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Research article

Study of implicit-impulsive differential equations involving Caputo-Fabrizio fractional derivative

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Abstract: This article is devoted to investigate a class of non-local initial value problem of implicit-impulsive fractional differential equations (IFDEs) with the participation of the Caputo-Fabrizio fractional derivative (CFFD). By means of Krasnoselskii's fixed-point theorem and Banach's contraction principle, the results of existence and uniqueness are obtained. Furthermore, we establish some results of Hyers-Ulam (H-U) and generalized Hyers-Ulam (g-H-U) stability. Finally, an example is provided to demonstrate our results.

Keywords: Caputo-Fabrizio fractional derivative; stability results; fixed point results; g-H-U stability
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1. Introduction

In recent decades, fractional differential equations have received great attention from the researchers in many applied fields such as physics, biology, chemistry and other fields of sciences and engineering [1–7, 32]. Due to many applications, this area has been studied with different fractional derivatives such as Riemman-Liouvile, Caputo, Hilfer and Hadamard type fractional derivatives [8–12, 30, 31]. Further, the fractional time derivatives are importance reactive-transport, since solutes may interact immobile porous medium in highly non-linear ways, some of the investigator using fractional time derivatives for the solution of space-time fractional diffusion equations [41, 42]. But now a days the researchers are studying a new type of fractional derivative which is called Caputo-Fabrizio fractional derivative. This fractional derivative is also known as a non-singular kernel or exponential kernel type derivative. In 2015, Caputo and Fabrizio together introduced this derivative [13]. Latter on, Caputo-Fabrizio derivatives was used by many researchers for modeling various problems in engineering sciences (look for example some articles [35, 36]. Further, this type of derivative have many applications. Such as it is use an exponential decay kernel to a novel HIV/AIDS epidemic model that includes an anti-retrovirus treatment compartment [37], and also some researcher apply this new type of the fractional derivative for the dynamical system with both chaotic and non-chaotic behaviors [38], hyper-chaotic behaviors, optimal control and synchronization [39], nonstandard finite difference scheme and non-identical synchronization of a novel fractional chaotic system [40]. Furthermore, the researchers studied the aforementioned area looking for results of existence, uniqueness and stability. Some of the articles we refer to see the reader to earlier works [14–18].

On the other hand, impulsive differential equations (IDEs) have played an important role in the modeling of phenomena, chiefly in the description of dynamics to sudden changes as well as other phenomena such as crops, diseases, etc. The said differential systems have been used to designate the model since the previous century. For the fundamentals theory on IDEs the reader can consult the monographs of Burton and Simeonov [19], Lakshmikantham et al. [20], Benchohra et al. [21]. Recently, impulsive FDEs are increasingly used to constitute an impulsive control theory. This theory is used to model some physical phenomena. The said area has become a very important direction in IDEs theory. Further numerous applications of IDEs to problems arising in satellite orbital transfer [24], ecosystem management [25], electrical engineering [26], etc. Here refer for further applications on IDEs [28, 29, 33]. When reviewing the existence literature, we see that very rarely it has been investigated IFDEs with the participation of the CFFD. For instance recently author [22] investigated the following problem of IDE with CFFD as

$$\begin{cases} {}_{0}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}\mathfrak{u}(r) = \mathfrak{f}(r, \mathfrak{u}(r)), \quad r \in \mathcal{J}, \quad r \neq r_k, \\ \Delta\mathfrak{u}(r_k) = \mathcal{I}_k(\mathfrak{u}(r_k)), \quad k = 1, 2, 3, ..., n, \\ \mathfrak{u}(0) = \mathfrak{u}_0, \end{cases} \tag{1.1}$$

where ${}_{0}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}$ represent CFFD of order ω, $\mathcal{J} = [0, \mathcal{T}]$, $\mathfrak{u}_0 \in \mathcal{R}$, the given function $\mathfrak{f} : \mathcal{J} \times \mathcal{R} \to \mathcal{R}$, $\mathcal{I}_k : \mathcal{R} \to \mathcal{R}$ are continuous. Where $\Delta\mathfrak{u}(r_k)$ represent change of right and left hand limit of the discontinuity points r_k, it is define as $\Delta\mathfrak{u}(r_k) = \mathfrak{u}(r_k^+) - \mathfrak{u}(r_k^-)$.

Inspired by the research work as mentioned above, we intend to work on implicit-IFDEs involving CFFD of the form:

$$\begin{cases} {}_{0}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}\mathfrak{u}(r) = \mathfrak{g}(r, \mathfrak{u}(r), {}_{0}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}\mathfrak{u}(r)), \; r \in \mathcal{J}, \quad r \neq r_k, \\ \Delta\mathfrak{u}(r_k) = \mathcal{I}_k(\mathfrak{u}(r_k)), \quad k = 1, 2, 3, ..., n, \\ \mathfrak{u}(0) = \mathfrak{f}(\mathfrak{u}), \end{cases} \tag{1.2}$$

where ${}_{0}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}$ represents the CFFD of order $0 < \omega < 1$, $\mathfrak{g} : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$, $\mathcal{I}_k : \mathcal{R} \longrightarrow \mathcal{R}$ and

$\mathfrak{f}:\mathcal{R}\longrightarrow\mathcal{R}$ are continuous function. Where $\Delta\mathfrak{u}(r_k)$ represent change of right and left hand limit of the discontinuity points r_k, it is define as $\Delta\mathfrak{u}(r_k)=\mathfrak{u}(r_k^+)-\mathfrak{u}(r_k^-)$. By using Krasnoselskii's and Banach fixed point theorems, we establish the existence theory for the considered problem. Also we develop some results for Hyers-Ulam (H-U) and generalized (G-H-U) stability. Pertinent example is given to verify our results. Further keeping in mind that right hand side of problem (1.2) vanish at $r=0$ as suggested in [43].

In this article, we use a new type of fractional derivative with non-singular kernel involving non-local initial condition and implicit functions is proposed. The introduced fractional derivative includes as a special case Caputo-Fabrizio fractional derivative, and also study the implicit-FDEs with using impulsive condition for the solution of existence, uniqueness and stability results. Next take two counter examples to verify the necessary results.

In section 2, some basics preliminaries on fractional calculus are presented. In section 3, We develop a results and discussion of implicit-IFDEs using an arbitrary non-singular kernel, such as Caputo-Fabrizio fractional derivative. In section 4, we will investigate the stability results of Hyers-Ulam and generalize Hyers-Ulam stability for the proposed problem of implicit-IFDEs. In section 5, take some counter examples and its graphs to verify the necessary results. In the last section 6, take concluding remarks of our article.

2. Basic results

In this part of our article, we need to provide some basic results and definitions of fractional calculus. We derived our main results through using these basic results.

Definition 1. [27] For $0<\omega<1$, $\mathfrak{u}\in\mathcal{H}^1(0,a)$. The CFFD for a function $\mathfrak{u}$ of order ω is defined as

$$ {}_{0}^{\mathcal{CF}}\mathcal{D}_r^{\omega}\mathfrak{u}(r)=\frac{(2-\omega)\mathcal{M}(\omega)}{2(1-\omega)}\int_0^r\exp\Big(-\frac{\omega}{1-\omega}(r-\theta)\Big)\mathfrak{u}'(\theta)d\theta, \tag{2.1}$$

where $\mathcal{M}(\omega)$ is a normalization constant depending on ω.

Definition 2. [27] For $0<\omega<1$, the fractional integral for a function $\mathfrak{u}$ is given by

$$ {}_{0}^{\mathcal{CF}}\mathcal{I}_r^{\omega}\mathfrak{u}(r)=\frac{2(1-\omega)}{(2-\omega)\mathcal{M}(\omega)}\mathfrak{u}(r)+\frac{2\omega}{(2-\omega)\mathcal{M}(\omega)}\int_0^r\mathfrak{u}(\theta)d\theta. \tag{2.2}$$

When $\omega=1$, then we get first order classical integral using Remark 1. This convergent has been proved in [34].

