin{aligned}&M(t\theta+(1-t)\xi)\\&\quad=\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)+m\left[t\theta+(1-t)\xi-\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right]\\&\quad\le\phi(t\theta+(1-t)\xi),\end{aligned} \tag{2.3}$$

for all $t \in (0, 1)$.
Multiplying both sides of (2.3) with $e^{-\kappa t}$ and integrating the resulting inequality with regard to t over $[0, 1]$, we find that

$$\begin{aligned}&\int_0^1 e^{-\kappa t}M(t\theta+(1-t)\xi)\mathrm{d}t\\&\quad=\int_0^1 e^{-\kappa t}\Bigg\{\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\\&\qquad+m\left[t\theta+(1-t)\xi-\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right]\Bigg\}\mathrm{d}t\\&\quad=\int_0^1 e^{-\kappa t}\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\mathrm{d}t\\&\qquad+m\int_0^1 e^{-\kappa t}\left[t\theta+(1-t)\xi-\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right]\mathrm{d}t\\&\quad=\int_0^1 e^{-\kappa t}\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\mathrm{d}t\\&\qquad+m\left[\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa^2}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa^2}\xi-\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa^2}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa^2}\xi\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 e^{-\kappa t}\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta + \frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\mathrm{d}t \\
&= \frac{1}{\kappa}(1-e^{-\kappa})\,\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta + \frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right) \\
&\le \int_0^1 e^{-\kappa t}\phi(t\theta+(1-t)\xi)\mathrm{d}t \\
&= \frac{1}{\xi-\theta}\int_\theta^\xi e^{-\frac{1-\mu}{\mu}(\xi-\omega)}\phi(\omega)\mathrm{d}\omega \\
&= \frac{1-\mu}{\kappa}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi).
\end{aligned} \tag{2.4}
$$

On the other hand, in accordance with the convexity of ϕ defined on the interval $[\theta,\xi]$, we know that

$$
\phi(t\theta+(1-t)\xi) \le t\phi(\theta)+(1-t)\phi(\xi), \tag{2.5}
$$

for all $t \in [0,1]$.
Multiplying both sides of (2.5) with $e^{-\kappa t}$ and integrating the resulting inequality regarding t on the interval $[0,1]$, we have that

$$
\begin{aligned}
\frac{1-\mu}{\kappa}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi) &= \int_0^1 e^{-\kappa t}\phi\,(t\theta+(1-t)\xi)\,\mathrm{d}t \\
&\le \phi(\theta)\int_0^1 te^{-\kappa t}\mathrm{d}t + \phi(\xi)\int_0^1(1-t)e^{-\kappa t}\mathrm{d}t \\
&= \phi(\theta)\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa^2}\right) + \phi(\xi)\left(\frac{\kappa+e^{-\kappa}-1}{\kappa^2}\right).
\end{aligned} \tag{2.6}
$$

Making use of (2.4) and (2.6), we deduce the required inequalities. This finishes the proof.

Remark 1. If one considers to require $\mu \to 1$, within Theorem 5, then one has the extraordinary Hermite–Hadamard's inequalities (1.1).

3. Identities

We need to prove two fractional integral identities as below, which are relevant to Lemma 1 and Lemma 2.

Lemma 3. *Assume that the mapping $\phi : \Lambda^\circ \to \mathbb{R}$ is a differentiable mapping defined on Λ°, $\theta,\xi \in \Lambda^\circ$ together with $\theta < \xi$. If the mapping $\phi' \in L^1([\theta,\xi])$, then we deduce the following trapezoid type identity for the left-sided fractional integrals having exponential kernels*

$$
\begin{aligned}
&\left[\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta) + \frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi)\right] - \frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi) \\
&\quad = (\xi-\theta)\int_0^1 \frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}\phi'(t\theta+(1-t)\xi)\mathrm{d}t,
\end{aligned} \tag{3.1}
$$

where $\kappa = \frac{1-\mu}{\mu}(\xi-\theta)$ and $\mu \in (0,1)$.

Proof. Integrating by parts for the right side of (3.1), we know that

$$\begin{aligned}
&(\xi-\theta)\int_0^1 \frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}\phi'(t\theta+(1-t)\xi)\mathrm{d}t\\
&=(\xi-\theta)\left[\frac{e^{-\kappa}-1}{\kappa(1-e^{-\kappa})(\theta-\xi)}\phi(t\theta+(1-t)\xi)\Big|_0^1+\frac{\kappa}{\kappa(1-e^{-\kappa})(\theta-\xi)}\int_0^1 e^{-\kappa t}\mathrm{d}\phi(t\theta+(1-t)\xi)\right]\\
&=\frac{1-e^{-\kappa}}{\kappa(1-e^{-\kappa})}\left[\phi(\theta)-\phi(\xi)\right]-\frac{\kappa}{\kappa(1-e^{-\kappa})}\left[e^{-\kappa}\phi(\theta)-\phi(\xi)+(1-\mu)\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right]\\
&=\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi)-\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi),
\end{aligned}$$

which is the identity asserted in Lemma 3. This ends the proof.

Remark 2. If one considers to require $\mu\to 1$, within Lemma 3, then one achieves Lemma 1.

Lemma 4. *With the same assumptions mentioned in Lemma 3, we derive the following midpoint type identity for the left-sided fractional integrals having exponential kernels*

$$\begin{aligned}
&\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)-\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\\
&=(\xi-\theta)\left[\int_0^{\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}\phi'(t\theta+(1-t)\xi)\mathrm{d}t\right. \qquad (3.2)\\
&\quad\left.+\int_{\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}}^1\frac{e^{-\kappa}-e^{-\kappa t}}{1-e^{-\kappa}}\phi'(t\theta+(1-t)\xi)\mathrm{d}t\right],
\end{aligned}$$

where $\kappa=\frac{1-\mu}{\mu}(\xi-\theta)$ *and* $\mu\in(0,1)$.

