AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 4068–4093.
DOI: 10.3934/math.2022225
Received: 07 October 2021
Accepted: 29 November 2021
Published: 14 December 2021

Research article

Dynamics and optimal harvesting of a stochastic predator-prey system with regime switching, S-type distributed time delays and Lévy jumps

Yuanfu Shao*

College of Science, Guilin University of Technology, Guilin, Guangxi 541004, China

* **Correspondence:** Email: shaoyuanfu@163.com.

Abstract: This work is concerned with a stochastic predator-prey system with S-type distributed time delays, regime switching and Lévy jumps. By use of the stochastic differential comparison theory and some inequality techniques, we study the extinction and persistence in the mean for each species, asymptotic stability in distribution and the optimal harvesting effort of the model. Then we present some simulation examples to illustrate the theoretical results and explore the effects of regime switching, distributed time delays and Lévy jumps on the dynamical behaviors, respectively.

Keywords: regime switching; Lévy jumps; persistence in the mean; asymptotic stability in distribution
Mathematics Subject Classification: 60H10, 92B05

1. Introduction

The natural world is full of environmental perturbations almost everywhere, which play an important role in the ecological system [1–5]. As May [6] said that the growth rates in ecological systems should be stochastic due to the influences of random noises, so the solution would fluctuate around some average values, not being a steady positive point. Therefore, it is more rational by considering a stochastic environmental perturbation (white noise) to better investigate the real population systems. In mathematical modelling, the white noise has been extensively introduced to reveal the stochastic population dynamics, see e.g. [3–5,7,8] and references cited therein.

In real world, population systems may inevitably suffer some abrupt changes. It is well known that the growth rates of some species are affected by seasonal factors, which are much different in summer from those in winter. These phenomena can be described by a continuous-time Markovian process with a finite state space, i.e. colorful noise in mathematical modelling. The colorful noise may take several values and switch among different regimes of environment, which is memoryless, and the waiting time for the next switching follows an exponential distribution. The effect of colorful noise on population dynamics has attracted many researchers [9–14]. For example, by using a Markovian switching process to model the colorful noise in environment, Liu, He and Yu [14] proposed the following stochastic system with harvesting and regime-switching:

$$\begin{cases} dx_1(t) = x_1(t)\Big(r_1(\alpha(t)) - h_1 - a_{11}x_1(t) - a_{12}\displaystyle\int_{-\tau_1}^{0} x_2(t+s)d\mu_1(s)\Big)dt + \sigma_1(\alpha(t))x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)\Big(r_2(\alpha(t)) - h_2 + a_{21}\displaystyle\int_{-\tau_2}^{0} x_1(t+s)d\mu_2(s) - a_{22}x_2(t)\Big)dt + \sigma_2(\alpha(t))x_2(t)dB_2(t), \end{cases} \tag{1.1}$$

where $r_1(\cdot) > 0$ is the growth rate of prey, $r_2(\cdot) < 0$ is the death rate of predator, $a_{ii} > 0$ is the intra-specific competition rate, $a_{12} > 0$ is the capture rate and $a_{21} > 0$ is the conversion rate of food; h_i is the harvesting constant; $B_1(t)$ and $B_2(t)$ are standard independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration. It is right continuous and $\mathcal{F}_0$ contains all P-null sets; $\sigma_i^2(\cdot)$ is the intensity of stochastic noise, $i = 1, 2$. The regime of switching $\alpha(t)$ is a Markovian chain in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$. The generator of $\alpha(t)$ is defined as $\Gamma = (\gamma_{ij})_{N\times N}$ with

$$P\{\alpha(t+\triangle t = j|\alpha(t) = i)\} = \begin{cases} \gamma_{ij} \triangle t + o(\triangle t), i \neq j, \\ 1 + \gamma_{ii} \triangle t + o(\triangle t), i = j, \end{cases}$$

where $\triangle t > 0, \gamma_{ij}$ is the transition rate from the ith stage to the jth stage and $\gamma_{ij} \geq 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}, i, j \in \mathbb{S}$. It is often assumed that every sample of $\alpha(t)$ is a right continuous step function and irreducible with a finite simple jumps in any finite subinterval of $R_+ = [0, \infty)$. It obeys a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ satisfying $\pi\Gamma = 0$ and $\sum_{k=1}^{N}\pi_k = 1, \pi_k > 0, k \in \mathbb{S}$. The switching mechanism of the hybrid system is referred to [9].

In the study of ecological dynamics, the current growth of populations is usually influenced by its past history, that is, time delay is often inevitable in the natural ecosystems [15]. S-type distributed time delay is such a distributed delay that the integral is Lebesgue-Stieltjes. Just as the authors [16] said, "systems with discrete time delays and those with continuously distributed time delays do not contain each other. However, systems with S-type distributed time delays contain both." So it is interesting to consider the impact of S-type distributed delay on the ecological dynamics. Models with S-type distributed time delays have been studied by many authors [17, 18]. On the other hand, earthquake, harvesting and epidemics often happen in natural world. These sudden environmental perturbations are so strong and can change the population size in a very short time, which can not be described by white noise [19, 20]. Many experiments show that, due to the influence of environmental disturbance, the distribution of many species exhibits a scale-free characteristic, and hence the biologists introduce a non-Gaussian Lévy jump to characterize it. Models with Lévy jump are studied by many researchers and many nice results have been obtained [21–23].

Considering the effects of S-type time delays and Lévy jumps on system (1.1), we establish the following stochastic model

$$\begin{cases} dx_1(t) &= x_1(t^-)\Big(r_1(\alpha(t^-)) - h_1 - a_{11}x_1(t^-) - \int_{-\tau_{11}}^{0} x_1(t+\theta)d\mu_{11}(\theta) - a_{12}x_2(t^-) \\ &\quad - \int_{-\tau_{12}}^{0} x_2(t+\theta)d\mu_{12}(\theta)\Big)dt + \sigma_1(\alpha(t^-))x_1(t^-)dB_1(t) + \int_{\mathbb{Z}} \gamma_1(u,\alpha(t^-))x_1(t^-)\widetilde{N}(dt,du), \\ dx_2(t) &= x_2(t^-)\Big(r_2(\alpha(t^-)) - h_2 + a_{21}x_1(t^-) + \int_{-\tau_{21}}^{0} x_1(t+\theta)d\mu_{21}(\theta) - a_{22}x_2(t^-) \\ &\quad - \int_{-\tau_{22}}^{0} x_2(t+\theta)d\mu_{22}(\theta)\Big)dt + \sigma_2(\alpha(t^-))x_2(t^-)dB_2(t) + \int_{\mathbb{Z}} \gamma_2(u,\alpha(t^-))x_2(t^-)\widetilde{N}(dt,du), \end{cases} \tag{1.2}$$

with initial value

$$x(\theta) = (x_1(\theta), x_2(\theta))^T = (\phi_1(\theta), \phi_2(\theta))^T = \phi(\theta) \in C([-\tau,0]; R_+^2), \alpha(\theta) = \varsigma,$$

where $x_i(t^-)$ stands for the left limit of $x_i(t)$; $\int_{-\tau_{ij}}^{0} x_i(t+\theta)d\mu_{ij}(\theta)$ is Lebesgue-Stieltjes integral; $\tau_{ij} > 0$ is time delay; $\mu_{ij}(\theta)$ is bounded, nondecreasing variation function defined on $[-\tau, 0]$ with $\tau = \max_{i,j=1,2}\{\tau_{ij}\}, \varsigma \in \mathbb{S}, i, j = 1, 2$; $R_+^2 = \{x = (x_1, x_2) \in R^2, x_i > 0, i = 1, 2\}$ with $|x(t)| = \sqrt{\sum_{i=1}^{2} x_i^2}$; N is a Poisson counting measure, $\widetilde{N}(dt, du) = N(dt, du) - \lambda(du)$ is the component of N, where λ is the characteristic measure on a measurable subset $\mathbb{Z} \subset R_+ = [0, \infty)$ such that $\lambda(\mathbb{Z}) < \infty$. The Markov chain, Brownian motion and Lévy jumps are mutually independent.

Our main goal of this paper is as follows. First, since the study of dynamical behaviors of predator-prey system is an important topic [7, 24], we establish some sufficient conditions assuring the extinction, persistence in the mean for all species of system (1.2).

Second, in the study of long-term behaviors of species, the existence of a unique probability measure plays an important role in stochastic models with Lévy jumps (see [25–27]). Hence, it is very interesting to analyze the asymptotic stability in distribution of (1.2).

Third, in mathematical biology, it is valuable to keep the species persistent to maintain the biological balance. Consequently, the optimal harvesting strategy of renewable resources becomes more and more important [8, 27–29]. By ergodic method, we will study the optimal harvesting strategy of system (1.2).

The rest of this paper is organized as follows. Section 2 begins with some definitions, important lemmas and notations. Section 3 is devoted to the extinction and persistence in the mean for species. Section 4 and Section 5 focus on the asymptotic stability in distribution and the optimal harvesting strategy of (1.2), respectively. Section 6 gives some numerical simulations to verify the main results. Finally, Section 7 presents a brief discussion to conclude this paper.

2. Preliminaries

To begin with this section, we introduce some notations of the Itô's integral for a stochastic differential equation with Markovian switching and Lévy jumps [19, 23]. Let

$$dx(t) = f(x(t^-), t^-, \alpha(t^-))dt + g(x(t^-), t^-, \alpha(t^-))dB(t) + \int_{\mathbb{Z}} h(x(t^-), t^-, \alpha(t^-), \mu)\widetilde{N}(dt, d\mu),$$

where f and g : $R^2 \times R_+ \times \mathbb{S} \to R^2, h : R^2 \times R_+ \times \mathbb{S} \times \mathbb{Z} \to R^2$ are measurable functions. Let $V \in C^{2,1}(R^2 \times R_+ \times \mathbb{S}, R^2)$. Define the operator L as follows:

$$\begin{aligned} LV(x,t,k) &= V_t(x,t,k) + V_x(x,t,k)f(x,t,k) + \frac{1}{2}\text{trace}[g^T(x,t,k)V_{xx}(x,t,k)g(x,t,k)] \\ &\quad + \int_{\mathbb{Z}} \{V(x + h(x,t,k,u),t,k) - V(x,t,k) - V_x(x,t,k)h(x,t,k,u)\}\lambda(du) + \sum_{j=1}^{N} \gamma_{kj} V(x,t,j), \end{aligned}$$

where $V_t(x,t,k) = \frac{\partial V(x,t,k)}{\partial t}, V_x(x,t,k) = (\frac{\partial V(x,t,k)}{\partial x_1}, \frac{\partial V(x,t,k)}{\partial x_2}), V_{xx}(x,t,k) = (\frac{\partial^2 V(x,t,k)}{\partial x_i \partial x_j})_{2\times 2}, i,j = 1,2$. The generalized Itô's formula with jumps (see, for example[19, 23]) is defined as

$$dV(x,t,k) = LV(x,t,k)dt + V_x(x,t,k)g(x,t,k)dB(t) + \int_{\mathbb{Z}} \{V(x + h(x,t,k,u),t,k) - V(x,t,k)\}\widetilde{N}(dt,du).$$

Now we give the definitions of extinction and persistence in the mean of each species, and an important comparison theorem.

Definition 2.1 ([27]). *Let $x(t) = (x_1(t), x_2(t))^T \in R_+^2$ be a solution of system* (1.2)*. Then,*

(a) the population $x_i(t)$ is said to be extinct if $\lim_{t\to\infty} x_i(t) = 0, i = 1,2$;

(b) the population $x_i(t)$ is said to be persistent in the mean if $\lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds = K$ a.s., where K is a positive constant, $i = 1,2$.

Lemma 2.1 ([4]). *Suppose that $Z(t) \in C[\Omega \times [0,+\infty), R_+]$ and $\lim_{t\to\infty} F(t)/t = 0, a.s.$*

(a) *If there exist two positive constants T and λ_0 such that, for all $t > T$,*

$$\ln Z(t) \leq \lambda t - \lambda_0 \int_0^t Z(s)ds + F(t), \quad a.s.,$$

then

$$\begin{cases} \limsup_{t\to\infty} \frac{1}{t}\int_0^t Z(s)ds \leq \lambda/\lambda_0, & a.s., \quad if \quad \lambda \geq 0, \\ \lim_{t\to\infty} Z(t) = 0, & a.s., \quad if \quad \lambda < 0. \end{cases}$$

(b) *If there exist three positive constants T, λ_0 and λ such that, for all $t > T$,*

$$\ln Z(t) \geq \lambda t - \lambda_0 \int_0^t Z(s)ds + F(t), \quad a.s,$$

then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t Z(s)ds \geq \lambda/\lambda_0, \quad a.s.$$

Further, for the need of our discussion, we give some technical assumptions.