Remark 1. *[22] Note that according to the previous definition, the fractional integral of a function* $\mathfrak{u}$ *with order* $0<\omega<1$ *is an average between function* $\mathfrak{u}$ *and its integral of one. Imposing*

$$\frac{2(1-\omega)}{(2-\omega)\mathcal{M}(\omega)}+\frac{2\omega}{(2-\omega)\mathcal{M}(\omega)}=1,$$

it can be concluded that

$$\mathcal{M}(\omega)=\frac{2}{2-\omega},\quad 0\le\omega\le 1.$$

Lemma 1. *[22] The unique solution of the given initial value problem*

$$\begin{cases} {}_{0}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}\mathfrak{u}(r) = y(r), \quad 0 < \omega < 1, \\ \mathfrak{u}(0) = \mathfrak{u}_0 \in \mathcal{R}, \end{cases} \tag{2.3}$$

is given by

$$\mathfrak{u}(r) = \mathfrak{u}_0 + \frac{2(1-\omega)}{(2-\omega)\mathcal{M}(\omega)}\Big(y(r) - y(0)\Big) + \frac{2\omega}{(2-\omega)\mathcal{M}(\omega)}\int_0^r y(\theta)d\theta,$$

where

$$G_\omega = \frac{2(1-\omega)}{(2-\omega)\mathcal{M}(\omega)} = 1-\omega, \quad Q_\omega = \frac{2\omega}{(2-\omega)\mathcal{M}(\omega)} = \omega.$$

Let $C(\mathcal{J}, \mathcal{R})$ be the space of all continuous functions defined on the interval $\mathcal{J}$ endowed with the usual supremum norm, that as:

$$\| \mathfrak{u} \|_C = \sup_{r\in\mathcal{J}} | \mathfrak{u}(r) | .$$

Let the set of functions

$$\mathcal{X} = PC(\mathcal{J}, \mathcal{R}) = \{\mathfrak{u} : \mathcal{J} \to \mathcal{R} \mid \mathfrak{u} \in C((r_k, r_{k+1}], \mathcal{R})\}, \ k = 0, 1, 2, 3, ..., m$$

and there exist $\mathfrak{u}(r_k^+)$ and $\mathfrak{u}(r_k^-)$, $\quad k = 1, 2, 3, 4, ..., m$. The given set is Banach space with the norm is defined as:

$$\|\mathfrak{u}\|_{PC} = \max_{r\in\mathcal{J}} |\mathfrak{u}(r)|.$$

Theorem 1. *[23] Let $\mathcal{Y}$ be non empty, convex and closed subset of $\mathcal{X}$. Consider two operators $\mathcal{T}$, $\mathcal{S}$ such that*
1) $\mathcal{T}(y_1) + \mathcal{S}(y_2) \in \mathcal{Y}$, for all $y_1, y_2 \in \mathcal{Y}$.
2) $\mathcal{T}$ is contraction operator.
3) $\mathcal{S}$ is continuous and compact.
then there exists at least one solution $y \in \mathcal{X}$ such that $\mathcal{T}(y) + \mathcal{S}(y) = y$.

3. Results and discussion

The present section of our paper is reserved to investigate the existence and uniqueness for the solution of the implicit-IFDEs by Krasnoselskii's and Banach fixed point theorems.

Lemma 2. *Suppose $0 < \omega < 1$ and $\tau : \mathcal{J} \to \mathcal{R}$ be continuous. A function $\mathfrak{u} \in \mathcal{X}$ is the solution of the given impulsive problem:*

$$\begin{cases} {}_{0}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}\mathfrak{u}(r) = \tau(r), \quad 0 < \omega < 1, \quad r \in \mathcal{J}, \quad r \neq r_k, \\ \Delta\mathfrak{u}(r_k) = \mathcal{I}_k(\mathfrak{u}(r_k)), \quad k = 1, 2, 3, ..., n, \\ \mathfrak{u}(0) = \mathfrak{f}(\mathfrak{u}). \end{cases} \tag{3.1}$$

If and only if it satisfies

$$
\mathfrak{u}(r)=\begin{cases}
\mathfrak{f}(\mathfrak{u})+G_{\omega}[\tau(r)-\tau(0)]+Q_{\omega}\int_{0}^{r}\tau(\eta)d\eta, \ \ for \ \ r\in[0,r_1],\\
\mathfrak{f}(\mathfrak{u})+G_{\omega}[\tau(r)-\tau(0)]+\mathcal{I}_1(\mathfrak{u}(r_1^-))+Q_{\omega}\int_{0}^{r_1}\tau(\eta)d\eta\\
\ +Q_{\omega}\int_{0}^{r}\tau(\eta)d\eta, \ \ for \ \ r\in[r_1,r_2],\\
\mathfrak{f}(\mathfrak{u})+G_{\omega}[\tau(r)-\tau(0)]+\mathcal{I}_1(\mathfrak{u}(r_1^-))+\mathcal{I}_2(\mathfrak{u}(r_2^-))+Q_{\omega}\int_{0}^{r_1}\tau(\eta)d\eta\\
\ +Q_{\omega}\int_{0}^{r_2}\tau(\eta)d\eta+Q_{\omega}\int_{0}^{r}\tau(\eta)d\eta, \ \ for \ \ r\in[r_2,r_3],\\
...,\\
\mathfrak{f}(\mathfrak{u})+G_{\omega}[\tau(r)-\tau(0)]+\sum_{i=1}^{k}\mathcal{I}_i(\mathfrak{u}(r_i^-))\\
\ +Q_{\omega}\sum_{i=1}^{k}\int_{0}^{r_i}\tau(\eta)d\eta+Q_{\omega}\int_{0}^{r}\tau(\eta)d\eta, \ \ for \ \ r\in[r_k,r_{k+1}], \quad where \ \ k=1,2,...,m.
\end{cases}\tag{3.2}
$$

Proof. Suppose $\mathfrak{u}(r)$ satisfies (3.1). If $r\in[0,r_1]$, then

$$
{}_{0}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}\mathfrak{u}(r)=\tau(r), \quad r\in(0,r_1] \ \text{ with } \ \mathfrak{u}(0)=\mathfrak{f}(\mathfrak{u}).
$$

Using Lemma 1, we get

$$
\mathfrak{u}(r)=\mathfrak{f}(\mathfrak{u})+G_{\omega}[\tau(r)-\tau(0)]+Q_{\omega}\int_{0}^{r}\tau(\eta)d\eta.\tag{3.3}
$$

Now applying impulsive condition $\mathfrak{u}(r_1^-)$, one has

$$
\mathfrak{u}(r_1^-)=\mathfrak{f}(\mathfrak{u})+G_{\omega}[\tau(r_1)-\tau(0)]+Q_{\omega}\int_{0}^{r_1}\tau(\eta)d\eta.\tag{3.4}
$$

Again if $r\in[r_1,r_2]$, then

$$
{}_{r_1}^{C\mathcal{F}}\mathcal{D}_{r}^{\omega}\mathfrak{u}(r)=\tau(r), \quad r\in(r_1,r_2] \ \text{ with } \ \mathfrak{u}(r_1^+)=\mathfrak{u}(r_1^-)+\mathcal{I}_1(\mathfrak{u}(r_1^-)).
$$

Again using Lemma 1, we can obtain

$$
\begin{aligned}
\mathfrak{u}(r)&=\mathfrak{u}(r_1^+)+G_{\omega}[\tau(r)-\tau(r_1)]+Q_{\omega}\int_{0}^{r}\tau(\eta)d\eta\\
&=\mathfrak{u}(r_1^-)+\mathcal{I}_1(\mathfrak{u}(r_1^-))+G_{\omega}[\tau(r)-\tau(r_1)]+Q_{\omega}\int_{0}^{r}\tau(\eta)d\eta.
\end{aligned}
$$

Further using (3.4), we get

$$
\begin{aligned}
\mathfrak{u}(r)=&\ \mathfrak{f}(\mathfrak{u})+G_{\omega}[\tau(r_1)-\tau(0)]+Q_{\omega}\int_{0}^{r_1}\tau(\eta)d\eta\\
&+\mathcal{I}_1(\mathfrak{u}(r_1^-))+G_{\omega}[\tau(r)-\tau(r_1)]+Q_{\omega}\int_{0}^{r}\tau(\eta)d\eta.
\end{aligned}
$$

Upon further simplification, we have

$$\begin{aligned}\mathfrak{u}(r) &= \mathfrak{f}(\mathfrak{u}) + G_\omega[\tau(r) - \tau(0)] + \mathcal{I}_1(\mathfrak{u}(r_1^-)) \\ &+ Q_\omega \int_0^{r_1} \tau(\eta)d\eta + Q_\omega \int_0^{r} \tau(\eta)d\eta.\end{aligned} \tag{3.5}$$

Again using impulsive condition $\mathfrak{u}(r_2^-)$ in (3.5), we obtain

$$\begin{aligned}\mathfrak{u}(r_2^-) &= \mathfrak{f}(\mathfrak{u}) + G_\omega[\tau(r_2) - \tau(0)] + \mathcal{I}_1(\mathfrak{u}(r_1^-)) \\ &+ Q_\omega \int_0^{r_1} \tau(\eta)d\eta + Q_\omega \int_0^{r_2} \tau(\eta)d\eta.\end{aligned} \tag{3.6}$$