Proof. Integrating by parts for the right side of (3.2), we find that

$$\begin{aligned}
&(\xi-\theta)\left[\int_0^{\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}\phi'(t\theta+(1-t)\xi)\mathrm{d}t+\int_{\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}}^1\frac{e^{-\kappa}-e^{-\kappa t}}{1-e^{-\kappa}}\phi'(t\theta+(1-t)\xi)\mathrm{d}t\right]\\
&=(\xi-\theta)\left\{\frac{1}{1-e^{-\kappa}}\int_0^{\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}}\phi'(t\theta+(1-t)\xi)\mathrm{d}t-\int_0^1\frac{e^{-\kappa t}}{1-e^{-\kappa}}\phi'(t\theta+(1-t)\xi)\mathrm{d}t\right.\\
&\quad\left.+\frac{e^{-\kappa}}{1-e^{-\kappa}}\int_{\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}}^1\phi'(t\theta+(1-t)\xi)\mathrm{d}t\right\}\\
&=(\xi-\theta)\left\{\frac{1}{(1-e^{-\kappa})(\theta-\xi)}\left[\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)-\phi(\xi)\right]\right.\\
&\quad-\frac{1}{(1-e^{-\kappa})(\theta-\xi)}\left[e^{-\kappa}\phi(\theta)-\phi(\xi)+(1-\mu)\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right]\\
&\quad\left.+\frac{e^{-\kappa}}{(1-e^{-\kappa})(\theta-\xi)}\left[\phi(\theta)-\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right]\right\}
\end{aligned}$$

$$= (\xi-\theta)\left[\frac{1-e^{-\kappa}}{(1-e^{-\kappa})(\theta-\xi)}\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)-\frac{1-\mu}{(1-e^{-\kappa})(\theta-\xi)}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right]$$
$$= \frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)-\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right),$$

which is the identity asserted in Lemma 4. The proof is completed.

Remark 3. If one considers to require $\mu \to 1$, within Lemma 4, then one acquires Lemma 2.

4. The left-sided fractional trapezoid and midpoint type inequalities

In this section, in accordance with Lemma 3 and Lemma 4, we will present the left-sided fractional trapezoid, as well as midpoint type inequalities having exponential kernels.

Theorem 6. *Assume that $\phi : \Lambda^\circ \to \mathbb{R}$ is a differentiable mapping defined on Λ°, $\theta, \xi \in \Lambda^\circ$ along with $\theta < \xi$. If the mapping $|\phi'|$ is convex on $[\theta, \xi]$, then we have the following trapezoid type inequality for the left-sided fractional integrals including exponential kernels*

$$\begin{aligned}&\left|\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi)-\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right|\\&\quad\le(\xi-\theta)\Big[(\Delta_1+\Delta_3)|\phi'(\theta)|+(\Delta_2+\Delta_4)|\phi'(\xi)|\Big],\end{aligned}\tag{4.1}$$

where

$$\Delta_1 = \frac{1}{\kappa(1-e^{-\kappa})}\left[\frac{1}{\kappa}-\left(t_0+\frac{1}{\kappa}\right)e^{-\kappa t_0}\right]-\frac{1}{2\kappa}t_0^2,$$
$$\Delta_2 = \frac{1}{\kappa(1-e^{-\kappa})}\left[\left(t_0+\frac{1}{\kappa}-1\right)e^{-\kappa t_0}-\frac{1}{\kappa}+1\right]+\frac{1}{2\kappa}t_0^2-\frac{1}{\kappa}t_0,$$
$$\Delta_3 = \frac{1}{\kappa(1-e^{-\kappa})}\left[\left(-t_0-\frac{1}{\kappa}\right)e^{-\kappa t_0}+\left(1+\frac{1}{\kappa}\right)e^{-\kappa}\right]+\frac{1}{2\kappa}\left(1-t_0^2\right),$$

and

$$\Delta_4 = \frac{1}{\kappa(1-e^{-\kappa})}\left[\left(t_0+\frac{1}{\kappa}-1\right)e^{-\kappa t_0}-\frac{1}{\kappa}e^{-\kappa}\right]+\frac{1}{\kappa}\left(\frac{1}{2}t_0^2-t_0+\frac{1}{2}\right),$$

for $t_0 = -\frac{1}{\kappa}\ln\frac{1-e^{-\kappa}}{\kappa}$, $\kappa = \frac{1-\mu}{\mu}(\xi-\theta)$ with $\mu \in (0, 1)$.

Proof. Taking advantage of Lemma 3 and the convexity of the mapping $|\phi'|$ defined on $[\theta, \xi]$, we find that

$$\begin{aligned}&\left|\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi)-\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right|\\&\quad\le(\xi-\theta)\int_0^1\frac{\left|e^{-\kappa}-1+\kappa e^{-\kappa t}\right|}{\kappa(1-e^{-\kappa})}\left|\phi'(t\theta+(1-t)\xi)\right|\mathrm{d}t\\&\quad\le(\xi-\theta)\left\{\int_0^1\frac{\left|e^{-\kappa}-1+\kappa e^{-\kappa t}\right|}{\kappa(1-e^{-\kappa})}\left[t\left|\phi'(\theta)\right|+(1-t)\left|\phi'(\xi)\right|\right]\mathrm{d}t\right\}\end{aligned}$$

$$= (\xi-\theta)\bigg\{|\phi'(\theta)|\int_0^{t_0}\frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}t\mathrm{d}t+|\phi'(\xi)|\int_0^{t_0}\frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}(1-t)\mathrm{d}t$$
$$+|\phi'(\theta)|\int_{t_0}^{1}\frac{1-e^{-\kappa}-\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}t\mathrm{d}t+|\phi'(\xi)|\int_{t_0}^{1}\frac{1-e^{-\kappa}-\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}(1-t)\mathrm{d}t\bigg\}.$$

Direct computation yields that

$$\int_0^{t_0}\frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}t\mathrm{d}t=\frac{1}{\kappa(1-e^{-\kappa})}\left[\frac{1}{\kappa}-\left(t_0+\frac{1}{\kappa}\right)e^{-\kappa t_0}\right]-\frac{1}{2\kappa}t_0^2,$$

$$\int_0^{t_0}\frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}(1-t)\mathrm{d}t=\frac{1}{\kappa(1-e^{-\kappa})}\left[\left(t_0+\frac{1}{\kappa}-1\right)e^{-\kappa t_0}-\frac{1}{\kappa}+1\right]+\frac{1}{2\kappa}t_0^2-\frac{1}{\kappa}t_0,$$

$$\int_{t_0}^{1}\frac{1-e^{-\kappa}-\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}t\mathrm{d}t=\frac{1}{\kappa(1-e^{-\kappa})}\left[\left(-t_0-\frac{1}{\kappa}\right)e^{-\kappa t_0}+\left(1+\frac{1}{\kappa}\right)e^{-\kappa}\right]+\frac{1}{2\kappa}\left(1-t_0^2\right),$$

and

$$\int_{t_0}^{1}\frac{1-e^{-\kappa}-\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}(1-t)\mathrm{d}t=\frac{1}{\kappa(1-e^{-\kappa})}\left[\left(t_0+\frac{1}{\kappa}-1\right)e^{-\kappa t_0}-\frac{1}{\kappa}e^{-\kappa}\right]+\frac{1}{\kappa}\left(\frac{1}{2}t_0^2-t_0+\frac{1}{2}\right).$$

Thus, the proof is completed.