Assumption 1. For any $\alpha \in \mathbb{S}$ and $i = 1, 2$, we assume $\gamma_i(\alpha, u) > -1$ and

$$\int_{\mathbb{Z}} \{\gamma_i(\alpha, u) - \ln(1 + \gamma_i(\alpha, u))\}\lambda(du) \le K,$$

$$\int_{\mathbb{Z}} \max\{|\gamma_i(\alpha, u)|^2, [\ln(1 + \gamma_i(\alpha, u))]^2\}\lambda(du) \le K,$$

where $K > 0$ denotes a positive and finite constant unless otherwise stated, which may be different in different places.

Assumption 2. For any $p > 0$, we assume

$$\int_{\mathbb{Z}} \{|1 + \gamma_i(\alpha, u)|^p - 1 - p\gamma_i(\alpha, u)\}\lambda(du) \le K(p).$$

Assumptions 1 and 2 imply that the intensity of Lévy noise cannot be too strong, otherwise, the solution of system (1.2) may explode in some finite time [23]. About the existence of nontrivial positive solutions of (1.2), we have the following lemma.

Lemma 2.2. *Let Assumptions 1 and 2 hold. For any given initial value* $x(\theta) \in C([-\tau, 0]; R_+^2)$, $\alpha(\theta) = \varsigma$, *system (1.2) has a unique solution* $x(t)$ *on* $t \in [-\tau, \infty)$, *and the solution remains in* R_+^2 *with probability one. Moreover, for any* $p > 0$, *there is a* $K(p) > 0$ *such that* $\limsup_{t\to\infty} \mathbb{E}|x(t)|^p \le K(p)$.

Proof. The proof of the existence of solutions is straightforward, one may refer to [14, 23]. As to the proof of the boundedness of expectation, one can get it by using the generalized Itô's formular with jumps to function $e^t \sum_{i=1}^2 x_i^p(t)$. The process is similar to reference [14] and is omitted.

For simplicity, we introduce some notations to end this section.

$$\xi_i(\alpha(t)) = r_i(\alpha(t)) - \frac{\sigma_i^2(\alpha(t))}{2} - \int_{\mathbb{Z}} [\gamma_i(u, \alpha(t)) - \ln(1 + \gamma_i(u, \alpha(t)))]\lambda(du), \quad \overline{\xi}_i = \sum_{k=1}^N \pi_k \xi_i(k),$$

$$\eta_i(\alpha(t)) = \xi_i(\alpha(t)) - h_i, \qquad \overline{\eta}_i = \sum_{k=1}^N \pi_k \eta_i(k), \qquad A_{ij} = a_{ij} + \int_{-\tau_{ij}}^0 d\mu_{ij}(\theta), \qquad i, j = 1, 2,$$

$$\Delta = \begin{vmatrix} A_{11} & A_{12} \\ -A_{21} & A_{22} \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} \overline{\eta}_1 & A_{12} \\ \overline{\eta}_2 & A_{22} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} A_{11} & \overline{\eta}_1 \\ -A_{21} & \overline{\eta}_2 \end{vmatrix}.$$

3. Extinction and persistence in the mean of species

In this section, we study the long-term behaviors of (1.2). Consider the following system,

$$\begin{aligned} dW(t) &= W(t^-)\left(r(\alpha(t^-)) - aW(t^-) - \int_{-\tau}^0 W(t^- + \theta)d\mu(\theta)\right)dt + \sigma(\alpha(t^-))W(t^-)dB(t) \\ &\quad + \int_{\mathbb{Z}} \gamma(\alpha(t^-), u)W(t^-)\widetilde{N}(dt, du). \end{aligned} \tag{3.1}$$

Lemma 3.1. *For system (3.1), the following statements hold.*

(i) If $\sum_{k=1}^{N}\pi_k\xi(k)<0$*, then* $\lim_{t\to\infty}W(t)=0$.

(ii) If $\sum_{k=1}^{N}\pi_k\xi(k)>0$*, then* $\lim_{t\to\infty}\frac{1}{t}\int_0^t W(s)ds=\frac{\sum_{k=1}^{N}\pi_k\xi(k)}{a+\int_{-\tau}^{0}d\mu(\theta)}$,

where $\xi(k)=r(k)-\frac{\sigma^2(k)}{2}-\int_Z[\gamma(k,u)-\ln(1+\gamma(k,u))]\lambda(du)$.

Proof. Using the Itô's formula to $\ln W(t)$ and computing the stochastic differential along the solution of (3.1), we have

$$\begin{aligned} d\ln W(t) \;=\; & \Big(r(\alpha(t))-aW(t)-\int_{-\tau}^{0}W(t+\theta)d\mu(\theta)-\int_{\mathbb{Z}}[\gamma(u,\alpha(t))-\ln(1+\gamma(\alpha(t)))]\lambda(du) \\ & -\tfrac{1}{2}\sigma^2(\alpha(t))\Big)dt+\sigma(\alpha(t))dB(t)+\int_{\mathbb{Z}}\ln(1+\gamma(u,\alpha(t)))\widetilde{N}(dt,du). \end{aligned} \tag{3.2}$$

Integrating both sides of (3.2) from 0 to t, and divided by t from both sides, then

$$\begin{aligned} \frac{1}{t}\ln\frac{W(t)}{W(0)} \;=\; & \frac{1}{t}\int_0^t\xi(\alpha(s))ds-a\frac{1}{t}\int_0^t W(s)ds-\frac{1}{t}\int_{-\tau}^{0}\left[\int_\theta^0+\int_0^t+\int_t^{t+\theta}\right]W(s)dsd\mu(\theta) \\ & +\frac{1}{t}\int_0^t\sigma(\alpha(s))dB(s)+\frac{1}{t}\int_0^t\int_{\mathbb{Z}}\ln(1+\gamma(u,\alpha(s)))\tilde{N}(ds,du). \end{aligned} \tag{3.3}$$

Since

$$\lim_{t\to\infty}\frac{1}{t}\int_{-\tau}^{0}\int_0^{t+\theta}W(s)dsd\mu(\theta)=\lim_{t\to\infty}\int_{-\tau}^{0}d\mu(\theta)\times\frac{1}{t}\int_0^t W(s)ds,$$

it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_{-\tau}^{0}\int_t^{t+\theta}W(s)dsd\mu(\theta)=\lim_{t\to\infty}\frac{1}{t}\int_{-\tau}^{0}\int_0^{t+\theta}W(s)dsd\mu(\theta)-\lim_{t\to\infty}\frac{1}{t}\int_{-\tau}^{0}d\mu(\theta)\int_0^t W(s)ds=0.$$

Then

$$\lim_{t\to\infty}\frac{1}{t}\int_{-\tau}^{0}\left[\int_\theta^0 W(s)ds-\int_{t+\theta}^{t}W(s)ds\right]d\mu(\theta)=\lim_{t\to\infty}\frac{1}{t}\int_{-\tau}^{0}\left[\int_\theta^0\psi(s)ds-\int_{t+\theta}^{t}W(s)ds\right]d\mu(\theta)=0.$$

Therefore (3.3) leads to (3.4) as follows.

$$\begin{aligned} \frac{1}{t}\ln\frac{W(t)}{W(0)} \;=\; & \frac{1}{t}\int_0^t\xi(\alpha(s))ds-\left(a+\int_{-\tau}^{0}d\mu(\theta)\right)\frac{1}{t}\int_0^t W(s)ds+\frac{1}{t}\int_0^t\sigma(\alpha(s))dB(s) \\ & +\frac{1}{t}\int_0^t\int_{\mathbb{Z}}\ln(1+\gamma(u,\alpha(s)))\tilde{N}(ds,du). \end{aligned} \tag{3.4}$$

On the other hand, by the ergodicity of Markovian chain, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\xi(\alpha(s))ds=\sum_{k=1}^{N}\pi_k\xi(k).$$

Now we give the proof of (i) and (ii).

(i) By comparison method, using Lemma 2.1, we can easily derive from (3.4) that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t W(s)ds \le \frac{\sum_{k=1}^{N}\pi_k\xi(k)}{a+\int_{-\tau}^{0}d\mu(\theta)}.$$

If $\sum_{k=1}^{N}\pi_k\xi(k) < 0$, it is clear that $\lim_{t\to\infty} W(t) = 0$.

(ii) If $\sum_{k=1}^{N}\pi_k\xi(k) > 0$, by Lemma 2.1 again, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t W(s)ds \ge \frac{\sum_{k=1}^{N}\pi_k\xi(k)}{a+\int_{-\tau}^{0}d\mu(\theta)}.$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t W(s)ds = \frac{\sum_{k=1}^{N}\pi_k\xi(k)}{a+\int_{-\tau}^{0}d\mu(\theta)}.$$

The proof is completed.

Next, we consider the following comparison system of (1.2):

$$\begin{cases} dy_1(t) &=& y_1(t)\Big(r_1(\alpha(t)) - h_1 - a_{11}y_1(t) - \int_{-\tau_{11}}^{0} y_1(t+s)d\mu_{11}(s)\Big)dt + \sigma_1(\alpha(t))y_1(t)dB_1(t) \\ && + \int_{\mathbb{Z}}\gamma_1(u,\alpha(t))y_1(t)\widetilde{N}(dt,du), \\ dy_2(t) &=& y_2(t)\Big(r_2(\alpha(t)) - h_2 + a_{21}y_1(t) + \int_{-\tau_{21}}^{0} y_1(t+s)d\mu_{21}(s) - a_{22}y_2(t) \\ && - \int_{-\tau_{22}}^{0} y_2(t+s)d\mu_{22}(s)\Big)dt + \sigma_2(\alpha(t))y_2(t)dB_2(t) + \int_{\mathbb{Z}}\gamma_2(u,\alpha(t))y_2(t)\widetilde{N}(dt,du). \end{cases} \tag{3.5}$$

By Lemma 3.1, we derive from (3.5) that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t y_1(s)ds = \frac{\sum_{k=1}^{N}\pi_k\eta_1(k)}{a_{11}+\int_{-\tau_{11}}^{0}d\mu_{11}(\theta)} \triangleq \frac{\bar{\eta}_1}{A_{11}}.$$

Consider the following comparison system of the second equation of (3.5),

$$d\hat{y}_2(t) \;=\; \hat{y}_2(t^-)\left(r_2(\alpha(t)) - h_2 + a_{21}y_1(t^-) + \int_{-\tau_{21}}^{0} y_1(t+s)d\mu_{21}(s) - a_{22}\hat{y}_2(t^-)\right)dt$$

$$+\sigma_2(\alpha(t^-))\hat{y}_2(t^-)dB(t)+\int_{\mathbb{Z}}\gamma_2(u,\alpha(t^-))\hat{y}_2(t^-)\tilde{N}(dt,du).$$

Similar with the previous reasoning, using Lemma 3.1 again, we can derive that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \hat{y}_2(s)ds=\frac{A_{11}\bar{\eta}_2+A_{21}\bar{\eta}_1}{A_{11}A_{22}}\triangleq\frac{\Delta_2}{A_{11}A_{22}}.$$

By the comparison theory [1, 9], it is clear that $x_1 \le y_1$ and $y_2 \le \hat{y}_2$. Consequently, we have the following results.

Lemma 3.2. *For system (3.5), the following statements of Table 1 hold.*

Table 1. Dynamics of system (3.5).

Conditions	Species y_1	Species y_2
$\bar{\eta}_1 < 0$	Extinction	Extinction
$\bar{\eta}_1 > 0$ and $\Delta_2 < 0$	$\lim_{t\to\infty}\frac{1}{t}\int_0^t y_1(s)ds=\frac{\bar{\eta}_1}{A_{11}}$	Extinction
$\Delta_2 > 0$	$\lim_{t\to\infty}\frac{1}{t}\int_0^t y_1(s)ds=\frac{\bar{\eta}_1}{A_{11}}$	$\lim_{t\to\infty}\frac{1}{t}\int_0^t y_2(s)ds=\frac{\Delta_2}{A_{11}A_{22}}$

Now we give the main result on the extinction and persistence in the mean of the species of system (1.2).

Theorem 3.1. *For system (1.2), the following statements hold (see the Table 2).*

Table 2. Dynamics of system (1.2).