If $r \in [r_2, r_3]$, then

$${}^{\mathcal{CF}}_{r_2}\mathcal{D}^{\omega}_{r}\mathfrak{u}(r) = \tau(r), \quad r \in (r_2, r_3] \ \text{ with } \ \mathfrak{u}(r_2^+) = \mathfrak{u}(r_2^-) + \mathcal{I}_2(\mathfrak{u}(r_2^-)).$$

Again using Lemma 1, we can obtain

$$\begin{aligned}\mathfrak{u}(r) &= \mathfrak{u}(r_2^+) + G_\omega[\tau(r) - \tau(r_2)] + Q_\omega \int_0^{r} \tau(\eta)d\eta \\ &= \mathfrak{u}(r_2^-) + \mathcal{I}_2(\mathfrak{u}(r_2^-)) + G_\omega[\tau(r) - \tau(r_2)] + Q_\omega \int_0^{r} \tau(\eta)d\eta.\end{aligned}$$

By using (3.6), we get

$$\begin{aligned}\mathfrak{u}(r) &= \mathfrak{f}(\mathfrak{u}) + G_\omega[\tau(r_2) - \tau(0)] + \mathcal{I}_1(\mathfrak{u}(r_1^-)) \\ &+ Q_\omega \int_0^{r_1} \tau(\eta)d\eta + Q_\omega \int_0^{r_2} \tau(\eta)d\eta + \mathcal{I}_2(\mathfrak{u}(r_2^-)) + G_\omega[\tau(r) - \tau(r_2)] + Q_\omega \int_0^{r} \tau(\eta)d\eta.\end{aligned}$$

Further simplify, we get

$$\begin{aligned}\mathfrak{u}(r) &= \mathfrak{f}(\mathfrak{u}) + G_\omega[\tau(r) - \tau(0)] + \mathcal{I}_1(\mathfrak{u}(r_1^-)) + \mathcal{I}_2(\mathfrak{u}(r_2^-)) \\ &+ Q_\omega \int_0^{r_1} \tau(\eta)d\eta + Q_\omega \int_0^{r_2} \tau(\eta)d\eta + Q_\omega \int_0^{r} \tau(\eta)d\eta.\end{aligned}$$

Furthermore, continue this process we obtain for $r \in [r_k, r_{k+1}]$ as

$$\begin{aligned}\mathfrak{u}(r) &= \mathfrak{f}(\mathfrak{u}) + G_\omega[\tau(r) - \tau(0)] + \sum_{i=1}^{k} \mathcal{I}_i(\mathfrak{u}(r_i^-)) \\ &+ Q_\omega \sum_{i=1}^{k} \int_0^{r_i} \tau(\eta)d\eta + Q_\omega \int_0^{r} \tau(\eta)d\eta, \quad \text{where } \ k = 1, 2, ..., m.\end{aligned}$$

Similarly, if $\mathfrak{u}(r)$ satisfies (3.2), then we can prove that $\mathfrak{u}(r)$ is the solution of (3.1). This complete the proof. □

Corollary 1. *In view of Lemma 2, the solution of the said problem* (1.2) *is given by*

$$\mathfrak{u}(r)=\begin{cases}\mathfrak{f}(\mathfrak{u})+G_\omega[\mathfrak{g}(r,\mathfrak{u}(r),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(r))-\mathfrak{g}(0,\mathfrak{u}(0),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(0))]\\ +Q_\omega\int_0^r\mathfrak{g}(\eta,\mathfrak{u}(\eta),\ {}_0^{C\mathcal{F}}\mathcal{D}_\eta^\omega\mathfrak{u}(\eta))d\eta,\ \ \textit{for}\ \ r\in[0,r_1],\\ \mathfrak{f}(\mathfrak{u})+G_\omega[\mathfrak{g}(r,\mathfrak{u}(r),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(r))-\mathfrak{g}(0,\mathfrak{u}(0),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(0))]+\mathcal{I}_1(\mathfrak{u}(r_1^-))\\ +Q_\omega\int_0^{r_1}\mathfrak{g}(\eta,\mathfrak{u}(\eta),\ {}_0^{C\mathcal{F}}\mathcal{D}_\eta^\omega\mathfrak{u}(\eta))d\eta+Q_\omega\int_0^r\mathfrak{g}(\eta,\mathfrak{u}(\eta),\ {}_0^{C\mathcal{F}}\mathcal{D}_\eta^\omega\mathfrak{u}(\eta))d\eta,\ \ \textit{for}\ \ r\in[r_1,r_2],\\ \mathfrak{f}(\mathfrak{u})+G_\omega[\mathfrak{g}(r,\mathfrak{u}(r),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(r))-\mathfrak{g}(0,\mathfrak{u}(0),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(0))]\\ +\mathcal{I}_1(\mathfrak{u}(r_1^-))+\mathcal{I}_2(\mathfrak{u}(r_2^-))+Q_\omega\int_0^{r_1}\mathfrak{g}(\eta,\mathfrak{u}(\eta),\ {}_0^{C\mathcal{F}}\mathcal{D}_\eta^\omega\mathfrak{u}(\eta))d\eta\\ +Q_\omega\int_0^{r_2}\mathfrak{g}(\eta,\mathfrak{u}(\eta),\ {}_0^{C\mathcal{F}}\mathcal{D}_\eta^\omega\mathfrak{u}(\eta))d\eta+Q_\omega\int_0^r\mathfrak{g}(\eta,\mathfrak{u}(\eta),\ {}_0^{C\mathcal{F}}\mathcal{D}_\eta^\omega\mathfrak{u}(\eta))d\eta,\ \ \textit{for}\ \ r\in[r_2,r_3],\\ \ldots,\\ \mathfrak{f}(\mathfrak{u})+G_\omega[\mathfrak{g}(r,\mathfrak{u}(r),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(r))-\mathfrak{g}(0,\mathfrak{u}(0),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(0))]+\sum_{i=1}^k\mathcal{I}_i(\mathfrak{u}(r_i^-))\\ +Q_\omega\sum_{i=1}^k\int_0^{r_i}\mathfrak{g}(\eta,\mathfrak{u}(\eta),\ {}_0^{C\mathcal{F}}\mathcal{D}_\eta^\omega\mathfrak{u}(\eta))d\eta+Q_\omega\int_0^r\mathfrak{g}(\eta,\mathfrak{u}(\eta),\ {}_0^{C\mathcal{F}}\mathcal{D}_\eta^\omega\mathfrak{u}(\eta))d\eta,\ \ \textit{for}\ \ r\in[r_k,r_{k+1}].\end{cases}$$

For the sake of simplicity, we use $\mathfrak{g}(r,\mathfrak{u}(r),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\mathfrak{u}(r))=\delta_\mathfrak{u}(r)$ and $\mathfrak{g}(r,\bar{\mathfrak{u}}(r),\ {}_0^{C\mathcal{F}}\mathcal{D}_r^\omega\bar{\mathfrak{u}}(r))=\bar{\delta}_\mathfrak{u}(r)$ also at $r=0$, we use $\delta_\mathfrak{u}(0)=\delta_0$. Further, for qualitative results, we need to transform the proposed problem (1.2) to fixed point problem, we need to define an operator $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ defined as:

$$\begin{aligned}\mathcal{T}\mathfrak{u}(r)&=\mathfrak{f}(\mathfrak{u})+G_\omega[\delta_\mathfrak{u}(r)-\delta_0]+\sum_{i=1}^k\mathcal{I}_i(\mathfrak{u}(r_i^-))\\&+Q_\omega\sum_{i=1}^k\int_0^{r_i}\delta_\mathfrak{u}(\eta)d\eta+Q_\omega\int_0^r\delta_\mathfrak{u}(\eta)d\eta.\end{aligned}\tag{3.7}$$

First of all we introduce some hypothesis which are needed:

(H_1) The function $\mathfrak{g}:\mathcal{J}\times\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ is continuous.

(H_2) There exist positive constants $C_\mathfrak{g}>0$ and $0<C_\mathfrak{g}^*<1$, such that

$$|\mathfrak{g}(r,\mathfrak{u}(r),\delta_\mathfrak{u}(r))-\mathfrak{g}(r,\bar{\mathfrak{u}}(r),\bar{\delta}_\mathfrak{u}(r))|\leq C_\mathfrak{g}|\mathfrak{u}(r)-\bar{\mathfrak{u}}(r)|+C_\mathfrak{g}^*|\delta_\mathfrak{u}(r)-\bar{\delta}_\mathfrak{u}(r)|.$$

(H_3) The function $\mathcal{I}_k:\mathcal{R}\to\mathcal{R}$ are continuous and there exists positive constant $0<L_k<1$ with $\sum_{k=1}^n L_k<1$, such that

$$\sum_{k=1}^n|\mathcal{I}_k(\mathfrak{u}(r))-\mathcal{I}_k(\bar{\mathfrak{u}}(r))|\leq\sum_{k=1}^n L_k|\mathfrak{u}(r)-\bar{\mathfrak{u}}(r)|,$$

for all $\mathfrak{u},\bar{\mathfrak{u}}\in\mathcal{R}$, $k=1,2,3,...,n$. Further we use $\sum_{k=1}^n L_k=L_I$ throughout the paper.