Remark 4. If one attempts to require $\mu\to 1$, within Theorem 6, then one gains Theorem 2.2, testified by Dragomir and Agarwal in the published article [9].

Theorem 7. *Suppose that the mapping $\phi:\Lambda^\circ\to\mathbb{R}$ is differentiable defined on the interval Λ°, $\theta,\xi\in\Lambda^\circ$ together with $\theta<\xi$. If the mapping $|\phi'|^\tau$ is convex on $[\theta,\xi]$ for $\tau>1$, then we obtain the following trapezoid type inequality for the left-sided fractional integrals containing exponential kernels*

$$\begin{aligned}&\left|\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi)-\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}_{\theta^+}^{\mu}\phi(\xi)\right|\\&\leq(\xi-\theta)\mathcal{S}^{1-\frac{1}{\tau}}\left((\Delta_1+\Delta_3)|\phi'(\theta)|^\tau+(\Delta_2+\Delta_4)|\phi'(\xi)|^\tau\right)^{\frac{1}{\tau}},\end{aligned}\tag{4.2}$$

where $\mathcal{S}=\frac{2t_0e^{-\kappa}-2e^{-\kappa t_0}-2t_0+2}{\kappa(1-e^{-\kappa})}$, $\Delta_i, i=1,2,3,4$, κ and t_0 are in line with Theorem 6 with $\mu\in(0,1)$.

Proof. Taking advantage of Lemma 3, the Power-mean integral inequality and the convexity of the mapping $|\phi'|^\tau$ defined on $[\theta, \xi]$ in order, we know that

$$
\begin{aligned}
&\left|\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi)-\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right| \\
&\leq (\xi-\theta)\int_0^1 \frac{\left|e^{-\kappa}-1+\kappa e^{-\kappa t}\right|}{\kappa(1-e^{-\kappa})}\left|\phi'(t\theta+(1-t)\xi)\right|\mathrm{d}t \\
&\leq (\xi-\theta)\left[\left(\int_0^1 \frac{\left|e^{-\kappa}-1+\kappa e^{-\kappa t}\right|}{\kappa(1-e^{-\kappa})}\mathrm{d}t\right)^{1-\frac{1}{\tau}}\left(\int_0^1 \frac{\left|e^{-\kappa}-1+\kappa e^{-\kappa t}\right|}{\kappa(1-e^{-\kappa})}\left|\phi'(t\theta+(1-t)\xi)\right|^\tau\mathrm{d}t\right)^{\frac{1}{\tau}}\right] \\
&\leq (\xi-\theta)\left[\left(\int_0^{t_0} \frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}\mathrm{d}t+\int_{t_0}^1 \frac{1-e^{-\kappa}-\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}\mathrm{d}t\right)^{1-\frac{1}{\tau}}\right. \\
&\quad\times\left(|\phi'(\theta)|^\tau\int_0^{t_0}\frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}t\mathrm{d}t+|\phi'(\xi)|^\tau\int_0^{t_0}\frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}(1-t)\mathrm{d}t\right. \\
&\quad\left.\left.+|\phi'(\theta)|^\tau\int_{t_0}^1\frac{1-e^{-\kappa}-\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}t\mathrm{d}t+|\phi'(\xi)|^\tau\int_{t_0}^1\frac{1-e^{-\kappa}-\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}(1-t)\mathrm{d}t\right)^{\frac{1}{\tau}}\right].
\end{aligned}
\tag{4.3}
$$

Let us evaluate the integrals involved in (4.3). We observe that

$$
\int_0^{t_0}\frac{e^{-\kappa}-1+\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}\mathrm{d}t+\int_{t_0}^1\frac{1-e^{-\kappa}-\kappa e^{-\kappa t}}{\kappa(1-e^{-\kappa})}\mathrm{d}t=\frac{2t_0e^{-\kappa}-2e^{-\kappa t_0}-2t_0+2}{\kappa(1-e^{-\kappa})}.
$$

Thus, this finishes the proof.

Remark 5. If one attempts to require $\mu \to 1$, within Theorem 7, then one obtains Theorem 1, proposed by Pearce in [30].

Theorem 8. *Postulating that $\phi : \Lambda^\circ \to \mathbb{R}$ is a differentiable mapping considered on the interval Λ°, $\theta, \xi \in \Lambda^\circ$ along with $\theta < \xi$. For $\tau > 1$ with $r^{-1} + \tau^{-1} = 1$, if the mapping $|\phi'|^\tau$ is convex on $[\theta, \xi]$, then we gain the following trapezoid type inequality for the left-sided fractional integrals including exponential kernels*

$$
\begin{aligned}
&\left|\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi)-\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right| \\
&\leq \frac{\xi-\theta}{\kappa(1-e^{-\kappa})}(\Delta_5+\Delta_6)^{\frac{1}{r}}\left(\frac{|\phi'(\theta)|^\tau+|\phi'(\xi)|^\tau}{2}\right)^{\frac{1}{\tau}},
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
\Delta_5 &= (2^{r-1}e^{-r\kappa}-1)t_0+\frac{(2\kappa)^{r-1}}{r}(1-e^{-r\kappa t_0}), \\
\Delta_6 &= 1-t_0-e^{-r\kappa}(1-t_0-r+re^{-\kappa t_0+\kappa}),
\end{aligned}
$$

for t_0 and κ are in line with Theorem 6 with $\mu \in (0, 1)$.

