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Research article

Cospectral graphs for the normalized Laplacian

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Abstract: Let $G(a_1, a_2, \ldots, a_k)$ be a simple graph with vertex set $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ and edge set $E(G) = \{(u, v)|u \in V_i, v \in V_{i+1}, i = 1, 2, \ldots, k-1\}$, where $|V_i| = a_i > 0$ for $1 \leq i \leq k$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Given two positive integers k and n, and $k-2$ positive rational numbers $t_2, t_3, \ldots, t_{\lceil k/2 \rceil}$ and $t'_2, t'_3, \ldots, t'_{\lfloor k/2 \rfloor}$, let $\Upsilon(n; k)_t^{t'} = \{G(a_1, a_2, \ldots, a_k)| \sum_{i=1}^{k} a_i = n, a_{2i-1} = t_i a_1, a_{2j} = t'_j a_2, i = 2, 3, \ldots, \lceil k/2 \rceil, j = 2, 3, \ldots, \lfloor k/2 \rfloor; t = (t_2, t_3, \ldots, t_{\lceil k/2 \rceil}), t' = (t'_2, t'_3, \ldots, t'_{\lfloor k/2 \rfloor}); a_s \in N, 1 \leq s \leq k\}$, where N is the set of positive integers. In this paper, we prove that all graphs in $\Upsilon(n; k)_t^{t'}$ are cospectral with respect to the normalized Laplacian if it is not an empty set.

Keywords: normalized Laplacian; cospectral; tridiagonal matrix; normalized Laplacian characteristic polynomial
Mathematics Subject Classification: 05C31, 05C50

1. Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge $E(G)$. Let $d(v)$ denote the degree of vertex v of G and $D(G) = diag(d(v_1), d(v_2), \ldots, d(v_n))$ the diagonal matrix of vertex degrees. The adjacency matrix of G is defined to be the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The normalized Laplacian matrix is defined to be $\mathcal{L}(G) = D(G)^{-1/2}(D(G) - A(G))D(G)^{-1/2}$ (with the convention that if the degree of v is 0 then $d(v)^{-1/2} = 0$) by Chung [7]. So its entries are defined by

$$\mathcal{L}_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -1/\sqrt{d(v_i)d(v_j)}, & \text{if } (v_i, v_j) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The normalized Laplacian characteristic polynomial of a graph is the characteristic polynomial of its normalized Laplacian matrix. Denote by $\Phi(B; x) = \det(xI - B)$ the characteristic polynomial of the square matrix B. Hence $\Phi(\mathcal{L}(G); x) = \det(xI - \mathcal{L}(G))$ is the normalized Laplacian characteristic polynomial of a graph G.

Spectral graph theory examines relationships between the structure of a graph and the eigenvalues (or spectrum) of a matrix associated with that graph. Different matrices are able to give different information, but all the common matrices have limitations. This is because there are graphs which have the same spectrum for a certain matrix but different structure–such graphs are called cospectral with respect to that matrix [4].

Cospectral graphs for the adjacency matrix (see for example [8, 10–13]) and the Laplacian matrix (see for example, [12, 17, 19]) have been studied extensively, particularly for graphs with few vertices. But little is also known about cospectral graphs with respect to the normalized Laplacian since the normalized Laplacian is a rather new tool which has rather recently (mid 1990s) been popularized by Chung [7]. One of the original motivations for defining the normalized Laplacian was to be able to deal more naturally with non-regular graphs. In some situations the normalized Laplacian is a more natural tool that works better than the adjacency matrix or Laplacian matrix. In particular, when dealing with random walks, the normalized Laplacian is a natural choice. This is because $D(G)^{-1}A(G)$ is the transition matrix of a Markov chain which has the same eigenvalues as $I - \mathcal{L}(G)$. Previously, the only cospectral graphs with respect to normalized Laplacian were bipartite (complete bipartite graphs [19] and bipartite graphs found by "unfolding" a small bipartite graph in two ways [3]). Some recent studies on cospectral graphs were carried out in [1, 2, 5, 6, 14–16, 18].