(H_4) The function $\mathfrak{f} : \mathcal{R} \to \mathcal{R}$ is continuous and there exists constant $0 < K_f < 1$, such that

$$|\mathfrak{f}(\mathfrak{u}) - \mathfrak{f}(\bar{\mathfrak{u}})| \leq K_f|\mathfrak{u} - \bar{\mathfrak{u}}|.$$

Theorem 2. *Under the hypothesis H_1–H_4, the impulsive problem* (1.2) *has a unique solution if*

$$\left[K_f + L_I + \left(G_\omega + (n+1)Q_\omega\right)T\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right] < 1.$$

Proof. Suppose for each $r \in \mathcal{J}$ and any $\mathfrak{u}(r), \bar{\mathfrak{u}}(r) \in \mathcal{X}$, we have in view of (3.7)

$$\begin{aligned}|\mathcal{T}\mathfrak{u}(r) - \mathcal{T}\bar{\mathfrak{u}}(r)| \leq& |\mathfrak{f}(\mathfrak{u}) - \mathfrak{f}(\bar{\mathfrak{u}})| + G_\omega|\delta_\mathfrak{u}(r) - \bar{\delta}_\mathfrak{u}(r)| \\ &+ \sum_{k=1}^{n}|\mathcal{I}_k(\mathfrak{u}(r)) - \mathcal{I}_k(\bar{\mathfrak{u}}(r))| \\ &+ Q_\omega\sum_{k=1}^{n}\int_0^{r_k}|\delta_\mathfrak{u}(\eta) - \bar{\delta}_\mathfrak{u}(\eta)|d\eta + Q_\omega\int_0^r|\delta_\mathfrak{u}(\eta) - \bar{\delta}_\mathfrak{u}(r)(\eta)|d\eta,\end{aligned} \tag{3.8}$$

where $\delta_\mathfrak{u}(r) = \mathfrak{g}(r, \mathfrak{u}(r), \delta_\mathfrak{u}(r))$, so

$$\begin{aligned}|\delta_\mathfrak{u}(r) - \bar{\delta}_\mathfrak{u}(r)(r)| =& |\mathfrak{g}(r, \mathfrak{u}(r), \delta_\mathfrak{u}(r)) - \mathfrak{g}(r, \bar{\mathfrak{u}}(r), \bar{\delta}_\mathfrak{u}(r))| \\ \leq& C_\mathfrak{g}|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)| + C^*_\mathfrak{g}|\delta_\mathfrak{u}(r) - \bar{\delta}_\mathfrak{u}(r)|.\end{aligned}$$

Continuing the above process, so we obtain

$$|\delta_\mathfrak{u}(r) - \bar{\delta}_\mathfrak{u}(r)| \leq \frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)|. \tag{3.9}$$

Using the hypothesis (H_3), (H_4) and (3.9) in Eq (3.8), we get

$$\begin{aligned}|\mathcal{T}\mathfrak{u}(r) - \mathcal{T}\bar{\mathfrak{u}}(r)| \leq& K_f|\mathfrak{u} - \bar{\mathfrak{u}}| + G_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right)\left|\mathfrak{u} - \bar{\mathfrak{u}}\right| + \sum_{k=1}^{n}L_k|\mathfrak{u} - \bar{\mathfrak{u}}| \\ &+ Q_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right)\sum_{k=1}^{n}\int_0^{r_k}|\mathfrak{u} - \bar{\mathfrak{u}}|d\eta + Q_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right)\int_0^r|\mathfrak{u} - \bar{\mathfrak{u}}|d\eta.\end{aligned}$$

Taking maximum on both side, we get

$$\begin{aligned}\max_{r\in\mathcal{J}}|\mathcal{T}\mathfrak{u}(r) - \mathcal{T}\bar{\mathfrak{u}}(r)| \leq& K_f\max_{r\in\mathcal{J}}|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)| + G_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right)\max_{r\in\mathcal{J}}\left|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)\right| \\ &+ \sum_{k=1}^{n}L_k\max_{r\in\mathcal{J}}|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)| + Q_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right)\sum_{k=1}^{n}\max_{r\in\mathcal{J}}\int_0^{r_k}|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)|d\eta \\ &+ Q_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right)\max_{r\in\mathcal{J}}\int_0^r|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)|d\eta.\end{aligned}$$

$$\begin{aligned}\|\mathcal{T}\mathfrak{u} - \mathcal{T}\bar{\mathfrak{u}}\|_{PC} \leq& \Bigg[K_f + G_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right) + L_I + nTQ_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right) \\ &+ TQ_\omega\left(\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right)\Bigg]\left\|\mathfrak{u} - \bar{\mathfrak{u}}\right\|_{PC}, \\ \leq& \left[K_f + L_I + \left(G_\omega + (n+1)TQ_\omega\right)\frac{C_\mathfrak{g}}{1 - C^*_\mathfrak{g}}\right]\left\|\mathfrak{u} - \bar{\mathfrak{u}}\right\|_{PC}.\end{aligned}$$

Hence the constant given as

$$\left[K_f + L_I + \left(G_\omega + (n+1)TQ_\omega\right)\frac{C_{\mathfrak{g}}}{1 - C_{\mathfrak{g}}^*}\right] < 1.$$

Therefore, the operator $\mathcal{T}$ is contraction, so the operator has a unique fixed point, therefore the said problem (1.2) has a unique solution. □

Next second main result is based on the Krasnoselskii's fixed-point theorem. For this results we need some hypothesis which is given below.

(H_5) There exist some positive constants $P_{\mathfrak{g}}, Q_{\mathfrak{g}}, R_{\mathfrak{g}} > 0$ and $0 < Q_{\mathfrak{g}} < 1$, such that

$$|\mathfrak{g}(r, \mathfrak{u}(r), \delta_{\mathfrak{u}}(r))| \leq P_{\mathfrak{g}} + Q_{\mathfrak{g}}|\mathfrak{u}(r)| + R_{\mathfrak{g}}|\delta_{\mathfrak{u}}(r)|,$$

for each $r \in \mathcal{J}$ and $\mathfrak{u}(r), \delta_{\mathfrak{u}}(r) \in \mathcal{R}$.

(H_6) There exists positive constant $K_f^* > 0$, such that

$$|\mathfrak{f}(\mathfrak{u})| \leq K_f^*|\mathfrak{u}(r)|,$$

for $\mathfrak{u}(r) \in \mathcal{R}$.

Theorem 3. *Under the hypothesis* (H_2)–(H_6) *are satisfied, then the implicit-impulsive problem* (1.2) *has at least one solution if*

$$0 < \left(K_f + G_\omega\frac{C_{\mathfrak{g}}}{1 - C_{\mathfrak{g}}^*} + L_I\right) < 1.$$

Proof. For the proof of this theorem, we need to define two operators from (3.7), we have

$$\mathcal{T}_1\mathfrak{u}(r) = \mathfrak{f}(\mathfrak{u}) + G_\omega[\delta_{\mathfrak{u}}(r) - \delta_0] + \sum_{i=1}^{k}\mathcal{I}_i(\mathfrak{u}(r_i^-))$$

and

$$\mathcal{T}_2\mathfrak{u}(r) = Q_\omega\sum_{i=1}^{k}\int_0^{r_i}\delta(\eta)d\eta + Q_\omega\int_0^{r}\delta(\eta)d\eta.$$

Let us define a set for a real number $q > 0$ as $H = \{\mathfrak{u}(r) \in \mathcal{X} : \|\mathfrak{u}\|_{PC} \leq q\}$, we need to show the operator $\mathcal{T}_1$, is contraction. For this suppose $\mathfrak{u}(r), \bar{\mathfrak{u}}(r) \in \mathcal{X}$, we have

$$\begin{aligned}|\mathcal{T}_1\mathfrak{u}(r) - \mathcal{T}_1\bar{\mathfrak{u}}(r)| \leq& |\mathfrak{f}(\mathfrak{u}) - \mathfrak{f}(\bar{\mathfrak{u}})| + G_\omega|\delta_{\mathfrak{u}}(r) - \bar{\delta}_{\mathfrak{u}}(r)| \\ &+ \sum_{k=1}^{n}|\mathcal{I}_k(\mathfrak{u}(r)) - \mathcal{I}_k(\bar{\mathfrak{u}}(r))|.\end{aligned}$$