Cases	Conditions	Species x_1	Species x_2
i	$\bar{\eta}_1 < 0$	Extinction	Extinction
ii	$\bar{\eta}_1 > 0$ and $\Delta_2 < 0$	$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)ds=\frac{\bar{\eta}_1}{A_{11}}$	Extinction
iii	$\Delta_2 > 0$	$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)ds=\frac{\Delta_1}{\Delta}$	$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)ds=\frac{\Delta_2}{\Delta}$

Proof. For system (1.2), by using the Itô's formula to $\ln x_i(t), i=1,2$, we have

$$\begin{aligned} d\ln x_1(t) &= \Big(r_1(\alpha(t))-h_1-a_{11}x_1(t)-\int_{-\tau_{11}}^0 x_1(t+\theta)d\mu_{11}(\theta)-a_{12}x_2(t)-\int_{-\tau_{12}}^0 x_2(t+\theta)d\mu_{12}(\theta) \\ &\quad -\tfrac{1}{2}\sigma_1^2(\alpha(t))-\int_{\mathbb{Z}}[\gamma_1(u,\alpha(t))-\ln(1+\gamma_1(\alpha(t)))]\lambda(du)\Big)dt+\sigma_1(\alpha(t))dB_1(t) \\ &\quad +\int_{\mathbb{Z}}\ln(1+\gamma_1(u,\alpha(t)))\widetilde{N}(dt,du), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} d\ln x_2(t) &= \Big(r_2(\alpha(t)) - h_2 + a_{21}x_1(t) + \int_{-\tau_{21}}^{0} x_1(t+\theta)d\mu_{21}(\theta) - a_{22}x_2(t) - \int_{-\tau_{22}}^{0} x_2(t+\theta)d\mu_{22}(\theta) \\ &\quad -\tfrac{1}{2}\sigma_2^2(\alpha(t)) - \int_{\mathbb{Z}}[\gamma_2(u,\alpha(t)) - \ln(1+\gamma_2(\alpha(t)))]\lambda(du)\Big)dt + \sigma_2(\alpha(t))dB_2(t) \\ &\quad + \int_{\mathbb{Z}} \ln(1+\gamma_2(u,\alpha(t)))\widetilde{N}(dt,du). \end{aligned} \tag{3.7}$$

Similar with the proof of Lemma 3.1, we have

$$\begin{aligned} &\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{11}}^{0}\left[\int_{\theta}^{0} + \int_{0}^{t} + \int_{t}^{t+\theta}\right] x_1(s)dsd\mu_{11}(\theta) \\ =\ &\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{11}}^{0}\left[\int_{\theta}^{0} x_1(s)ds - \int_{t+\theta}^{t} x_1(s)ds\right]d\mu_{11}(\theta) + \lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{11}}^{0}\int_{0}^{t} x_1(s)dsd\mu_{11}(\theta) \\ =\ &\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{11}}^{0} d\mu_{11}(\theta)\int_{0}^{t} x_1(s)ds, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{12}}^{0}\left[\int_{\theta}^{0} + \int_{0}^{t} + \int_{t}^{t+\theta}\right] x_2(s)dsd\mu_{12}(\theta) \\ =\ &\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{12}}^{0}\left[\int_{\theta}^{0} x_2(s)ds - \int_{t+\theta}^{t} x_2(s)ds\right]d\mu_{12}(\theta) + \lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{12}}^{0}\int_{0}^{t} x_2(s)dsd\mu_{12}(\theta) \\ =\ &\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{0}^{t} x_2(s)ds. \end{aligned} \tag{3.9}$$

Integrating both sides of (3.6) from 0 to t, and combining (3.8) and (3.9), then

$$\begin{aligned} \frac{1}{t}\ln\frac{x_1(t)}{x_1(0)} &= \frac{1}{t}\int_0^t \eta_1(\alpha(s))ds - a_{11}\frac{1}{t}\int_0^t x_1(s)ds - \frac{1}{t}\int_{-\tau_{11}}^{0}\left[\int_{\theta}^{0} + \int_{0}^{t} + \int_{t}^{t+\theta}\right] x_1(s)dsd\mu_{11}(\theta) \\ &\quad -a_{12}\frac{1}{t}\int_0^t x_2(s)ds - \frac{1}{t}\int_{-\tau_{12}}^{0}\left[\int_{\theta}^{0} + \int_{0}^{t} + \int_{t}^{t+\theta}\right] x_2(s)dsd\mu_{12}(\theta) \\ &\quad +\frac{1}{t}\int_0^t \sigma_1(\alpha(s))dB_1(s) + \frac{1}{t}\int_0^t\int_{\mathbb{Z}} \ln(1+\gamma_1(u,\alpha(s)))\tilde{N}(ds,du). \\ &= \frac{1}{t}\int_0^t \eta_1(\alpha(s))ds - \frac{a_{11}}{t}\int_0^t x_1(s)ds - \frac{1}{t}\int_{-\tau_{11}}^{0} d\mu_{11}(\theta)\int_0^t x_1(s)ds - \frac{a_{12}}{t}\int_0^t x_2(s)ds \\ &\quad -\frac{1}{t}\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_0^t x_2(s)ds + \frac{1}{t}\int_0^t \sigma_1(\alpha(s))dB_1(s) + \frac{1}{t}\int_0^t\int_{\mathbb{Z}} \ln(1+\gamma_1(u,\alpha(s)))\tilde{N}(ds,du). \end{aligned} \tag{3.10}$$

By the same argumentation, we have

$$\begin{aligned} \frac{1}{t}\ln\frac{x_2(t)}{x_2(0)} &= \frac{1}{t}\int_0^t \eta_2(\alpha(s))ds + \frac{a_{21}}{t}\int_0^t x_1(s)ds + \frac{1}{t}\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\int_0^t x_1(s)ds - \frac{a_{22}}{t}\int_0^t x_2(s)ds \\ &\quad -\frac{1}{t}\int_{-\tau_{22}}^{0} d\mu_{22}(\theta)\int_0^t x_2(s)ds + \frac{1}{t}\int_0^t \sigma_2(\alpha(s))dB_2(s) + \frac{1}{t}\int_0^t\int_{\mathbb{Z}} \ln(1+\gamma_2(u,\alpha(s)))\tilde{N}(ds,du). \end{aligned} \tag{3.11}$$

By the ergodicity of Markovian chain, then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \eta_i(\alpha(s))ds = \sum_{k=1}^{N}\pi_k\eta_i(k), i = 1, 2.$$

Now we prove the conclusion of Theorem 3.1.

(i) If $\bar{\eta}_1 < 0$, then by Lemma 3.2, we have $\lim_{t\to\infty} x_1(t) = 0$. Since $\bar{\eta}_2 < 0$, it follows from (3.11) that $\lim_{t\to\infty} x_2(t) = 0$. Hence $\lim_{t\to\infty} x_i(t) = 0, i = 1, 2$.

(ii) If $\bar{\eta}_1 > 0$, then we derive from Lemma 3.2 that $\limsup_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds \le \frac{\bar{\eta}_1}{A_{11}}$. Substituting it into (3.11) and using Lemma 2.1 gives

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)ds \le \frac{\bar{\eta}_2 + A_{21}\frac{\bar{\eta}_1}{A_{11}}}{A_{22}} = \frac{A_{11}\bar{\eta}_2 + A_{21}\bar{\eta}_1}{A_{11}A_{22}} = \frac{\Delta_2}{A_{11}A_{22}}.$$

By the condition $\Delta_2 < 0$, then $\lim_{t\to\infty} x_2(t) = 0$.

From (3.11) again, by the comparison theorem [4, 14, 27], we have $\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds \ge \frac{\bar{\eta}_1}{A_{11}}$. Therefore,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds = \frac{\bar{\eta}_1}{A_{11}}.$$

(iii) If $\Delta_2 > 0$, it is clear that $\bar{\eta}_1 > 0$. We compute (3.10) $\times A_{21}$ + (3.11) $\times A_{11}$, then

$$\begin{aligned}
& A_{21}\frac{1}{t}\ln\frac{x_1(t)}{x_1(0)} + A_{11}\frac{1}{t}\ln\frac{x_2(t)}{x_2(0)} \\
= \ & \frac{A_{21}}{t}\int_0^t (\eta_1(\alpha(s))ds + \frac{A_{11}}{t}\int_0^t (\eta_2(\alpha(s))ds - \frac{(A_{12}A_{21} + A_{11}A_{22})}{t}\int_0^t x_2(s)ds \\
& + \frac{1}{t}[A_{21}M_1(t) + A_{11}M_2(t)],
\end{aligned}$$

where $M_i(t) = \int_0^t \sigma_i(\alpha(s))dB_i(s) + \int_0^t\int_{\mathbb{Z}} \ln(1 + \gamma_i(u, \alpha(s)))\tilde{N}(ds, du), i = 1, 2$. By the strong law of large numbers [9], we have

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = 0, i = 1, 2.$$

On the other hand, by (3.10) and (ii), we can derive that $\lim_{t\to\infty} \frac{\ln x_1(t)}{t} = 0$. By comparison method (using Lemma 2.1 again), then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)ds = \frac{A_{21}\bar{\eta}_1 + A_{11}\bar{\eta}_2}{A_{12}A_{21} + A_{11}A_{22}} = \frac{\Delta_2}{\Delta}. \tag{3.12}$$

Substituting (3.12) into (3.10) and using Lemma 2.1 again, we can obtain that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds = \frac{\bar{\eta}_1 - A_{12}\frac{\Delta_2}{\Delta}}{A_{11}} = \frac{\Delta_1}{\Delta}. \tag{3.13}$$

Consequently, we have $\lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds = \frac{\Delta_i}{\Delta}, i = 1, 2$. The proof is completed.

Remark 3.1. *Compared with (1.1), model (1.2) is more popular and contains (1.1) as its special case. Another difference between (1.1) and (1.2) is that the Lévy jump is considered in (1.2) while it is not considered in (1.1).*

Remark 3.2. *Theorem 3.1 implies that the regime switching, time delays and Lévy jumps will bring large influence to the dynamics of (1.2). By changing their values, the species of (1.2) may change from persistence in the mean to extinction, and vice versa, which is analyzed and verified well by numerical simulations in Section 6.*

4. Asymptotically stable in distribution of (1.2)

For simplicity, we denote the solution of (1.2) with initial data $x(\theta) = \phi(\theta), \alpha(\theta) = \varsigma$ by $x(t, \phi, \varsigma)$. Let $x(t, \phi, \varsigma)$ be the $R^2_+ \times \mathbb{S}-$valued stochastic Markovian process. Let $\mathbb{B} \subseteq R^2_+$ be a Borel measurable set and $\mathbb{D} \subseteq \mathbb{S}$, and we denote the transmission probability of the event $\{x(t, \phi, \varsigma) \in \mathbb{B} \times \mathbb{D}\}$ by $P(t, \phi, \varsigma, \mathbb{B} \times \mathbb{D})$, that is, $P(t, \phi, \varsigma, \mathbb{B} \times \mathbb{D}) = \sum_{u\in\mathbb{D}} \int_{\mathbb{B}} P(t, \phi, \varsigma, dx \times \{u\})$. Denote by $P(R^2_+, \mathbb{S})$ all the probability measures on $R^2_+ \times \mathbb{S}$, and for any two $P_1, P_2 \in P(R^2_+, \mathbb{S})$, we define the metric d_L as follows:

$$d_L(P_1, P_2) = \sup_{f\in L} \left| \sum_{k=1}^{N} f(x, k)P_1(dx, k) - \sum_{k=1}^{N} f(x, k)P_2(dx, k) \right|, \tag{4.1}$$

where $L = \{f : C([-\tau, 0], R^2_+) \times \mathbb{S} \to R : |f(x_1, k) - f(x_2, \tilde{k}) \leq |x_1 - x_2| + |k - \tilde{k}|, |f(\cdot, \cdot)| \leq 1\}$.

Definition 4.1 ([9])**.** *The process $x(t, \phi, \varsigma)$ is said to be asymptotically stable in distribution if there exists a probability measure $u(\cdot\times\cdot)$ on $R^2_+\times\mathbb{S}$ such that the transmission probability $P(t, \phi, \varsigma, dx\times\{k\})$ of $x(t, \phi, \varsigma)$ converges weakly to $u(dx\times\{k\})$ as $t \to \infty$ for every $(\phi, \varsigma) \in C([-\tau, 0], R^2_+)\times\mathbb{S}$. System (1.2) is said to be asymptotically stable in distribution if the solution $x(t, \phi, \varsigma)$ of (1.2) is asymptotically stable in distribution.*

Lemma 4.1 ([30])**.** *Let $f(t)$ be a nonnegative function defined on $[0, \infty)$ such that $f(t)$ is integrable on $[0, \infty)$ and is uniformly continuous on $[0, \infty)$, then $\lim_{t\to\infty} f(t) = 0$.*

For the need of discussion, we give the following technical assumption.