In this paper, a particular class of graphs are constructed as follows. Let $G(a_1, a_2, \ldots, a_k)$ be a simple graph with vertex set $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ and edge set $E(G) = \{(u, v) | u \in V_i, v \in V_{i+1}, i = 1, 2, \ldots, k-1\}$, where $|V_i| = a_i > 0$ for $1 \le i \le k$ and $V_i \cap V_j = \emptyset$ for $i \ne j$. The graph $G(3, 1, 2, 4)$ is illustrated in Figure 1.

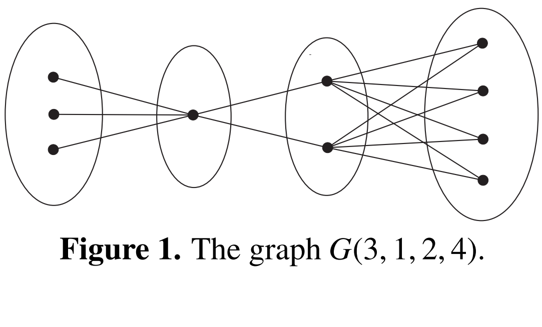

Figure 1. The graph $G(3, 1, 2, 4)$.

Given two positive integers k and n, and $k-2$ positive rational numbers $t_2, t_3, \ldots, t_{\lceil k/2 \rceil}$ and $t'_2, t'_3, \ldots, t'_{\lfloor k/2 \rfloor}$, let Λ be the set of the positive integer solutions $(a_1, a_2, \ldots, a_k)$ of the following equations:

$$\begin{cases} a_1 + a_2 + \cdots + a_k = n, \\ a_{2i-1} = t_i a_1 \text{ for } i \geq 2, \\ a_{2j} = t'_j a_2 \text{ for } j \geq 2. \end{cases}$$

It is not difficult to show that $\Lambda = \{(a_1, a_2, \ldots, a_k) \in N \times N \times \cdots \times N =: N^k | (1 + t_2 + t_3 + \cdots + t_{\lceil k/2 \rceil}) a_1 + (1 + t'_2 + t'_3 + \cdots + t'_{\lfloor k/2 \rfloor}) a_2 = n; a_{2i-1} = t_i a_1, a_{2j} = t'_j a_2, i = 2, 3, \ldots, \lceil k/2 \rceil, j = 2, 3, \ldots, \lfloor k/2 \rfloor\}$, where N is the set of positive integers. Define $\Upsilon(n; k)_t^{t'} = \{G(a_1, a_2, \ldots, a_k) | (a_1, a_2, \ldots, a_k) \in \Lambda\}$, where $t = (t_2, t_3, \ldots, t_{\lceil k/2 \rceil}), t' = (t'_2, t'_3, \ldots, t'_{\lfloor k/2 \rfloor})$.

Example 1.1. If $n = 20, k = 5$, and $t_2 = t_3 = \frac{1}{2}$ and $t'_2 = \frac{1}{3}$, then the set of positive integer solutions (a_1, a_2) of the Eq $(1 + \frac{1}{2} + \frac{1}{2})a_1 + (1 + \frac{1}{3})a_2 = 20$ is $\{(2, 12), (4, 9), (6, 6), (8, 3)\}$. Hence

$$\Lambda = \{(2, 12, 1, 4, 1), (4, 9, 2, 3, 2), (6, 6, 3, 2, 3), (8, 3, 4, 1, 4)\}$$

and

$$\Upsilon(20; 5)_{1/2,1/2}^{1/3} = \{G(2, 12, 1, 4, 1), G(4, 9, 2, 3, 2), G(6, 6, 3, 2, 3), G(8, 3, 4, 1, 4)\}.$$

In this paper, we prove that all graphs in $\Upsilon(n; k)_t^{t'}$ are cospectral with respect to the normalized Laplacian if it is not an empty set.

2. Main results

Given $2n-1$ real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{n-1}$, define $\mathcal{T}_{a_1,a_2,\ldots,a_n}^{b_1,b_2,\ldots,b_{n-1}}$ to be the set of tridigonal matrices $Q = (q_{ij})_{n \times n}$ with form of

$$Q = \begin{pmatrix} a_1 & x_1 & & & & \\ y_1 & a_2 & x_2 & & & \\ & y_2 & a_3 & x_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & y_{n-2} & a_{n-1} & x_{n-1} \\ & & & & y_{n-1} & a_n \end{pmatrix},$$

where $x_i y_i = b_i, i = 1, 2, \ldots, n-1$.