Proof. Utilizing Lemma 3, the Hölder's integral inequality and the convexity of the mapping $|\phi'|^{\tau}$ defined on $[\theta, \xi]$, it follows that

$$\begin{aligned}
&\left|\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi)-\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right| \\
&\leq \frac{\xi-\theta}{\kappa(1-e^{-\kappa})}\int_0^1\left|e^{-\kappa}-1+\kappa e^{-\kappa t}\right|\left|\phi'(t\theta+(1-t)\xi)\right|\mathrm{d}t \\
&\leq \frac{\xi-\theta}{\kappa(1-e^{-\kappa})}\left(\int_0^1\left|e^{-\kappa}-1+\kappa e^{-\kappa t}\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^1|\phi'(t\theta+(1-t)\xi)|^{\tau}\mathrm{d}t\right)^{\frac{1}{\tau}} \\
&\leq \frac{\xi-\theta}{\kappa(1-e^{-\kappa})}\left(\int_0^{t_0}(e^{-\kappa}-1+\kappa e^{-\kappa t})^r\mathrm{d}t+\int_{t_0}^1(1-e^{-\kappa}-\kappa e^{-\kappa t})^r\mathrm{d}t\right)^{\frac{1}{r}} \\
&\quad\times\left(\frac{|\phi'(\theta)|^{\tau}+|\phi'(\xi)|^{\tau}}{2}\right)^{\frac{1}{\tau}}.
\end{aligned} \tag{4.5}$$

By virtue of the inequality $(\mathcal{A}-\mathcal{B})^{\gamma}\leq\mathcal{A}^{\gamma}-\mathcal{B}^{\gamma}$ for $\mathcal{A}\geq\mathcal{B}\geq 0$ with $\gamma\geq 1$, we deduce that

$$\begin{aligned}
\int_0^{t_0}(e^{-\kappa}-1+\kappa e^{-\kappa t})^r\mathrm{d}t &\leq \int_0^{t_0}\left[(e^{-\kappa}+\kappa e^{-\kappa t})^r-1\right]\mathrm{d}t \\
&\leq \int_0^{t_0}\left[2^{r-1}(e^{-r\kappa}+\kappa^r e^{-r\kappa t})-1\right]\mathrm{d}t \\
&= (2^{r-1}e^{-r\kappa}-1)t_0+\frac{(2\kappa)^{r-1}}{r}(1-e^{-r\kappa t_0}),
\end{aligned} \tag{4.6}$$

in which we take advantage of the inequality $(\mathcal{A}+\mathcal{B})^{\gamma}\leq 2^{\gamma-1}(\mathcal{A}^{\gamma}+\mathcal{B}^{\gamma})$ with $\mathcal{A}>0, \mathcal{B}>0$ and $\gamma\geq 1$ for the second inequality above.
Analogously, we deduce that

$$\begin{aligned}
\int_{t_0}^1(1-e^{-\kappa}-\kappa e^{-\kappa t})^r\mathrm{d}t &\leq \int_{t_0}^1\left[1-(e^{-\kappa}+\kappa e^{-\kappa t})^r\right]\mathrm{d}t \\
&= \int_{t_0}^1\left[1-\left[e^{-\kappa}(1+\kappa e^{-\kappa t+\kappa})\right]^r\right]\mathrm{d}t \\
&\leq \int_{t_0}^1\left[1-e^{-r\kappa}(1+r\kappa e^{-\kappa t+\kappa})\right]\mathrm{d}t \\
&= 1-t_0-e^{-r\kappa}(1-t_0-r+re^{-\kappa t_0+\kappa}),
\end{aligned} \tag{4.7}$$

in which we make use of the inequality $(1+\omega)^{\lambda}\geq 1+\lambda\omega$, if $\omega>-1$ with $\lambda<0$ or $\lambda>1$.
Employing (4.6) and (4.7) in (4.5), one obtains the required inequality. This fulfills the proof.

Theorem 9. *Suppose that the mapping $\phi:\Lambda^{\circ}\to\mathbb{R}$ is differentiable on the interval Λ°, $\theta, \xi\in\Lambda^{\circ}$ along with $\theta<\xi$. If the mapping $|\phi'|$ is convex on $[\theta, \xi]$, then the following midpoint type inequality for the left-sided fractional integrals involving exponential kernels holds true*

$$\left|\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)-\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right| \le (\xi-\theta)\Big[(\Delta_7+\Delta_9)\left|\phi'(\theta)\right|+(\Delta_8+\Delta_{10})\left|\phi'(\xi)\right|\Big], \tag{4.8}$$

where

$$\Delta_7=\frac{1}{\kappa(1-e^{-\kappa})}\left[\frac{1}{2}\kappa t_1^2+(t_1+\frac{1}{\kappa})e^{-\kappa t_1}-\frac{1}{\kappa}\right],$$
$$\Delta_8=\frac{1}{\kappa(1-e^{-\kappa})}\left[-\frac{1}{2}\kappa t_1^2+(1-t_1-\frac{1}{\kappa})e^{-\kappa t_1}+\kappa t_1+\frac{1}{\kappa}-1\right],$$
$$\Delta_9=\frac{1}{\kappa(1-e^{-\kappa})}\left[(t_1+\frac{1}{\kappa})e^{-\kappa t_1}-\left(\frac{1}{2}\left(1-t_1^2\right)\kappa+\frac{1}{\kappa}+1\right)e^{-\kappa}\right],$$

and

$$\Delta_{10}=\frac{1}{\kappa(1-e^{-\kappa})}\left[(1-t_1-\frac{1}{\kappa})e^{-\kappa t_1}+\left(\frac{1}{2}\left(1-t_1^2\right)\kappa+\frac{1}{\kappa}+\kappa t_1-\kappa\right)e^{-\kappa}\right],$$

for $t_1=\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}$, $\kappa=\frac{1-\mu}{\mu}(\xi-\theta)$ *along with* $\mu\in(0,1)$.