Using hypothesis (H_3), (H_4) and (3.9), then taking maximum on both side, we get

$$\begin{aligned}\max_{r\in\mathcal{J}}|\mathcal{T}_1\mathfrak{u}(r) - \mathcal{T}_1\bar{\mathfrak{u}}(r)| \leq& K_f\max_{r\in\mathcal{J}}|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)| + G_\omega\frac{C_{\mathfrak{g}}}{1 - C_{\mathfrak{g}}^*}\max_{r\in\mathcal{J}}|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)| \\ &+ \sum_{k=1}^{n}L_k\frac{C_{\mathfrak{g}}}{1 - C_{\mathfrak{g}}^*}\max_{r\in\mathcal{J}}|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)|.\end{aligned}$$

$$\|\mathcal{T}_1\mathfrak{u} - \mathcal{T}_1\bar{\mathfrak{u}}\|_{PC} \leq \Big(K_f + G_\omega \frac{C_\mathfrak{g}}{1 - C_\mathfrak{g}^*} + L_I\Big)\|\mathfrak{u} - \bar{\mathfrak{u}}\|_{PC}.$$

Here the given constant is

$$\Big(K_f + G_\omega \frac{C_\mathfrak{g}}{1 - C_\mathfrak{g}^*} + L_I\Big) < 1.$$

Hence the operator $\mathcal{T}_1$ is contraction. Next we need to prove that the operator $\mathcal{T}_2$ is compact and continuous, for this $\mathfrak{u}(r) \in \mathcal{X}$, we have

$$|\mathcal{T}_2\mathfrak{u}(r)| \leq Q_\omega \sum_{k=1}^{n} \int_0^{r_k} |\delta(\eta)| d\eta + Q_\omega \int_0^{r} |\delta(\eta)| d\eta, \tag{3.10}$$

where

$$\begin{aligned} |\delta_\mathfrak{u}(r)| &= |\mathfrak{g}(r, \mathfrak{u}(r), \delta_\mathfrak{u}(r))| \\ &\leq P_\mathfrak{g} + Q_\mathfrak{g}|\mathfrak{u}(r)| + R_\mathfrak{g}|\delta_\mathfrak{u}(r)|. \end{aligned}$$

Upon further simplification, we have

$$|\delta_\mathfrak{u}(r)| \leq \frac{P_\mathfrak{g}}{1 - R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1 - R_\mathfrak{g}}\Big)|\mathfrak{u}(r)|. \tag{3.11}$$

Using (3.11) in (3.10) and then taking maximum, we get

$$\begin{aligned} \max_{r\in\mathcal{J}} |\mathcal{T}_2\mathfrak{u}(r)| &\leq nQ_\omega\Big[\frac{P_\mathfrak{g}}{1 - R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1 - R_\mathfrak{g}}\Big)\Big] \max_{r\in\mathcal{J}} \Big(r_k|\mathfrak{u}(r)|\Big) \\ &+ Q_\omega\Big[\frac{P_\mathfrak{g}}{1 - R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1 - R_\mathfrak{g}}\Big)\Big] \max_{r\in\mathcal{J}} \Big(r|\mathfrak{u}(r)|\Big). \end{aligned}$$

$$\begin{aligned} \|\mathcal{T}_2\mathfrak{u}\|_{PC} &\leq (n + 1)Q_\omega T\Big[\frac{P_\mathfrak{g}}{1 - R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1 - R_\mathfrak{g}}\Big)\Big]\|\mathfrak{u}\|_{PC}, \\ \|\mathcal{T}_2\mathfrak{u}\|_{PC} &\leq A^*, \end{aligned}$$

where

$$A^* = (n + 1)Q_\omega T\Big[\frac{P_\mathfrak{g}}{1 - R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1 - R_\mathfrak{g}}\Big)\Big]q.$$

Hence the operator $\mathcal{T}_2$ is bounded. Further suppose $r_1 < r_2$ in $\mathcal{J}$, we have

$$\begin{aligned} |\mathcal{T}_2\mathfrak{u}(r_2) - \mathcal{T}_2\mathfrak{u}(r_1)| &= \Big|Q_\omega \sum_{k=1}^{n} \int_0^{r_k} \delta(\eta) d\eta + Q_\omega \int_0^{r_2} \delta(\eta) d\eta \\ &- Q_\omega \sum_{k=1}^{n} \int_0^{r_k} \delta(\eta) d\eta - Q_\omega \int_0^{r_1} \delta(\eta) d\eta\Big|, \\ &\leq Q_\omega \int_0^{r_2} |\delta(\eta)| d\eta + Q_\omega \int_{r_1}^{0} |\delta(\eta)| d\eta. \end{aligned} \tag{3.12}$$

Using (3.11) in (3.12), we get

$$|\mathcal{T}_2\mathfrak{u}(r_2) - \mathcal{T}_2\mathfrak{u}(r_1)| \leq Q_\omega\Big[\frac{P_\mathfrak{g}}{1-R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1-R_\mathfrak{g}}\Big)\Big|\mathfrak{u}(r)\Big|\Big]\Big(r_2 - r_1\Big).$$

Taking maximum on right hand side, we have

$$\begin{aligned}|\mathcal{T}_2\mathfrak{u}(r_2) - \mathcal{T}_2\mathfrak{u}(r_1)| &\leq Q_\omega\Big[\frac{P_\mathfrak{g}}{1-R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1-R_\mathfrak{g}}\Big)\max_{r\in\mathcal{J}}\Big|\mathfrak{u}(r)\Big|\Big]\Big(r_2 - r_1\Big)\\ &\leq Q_\omega\Big[\frac{P_\mathfrak{g}}{1-R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1-R_\mathfrak{g}}\Big)\Big\|\mathfrak{u}\Big\|_{PC}\Big]\Big(r_2 - r_1\Big).\end{aligned}$$

Further

$$|\mathcal{T}_2\mathfrak{u}(r_2) - \mathcal{T}_2\mathfrak{u}(r_1)| \leq Q_\omega\Big[\frac{P_\mathfrak{g}}{1-R_\mathfrak{g}} + \Big(\frac{Q_\mathfrak{g}}{1-R_\mathfrak{g}}\Big)q\Big]\Big(r_2 - r_1\Big). \tag{3.13}$$

Obviously, from (3.13), we look that if $r_1 \to r_2$, then the right hand side of the Eq (3.13) goes to zero, so $|\mathcal{T}_2\mathfrak{u}(r_2) - \mathcal{T}_2\mathfrak{u}(r_1)| \to 0$ as if $r_1 \to r_2$. Hence we observe that the right-hand side of (3.13) goes to zero uniformly. Therefore, the operator $\mathcal{T}_2$ is equicontinuou. Therefore the operator $\mathcal{T}_2$ is compact by Arzelá-Ascoli theorem. Hence in view of Krasnoselskii theorem, we conclude that (1.2) has at least one solution. □

4. Stability results

In this part of our article, we will investigate the stability of H-U and g-U-H stability for the problem of implicit-IFDEs.

Definition 3. ⟨ *H-U stable* ⟩
The said implicit-impulsive problem (1.2), is H-U stable if any $\epsilon > 0$ for the given inequality

$$|{}^{C\mathcal{F}}_0\mathcal{D}^\omega_r\mathfrak{u}(r) - \mathfrak{g}(r, \mathfrak{u}(r), {}^{C\mathcal{F}}_0\mathcal{D}^\omega_r\mathfrak{u}(r))| \leq \epsilon, \quad \forall \ \ r \in \mathcal{J}.$$

Then, there exists unique solution $\bar{\mathfrak{u}}(r)$ with a constant $\mathcal{Z}$ such that

$$|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)| \leq \mathcal{Z}\epsilon, \quad \forall \ \ r \in \mathcal{J}.$$

Definition 4. ⟨ *Generalized H-U stable* ⟩ Our implicit-impulsive problem is g-H-U stable if there exists non-decreasing function $\phi : (0, T) \to (0, \infty)$, such that

$$|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)| \leq \mathcal{Z}\phi(\epsilon), \quad \forall \ \ r \in \mathcal{J}.$$

With $\phi(0) = 0, \ \phi(T) = 0$.