Lemma 2.1. *Keeping the above notation, then, for arbitrary two matrices $Q_1, Q_2 \in \mathcal{T}_{a_1,a_2,\ldots,a_n}^{b_1,b_2,\ldots,b_{n-1}}$, we have $\Phi(Q_1; x) = \Phi(Q_2; x)$. That is, all matrices in $\mathcal{T}_{a_1,a_2,\ldots,a_n}^{b_1,b_2,\ldots,b_{n-1}}$ are cospectral.*

Proof. We prove the lemma by induction on n. if n=2, then matrix

$$Q = \begin{pmatrix} a_1 & x_1 \\ y_1 & a_2 \end{pmatrix} \in \mathcal{T}_{a_1,a_2}^{b_1}.$$

Note that

$$\Phi(Q; x) = (x - a_1)(x - a_2) - x_1 y_1 = (x - a_1)(x - a_2) - b_1.$$

So, for arbitrary $Q_1, Q_2 \in \mathcal{T}_{a_1,a_2}^{b_1}$, $\Phi(Q_1; x) = \Phi(Q_2; x)$.

Now we assume that $n > 2$. Let

$$Q_n = \begin{pmatrix} a_1 & x_1 & & & & \\ y_1 & a_2 & x_2 & & & \\ & y_2 & a_3 & x_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & y_{n-2} & a_{n-1} & x_{n-1} \\ & & & & y_{n-1} & a_n \end{pmatrix} \in \mathcal{T}_{a_1,a_2,\ldots,a_n}^{b_1,b_2,\ldots,b_{n-1}}.$$

Hence,

$$\Phi(Q_n; x) = \det(xI_n - Q_n) = (x - a_1)\det(xI_{n-1} - Q_{n-1}) - b_1 \det(xI_{n-2} - Q_{n-2}), \tag{1}$$

where Q_{n-1} and Q_{n-2} are the matrices obtained from Q_n by deleting the first row and column and first two rows and columns, respectively. Obviously, $Q_{n-1} \in \mathcal{T}_{a_2,a_3,\ldots,a_n}^{b_2,b_3,\ldots,b_{n-1}}$ and $Q_{n-2} \in \mathcal{T}_{a_3,\ldots,a_n}^{b_3,\ldots,b_{n-1}}$. By induction, all matrices in $T_{a_2,a_3,\ldots,a_n}^{b_2,b_3,\ldots,b_{n-1}}$ (resp. $T_{a_3,a_4,\ldots,a_n}^{b_3,b_4,\ldots,b_{n-1}}$) are cospectral. Hence, by Eq (1), it is not difficult to see that all matrices in $\mathcal{T}_{a_1,a_2,\ldots,a_n}^{b_1,b_2,\ldots,b_{n-1}}$ are cospectral. The lemma has thus been proved.

□

Now we use a similar method to that in [9] to prove the following theorem.

Theorem 2.2. *The characteristic polynomial of normalized Laplacian matrix of graph $G(a_1, a_2, \ldots, a_k)$ with $\sum_{i=1}^{k} a_i = n$ is*

$$\Phi(\mathcal{L}(G(a_1, a_2, \ldots, a_k)); x) = (x - 1)^{n-k}\Phi(M; x), \tag{2}$$

where $M = (m_{ij})_{k\times k}$ is the tridigonal matrix satisfying $m_{ij} = 1$ if $i = j$ and $m_{ij} = -a_j/\sqrt{d_i d_j}$ if $i = j - 1$ or $i = j + 1$, and $m_{ij} = 0$ otherwise, $d_1 = a_2, d_2 = a_1 + a_3, d_3 = a_2 + a_4, \ldots, d_{k-1} = a_{k-2} + a_k, d_k = a_{k-1}$.