Proof. On account of Lemma 4 and the convexity of $|\phi'|$ defined on the interval $[\theta,\xi]$, it follows that

$$\begin{aligned}&\left|\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)-\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right|\\
&\le(\xi-\theta)\left[\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}\left|\phi'(t\theta+(1-t)\xi)\right|\mathrm{d}t+\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}\left|\phi'(t\theta+(1-t)\xi)\right|\mathrm{d}t\right]\\
&\le(\xi-\theta)\Big[\left|\phi'(\theta)\right|\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}t\mathrm{d}t+\left|\phi'(\xi)\right|\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}(1-t)\mathrm{d}t\\
&\quad+\left|\phi'(\theta)\right|\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}t\mathrm{d}t+\left|\phi'(\xi)\right|\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}(1-t)\mathrm{d}t\Big].\end{aligned}$$

Direct computation yields that

$$\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}t\mathrm{d}t=\frac{1}{\kappa(1-e^{-\kappa})}\left[\frac{1}{2}\kappa t_1^2+(t_1+\frac{1}{\kappa})e^{-\kappa t_1}-\frac{1}{\kappa}\right],$$
$$\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}(1-t)\mathrm{d}t=\frac{1}{\kappa(1-e^{-\kappa})}\left[-\frac{1}{2}\kappa t_1^2+(1-t_1-\frac{1}{\kappa})e^{-\kappa t_1}+\kappa t_1+\frac{1}{\kappa}-1\right],$$
$$\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}t\mathrm{d}t=\frac{1}{\kappa(1-e^{-\kappa})}\left[(t_1+\frac{1}{\kappa})e^{-\kappa t_1}-\left(\frac{1}{2}\left(1-t_1^2\right)\kappa+\frac{1}{\kappa}+1\right)e^{-\kappa}\right],$$

and

$$\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}(1-t)\mathrm{d}t=\frac{1}{\kappa(1-e^{-\kappa})}\left[(1-t_1-\frac{1}{\kappa})e^{-\kappa t_1}+\left(\frac{1}{2}\left(1-t_1^2\right)\kappa+\frac{1}{\kappa}+\kappa t_1-\kappa\right)e^{-\kappa}\right].$$

Thus, this ends the proof.

Remark 6. If one attempts to require $\mu \to 1$, within Theorem 9, then one obtains Theorem 2.2 provided by Kirmaci in [20].

Theorem 10. *Postulating that the mapping $\phi : \Lambda^\circ \to \mathbb{R}$ is differentiable defined on the interval Λ°, $\theta, \xi \in \Lambda^\circ$ with $\theta < \xi$. If the mapping $|\phi'|^\tau$ is convex on $[\theta, \xi]$ for $\tau > 1$, then we obtain the following midpoint type inequality for the left-sided fractional integrals containing exponential kernels*

$$\begin{aligned}&\left|\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi) - \phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta + \frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right| \\ &\quad\le (\xi-\theta)\eta^{1-\frac{1}{\tau}}\left[\left(\Delta_7\,|\phi'(\theta)|^\tau + \Delta_8\,|\phi'(\xi)|^\tau\right)^{\frac{1}{\tau}} + \left(\Delta_9\,|\phi'(\theta)|^\tau + \Delta_{10}\,|\phi'(\xi)|^\tau\right)^{\frac{1}{\tau}}\right],\end{aligned} \tag{4.9}$$

where

$$\eta = \frac{e^{\frac{\kappa e^{-\kappa}+e^{-\kappa}-1}{1-e^{-\kappa}}} - \frac{\kappa e^{-\kappa}}{1-e^{-\kappa}}}{\kappa(1-e^{-\kappa})},$$

and $\Delta_i, i = 7, 8, 9, 10$, κ are in line with Theorem 9 with $\mu \in (0, 1)$.

Proof. Taking advantage of Lemma 4, the Power-mean integral inequality and the convexity of the mapping $|\phi'|^\tau$ on the interval $[\theta, \xi]$ in order, we state that

$$\begin{aligned}&\left|\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi) - \phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta + \frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right| \\
&\le (\xi-\theta)\left[\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}\,|\phi'(t\theta+(1-t)\xi)|\,\mathrm{d}t + \int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}\,|\phi'(t\theta+(1-t)\xi)|\,\mathrm{d}t\right] \\
&\le (\xi-\theta)\Bigg[\left(\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}\mathrm{d}t\right)^{1-\frac{1}{\tau}}\left(\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}\,|\phi'(t\theta+(1-t)\xi)|^\tau\,\mathrm{d}t\right)^{\frac{1}{\tau}} \\
&\quad+\left(\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}\mathrm{d}t\right)^{1-\frac{1}{\tau}}\left(\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}\,|\phi'(t\theta+(1-t)\xi)|^\tau\,\mathrm{d}t\right)^{\frac{1}{\tau}}\Bigg] \\
&\le (\xi-\theta)\Bigg[\left(\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}\mathrm{d}t\right)^{1-\frac{1}{\tau}}\left(|\phi'(\theta)|^\tau\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}t\mathrm{d}t + |\phi'(\xi)|^\tau\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}(1-t)\mathrm{d}t\right)^{\frac{1}{\tau}} \\
&\quad+\left(\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}\mathrm{d}t\right)^{1-\frac{1}{\tau}}\left(|\phi'(\theta)|^\tau\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}t\mathrm{d}t + |\phi'(\xi)|^\tau\int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}(1-t)\mathrm{d}t\right)^{\frac{1}{\tau}}\Bigg].\end{aligned}$$

Direct computation yields that

$$\int_0^{t_1}\frac{1-e^{-\kappa t}}{1-e^{-\kappa}}\mathrm{d}t = \int_{t_1}^1\frac{e^{-\kappa t}-e^{-\kappa}}{1-e^{-\kappa}}\mathrm{d}t = \frac{e^{\frac{\kappa e^{-\kappa}+e^{-\kappa}-1}{1-e^{-\kappa}}} - \frac{\kappa e^{-\kappa}}{1-e^{-\kappa}}}{\kappa(1-e^{-\kappa})},$$

in which t_1 is the same as in Theorem 9. Thus, this ends the proof.