Assumption 3. $a_{ii} - \int_{-\tau_{ii}}^{0} d\mu_{ii}(\theta) - \int_{-\tau_{ji}}^{0} d\mu_{ji}(\theta) > a_{ji}, i, j = 1, 2, i \neq j.$

Assumption 3 means that under the effect of time delays, the intraspecific competition rate is still greater than the interaction rate between different species.

Theorem 4.1. *Let $x(t) = x(t, \phi, \varsigma)$ and $\widetilde{x}(t) = \widetilde{x}(t, \varphi, \varsigma)$ be two solutions of (1.2) with initial value $x(\theta) = \phi, \widetilde{x}(\theta) = \varphi$ and $\alpha(\theta) = \varsigma$, respectively. If Assumption 3 holds, then we have*

$$\lim_{t\to\infty} \mathbb{E}|x_i(t) - \widetilde{x}_i(t)| = 0, i = 1, 2.$$

Proof. Define Lyapunov function $V_i(t) = |\ln x_i(t) - \ln \widetilde{x}_i(t)|, t \geq 0, i = 1, 2$. We calculate the right differential of $V_1(t)$ along the solutions of (1.2), then

$$\begin{aligned} LV_1(t) \;\; = \;\; & sgn(x_1(t) - \widetilde{x}_1(t))\bigg(-a_{11}(x_1(t) - \widetilde{x}_1(t)) - \int_{-\tau_{11}}^{0} (x_1(t + \theta) - \widetilde{x}_1(t + \theta))d\mu_{11}(\theta) \\ & -a_{12}(x_2(t) - \widetilde{x}_2(t)) - \int_{-\tau_{12}}^{0} (x_2(t + \theta) - \widetilde{x}_2(t + \theta))d\mu_{12}(\theta)\bigg) \end{aligned}$$

$$
\begin{aligned}
&\leq -a_{11}|x_1(t)-\widetilde{x}_1(t)| + a_{12}|x_2(t)-\widetilde{x}_2(t)| + \int_{-\tau_{11}}^{0} |x_1(t+\theta)-\widetilde{x}_1(t+\theta)| d\mu_{11}(\theta) \\
&\quad + \int_{-\tau_{12}}^{0} |x_2(t+\theta)-\widetilde{x}_2(t+\theta)| d\mu_{12}(\theta).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
LV_2(t) &\leq -a_{22}|x_2(t)-\widetilde{x}_2(t)| + a_{21}|x_1(t)-\widetilde{x}_1(t)| + \int_{-\tau_{21}}^{0} |x_1(t+\theta)-\widetilde{x}_1(t+\theta)| d\mu_{21}(\theta) \\
&\quad + \int_{-\tau_{22}}^{0} |x_2(t+\theta)-\widetilde{x}_2(t+\theta)| d\mu_{22}(\theta).
\end{aligned}
$$

Define $V_3(t)$ as follows:

$$
\begin{aligned}
V_3(t) &= \int_{-\tau_{11}}^{0}\int_{t+\theta}^{t} |x_1(s)-\widetilde{x}_1(s)| ds d\mu_{11}(\theta) + \int_{-\tau_{12}}^{0}\int_{t+\theta}^{t} |x_2(s)-\widetilde{x}_2(s)| ds d\mu_{12}(\theta) \\
&\quad + \int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} |x_1(s)-\widetilde{x}_1(s)| ds d\mu_{21}(\theta) + \int_{-\tau_{22}}^{0}\int_{t+\theta}^{t} |x_2(s)-\widetilde{x}_2(s)| ds d\mu_{22}(\theta).
\end{aligned}
$$

Let $V(t) = V_1(t) + V_2(t) + V_3(t)$, then by Assumption 3, after computation, we have

$$
\begin{aligned}
LV(t) &\leq -\left(a_{11} - \int_{-\tau_{11}}^{0} d\mu_{11}(\theta) - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - a_{21}\right)|x_1(t)-\widetilde{x}_1(t)| \\
&\quad -\left(a_{22} - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - \int_{-\tau_{22}}^{0} d\mu_{22}(\theta) - a_{12}\right)|x_2(t)-\widetilde{x}_2(t)| \\
&< 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 \leq \mathbb{E}V(t) &\leq \mathbb{E}V(0) - \left(a_{11} - \int_{-\tau_{11}}^{0} d\mu_{11}(\theta) - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - a_{21}\right)\mathbb{E}|x_1(t)-\widetilde{x}_1(t)| \\
&\quad -\left(a_{22} - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - \int_{-\tau_{22}}^{0} d\mu_{22}(\theta) - a_{12}\right)\mathbb{E}|x_2(t)-\widetilde{x}_2(t)|.
\end{aligned}
$$

Hence,

$$
\mathbb{E}|x_i(t)-\widetilde{x}_i(t)| \leq \frac{\mathbb{E}V(0)}{a_{ii} - \int_{-\tau_{ii}}^{0} d\mu_{ii}(\theta) - \int_{-\tau_{ij}}^{0} d\mu_{ij}(\theta)} \in L^1[0,\infty)
$$

for $i, j = 1, 2, i \neq j$.

On the other hand, by (1.2) we have,

$$
x_i(t) = x_i(0) + \int_0^t f_i(x_i(s),\alpha(s))ds + \int_0^t g_i(x_i(s),\alpha(s))dB(s) + \int_0^t\int_{\mathbb{Z}} h_i(x_i(s),\alpha(s),u)\widetilde{N}(ds,du), \quad (4.2)
$$

where

$$f_i(x_i,\alpha(s)) = x_i\Big(r_i(\alpha(s)) - h_i - a_{ii}x_i - \int_{-\tau_{ii}}^{0} x_i(s+\theta)d\mu_{ii}(\theta) - a_{ij}x_j - \int_{-\tau_{ij}}^{0} x_j(s+\theta)d\mu_{ij}(\theta)\Big),$$

$$g_i(x_i(s),\alpha(s)) = x_i(s)\sigma_i(\alpha(s)), \qquad h_i(x_i(s),\alpha(s),u) = x_i(s)\gamma_i(\alpha(s),u), i,j=1,2, i\neq j.$$

Taking expectation from both sides of (4.2), then

$$\begin{aligned}\mathbb{E}x_i(t) &= x_i(0) + \int_0^t (r_i(\alpha(s)) - h_i)\mathbb{E}(x_i) - a_{ii}\mathbb{E}(x_i^2) - \int_{-\tau_{ii}}^{0} \mathbb{E}(x_i)\mathbb{E}(x_i(s+\theta))d\mu_{ii}(\theta) \\ &\quad - a_{ij}\mathbb{E}(x_i)\mathbb{E}(x_j) - \int_{-\tau_{ij}}^{0} \mathbb{E}(x_i)\mathbb{E}(x_j(s+\theta))d\mu_{ij}(\theta)ds.\end{aligned}$$

Consequently, $\mathbb{E}(x_i(t))$ is continuously differentiable with respect to t. Moreover,

$$\frac{d\mathbb{E}(x_i(t))}{dt} \leq (r_i(\alpha(s)) - h_i)\mathbb{E}(x_i(t)) \leq K.$$

That is to say, $\mathbb{E}(x_i(t))$ is uniformly continuous. An application of Lemma 4.1 gives $\lim_{t\to\infty}\mathbb{E}|x_i(t) - \widetilde{x_i}(t)| = 0$. This completes the proof.

Remark 4.1. *Theorem 4.1 will be applied later to prove the stability in distribution of (1.2). Assumption 3 is necessary in our proof. If without S-type time delays and Lévy jumps, Reference [14] implies that Assumption 3 is unnecessary and may be dropped.*

Lemma 4.2. *For any compact subset $\mathbb{B} \subseteq R_+^2$ and $(\phi,\varsigma) \in C([-\tau,0],\mathbb{B})\times\mathbb{S}$, the family of transmission probability of the solution $P(t,\phi,\varsigma,dx\times\{u\})$ is tight.*

Proof. For (4.2), by use of the Hölder inequality and the moment inequality of stochastic integrals, there exist $k = 1,2,...$ such that

$$\begin{aligned}\mathbb{E}\left[\sup_{(k-1)\delta\leq s\leq k\delta}|x_i|^p\right] &\leq 4^{p-1}\Bigg\{\mathbb{E}\left|\int_{(k-1)\delta}^{k\delta} f_i(x_i,\alpha(s))ds\right|^p + \mathbb{E}\sup_{(k-1)\delta\leq s\leq k\delta}\left|\int_{(k-1)\delta}^{k\delta} g_i(x_i,\alpha(s))ds\right|^p \\ &\quad + |x_i((k-1)\delta)|^p + \mathbb{E}\sup_{(k-1)\delta\leq s\leq k\delta}\left|\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{Z}} h_i(x_i,\alpha(s))\widetilde{N}(ds,du)\right|^p\Bigg\}. \qquad (4.3)\end{aligned}$$

By Lemma 2.2 (i.e., $\mathbb{E}|x_i(t)|^p \leq K(p)$), then

$$\begin{aligned}&\mathbb{E}\left|\int_{(k-1)\delta}^{k\delta} f_i(x_i,\alpha(s))ds\right|^p \\ \leq\ &\mathbb{E}\left[\delta^p \sup_{(k-1)\delta\leq s\leq k\delta} x_i^p\left(r_i(\alpha(s)) - h_i - a_{ii}x_i - \int_{-\tau_{ii}}^{0} x_i(t+\theta)d\mu_{ii}(\theta) - a_{ij}x_j - \int_{-\tau_{ij}}^{0} x_j(t+\theta)d\mu_{ij}(\theta)\right)^p\right] \\ \leq\ &2^{p-1}\delta^p\mathbb{E}\left|r_i(\alpha(s)) - h_i - a_{ii}x_i - \int_{-\tau_{ii}}^{0} x_i(t+\theta)d\mu_{ii}(\theta) - a_{ij}x_j - \int_{-\tau_{ij}}^{0} x_j(t+\theta)d\mu_{ij}(\theta)\right|^p \\ &+2^{p-1}\delta^p\mathbb{E}|x_i|^p\end{aligned}$$

$$\begin{aligned}
&\leq 2^{p-1}\delta^p 5^{p-1}\left\{\mathbb{E}|r_i(\alpha(s)) - h_i|^p + a_{ii}\mathbb{E}|x_i|^p + \mathbb{E}\left|\int_{-\tau_{ii}}^0 x_i(t+\theta)d\mu_{ii}(\theta)\right|^p + a_{ij}\mathbb{E}|x_j|^p\right.\\
&\quad \left. +\mathbb{E}\left|\int_{-\tau_{ij}}^0 x_j(t+\theta)d\mu_{ij}(\theta)\right|^p\right\} + 2^{p-1}\delta^p\mathbb{E}|x_i|^p\\
&\leq 10^{p-1}\delta^p\left[|r_i^u - h_i|^p + a_{ii}K_i(p) + \left|\int_{-\tau_{ii}}^0 d\mu_{ii}(\theta)\right|^p K_i(p) + a_{ij}K_j(p) + \left|\int_{-\tau_{ij}}^0 d\mu_{ij}(\theta)\right|^p K_j(p)\right]\\
&\quad +2^{p-1}\delta^p K_i(p)\\
&\triangleq M_1(p)\delta^p, \qquad (4.4)
\end{aligned}$$

where $M_1(p)$ is a constant. On the other hand, using the Burkholder-Davis-Gundy inequality [19], there exists a constant C_p such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{(k-1)\delta\leq s\leq k\delta}\left|\int_{(k-1)\delta}^{k\delta} g_1(x_1,\alpha(s))ds\right|^p\right] &\leq C_p\mathbb{E}\left[\int_{(k-1)\delta}^{k\delta}|x_i(s)\sigma_i(\alpha(s))|^2\,ds\right]^{\frac{p}{2}}\\
&\leq C_p\delta^{\frac{p}{2}}(\sigma^u)^p\mathbb{E}|x_i(s)|^p \triangleq M_2(p)\delta^{\frac{p}{2}}. \qquad (4.5)
\end{aligned}$$

As to the semimartingale part, by applying the Kunita's first inequality [19], there exists a constant $D(p)$ such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{(k-1)\delta\leq s\leq k\delta}\left|\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{Z}} h_i(x_i(s),\alpha(s),u)\widetilde{N}(ds,du))\right|^p\right]\\
\leq\ & D(p)\mathbb{E}\left[\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{Z}}|x_i(s)\gamma_i(\alpha(s),u)|^2\lambda(du)ds\right]^{\frac{p}{2}} + \mathbb{E}\left[\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{Z}}|x_i(s)\gamma_i(\alpha(s),u)|^p\lambda(du)ds\right]\\
\leq\ & D(p)\delta^{\frac{p}{2}}K^{\frac{p}{2}}K(p) + D(p)\delta K^{\frac{p}{2}}K(p). \qquad (4.6)
\end{aligned}$$

Therefore, for any $s\in[0,t]$, we have

$$\sup_{(\phi,\varsigma)\in(C([-\tau,0],R_+^2)\times\mathbb{S})}\mathbb{E}\left[\sup_{0\leq s\leq t}|x(\phi,\varsigma,s)|^p\right] < \infty.$$

That is, the probability measure set $P(t,\phi,\varsigma,dx\times\{u\})$ is tight for $(\phi,\varsigma)\in C([-\tau,0],R_+^2)\times\mathbb{S}$. This completes the proof.