Proof. Note that if vertices v and w are in the same part of $G(a_1, a_2, \ldots, a_k)$, the transpose of the row vector β_i whose coordinates on v, w and elsewhere are respectively 1, −1 and 0 is an eigenvector for the eigenvalue 1 of the normalized Laplacian matrix $\mathcal{L}(G(a_1, a_2, \ldots, a_k))$, and there are $a_i - 1$ eigenvectors for the eigenvalue 1 ($1 \le i \le k$). So we can find $\sum_{i=1}^{k}(a_i - 1) = n - k$ linearly independent eigenvectors of matrix $\mathcal{L}(G(a_1, a_2, \ldots, a_k))$ which generate a linear subspace U of dimension $n - k$. Now we choose an orthogonal basis of the orthogonal complement of U. It is constituted by the transposes of k row vectors γ_i ($1 \le i \le k$), where γ_i is the vector whose coordinates on vertices $v \in V_i$ are 1 and elsewhere are 0, that is, $\gamma_i = (0, \ldots, 0, \overbrace{1, \ldots, 1}^{a_i}, 0, \ldots, 0)$. It is easy to find that $\mathcal{L}(G(a_1, a_2, \ldots, a_k))(\gamma_1^T, \gamma_2^T, \ldots, \gamma_k^T) = (\gamma_1^T, \gamma_2^T, \ldots, \gamma_k^T)M$, where $M = (m_{ij})$ is a $k \times k$ matrix such that $m_{ij} = 1$ if $i = j$, $m_{ij} = -a_j/\sqrt{d_i d_j}$ if $i = j - 1$ or $i = j + 1$, $m_{ij} = 0$ otherwise. Hence Eq (2) holds.

□

Theorem 2.3. *Given two positive integers k and n, and $k - 2$ positive rational numbers $t_2, t_3, \ldots, t_{\lceil k/2\rceil}$ and $t'_2, t'_3, \ldots, t'_{\lfloor k/2\rfloor}$, then all graphs in $\Upsilon(n; k)_t^{t'}$ are cospectral with respect to the normalized Laplacian if it is not an empty set.*

Proof. We only need to consider the case $|\Upsilon(n; k)_t^{t'}| \ge 2$. Let $G(a_1, a_2, \ldots, a_k)$ and $G(b_1, b_2, \ldots, b_k)$ be two graphs in $\Upsilon(n; k)_t^{t'}$. Then, by Theorem 2.2,

$$\Phi(\mathcal{L}(G(a_1, a_2, \ldots, a_k)); x) = (x - 1)^{n-k}\Phi(M_1; x)$$

and

$$\Phi(\mathcal{L}(G(b_1, b_2, \ldots, b_k)); x) = (x-1)^{n-k}\Phi(M_2; x),$$

where $M_1 = (m_{ij})_{k\times k}$ and $M_2 = (m'_{ij})_{k\times k}$ are two tridigonal matrices satisfying $m_{ij} = m'_{ij} = 1$ if $i = j$ and $m_{ij} = -a_j/\sqrt{d_i d_j}$ and $m'_{ij} = -b_j/\sqrt{d'_i d'_j}$ if $i = j-1$ or $i = j+1$, and $m_{ij} = m'_{ij} = 0$ otherwise, $d_1 = a_2, d_2 = a_1 + a_3, d_3 = a_2 + a_4, \ldots, d_{k-1} = a_{k-2} + a_k, d_k = a_{k-1}$, and $d'_1 = b_2, d'_2 = b_1 + b_3, d'_3 = b_2 + b_4, \ldots, d'_{k-1} = b_{k-2} + b_k, d'_k = b_{k-1}$. Hence we need to show that $\Phi(M_1; x) = \Phi(M_2; x)$.

Note that $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are two solutions of the following equations:

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n, \\ x_{2i-1} = t_i x_1 \text{ for } i \geq 2, \\ x_{2j} = t'_j x_2 \text{ for } j \geq 2. \end{cases}$$

It is not difficult to show that tridigonal matrices M_1 and M_2 satisfy $m_{i,i+1}m_{i+1,i} = m'_{i,i+1}m'_{i+1,i}$ for $i = 1, 2, \cdots, k-1$. For example,

$$m_{12}m_{21} = m'_{12}m'_{21} = \frac{1}{1+t_2},$$
$$m_{23}m_{32} = m'_{23}m'_{32} = \frac{t_2}{(1+t_2)(1+t'_2)},$$
$$m_{34}m_{43} = m'_{34}m'_{43} = \frac{t_2 t'_2}{(1+t'_2)(t_2+t_3)},$$

and so on. By Lemma 2.1, M_1 and M_2 are cospectral. Hence the theorem has been proved. □

3. Examples

In this section, by using Theorem 2.3, we give some examples of cospectral graphs with respect to the normalized Laplacian.