Remark 7. If one attempts to require $\mu \to 1$, within Theorem 10, then one derives the following midpoint type inequality

$$\begin{aligned}&\left|\frac{1}{\xi-\theta}\int_{\theta}^{\xi}\phi(\omega)\mathrm{d}\omega-\phi\left(\frac{\theta+\xi}{2}\right)\right|\\&\leq\frac{\xi-\theta}{8}\left[\left(\frac{|\phi'(\theta)|^{\tau}+2\,|\phi'(\xi)|^{\tau}}{3}\right)^{\frac{1}{\tau}}+\left(\frac{2\,|\phi'(\theta)|^{\tau}+|\phi'(\xi)|^{\tau}}{3}\right)^{\frac{1}{\tau}}\right].\end{aligned}\tag{4.10}$$

Theorem 11. *Assume that $\phi:\Lambda^{\circ}\to\mathbb{R}$ is a differentiable mapping considered on the interval Λ°, $\theta,\xi\in\Lambda^{\circ}$ along with $\theta<\xi$. If the mapping $|\phi'|^{\tau}$ is convex on $[\theta,\xi]$ for $\tau>1$ with $r^{-1}+\tau^{-1}=1$, then we obtain the following midpoint type inequality for the left-sided fractional integrals including exponential kernels*

$$\begin{aligned}&\left|\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^{+}}\phi(\xi)-\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right|\\&\leq\frac{\xi-\theta}{1-e^{-\kappa}}\Bigg[\zeta^{\frac{1}{r}}\left(\frac{t_1^2}{2}\,|\phi'(\theta)|^{\tau}+\frac{2t_1-t_1^2}{2}\,|\phi'(\xi)|^{\tau}\right)^{\frac{1}{\tau}}\\&\quad+\beta^{\frac{1}{r}}\left(\frac{1-t_1^2}{2}\,|\phi'(\theta)|^{\tau}+\frac{t_1^2-2t_1+1}{2}\,|\phi'(\xi)|^{\tau}\right)^{\frac{1}{\tau}}\Bigg],\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}\zeta&=t_1+\frac{1}{r\kappa}(e^{-r\kappa t_1}-1),\\\beta&=\frac{1}{r\kappa}(e^{-r\kappa t_1}-e^{-r\kappa})+e^{-r\kappa}(t_1-1),\end{aligned}$$

and κ, t_1 are in line with Theorem 9 with $\mu\in(0,1)$.

Proof. Taking advantage of Lemma 4, the Hölder's integral inequality and the convexity of the mapping $|\phi'|^{\tau}$ on the interval $[\theta,\xi]$, we know that

$$\begin{aligned}&\left|\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^{+}}\phi(\xi)-\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right)\right|\\&\leq\frac{\xi-\theta}{1-e^{-\kappa}}\left[\int_0^{t_1}(1-e^{-\kappa t})\left|\phi'(t\theta+(1-t)\xi)\right|\mathrm{d}t+\int_{t_1}^{1}(e^{-\kappa t}-e^{-\kappa})\left|\phi'(t\theta+(1-t)\xi)\right|\mathrm{d}t\right]\\&\leq\frac{\xi-\theta}{1-e^{-\kappa}}\left[\left(\int_0^{t_1}(1-e^{-\kappa t})^{r}\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{t_1}\left|\phi'(t\theta+(1-t)\xi)\right|^{\tau}\mathrm{d}t\right)^{\frac{1}{\tau}}\right.\\&\quad\left.+\left(\int_{t_1}^{1}(e^{-\kappa t}-e^{-\kappa})^{r}\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{t_1}^{1}\left|\phi'(t\theta+(1-t)\xi)\right|^{\tau}\mathrm{d}t\right)^{\frac{1}{\tau}}\right]\\&\leq\frac{\xi-\theta}{1-e^{-\kappa}}\left[\left(\int_0^{t_1}(1-e^{-\kappa t})^{r}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{t_1^2}{2}\left|\phi'(\theta)\right|^{\tau}+\frac{2t_1-t_1^2}{2}\left|\phi'(\xi)\right|^{\tau}\right)^{\frac{1}{\tau}}\right.\\&\quad\left.+\left(\int_{t_1}^{1}(e^{-\kappa t}-e^{-\kappa})^{r}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{1-t_1^2}{2}\left|\phi'(\theta)\right|^{\tau}+\frac{t_1^2-2t_1+1}{2}\left|\phi'(\xi)\right|^{\tau}\right)^{\frac{1}{\tau}}\right].\end{aligned}\tag{4.12}$$

Making use of the inequality $(\mathcal{A}-\mathcal{B})^{\gamma} \leq \mathcal{A}^{\gamma}-\mathcal{B}^{\gamma}$ for $\mathcal{A} \geq \mathcal{B} \geq 0$ with $\gamma \geq 1$, we observe that

$$\begin{aligned}\int_0^{t_1}(1-e^{-\kappa t})^r\mathrm{d}t &\leq \int_0^{t_1}(1-e^{-r\kappa t})\mathrm{d}t\\ &= t_1+\frac{1}{r\kappa}(e^{-r\kappa t_1}-1).\end{aligned} \tag{4.13}$$

Similarly,

$$\begin{aligned}\int_{t_1}^{1}(e^{-\kappa t}-e^{-\kappa})^r\mathrm{d}t &\leq \int_{t_1}^{1}(e^{-r\kappa t}-e^{-r\kappa})\mathrm{d}t\\ &= \frac{1}{r\kappa}(e^{-r\kappa t_1}-e^{-r\kappa})+e^{-r\kappa}(t_1-1).\end{aligned} \tag{4.14}$$

Employing (4.13) and (4.14) in (4.12). This completes the proof.

5. Examples

In order to check the correctness of the outcomes investigated in this study, we enumerate three examples in this section.

Example 1. *Let the mapping $\phi : (0,\infty) \longrightarrow \mathbb{R}$ be defined by $\phi(\omega) = -\ln\omega$. If we attempt to take $\theta = 1$, $\xi = 2$, $\mu = \frac{1}{2}$, then all postulations mentioned in Theorem 5 are met.*
Evidently, $\kappa = \frac{1-\mu}{\mu}(\xi-\theta) = 1$. The left side of (2.1) *is*

$$\begin{aligned}\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\theta+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\xi\right) &= \phi\left(\frac{1-2e^{-1}}{1-e^{-1}}+\frac{e^{-1}}{1-e^{-1}}\times 2\right)\\ &= -\ln\left(\frac{1-2e^{-1}}{1-e^{-1}}+\frac{e^{-1}}{1-e^{-1}}\times 2\right)\\ &\approx -0.4587.\end{aligned}$$

The middle-hand term of (2.1) *is*

$$\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi) = \frac{1}{1-e^{-1}}\int_1^2 e^{\omega-2}(-\ln\omega)\mathrm{d}\omega \approx -0.4416.$$