Lemma 4.3. *Let Assumptions 1–3 hold, then for any $(\phi,\varsigma)\in C([-\tau,0],R_+^2)\times\mathbb{S}$, the transmission probability $P(t,\phi,\varsigma,\cdot\times\cdot : t\geq 0)$ of the solution of (1.2) is Cauchy in the space $P(R_+^2\times\mathbb{S})$ with the metric d_L defined as before.*

Proof. Let $\mathbb{B} = \{x\in C([-\tau,0],R_+^2) : \sigma\leq|x|\leq\varrho\}$, where ϱ is a sufficiently large positive number, and σ is a sufficiently small positive number. By the tightness of the transmission probability of solutions of (1.2), for any $\varepsilon>0$, we have $P(t,x,\alpha,\mathbb{B}^C\times\mathbb{S})\leq\varepsilon, t\geq 0$, where $\mathbb{B}^C = R_+^2/\mathbb{B}$.

For any $f\in L$, $t,s>0$, and initial value $(\phi,\varsigma)\in C([-\tau,0],R_+^2)\times\mathbb{S}$, by the characteristic of condition expectation, we have

$$d_L(P(t+s,\phi,\varsigma,\cdot\times\cdot),P(t,\phi,\varsigma,\cdot\times\cdot))$$

$$
\begin{aligned}
&= \sup_{f\in L}|\mathbb{E}[f(x(t+s;\phi,\varsigma),\alpha_\varsigma(t+s))]-\mathbb{E}[f(x(t;\phi,\varsigma),\alpha_\varsigma(t))]| \\
&= \sup_{f\in L}|\mathbb{E}[\mathbb{E}[f(x(t+s;\phi,\varsigma),\alpha_\varsigma(t+s))|_{\mathcal{F}_S}]]-\mathbb{E}[f(x(t;\phi,\varsigma),\alpha_\varsigma(t))]| \\
&= \sup_{f\in L}|\sum_{k=1}^{N}\int_{R_+^2}\mathbb{E}[f(x(t;\varphi,k),\alpha_k(t))P(s,\phi,\varsigma,d\varphi\times\{k\})-\mathbb{E}[f(x(t;\phi,\varsigma),\alpha_\varsigma(t))]| \\
&\le \sup_{f\in L}\sum_{k=1}^{N}\int_{R_+^2}|\mathbb{E}[f(x(t;\varphi,k),\alpha_k(t))-\mathbb{E}[f(x(t;\phi,\varsigma),\alpha_\varsigma(t))]|P(s,\phi,\varsigma,d\varphi\times\{k\}) \\
&\le \sup_{f\in L}\sum_{k=1}^{N}\int_{R_+^2/\mathbb{B}}|\mathbb{E}[f(x(t;\varphi,k),\alpha_k(t))-\mathbb{E}[f(x(t;\phi,\varsigma),\alpha_\varsigma(t))]|P(s,\phi,\varsigma,d\varphi\times\{k\}) \\
&\quad+\sup_{f\in L}\sum_{k=1}^{N}\int_{\mathbb{B}}|\mathbb{E}[f(x(t;\varphi,k),\alpha_k(t))-E[f(x(t;\phi,\varsigma),\alpha_\varsigma(t))]|P(s,\phi,\varsigma,d\varphi\times\{k\}), \qquad (4.7)
\end{aligned}
$$

where

$$
\begin{aligned}
&\sup_{f\in L}\sum_{k=1}^{N}\int_{R_+^2/\mathbb{B}}|\mathbb{E}[f(x(t;\varphi,k),\alpha_k(t))-\mathbb{E}[f(x(t;\phi,\varsigma),\alpha_\varsigma(t))]|P(s,\phi,\varsigma,d\varphi\times\{k\}) \\
\le\ & 2P(s,\phi,\varsigma,R_+^2/\mathbb{B}\times\mathbb{S})\le 2\varepsilon. \qquad (4.8)
\end{aligned}
$$

By the proof of Theorem 4.6 in [23] (4.22 of page 106), for any $\varepsilon>0$ and sufficiently large t, we have

$$
f(x(t;\varphi,k),\alpha_k(t))-f(x(t;\phi,\varsigma),\alpha_\varsigma(t))\le\varepsilon.
$$

Then

$$
\begin{aligned}
&\sup_{f\in L}\sum_{k=1}^{N}\int_{\mathbb{B}}|\mathbb{E}[f(x(t;\varphi,k),\alpha_k(t))-\mathbb{E}[f(x(t;\phi,\varsigma),\alpha_\varsigma(t))]|P(s,\phi,\varsigma,d\varphi\times\{k\}) \\
\le\ & \sup_{f\in L}\sum_{k=1}^{N}\int_{\mathbb{B}}\varepsilon P(s,\phi,\varsigma,d\varphi\times\{k\}) \\
\le\ & \varepsilon. \qquad (4.9)
\end{aligned}
$$

Hence, $d_L(P(t+s,\phi,\varsigma,\cdot\times\cdot),P(t,\phi,\varsigma,\cdot\times\cdot))\le 3\varepsilon$ for $s>0$ and sufficiently large t, that is, $P(t,\phi,\varsigma,\cdot\times\cdot: t\ge 0)$ is Cauchy in the space $P(C([-\tau,0],R_+^2)\times\mathbb{S})$. The proof is completed.

Theorem 4.2. *Under the conditions of Lemma 4.3, the solution process $x(t)$ of (1.2) is asymptotically stable in distribution.*

Proof. By Lemma 4.3, the transmission probability $P(t,\phi,\varsigma,\cdot\times\cdot: t\ge 0)$ is Cauchy in the space $P(R_+^2\times\mathbb{S})$ with the metric d_L. Hence, for any fixed $\varphi\in C([-\tau,0],R_+^2)$, $P(t,\widetilde{x},\varsigma,\cdot\times\cdot: t\ge 0)$ is Cauchy in $P(R_+^2\times\mathbb{S})$. Then there exists a probability measure $u(\cdot\times\cdot)$ such that

$$
\lim_{t\to\infty} d_L(P(t,\varphi,\varsigma,\cdot\times\cdot),u(\cdot\times\cdot))=0. \qquad (4.10)
$$

On the other hand, by Theorem 4.1, we have

$$\begin{aligned}
& d_L P(t,\varphi,\varsigma,\cdot\times\cdot), P(t,\phi,\varsigma,\cdot\times\cdot))) \\
= \; & \sup_{f\in L} |\mathbb{E}f(x(t,\varphi,\varsigma) - \mathbb{E}f(x(t,\phi,\varsigma)| \\
\leq \; & \sup_{f\in L} |\mathbb{E}(f(x(t,\varphi,\varsigma)) - f(x(t,\phi,\varsigma)))| \\
\leq \; & \mathbb{E}|x(t,\varphi,\varsigma) - x(t,\phi,\varsigma)| \\
\to \; & 0.
\end{aligned} \tag{4.11}$$

Therefore, by the triangle inequality, we can derive from (4.10) and (4.11) that

$$\begin{aligned}
& \lim_{t\to\infty} d_L(P(t,\phi,\varsigma,\cdot\times\cdot), u(\cdot\times\cdot)) \\
\leq \; & \lim_{t\to\infty} d_L(P(t,\varphi,\varsigma,\cdot\times\cdot), u(\cdot\times\cdot)) + d_L(P(t,\varphi,\varsigma,\cdot\times\cdot), P(t,\phi,\varsigma,\cdot\times\cdot)) \\
= \; & 0.
\end{aligned} \tag{4.12}$$

Since the weak convergence of probability measure is a metric concept, (4.12) shows that for any initial value $(\phi,\varsigma) \in C([-\tau,0], R_+^2) \times \mathbb{S}$, the probability measure $P(t,\phi,\varsigma,\cdot\times\cdot : t \geq 0)$ of the solution of (1.2) converges weakly to the probability measure $u(\cdot\times\cdot)$. By Definition 4.1, then the solution process $x(t)$ of (1.2) is asymptotically stable in distribution. This completes the proof.

Remark 4.2. *The asymptotic stability in distribution of species reveals the existence and uniqueness of an invariant probability measure, which is the basis of discussing the optimal harvesting effort in Section 5.*

5. Optimal harvesting effort

In this section, we consider the optimal harvesting effort (OHE) of (1.2). By [8, 27–29], the OHE problem is to find a constant $h^* = (h_1^*, h_2^*)^T$ such that both x_1 and x_2 survive, and $Y(h^*) = \lim_{t\to\infty} \mathbb{E}(h_i^* x_i(t))$ is maximum.

Theorem 5.1. *Let $\lambda = (\lambda_1, \lambda_2)^T = (A(A^{-1})^T + I)^{-1}\xi$, where I is the unit matrix, $\xi = (\bar{\xi}_1, \bar{\xi}_2)^T$.*

(1) If $A + (A^{-1})^T$ is positive semi-definite, and $\Delta_2|_{h_i=\lambda_i} > 0, \lambda_i \geq 0, i = 1,2$, then the OHE is $h^* = \lambda$, and the maximum of the sustainable yield is

$$Y(h^*) = \lambda^T A^{-1}(\xi - \lambda).$$

(2) If the conditions of (1) fail, then there exists no OHE.

Proof. We define $\Xi = \{h = (h_1,h_2)^T \in R_+^2, \Delta_2 > 0\}$, then for any $h \in \Xi$, by Theorem 3.1, we have $\lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(t)dt = \frac{\Delta_i}{\Delta}, i = 1,2$. Conversely, if the h^* of OHE exists, obviously $h^* \in \Xi$.

(1) By the given conditions, it is clear that $\lambda \in \Xi$, that is, the set Ξ is not empty. For any $h \in \Xi$, by Theorem 3.1, after computation, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t h^T x(s)ds = \sum_{i=1}^2 h_i \lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds = \sum_{i=1}^2 h_i \frac{\Delta_i}{\Delta} = h^T A^{-1}(\xi - h). \tag{5.1}$$

Theorem 4.1 implies that the distribution of solutions of (1.2) is asymptotically stable, and hence suppose the stationary probability density is ρ, then

$$Y(h) = \lim_{t\to\infty} \mathbb{E}(h_i x_i(t)) = \lim_{t\to\infty} \mathbb{E}(h^T x(t)) = \sum_{k=1}^{N} \int_{R_+^2} h^T x\rho(x,k)dx. \tag{5.2}$$

The asymptotical stability in distribution means that there exists a unique invariant measure u. In view of the one-to-one correspondence between the stationary probability density ρ and the invariant measure u (see, e.g.[31], page 105), we have

$$\sum_{k=1}^{N} \int_{R_+^2} h^T x\rho(x,k)dx = \sum_{k=1}^{N} \int_{R_+^2} h^T xu(dx,k). \tag{5.3}$$

By (5.1)–(5.3), we observe that

$$Y(h) = h^T A^{-1}(\xi - h). \tag{5.4}$$

Clearly, by computing the derivative of the variable h, we have $\frac{d(Y(h))}{dh} = A^{-1}\xi - (A^{-1} + (A^{-1})^T)h$. Denote the solution of $\frac{d(Y(h))}{dh} = 0$ by λ, then $\lambda = (AA^{-1} + I)\xi$. Further, after computation, we have

$$\frac{d\frac{d(Y(h))}{dh}}{dh^T} = -(A^{-1} + (A^{-1})^T),$$

which is negative semi-definite for all h by the given conditions, then we derive from Theorem 4.1.5 of [32] that λ is a global maximum point of $Y(h)$. That is, if the conditions $\Delta_2|_{h_i=\lambda_i} > 0, \lambda_i \geq 0 (i = 1,2)$ hold, then the OHE is $h^* = \lambda$, and $Y(h^*) = \lambda^T A^{-1}(\xi - \lambda)$.