Note that the graphs with form of $G(a_1, a_2)$ or $G(b_1, b_2, b_3)$ are complete bipartite graphs. Using Theorem 2.2, it is not difficult to see that, if $a_1 + a_2 = n$ and $b_1 + b_2 + b_3 = n$, then

$$\Phi(\mathcal{L}(G(a_1, a_2)); x) = \Phi(\mathcal{L}(G(b_1, b_2, b_3)); x) = (x-1)^{n-2}(x^2 - 2x).$$

Hence we have the following.

Corollary 3.1 ([19]). *All complete bipartite graphs with n vertices are cospectral with respect to the normalized Laplacian.*

By Theorems 2.2 and 2.3, four graphs $G(2, 12, 1, 4, 1), G(4, 9, 2, 3, 2), G(6, 6, 3, 2, 3)$, and $G(8, 3, 4, 1, 4)$ in Example 1.1 are cospectral with respect to the normalized Laplacian. Their normalized Laplacian characteristic polynomial is

$$x^{20} - 20x^{19} + \frac{4523}{24}x^{18} - \frac{4449}{4}x^{17} + \frac{13829}{3}x^{16} - \frac{42764}{3}x^{15} + \frac{68215}{2}x^{14} -$$
$$\frac{193843}{3}x^{13} + \frac{295009}{3}x^{12} - 121264x^{11} + \frac{1459601}{12}x^{10} - \frac{595205}{6}x^{9} + 65481x^{8}$$
$$-\frac{103964}{3}x^{7} + \frac{86869}{6}x^{6} - 4661x^{5} + \frac{3340}{3}x^{4} - \frac{556}{3}x^{3} + \frac{153}{8}x^{2} - \frac{11}{12}x.$$

Example 3.2. It is not difficult to show that

$$\Upsilon(24;4)_1^1 = \{G(1,11,1,11), G(2,10,2,10), G(3,9,3,9), G(4,8,4,8), G(5,7,5,7), G(6,6,6,6)\}.$$

By Theorems 2.2 and 2.3, all six graphs in $\Upsilon(24;4)_1^1$ are cospectral with respect to the normalized Laplacian. Their normalized Laplacian characteristic polynomial is

$$x^{24} - 24x^{23} + \frac{1099}{4}x^{22} - \frac{3993}{2}x^{21} + \frac{20675}{2}x^{20} - 40584x^{19} + \frac{501999}{4}x^{18} - \frac{626943}{2}x^{17} +$$

$$643416x^{16} - 1098200x^{15} + \frac{3142467}{2}x^{14} - 1893749x^{13} + 1927341x^{12} - 1656344x^{11} + \frac{2398275}{2}x^{10} -$$

$$727719x^9 + 367251x^8 - 152304x^7 + \frac{204079}{4}x^6 - \frac{26925}{2}x^5 + \frac{5387}{2}x^4 - 384x^3 + \frac{139}{4}x^2 - \frac{3}{2}x.$$

Example 3.3. It is not difficult to show that

$$\Upsilon(60;5)_{2,3}^2 = \{G(1,18,2,36,3), G(2,16,4,32,6), G(3,14,6,28,9), G(4,12,8,24,12),$$

$$G(5,10,10,20,15), G(6,8,12,16,18), G(7,6,14,12,21), G(8,4,16,8,24), G(9,2,18,4,27)\}.$$

By Theorem 2.3, all nine graphs in $\Upsilon(60;5)_{2,3}^2$ are cospectral with respect to the normalized Laplacian.

4. Conclusions

In this paper, we construct a class of graph $\Upsilon(n;k)_t^{t'}$, and proved that all graphs in $\Upsilon(n;k)_t^{t'}$ are cospectral graphs with respect to the normalized Laplacian. We also give some examples to verify our results.

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Conflict of interest

There is no conflict interest for this paper.

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