The right side of (2.1) *is*

$$\begin{aligned}\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(\theta)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(\xi) &= \frac{1-2e^{-1}}{1-e^{-1}}(-\ln 1)+\frac{e^{-1}}{1-e^{-1}}(-\ln 2)\\ &\approx -0.4034.\end{aligned}$$

It is clear that $-0.4587 < -0.4416 < -0.4034$*, which confirms the correctness of the result described in Theorem 5.*

Furthermore, by plotting graph of the inequalities asserted in Theorem 5 for $\phi(\omega) = -\ln(\omega)$, $\omega \in [\theta,\xi]$, corresponding to $\theta = 1$, $\xi = 2$, we examine the correctness of the result.

$$\phi\left(\frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\times 2\right) \leq \frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{1^+}\phi(2) \leq \frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\phi(1)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\phi(2). \tag{5.1}$$

Three functions given by the inequalities (5.1) *with respect to the left, middle and right sides are plotted in Figure 1 against* $\mu \in [0.2, 0.9]$. *The graph of the functions shows the correctness of the inequalities.*

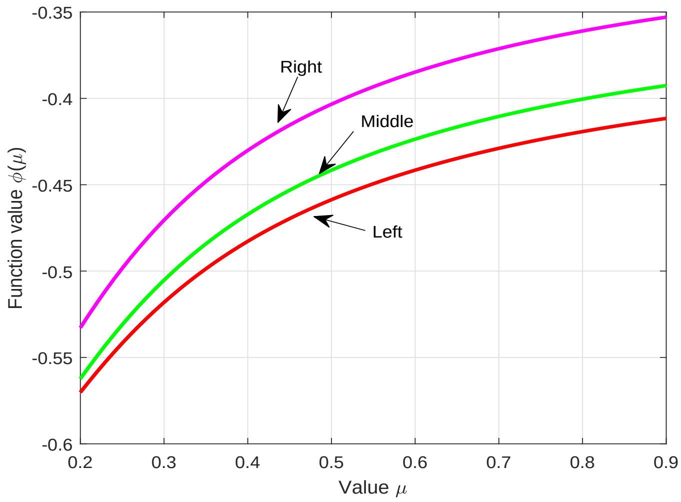

Figure 1. For the case $\theta = 1, \xi = 2$, the graphical representation for Example 1.

Example 2. *For the mapping* $\phi : [\theta, \xi] \longrightarrow \mathbb{R}$, *given by* $\phi(\omega) = e^{\omega}$. *It is straightforward to find that the mapping* $|\phi'|$ *is convex. If we attempt to take* $\theta = 1$, $\xi = 2$ *and* $\mu = \frac{1}{2}$, *then all postulations mentioned in Theorem 6 are met.*
Evidently, $\kappa = \frac{1-\mu}{\mu}(\xi - \theta) = 1$. *The left side of* (4.1) *is*

$$\begin{aligned} &\left|\frac{1 - e^{-\kappa} - \kappa e^{-\kappa}}{\kappa(1 - e^{-\kappa})}\phi(\theta) + \frac{\kappa + e^{-\kappa} - 1}{\kappa(1 - e^{-\kappa})}\phi(\xi) - \frac{1 - \mu}{1 - e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right| \\ &= \frac{1 - 2e^{-1}}{1 - e^{-1}}e + \frac{e^{-1}}{1 - e^{-1}}e^2 - \frac{1}{1 - e^{-1}}\int_1^2 e^{-(2-\omega)}e^{\omega}\mathrm{d}\omega \\ &\approx 0.3829. \end{aligned}$$

The right side of (4.1) *is*

$$\begin{aligned} &(\xi - \theta)\Big[(\Delta_1 + \Delta_3)|\phi'(\theta)| + (\Delta_2 + \Delta_4)|\phi'(\xi)|\Big] \\ &= \Big[(\Delta_1 + \Delta_3)e + (\Delta_2 + \Delta_4)e^2\Big] \\ &\approx 1.2701, \end{aligned}$$

where $\Delta_i, i = 1, 2, 3, 4$ *are in line with Theorem 6.*
It is clear that $0.3829 \le 1.2701$, *which confirms the correctness of the result described in Theorem 6.*

Furthermore, by plotting graph of the inequalities asserted in Theorem 6 for $\phi(\omega) = e^{\omega}$, $\omega \in [\theta, \xi]$, *corresponding to $\theta = 1$, $\xi = 2$, we examine the correctness of the desired inequalities.*

$$-[(\Delta_1 + \Delta_3)e + (\Delta_2 + \Delta_4)e^2] \leq \frac{1 - e^{-\kappa} - \kappa e^{-\kappa}}{\kappa(1 - e^{-\kappa})}e + \frac{\kappa + e^{-\kappa} - 1}{\kappa(1 - e^{-\kappa})}e^2 - \frac{1-\mu}{1 - e^{-\kappa}}\mathcal{I}^{\mu}_{1^+}\phi(2) \\ \leq [(\Delta_1 + \Delta_3)e + (\Delta_2 + \Delta_4)e^2]. \tag{5.2}$$

Three functions given by the inequalities (5.2) *with regard to the left, middle and right sides are plotted in Figure 2 that occur with the variation of the parameter $\mu \in [0.1, 0.9]$. The graph of the functions shows the correctness of the required inequalities.*

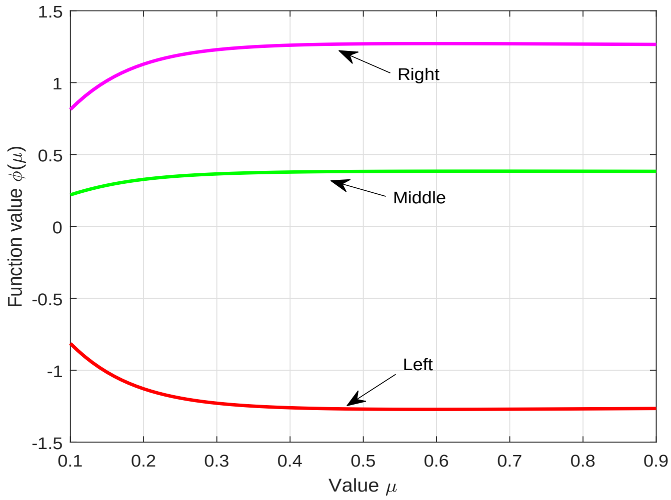

Figure 2. For the case $\theta = 1$, $\xi = 2$, the graphical representation for Example 2.