(2) Firstly we prove that there is no OHE if $\Delta_2|_{h_i=\lambda_i, i=1,2} > 0$, $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, but $A^{-1}+(A^{-1})^T$ is not positive semi-definite. By above conditions, clearly the set Ξ is not empty. Let $(\iota_{ij})_{2\times 2} = A^{-1} + (A^{-1})^T$, then $\iota_{11} = \frac{2A_{22}}{\Delta} > 0$. It implies that $A^{-1} + (A^{-1})^T$ is not negative semi-definite. Noting that $A^{-1} + (A^{-1})^T$ is not positive semi-definite, hence $A^{-1} + (A^{-1})^T$ is indefinite. The extreme value theory [32] shows that $Y(h)$ has no extreme points. Therefore, there is no OHE.

Secondly, under the conditions $\Delta_2|_{h_i=\lambda_i, i=1,2} < 0$, or $\lambda_1 \leq 0$, or $\lambda_2 \leq 0$, we prove there is also no OHE. Otherwise, suppose the OHE is $\tilde{h}^* = (\tilde{h}_1^*, \tilde{h}_2^*)^T$, then $\tilde{h}^* \in \Xi$. Thus $\Delta_2|_{h_i^*=\tilde{h}_i^*} > 0, \tilde{h}_i^* \geq 0, i = 1,2$. That means $\tilde{h}_i^*$ is also the solution of $\frac{d(Y(h))}{dh} = 0$, which contradicts with the uniqueness of the solution. This completes the proof.

Remark 5.1. *The existence and uniqueness of an invariant probability measure plays a key role to derive (5.4). With the invariant measure, by using the extremum theory, we find the maximum of $Y(h)$, which is very popular and can be applied to resolve some similar problems.*

6. Examples and simulations

In this section, by numerical analysis, we give some examples and apply the Milstein method [33] to illustrate our theoretical results, and explore the effects of regime switching, time delays and Lévy jumps on the system dynamics.

For simplicity, we assume that the continuous-time discrete state Markovian chain $\alpha(t)$ takes value in the space $\mathbb{S} = \{1, 2\}$, then system (1.2) reduces to the following subsystem (6.1) and (6.2):

$$\begin{cases} dx_1(t) &= x_1(t^-)\Big(r_1(1) - h_1 - a_{11}x_1(t^-) - \int_{-\tau_{11}}^{0} x_1(t+\theta)d\mu_{11}(\theta) - a_{12}x_2(t^-) \\ &\quad - \int_{-\tau_{12}}^{0} x_2(t+\theta)d\mu_{12}(\theta)\Big)dt + \sigma_1(1)x_1(t^-)dB_1(t) + \int_{\mathbb{Z}} \gamma_1(u,1)x_1(t^-)\widetilde{N}(dt,du), \\ dx_2(t) &= x_2(t^-)\Big(r_2(1) - h_2 + a_{21}x_1(t^-) + \int_{-\tau_{21}}^{0} x_1(t+\theta)d\mu_{21}(\theta) - a_{22}x_2(t^-) \\ &\quad - \int_{-\tau_{22}}^{0} x_2(t+\theta)d\mu_{22}(\theta)\Big)dt + \sigma_2(1)x_2(t^-)dB_2(t) + \int_{\mathbb{Z}} \gamma_2(u,1)x_2(t^-)\widetilde{N}(dt,du), \end{cases} \tag{6.1}$$

and

$$\begin{cases} dx_1(t) &= x_1(t^-)\Big(r_1(2) - h_1 - a_{11}x_1(t^-) - \int_{-\tau_{11}}^{0} x_1(t+\theta)d\mu_{11}(\theta) - a_{12}x_2(t^-) \\ &\quad - \int_{-\tau_{12}}^{0} x_2(t+\theta)d\mu_{12}(\theta)\Big)dt + \sigma_1(2)x_1(t^-)dB_1(t) + \int_{\mathbb{Z}} \gamma_1(u,2)x_1(t^-)\widetilde{N}(dt,du), \\ dx_2(t) &= x_2(t^-)\Big(r_2(2) - h_2 + a_{21}x_1(t^-) + \int_{-\tau_{21}}^{0} x_1(t+\theta)d\mu_{21}(\theta) - a_{22}x_2(t^-) \\ &\quad - \int_{-\tau_{22}}^{0} x_2(t+\theta)d\mu_{22}(\theta)\Big)dt + \sigma_2(2)x_2(t^-)dB_2(t) + \int_{\mathbb{Z}} \gamma_2(u,2)x_2(t^-)\widetilde{N}(dt,du). \end{cases} \tag{6.2}$$

For (6.1) and (6.2), unless otherwise stated, we always take $a_{11} = 0.8, a_{12} = 0.6, a_{21} = 0.4, a_{22} = 0.7, h_i = 0, \sigma_i^2 = 0.2, \int_{-\tau_{ii}}^{0} d\mu_{ii}(\theta) = 0.3, \int_{-\tau_{ij}}^{0} d\mu_{ij}(\theta) = 0.2, \gamma_i(1) = \gamma_i(2) = 0.4, i = 1, 2, i \neq j, \mathbb{Z} = [0, \infty)$, and $\lambda(\mathbb{Z}) = 1$. Then $A_{11} = 1.1, A_{12} = 0.8, -A_{21} = -0.6, A_{22} = 1, \Delta = 1.58$.

For (6.1), let $r_1(1) = 0.06, r_2(1) = -0.02$. It is easy to verify that Assumptions 1–3 hold, and $\eta_1(1) = -0.1035 < 0$. Theorem 3.1 implies both x_1 and x_2 are extinct (see Figure 1 (a)).

For (6.2), let $r_1(2) = 0.7, r_2(2) = -0.05$. Similarly Assumptions 1–3 hold, and $\eta_1(2) = 0.5365 > 0, \eta_2(2) = -0.2135, \Delta_1 = 0.7073, \Delta_2 = 0.087 > 0$.

By Theorem 3.1 again, we can obtain that both x_1 and x_2 are persistent in the mean, and

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t x_1(s)ds = \frac{\Delta_1}{\Delta} = 0.4477, \qquad \lim_{t\to\infty} \frac{1}{t} \int_0^t x_2(s)ds = \frac{\Delta_2}{\Delta} = 0.0551.$$

Figure 1 (b) reveals it clearly.

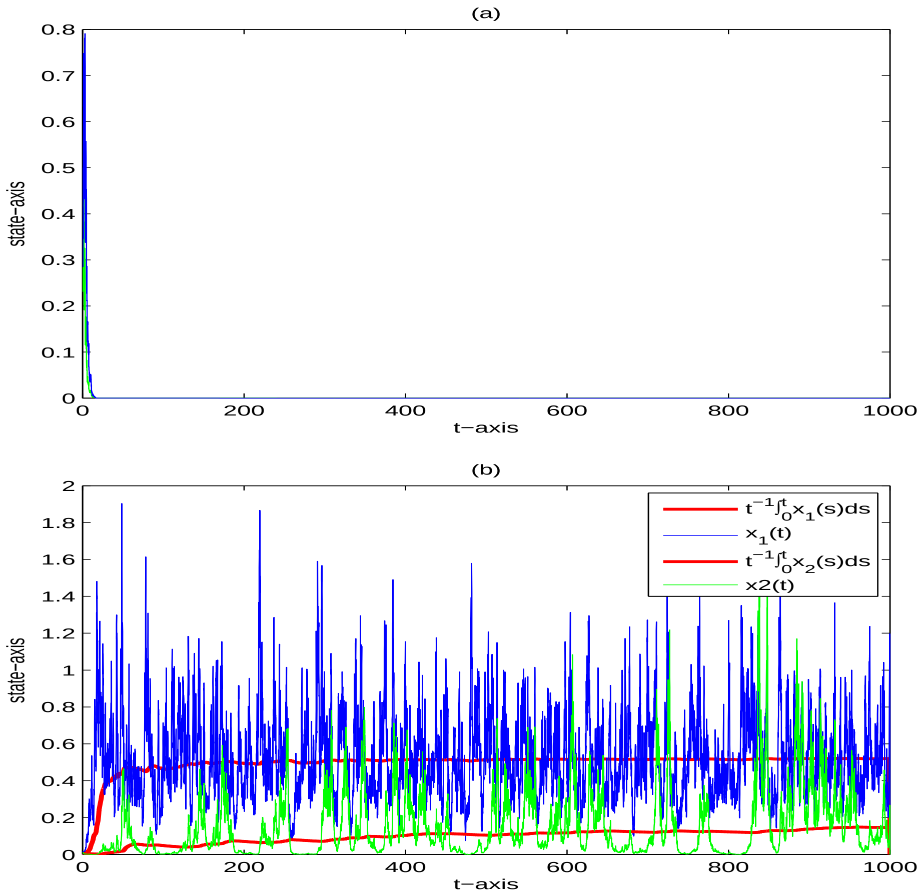


Figure 1. Dynamics of system (6.1) and (6.2), respectively. (a) Dynamics of (6.1) with $r_1(1) = 0.06, r_2(1) = -0.02$, where x_1 and x_2 are extinct. (b) Dynamics of (6.2) with $r_1(2) = 0.7, r_2(2) = -0.05$, where x_1 and x_2 are persistent in the mean with $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds = 0.4477, \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)ds = 0.0551$.

Next we will study the effect of regime switching, time delays and Lévy jumps on the extinction and persistence in the mean of all species, respectively. We discuss in three cases.

- Case (i) The effect of switching π.
  (1) Let the stationary distribution $\pi = (0.1, 0.9)$. By computation, then

  $$\bar{\eta}_1 = 0.1 * (-0.1035) + 0.9 * 0.5365 = 0.4725, \bar{\eta}_2 = 0.1 * (-0.1835) + 0.9 * (-0.2135) = -0.2105,$$

  and
  $$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds = \frac{\Delta_1}{\Delta} = 0.4056, \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)ds = \frac{\Delta_2}{\Delta} = 0.0328.$$
  That is, both $x_1(t)$ and $x_2(t)$ are persistent in the mean. We call it the "persistent case". Figure 2 (a) is the long-term behaviours and Figure 2 (b) is the probability densities of $x_1(t)$ and $x_2(t)$, respectively. Figure 3 gives the time series and histogram of Markov chain $\alpha(t)$ with stationary distribution $\pi = (0.1, 0.9)$.