Also we discuss important remark here which is used in this section as:

Remark 2. *Suppose there exists a function* $\Psi(r)$*, which is depend on* $\mathfrak{u} \in \mathcal{X}$ *with* $\Psi(0) = 0, \ \ \Psi(T) = 0$ *such that*
(1) $|\Psi(r)| \leq \epsilon, \ \forall \ r \in \mathcal{J}$,

(2) ${}^{C\mathcal{F}}_0\mathcal{D}^\omega_r\mathfrak{u}(r) = \mathfrak{g}(r, \mathfrak{u}(r), {}^{C\mathcal{F}}_0\mathcal{D}^\omega_r\mathfrak{u}(r)) + \Psi(r), \ \forall \ \ r \in \mathcal{J}$.

Lemma 3. *The solution of given proposed problem*

$$\begin{cases} {}_0^{C\mathcal{F}}\mathcal{D}_r^{\omega}\mathfrak{u}(r) = \mathfrak{g}(r, \mathfrak{u}(r), {}_0^{C\mathcal{F}}\mathcal{D}_r^{\omega}\mathfrak{u}(r)) + \Psi(r), \quad 0 < \omega < 1, \quad r \in \mathcal{J} = [0, T], \quad r \neq r_k, \\ \Delta\mathfrak{u}(r_k) = \mathcal{I}_k(\mathfrak{u}(r_k)), \quad k = 1, 2, 3, ..., n, \\ \mathfrak{u}(0) = \mathfrak{f}(\mathfrak{u}), \end{cases}$$

is

$$\begin{aligned} \mathfrak{u}(r) =& \mathfrak{f}(\mathfrak{u}) + G_\omega[\delta_{\mathfrak{u}}(r) - \delta_0] + G_\omega[\Psi(r) - \Psi(0)] \\ &+ \sum_{k=1}^{n} \mathcal{I}_k(\mathfrak{u}(r_k^-)) + Q_\omega \sum_{k=1}^{n} \int_0^{r_k} \delta(\eta)d\eta + Q_\omega \sum_{k=1}^{n} \int_0^{r_k} \Psi(\eta)d\eta \\ &+ Q_\omega \int_0^r \delta(\eta)d\eta + Q_\omega \int_0^r \Psi(\eta)d\eta, \ \ \forall \ \ r \in \mathcal{J}, \ \textit{where} \ \ k = 1, 2, 3, ..., n. \end{aligned} \tag{4.1}$$

where $\delta_{\mathfrak{u}}(r) = \mathfrak{g}(r, \mathfrak{u}(r), {}_0^{C\mathcal{F}}\mathcal{D}_r^{\omega}\mathfrak{u}(r))$ *and* $\Psi(0) = 0$. *Further, from the solution* (4.1)*, we get*

$$\begin{aligned} &\Bigg|\mathfrak{u}(r) - \Bigg[\mathfrak{f}(\mathfrak{u}) + G_\omega[\delta_{\mathfrak{u}}(r) - \delta_0] + \sum_{i=1}^{k} \mathcal{I}_i(\mathfrak{u}(r_i^-)) \\ &+ Q_\omega \sum_{i=1}^{k} \int_0^{r_i} \delta(\eta)d\eta + Q_\omega \int_0^r \delta(\eta)d\eta\Bigg]\Bigg| \leq \Big(G_\omega + Q_\omega T(n+1)\Big)\epsilon. \end{aligned} \tag{4.2}$$

Proof. The solution of (4.1) can be easily obtained through using Lemma 2. Although from the solution it is clear to become result (4.2), by using Remark 2. □

Theorem 4. *Under the Lemma 3, solution of the said implicit-impulsive problem* (1.2)*, is H-U and g-H-U stable if*

$$\mathcal{Z} = \frac{\Big(G_\omega + Q_\omega T(n+1)\Big)}{1 - \Big(K_f + L_I + \Big(G_\omega + (n+1)TQ_\omega\Big)\frac{C_{\mathfrak{g}}}{1 - C_{\mathfrak{g}}^*}\Big)} < 1.$$

Proof. Suppose $\mathfrak{u}(r) \in \mathcal{X}$ be any solution of the mentioned problem (1.2) and $\bar{\mathfrak{u}}(r) \in \mathcal{X}$ be unique solution of the said problem, then we need to consider

$$
\begin{aligned}
|\mathfrak{u}(r)-\bar{\mathfrak{u}}(r)| =& \Big|\mathfrak{u}(r)-\Big[\mathfrak{f}(\bar{\mathfrak{u}})+G_\omega[\bar{\delta}_{\mathfrak{u}}(r)-\delta_0]+\sum_{k=1}^{n}\mathcal{I}_k(\bar{\mathfrak{u}}(r_k^-))\\
&+Q_\omega\sum_{k=1}^{n}\int_0^{r_k}\bar{\delta}(\eta)d\eta+Q_\omega\int_0^{r}\bar{\delta}(\eta)d\eta\Big]\Big|,\\
=& \Big|\mathfrak{u}(r)-\Big[\mathfrak{f}(\mathfrak{u})+G_\omega[\delta_{\mathfrak{u}}(r)-\delta_0]+\sum_{i=1}^{k}\mathcal{I}_i(\mathfrak{u}(r_i^-))+Q_\omega\sum_{k=1}^{n}\int_0^{r_k}\delta(\eta)d\eta\\
&+Q_\omega\int_0^{r}\delta(\eta)d\eta\Big]+\Big[\mathfrak{f}(\mathfrak{u})+G_\omega[\delta_{\mathfrak{u}}(r)-\delta_0]+\sum_{k=1}^{n}\mathcal{I}_k(\mathfrak{u}(r_k^-))\\
&+Q_\omega\sum_{k=1}^{n}\int_0^{r_k}\delta(\eta)d\eta+Q_\omega\int_0^{r}\delta(\eta)d\eta\Big]\\
&-\Big[\mathfrak{f}(\bar{\mathfrak{u}})+G_\omega[\bar{\delta}_{\mathfrak{u}}(r)-\delta_0]+\sum_{k=1}^{n}\mathcal{I}_k(\bar{\mathfrak{u}}(r_k^-))+Q_\omega\sum_{k=1}^{n}\int_0^{r_k}\bar{\delta}(\eta)d\eta+Q_\omega\int_0^{r}\bar{\delta}(\eta)d\eta\Big]\Big|
\end{aligned}
$$

$$
\begin{aligned}
|\mathfrak{u}(r)-\bar{\mathfrak{u}}(r)| \leq& \Big|\mathfrak{u}(r)-\Big[\mathfrak{f}(\mathfrak{u})+G_\omega[\delta_{\mathfrak{u}}(r)-\delta_0]+\sum_{k=1}^{n}\mathcal{I}_k(\mathfrak{u}(r_k^-))\\
&+Q_\omega\sum_{k=1}^{n}\int_0^{r_k}\delta(\eta)d\eta+Q_\omega\int_0^{r}\delta(\eta)d\eta\Big]\Big|\\
&+|\mathfrak{f}(\mathfrak{u})-\mathfrak{f}(\bar{\mathfrak{u}})|+G_\omega|\delta_{\mathfrak{u}}(r)-\bar{\delta}_{\mathfrak{u}}(r)|+\sum_{k=1}^{n}|\mathcal{I}_k(\mathfrak{u}(r_k^-))-\mathcal{I}_k(\bar{\mathfrak{u}}(r_k^-))|\\
&+Q_\omega\sum_{k=1}^{n}\int_0^{r_k}|\delta(\eta)-\bar{\delta}(\eta)|d\eta+Q_\omega\int_0^{r}|\delta(\eta)-\bar{\delta}(\eta)|d\eta\Big].
\end{aligned}
$$

Using (4.2), (3.9) and hypothesis H_3, H_4 and then taking maximum on both side, we get

$$
\begin{aligned}
|\mathfrak{u}(r)-\bar{\mathfrak{u}}(r)| \leq& \Big(G_\omega+Q_\omega T(n+1)\Big)\epsilon\\
&+\Big(K_f+G_\omega\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^*}+L_I+nQ_\omega\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^*}r_k+Q_\omega\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^*}r\Big)|\mathfrak{u}(r)-\bar{\mathfrak{u}}(r)|.
\end{aligned}
$$

$$
\begin{aligned}
\max_{r\in\mathcal{J}}|\mathfrak{u}(r)-\bar{\mathfrak{u}}(r)| \leq& \Big(G_\omega+Q_\omega T(n+1)\Big)\epsilon\\
&+\max_{r\in\mathcal{J}}\Big(K_f+G_\omega\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^*}+L_I+nQ_\omega\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^*}r_k+Q_\omega\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^*}r\Big)\max_{r\in\mathcal{J}}|\mathfrak{u}(r)-\bar{\mathfrak{u}}(r)|.
\end{aligned}
$$

Upon further simplification

$$
\begin{aligned}
\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{PC} \leq& \Big(G_\omega+Q_\omega T(n+1)\Big)\epsilon\\
&+\Big(K_f+L_I+\Big(G_\omega+(n+1)TQ_\omega\Big)\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^*}\Big)\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{PC}.
\end{aligned}
$$

Hence from the above inequality, we get

$$\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{PC} \leq \frac{\Big(G_\omega + Q_\omega T(n+1)\Big)\epsilon}{1-\Big(K_f + L_I + \Big(G_\omega + (n+1)TQ_\omega\Big)\frac{C_\mathfrak{g}}{1-C_\mathfrak{g}^*}\Big)},$$

which gives

$$\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{PC} \leq \mathcal{Z}\epsilon. \tag{4.3}$$

Therefore, solution is H-U stable. And there exists non-decreasing function $\phi \in \mathcal{X}$. Then from Eq (4.3), we can be write as

$$\|\mathfrak{u}-\bar{\mathfrak{u}}\|_{PC} \leq \mathcal{Z}\phi(\epsilon),$$

with $\phi(0) = \phi(T) = 0$. Therefore, solution of the implicit-impulsive problem (1.2) is g-H-U stable. □

5. Illustrative example

In this section, we study counter example to verify our results.