Example 3. *For the mapping $\phi : [\theta, \xi] \longrightarrow \mathbb{R}$, given by $\phi(\omega) = \frac{1}{2}e^{\omega}$. It is straightforward to know that the mapping $|\phi'|^{\tau}$ is convex. If we attempt to take $\theta = 1$, $\xi = 2$, $\tau = 30$, $r = \tau/(\tau - 1) = \frac{30}{29}$ and $\mu = 0.85$, then all postulations mentioned in Theorem 8 are met.*
Evidently, $\kappa = \frac{1-\mu}{\mu}(\xi - \theta) = \frac{3}{17}$. The left side of (4.4) *is*

$$\begin{aligned} &\left|\frac{1 - e^{-\kappa} - \kappa e^{-\kappa}}{\kappa(1 - e^{-\kappa})}\phi(\theta) + \frac{\kappa + e^{-\kappa} - 1}{\kappa(1 - e^{-\kappa})}\phi(\xi) - \frac{1-\mu}{1 - e^{-\kappa}}\mathcal{I}^{\mu}_{\theta^+}\phi(\xi)\right| \\ &= \left|\frac{1 - e^{-\kappa} - \kappa e^{-\kappa}}{\kappa(1 - e^{-\kappa})}\left(\frac{1}{2}e\right) + \frac{\kappa + e^{-\kappa} - 1}{\kappa(1 - e^{-\kappa})}\left(\frac{1}{2}e^2\right) - \frac{1 - 0.85}{1 - e^{-\kappa}}\mathcal{I}^{0.85}_{1^+}\phi(2)\right| \\ &\approx 0.1919. \end{aligned}$$

The right side of (4.4) *is*

$$\begin{aligned} &\frac{\xi - \theta}{\kappa(1 - e^{-\kappa})}(\Delta_5 + \Delta_6)^{\frac{1}{r}}\left(\frac{|\phi'(\theta)|^{\tau} + |\phi'(\xi)|^{\tau}}{2}\right)^{\frac{1}{\tau}} \\ &= \frac{1}{\kappa(1 - e^{-\kappa})}(\Delta_5 + \Delta_6)^{\frac{1}{r}}\left(\frac{(\frac{1}{2}e)^{\tau} + (\frac{1}{2}e^2)^{\tau}}{2}\right)^{\frac{1}{\tau}} \\ &\approx 1.7397, \end{aligned}$$

where Δ_5 *and* Δ_6 *are the same in Theorem 8.*
It is clear that $0.1919 \le 1.7397$*, which confirms the correctness of the result asserted in Theorem 8.*

Furthermore, by plotting graph of the inequalities asserted in Theorem 8 for $\phi(\omega) = \frac{1}{2}e^{\omega}$*,* $\omega \in [\theta, \xi]$ *corresponding to* $\theta = 1$*,* $\xi = 2$*,* $\tau = 30$ *and* $r = \tau/(\tau - 1) = \frac{30}{29}$*, we examine the correctness of the required result.*

$$\begin{aligned}
&-\frac{1}{\kappa(1-e^{-\kappa})}(\Delta_5+\Delta_6)^{\frac{29}{30}}\left(\frac{(\frac{1}{2}e)^{30}+(\frac{1}{2}e^2)^{30}}{2}\right)^{\frac{1}{30}}\\
&\le \frac{1-e^{-\kappa}-\kappa e^{-\kappa}}{\kappa(1-e^{-\kappa})}\left(\frac{1}{2}e\right)+\frac{\kappa+e^{-\kappa}-1}{\kappa(1-e^{-\kappa})}\left(\frac{1}{2}e^2\right)-\frac{1-\mu}{1-e^{-\kappa}}\mathcal{I}^{\mu}_{1+}\phi(2) \qquad (5.3)\\
&\le \frac{1}{\kappa(1-e^{-\kappa})}(\Delta_5+\Delta_6)^{\frac{29}{30}}\left(\frac{(\frac{1}{2}e)^{30}+(\frac{1}{2}e^2)^{30}}{2}\right)^{\frac{1}{30}}.
\end{aligned}$$

Three functions given by the inequalities (5.3) *with respect to the left, middle and right sides are plotted in Figure 3 against* $\mu \in [0.8, 0.9]$*, which shows the correctness of the required inequalities.*

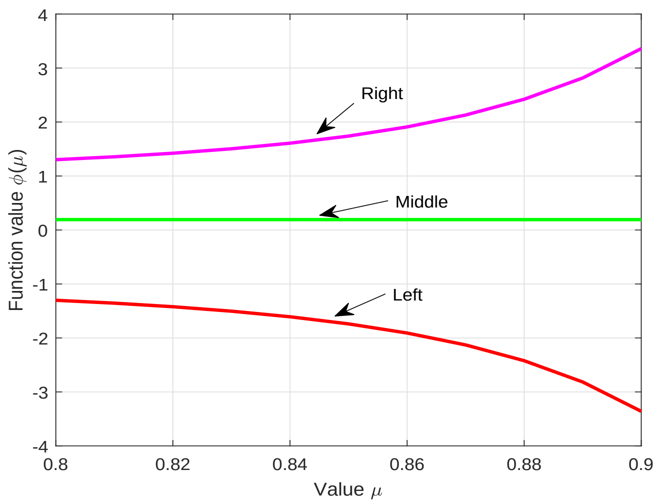

Figure 3. For the case $\theta = 1$, $\xi = 2$, the graphical representation for Example 3.

6. Conclusions

By using only the left-sided fractional integrals having exponential kernels, the study addresses certain Hermite–Hadamard type inequalities. To achieve this objective, we construct two fractional integral identities, which play a key role in proving our main inequalities. And several fractional trapezoid and midpoint type integral inequalities involving the differentiable convex mappings are presented here. The findings acquired in this work, in particular, extend and generalize previous inequalities in the literature regarding the Hermite–Hadamard type inequalities. What we want to emphasize here is that the fractional integral operators are widely utilized in applied mathematics, see [32, 33, 39]. This significant field is worth further exploration.

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Conflict of interest

The authors declare no conflicts of interest.

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