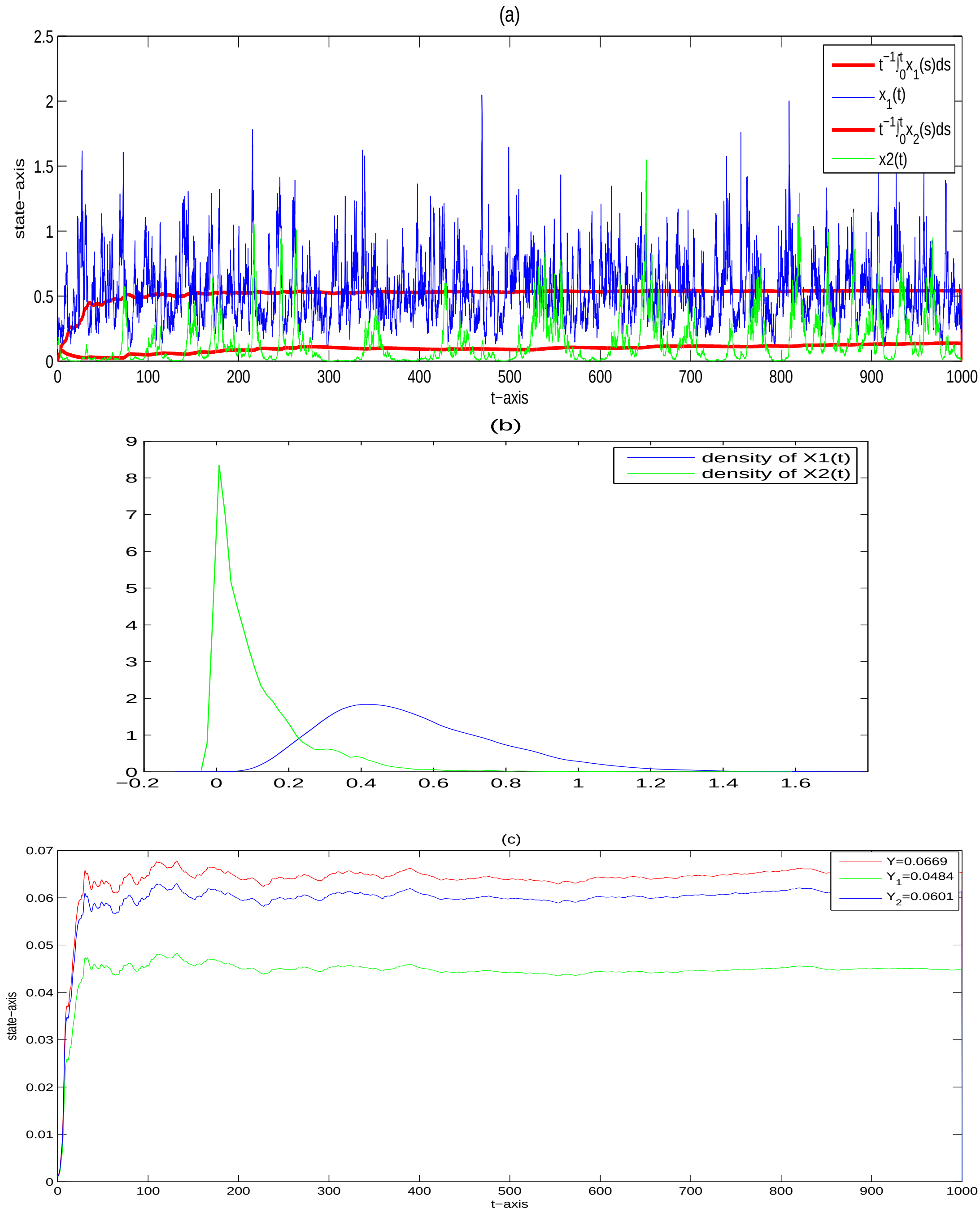


Figure 2. The "persistent case" of hybrid system of (6.1) and (6.2) with $\pi = (0.1, 0.9)$. (a) Both $x_1(t)$ and $x_2(t)$ are persistent in the mean with $\lim_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds = 0.4056$, $\lim_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)ds = 0.0328$. (b) The probability density of $x_1(t)$ and $x_2(t)$ respectively. (c) The OHE $Y(h^*) = 0.0669$ with $\lambda = (0.3258, 0.0532)^T$, depicted in red line. The green line represents the yield $Y_1 = 0.0484$ with $\lambda_1 = (0.1558, 0.0632)^T$, and the blue line represents the yield $Y_2 = 0.0601$ with $\lambda_2 = (0.4258, 0.0332)^T$. By comparison, the maximum of sustainable yield is $Y(h^*) = 0.0669$.

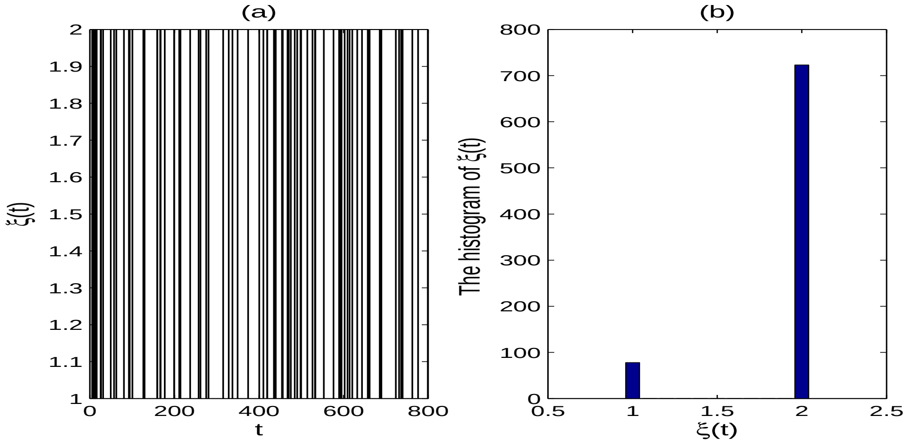


Figure 3. (a) Time series of the Markovian chain $\xi(t)$ switching between states 1 and 2. (b) The histogram of Markovian chain $\xi(t)$.

For the persistent case, Theorem 5.1 shows that there exists OHE, and

$$h^* = \lambda = (A(A^{-1})^T + I)^{-1}\overline{\eta} = (0.3258, 0.0532)^T.$$

The maximum of the sustainable yield is

$$Y(h^*) = \lambda^T A^{-1}(\xi - \lambda) = 0.0669.$$

By numerical simulation, we get Figure 2 (c). In Figure 2 (c), the red line represents the yield $Y = 0.0669$ with $\lambda = (0.3258, 0.0532)^T$, the green line represents the yield $Y_1 = 0.0484$ with $\lambda_1 = (0.1558, 0.0632)^T$, the blue line represents the yield $Y_2 = 0.0601$ with $\lambda_2 = (0.4258, 0.0332)^T$, respectively. Obviously, the maximum of sustainable yield is $Y(h^*) = 0.0669$. Figure 2 (c) verify it well. For the interpretation of the references to color, readers are referred to the web version of the article.

(2) Let $\pi = (0.9, 0.1)$. Then $\bar{\eta}_1 = 0.9 * (-0.1035) + 0.1 * 0.5365 = -0.0395 < 0$. Theorem 3.1 implies that both $x_1(t)$ and $x_2(t)$ are to be extinct. That is, the switching π leads to the extinction of $x_1(t)$ and $x_2(t)$ (see Figure 4 (a)).

(3) Let $\pi = (0.4, 0.6)$. Then by computing, we have

$$\bar{\eta}_1 = 0.2805 > 0 \qquad \text{and} \qquad \Delta_2 = -0.0534 < 0.$$

By Theorem 3.1 again, then x_2 is extinct and x_1 is persistent in the mean, and

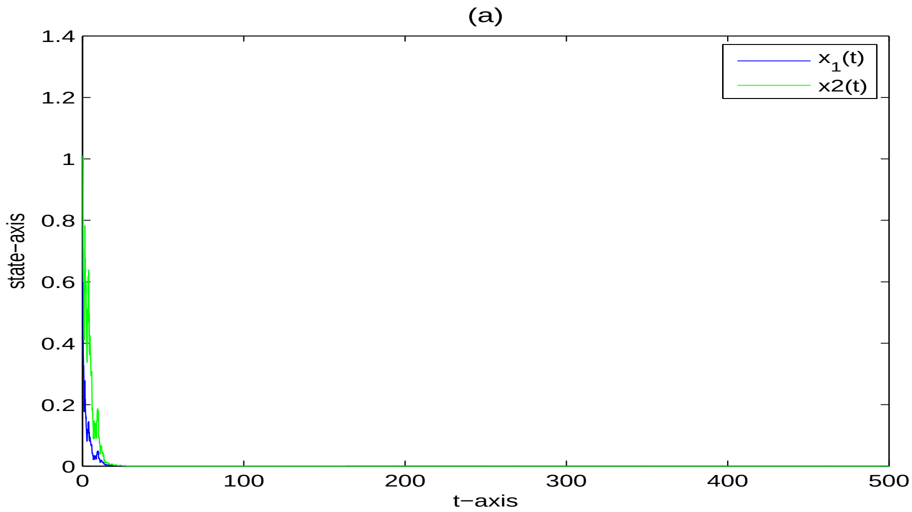
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)ds = \frac{\bar{\eta}_1}{A_{11}} = 0.255.$$

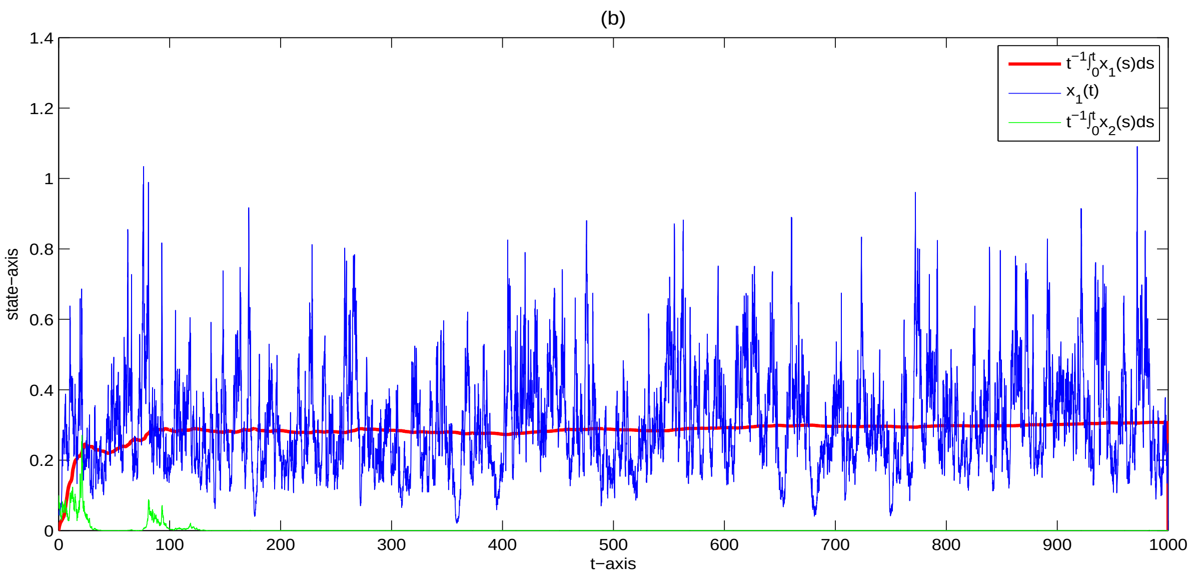
It shows the switching π leads to the extinction of $x_2(t)$ and the different persistence in the mean of $x_1(t)$ (see Figure 4 (b)).

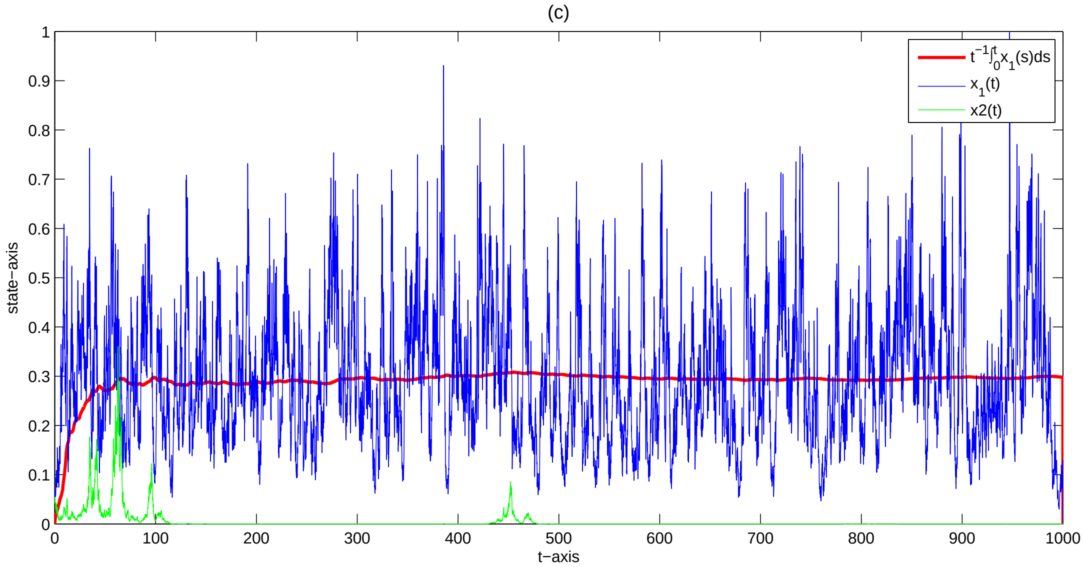
- Case (ii) The effect of time-delays.
  For the persistent case, if we take $\int_{-\tau_{ii}}^{0} d\mu_{ii}(\theta) = 0.87$, $\int_{-\tau_{ij}}^{0} d\mu_{ij}(\theta) = 0.3$, other parameters are same as before, then $\bar{\eta}_1 = 0.4725 > 0, \Delta_2 = -0.0208 < 0$, which leads to the extinction of $x_2(t)$ and

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t x_1(s)ds = 0.2829.$$

  That is, time delays destroy the persistent case, see Figure 4 (c).
- Case (iii)The effect of Lévy jumps.
  For the persistent case, if we take $\gamma_i(1) = \gamma_i(2) = 1, i = 1, 2, i \neq j$, then $\overline{\eta}_1 = -0.4189 < 0$. By Theorem 3.1, both x_1 and x_2 go to be extinct. It implies that too large Lévy jumps leads to the extinction of $x_1(t)$ and $x_2(t)$, see Figure 4 (d).