Example 1. *Considered the implicit-IFDEs problem*

$$\begin{cases} {}^{C\mathcal{F}}_{0}\mathcal{D}^{\frac{1}{5}}_{r}\mathfrak{u}(r) = \dfrac{r^2}{25} + \dfrac{\sin(\mathfrak{u}(r)) + \sin({}^{C\mathcal{F}}_{0}\mathcal{D}^{\frac{1}{5}}_{r}\mathfrak{u}(r))}{47+r^2}, \quad r \in [0,1], \\ \Delta\mathfrak{u}(\dfrac{1}{3}) = \dfrac{e^{-\mathfrak{u}(\frac{1}{3})}}{55}, \\ \mathfrak{u}(0) = \dfrac{\cos|\mathfrak{u}|}{15}. \end{cases} \tag{5.1}$$

Here $\omega = \frac{1}{5}$ *and* ${}^{C\mathcal{F}}_{0}\mathcal{D}^{\frac{1}{5}}_{r}\mathfrak{u}(r) = \delta_\mathfrak{u}(r)$, *we can set*

$$\mathfrak{g}(r, \mathfrak{u}(r), \delta_\mathfrak{u}(r)) = \frac{r^2}{25} + \frac{\sin(\mathfrak{u}(r)) + \sin(\delta_\mathfrak{u}(r))}{47+r^2}, \quad \textit{where} \;\; \mathfrak{u}(r) \in \mathcal{X}, \; \delta_\mathfrak{u}(r) \in \mathcal{R},$$

$$\mathcal{I}_1(\mathfrak{u}(\frac{1}{3})) = \frac{e^{-\mathfrak{u}(\frac{1}{3})}}{55},$$

and

$$\mathfrak{f}(\mathfrak{u}) = \frac{\cos|\mathfrak{u}|}{15}.$$

Clearly $\mathfrak{f}$ *and* $\mathfrak{g}$ *are continuous functions. Now for* $\mathfrak{u}(r),\ \bar{\mathfrak{u}}(r) \in \mathcal{X},\ \delta_\mathfrak{u}(r),\ \bar{\delta}_\mathfrak{u}(r) \in \mathcal{R}$ *and* $r \in [0,1]$. *Now, we consider*

$$\begin{aligned}\Big|\mathfrak{g}(r, \mathfrak{u}(r), \delta_\mathfrak{u}(r)) - \mathfrak{g}(r, \bar{\mathfrak{u}}(r), \bar{\delta}_\mathfrak{u}(r))\Big| \; &\leq \frac{\Big|\sin(\mathfrak{u}(r)) + \sin(\bar{\mathfrak{u}}(r))\Big|}{47+r^2} + \frac{\Big|\sin(\delta_\mathfrak{u}(r)) + \sin(\bar{\delta}_\mathfrak{u}(r))\Big|}{47+r^2}, \\ &\leq \frac{1}{47+r^2}\Big(|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)| + |\delta_\mathfrak{u}(r) - \bar{\delta}_\mathfrak{u}(r)|\Big).\end{aligned}$$

Applying maximum on both side, so we get

$$\max_{r\in[0,1]}\Big|\mathfrak{g}(r,\mathfrak{u}(r),\delta_{\mathfrak{u}}(r))-\mathfrak{g}(r,\bar{\mathfrak{u}}(r),\bar{\delta}_{\mathfrak{u}}(r))\Big| \le\frac{1}{48}\Big(\|\mathfrak{u}-\bar{\mathfrak{u}}\|+\|\delta-\bar{\delta}\|\Big).$$

Which satisfy hypothesis H_2, we have

$$C_{\mathfrak{g}}=C_{\mathfrak{g}}^{*}=\frac{1}{48}.$$

Now next we set for $\mathfrak{u}(r),\ \bar{\mathfrak{u}}(r)\in\mathcal{X}$, *we have*

$$\begin{aligned}\Big|\mathcal{I}_1(\mathfrak{u}(\frac{1}{3}))-\mathcal{I}_1(\bar{\mathfrak{u}}(\frac{1}{3}))\Big| &=\Big|\frac{e^{-\mathfrak{u}(\frac{1}{3})}}{55}-\frac{e^{-\bar{\mathfrak{u}}(\frac{1}{3})}}{55}\Big|\\ &\le\frac{1}{55}\Big(\Big|\mathfrak{u}(\frac{1}{3})-\bar{\mathfrak{u}}(\frac{1}{3})\Big|\Big)\\ &\le\frac{1}{55}\Big(\Big\|\mathfrak{u}-\bar{\mathfrak{u}}\Big\|_{PC}\Big).\end{aligned}$$

Hence, hypothesis H_3, is satisfied, so $L=\frac{1}{55}$. Next we consider a function $\mathfrak{f}(\mathfrak{u})=\frac{\cos|\mathfrak{u}|}{15}$, *for* $\mathfrak{u},\ \bar{\mathfrak{u}}\in\mathcal{R}$, *we have*

$$\begin{aligned}|\mathfrak{f}(\mathfrak{u})-\mathfrak{f}(\bar{\mathfrak{u}})| &=|\frac{\cos|\mathfrak{u}|}{15}-\frac{\cos|\bar{\mathfrak{u}}|}{15}|\\ &\le\frac{1}{15}|\mathfrak{u}-\bar{\mathfrak{u}}|.\end{aligned}$$

Therefore, H_4, is satisfied, so $K_f=\frac{1}{15}$. Further, we need to verify the condition of the theorems, for this we know that $\omega=\frac{1}{5}$ and $G_\omega=\frac{4}{5}$, $Q_\omega=\frac{1}{5}$, we have the condition of theorem 2 is

$$\begin{aligned}\Big[K_f+L_I+\Big(G_\omega+(n+1)Q_\omega\Big)\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^{*}}\Big] &= \Big[\frac{1}{15}+\frac{1}{55}+\Big(\frac{4}{5}+2\times\frac{1}{5}\Big)\frac{\frac{1}{48}}{1-\frac{1}{48}}\Big]\\ &= \frac{856}{7755}=0.11038<1.\end{aligned}$$

Therefore, the condition of Theorem 2, is satisfied, hence the mentioned implicit-impulsive problem (5.1) *has a unique solution. Further, we need to verify the condition of the theorem 3, we have*

$$0<\Big(K_f+G_\omega\frac{C_{\mathfrak{g}}}{1-C_{\mathfrak{g}}^{*}}+L_I\Big)=\frac{790}{7755}<1.$$

Also the condition of Theorem 3, holds, so the solution of the said problem (5.1)*, is at least one solution. In the last, we need to verify the stability results, for this, we verify the condition of the Theorem 4, we have*

$$\mathcal{Z}=0\cdot 2698<1.$$

Therefore, condition of the Theorem 4, is satisfied, hence the solution of the problem (5.1)*, has H-U and g- H-U stable.*

Remark 3. *Here we consider the given Example 1 and provide the graphical presentation in Figure 1. We present graphical presentation of solution at different values of fractional order* $\omega = 0.45, 0.65.0.85$ *and at the given values of impulsive points* $r_1 = \frac{1}{4},\ r_2 = \frac{1}{2},\ r_3 = \frac{3}{4}$.

From Figure 1, we see the stability behavior for different fractional order.