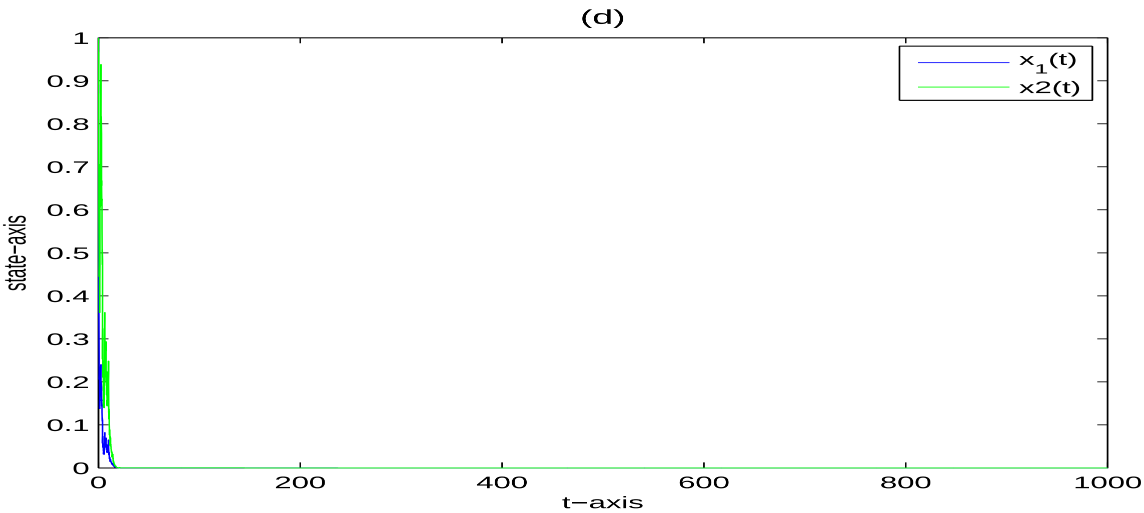


Figure 4. The effects of regime switching, time delays and Lévy jumps on the "persistent case", respectively. (a) The effect of regime switching with $\pi = (0.9, 0.1)$, where $x_1(t)$ and $x_2(t)$ are extinct. (b) The effect of regime switching with $\pi = (0.4, 0.6)$, where x_1 is persistent in the mean with $\lim_{t\to\infty} \frac{1}{t} \int_0^t x_1(s)ds = 0.255$ and x_2 is extinct. (c) The effect of time-delays with $\int_{-\tau_{ii}}^0 d\mu_{ii}(\theta) = 0.87, \int_{-\tau_{ij}}^0 d\mu_{ij}(\theta) = 0.3, i = 1, 2, i \neq j$, where $\lim_{t\to\infty} \frac{1}{t} \int_0^t x_1(s)ds = 0.2829$ and $x_2(t)$ is extinct. (d) The effect of Lévy jumps with $\gamma_i(1) = \gamma_i(2) = 1$, where x_1 and x_2 are extinct.

7. Conclusions and discussions

In this paper, we study the dynamics of a stochastic predator-prey system with S-type distributed time delays, regime switching and Lévy jumps. Theorem 3.1 gives the sufficient conditions assuring the extinction and persistence in the mean of each species. Theorem 4.2 shows that (1.2) is asymptotically stable in distribution. Theorem 5.1 gives the optimal harvesting effort. Finally, some examples are given and the effects of regime switching, distributed time delays and Lévy jumps are discussed by numerical analysis.

Figure 4 (a) and (b) imply that different regime switching may lead to very different long-term behaviours of species. Figure 2 (c) shows that the OHE is based on the suitable regime switching. Figure 4 (c) and (d) show that too large delays or Lévy jumps will destroy the persistent case of (1.2), respectively. All these show that regime switching, distributed time delays and Lévy jumps play key role in the dynamics of (1.2).

Further, (1.2) is very popular and contains many researched model as its special cases. Firstly, (1.1) is contained in (1.2), and hence, one can obtain the sufficient conditions of the extinction and persistence in the mean for species of (1.1), that is, Theorems 1 and 2 of Reference [14] are contained in our results. Secondly, if $\alpha(t) = 1, h_i = 0, \gamma_i = 0 (i = 1, 2)$, then we get the model studied by Wang et.al. [18]. Our result coincides with Theorem 2.2 of [18]. Thirdly, if $\mu_{ii}(\theta)$ are constant on $[-\tau, 0], a_{ij} = 0, i \neq j$, and

$$\mu_{12}(\theta) = \begin{cases} b_{12}, & -\tau_1 < \theta \leq 0, \\ 0, & -\tau_{12} \leq \theta \leq -\tau_1, \end{cases} \qquad \mu_{21}(\theta) = \begin{cases} b_{21}, & -\tau_2 < \theta \leq 0, \\ 0, & -\tau_{21} \leq \theta \leq -\tau_2, \end{cases}$$

then we get the discrete time delays model proposed in Reference [5, 8]. Similarly we can obtain Theorem 2.2 in [8] and Lemma 3 in [5]. As Liu et. al. stated in [14], the growth of the ith species at time t is often affected by the abundance of the jth species on the interval $[t - \tau_i, t]$, rather than only on the time $t - \tau_i$. Hence the S-type distributed delays can fit with some real biological systems better. In above sense, we improve and generalize the obtained conclusions of [5, 8, 14, 18].

However, the switching does not appear in all parameters and the control inputs are all constants, if they are dependent on time t, then the persistence in the mean and the optimal harvesting strategy can not be established and is still unknown. On the other hand, for predator-prey system, if the predator is provided with additional food [34], or the fear of prey induced by predator appears [35], what will happen is very interesting. All these will be our research work in the future.

Acknowledgements

The author is very grateful to the anonymous referees and the editor for their helpful and suggestive comments. This research was supported by the National Natural Science Foundation of China (Nos. 11861027 and 11871475).

Conflict of interest

The authors declare that they have no competing interests in this paper.

References

1. X. Mao, *Stochastic differential equations and applications*, England: Horwood Publishing Limited, 2007.

2. N. Ikeda, S. Watanable, *Stochastic differential equations and diffusion processes*, New York: North-Holland, 1989.
3. X. Mao, G. Marion, E. Renshaw, Environmental Brownian noise suppresses explosions in populations dynamics, *Stoch. Proc. Appl.*, **97** (2002), 95–110. doi: 10.1016/S0304-4149(01)00126-0.
4. M. Liu, K. Wang, Q. Wu, Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle, *Bull. Math. Biol.*, **73** (2011), 1969–2012. doi: 10.1007/s11538-010-9569-5.
5. M. Liu, H. Qiu, K. Wang, A remark on a stochastic predator-prey system with time delays, *Appl. Math. Lett.*, **26** (2013), 318–323. doi: 10.1016/j.aml.2012.08.015.
6. R. M. May, *Stability and complexity in model ecosystems*, Princeton: Princeton University Press, 1973.
7. R. Z. Khas'minskii, *Stochastic stability of differential equations*, Netherlands: Sijthoff Noordhoff, Alphen aan den Rijn, 1980.
8. M. Liu, Optimal harvesting policy of a stochastic predator-prey model with time delay, *Appl. Math. Lett.*, **48** (2015), 102–108. doi: 10.1016/j.aml.2014.10.007.
9. X. Mao, C. Yuan, *Stochastic differential equations with markovian switching*, London: Imperial College Press, 2006.
10. Q. Luo, X. Mao, Stochastic population dynamics under regime switching, *J. Math. Anal. Appl.*, **334** (2007), 69–84.
11. W. Kong, Y. Shao, Long-time behaviours of a stochastic predator-prey system with Holling-type II functional response and regime switching, *J. Math.*, **2021** (2021), 6672030.
12. J. Bao, J. Shao, Permanence and extinction of regime-switching predator-prey models, *SIAM J. Math. Anal.*, **48** (2016), 725–739. doi: 10.1137/15M1024512.
13. H. Chen, P. Shi, C. Lim, Stability analysis for neutral stochastic delay systems with Markovian switching, *Syst. Contr. Lett.*, **110** (2017), 38–48. doi: 10.1016/j.sysconle.2017.10.008.
14. M. Liu, X. He, J. Yu, Dynamics of a stochastic regime-switching predator-prey model with harvesting and distributed delays, *Nonlinear Anal.-Hybri.*, **28** (2018), 87–104. doi: 10.1016/j.nahs.2017.10.004.
15. Y. Kuang, *Delay differential equations: With applications in population dynamics*, Boston: Academic Press, 1993.
16. S. Wang, G. Hu, T. Wei, L. Wang, Stability in distribution of a stochastic predator-prey system with S-type distributed time delays, *Physica A*, **505** (2018), 919–930. doi: 10.1016/j.physa.2018.03.078.
17. L. Wang, D. Xu, Global asymptotic stability of bidirectional associative memory neural networks with S-type distributed delays, *Int. J. Syst. Sci.*, **33** (2002), 869–877. doi: 10.1080/00207720210161777.
18. L. Wang, R. Zhang, Y. Wang, Global exponential stability of reaction-diffusion cellular neural networks with S-type distributed time delays, *Nonlinear Anal.*, **10** (2009), 1101–1113.
19. D. Applebaum, *Lévy processes and stochastic Calculus*, 2nd ed., Lodon: Cambridge University Press, 2009.
20. M. Liu, K. Wang, Stochastic Lotka-Volterra systems with Lévy noise, *J. Math. Anal. Appl.*, **410** (2014), 750–763. doi: 10.1016/j.jmaa.2013.07.078.

21. X. Zhang, W. Li, M. Liu, K. Wang, Dynamics of a stochastic Holling II one-predator two-prey system with jumps, *Physica A*, **421** (2015), 571–582. doi: 10.1016/j.physa.2014.11.060.
22. S. Wang, L. Wang, T. Wei, Permanence and asymptotic behaviors of stochastic predator-prey system with Markovian switching and Lévy noise, *Physica A*, **495** (2018), 294–311. doi: 10.1016/j.physa.2017.12.088.
23. Y. Zhao, S. Yuan, Stability in distribution of a stochastic hybrid competitive Lotka-Volterra model with Lévy jumps, *Chaos, Solit. Fract.*, **85** (2016), 98–109. doi: 10.1016/j.chaos.2016.01.015.
24. Y. Zhang, Q. Zhang, Dynamic behavior in a delayed stage-structured population model with stochastic fluctuation and harvesting, *Nonlinear Dynam.*, **66** (2011), 231–245. doi: 10.1007/s11071-010-9923-z.
25. J. Yu, M. Liu, Stationary distribution and ergodicity of a stochastic food-chain model with Lévy jumps, *Physica A*, **482** (2017), 14–28. doi: 10.1016/j.physa.2017.04.067.
26. G. Hu, K. Wang, Stability in distribution of neutral stochastic functional differential equations with Markovian switching, *J. Math. Anal. Appl.*, **385** (2012), 757–769. doi: 10.1016/j.jmaa.2011.07.002.
27. Y. Shao, Y. Chen, B. X. Dai, Dynamical analysis and optimal harvesting of a stochastic three-species cooperative system with delays and Lévy jumps, *Adv. Differ. Equ.*, **2018** (2018), 423. doi: 10.1186/s13662-018-1874-6.
28. G. D. Liu, X. Z. Meng, Optimal harvesting strategy for a stochastic mutualism system in a polluted environment with regime switching, *Physica A*, **536** (2019), 120893. doi: 10.1016/j.physa.2019.04.129.
29. M. Liu, C. Z. Bai, Optimal harvesting of a stochastic mutualism model with regime-switching, *Appl. Math. Comput.*, **373** (2020), 125040. doi: 10.1016/j.amc.2020.125040.
30. I. Barbalat, Systems d"equations differentielles d"oscillations non lineaires, *Rev. Roum. Math. Pures Appl.*, **4** (1959), 267–270.
31. D. Prato, J. Zabczyk, *Ergodicity for infinite dimensional systems*, Cambridge: Cambridge University Press, 1996. doi: 10.1017/CBO9780511662829.
32. M. S. Bazaraa, C. M. Shetty, *Nonlinear Programming*, Wiley, New York: Academic Press, 1979.
33. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.,* **43** (2001), 525–546. doi: 10.1137/S0036144500378302.
34. D. Amartya, G. P. Samanta, Stochastic prey-predator model with additional food for predator, *Physica A,* **512** (2018), 121–141. doi: 10.1016/j.physa.2018.08.138.
35. D. Amartya, G. P. Samanta, Modelling the fear effect on a stochastic prey-predator system with additional food for predator, *J. Phys. A-Math. Theor.,* **51** (2018), 465601. doi: 10.1088/1751-8121/aae4c6.