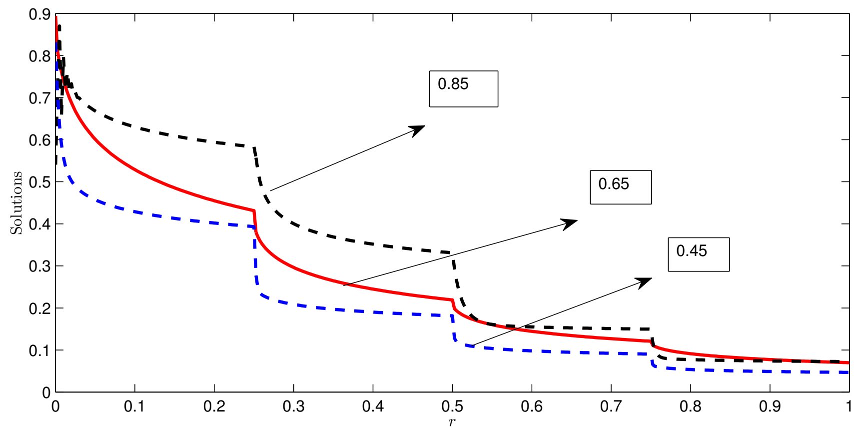

Figure 1. Graphical presentation for different fractional order and given impulsive points of Example 1.

Example 2. *Take another implicit-IFDEs problem*

$$\begin{cases} {}^{\mathcal{CF}}_{0}\mathcal{D}_r^{\frac{1}{7}}\mathfrak{u}(r) = \dfrac{r^3}{15} + \dfrac{\cos(u(r))}{35} + \dfrac{\cos({}^{\mathcal{CF}}_{0}\mathcal{D}_r^{\frac{1}{7}}\mathfrak{u}(r))}{25+r}, \quad r \in [0,1], \\ \Delta\mathfrak{u}(\dfrac{1}{5}) = \dfrac{e^{-\mathfrak{u}(\frac{1}{5})}}{35}, \\ \mathfrak{u}(0) = \dfrac{\sin(|\mathfrak{u}|)}{25}. \end{cases} \tag{5.2}$$

Here $\omega = \frac{1}{7}$ *and* ${}^{\mathcal{CF}}_{0}\mathcal{D}_r^{\frac{1}{7}}\mathfrak{u}(r) = \delta_{\mathfrak{u}}(r)$, *we can set*

$$\mathfrak{g}(r, \mathfrak{u}(r), \delta_{\mathfrak{u}}(r)) = \frac{r^3}{15} + \frac{\cos(u(r))}{35} + \frac{\cos({}^{\mathcal{CF}}_{0}\mathcal{D}_r^{\frac{1}{7}}\mathfrak{u}(r))}{25+r}, \quad \textit{where} \quad \mathfrak{u}(r) \in \mathcal{X},\ \delta_{\mathfrak{u}}(r) \in \mathcal{R},$$

$$\mathcal{I}_1(\mathfrak{u}(\frac{1}{5})) = \frac{e^{\mathfrak{u}(\frac{1}{5})}}{35},$$

and

$$\mathfrak{f}(\mathfrak{u}) = \frac{\sin(|\mathfrak{u}|)}{25}.$$

Clearly $\mathfrak{f}$ *and* $\mathfrak{g}$ *are continuous functions. For* $\mathfrak{u}(r),\ \bar{\mathfrak{u}}(r) \in \mathcal{X},\ \delta_{\mathfrak{u}}(r),\ \bar{\delta}_{\mathfrak{u}}(r) \in \mathcal{R}$ *and* $r \in [0,1]$. *Consider,*

$$\begin{aligned} \left|\mathfrak{g}(r, \mathfrak{u}(r), \delta_{\mathfrak{u}}(r)) - \mathfrak{g}(r, \bar{\mathfrak{u}}(r), \bar{\delta}_{\mathfrak{u}}(r))\right| &\le \frac{\left|\cos(\mathfrak{u}(r)) + \cos(\bar{\mathfrak{u}}(r))\right|}{35} + \frac{\left|\cos(\delta_{\mathfrak{u}}(r)) + \cos(\bar{\delta}_{\mathfrak{u}}(r))\right|}{25+r}, \\ &\le \frac{1}{35}\Big(|\mathfrak{u}(r) - \bar{\mathfrak{u}}(r)|\Big) + \frac{1}{25+r}\Big(|\delta_{\mathfrak{u}}(r) - \bar{\delta}_{\mathfrak{u}}(r)|\Big). \end{aligned}$$

Taking maximum on both side, we get

$$\max_{r\in[0,1]}\left|\mathfrak{g}(r,\mathfrak{u}(r),\delta_{\mathfrak{u}}(r))-\mathfrak{g}(r,\bar{\mathfrak{u}}(r),\bar{\delta}_{\mathfrak{u}}(r))\right| \leq\frac{1}{35}\Big(\|\mathfrak{u}-\bar{\mathfrak{u}}\|\Big)+\frac{1}{25}\Big(\|\delta-\bar{\delta}\|\Big).$$

One can see that the hypothesis H_2 is satisfy, we have

$$C_{\mathfrak{g}}=\frac{1}{35}\ \ and\ \ C^*_{\mathfrak{g}}=\frac{1}{25}.$$

Next consider for $\mathfrak{u}(r),\ \bar{\mathfrak{u}}(r)\in\mathcal{X}$*, we have*

$$\begin{aligned}\left|\mathcal{I}_1(\mathfrak{u}(\frac{1}{5}))-\mathcal{I}_1(\bar{\mathfrak{u}}(\frac{1}{5}))\right| &=\left|\frac{e^{-\mathfrak{u}(\frac{1}{5})}}{35}-\frac{e^{-\bar{\mathfrak{u}}(\frac{1}{5})}}{35}\right|\\ &\leq\frac{1}{35}\left(\left|\mathfrak{u}(\frac{1}{5})-\bar{\mathfrak{u}}(\frac{1}{5})\right|\right)\\ &\leq\frac{1}{35}\left(\left\|\mathfrak{u}-\bar{\mathfrak{u}}\right\|\right).\end{aligned}$$

Hence, the hypothesis H_3, is satisfied, where $L_I=\frac{1}{35}$.
Next we consider a function $\mathfrak{f}(\mathfrak{u})=\frac{\sin(|\mathfrak{u}|)}{25}$*, for* $\mathfrak{u},\ \bar{\mathfrak{u}}\in\mathcal{R}$*, we have*

$$\begin{aligned}|\mathfrak{f}(\mathfrak{u})-\mathfrak{f}(\bar{\mathfrak{u}})| &=\left|\frac{\sin(|\mathfrak{u}|)}{25}-\frac{\sin(|\bar{\mathfrak{u}}|)}{25}\right|\\ &\leq\frac{1}{25}\left|\mathfrak{u}-\bar{\mathfrak{u}}\right|.\end{aligned}$$

Here $K_f=\frac{1}{25}$, so the hypothesis H_3, is satisfied.
Moreover, we need to verify the sufficient conditions of the theorems. For this we have $\omega=\frac{1}{7}$ and $G_\omega=\frac{6}{7}, Q_\omega=\frac{1}{7}$. First we have to verify the condition of the Theorem 2, is

$$\begin{aligned}&\left[K_f+L_I+\Big(G_\omega+(n+1)Q_\omega\Big)\frac{C_{\mathfrak{g}}}{1-C^*_{\mathfrak{g}}}\right]=\left[\frac{1}{25}+\frac{1}{35}+\Big(\frac{6}{7}+2\times\frac{1}{7}\Big)\frac{\frac{1}{35}}{1-\frac{1}{35}}\right]\\ &=\frac{161}{525}=0.3067<1.\end{aligned}$$

One can see that condition of the Theorem 2, is satisfied. Therefore, the implicit-impulsive problem (5.2) *has a unique solution.*
Further, verify condition of the Theorem 3, we have

$$0<\left(K_f+G_\omega\frac{C_{\mathfrak{g}}}{1-C^*_{\mathfrak{g}}}+L_I\right)=\frac{73}{700}<1.$$

Also holds condition of the Theorem 3, hence solution of the said problem (5.2)*, is at least one solution.*
In the last, we need to verify the stability results, for this, we have to verify condition of the Theorem 4, we get

$$\mathcal{Z}=0\cdot 792<1.$$

Therefore, condition of the Theorem 4, is satisfied, hence the solution of the problem (5.2)*, has H-U and g-H-U stable.*

Remark 4. *Here we consider the given Example 2 and provide the graphical presentation in Figure 2. We present graphical presentation of solution at different values of fractional order* $\omega = 0.25, 0.45.0.99$ *and at the given values of impulsive points* $r_1 = \frac{1}{4}$, $r_2 = \frac{1}{2}$, $r_3 = \frac{3}{4}$. *From Figure 2, we see the stability behavior for different fractional order.*

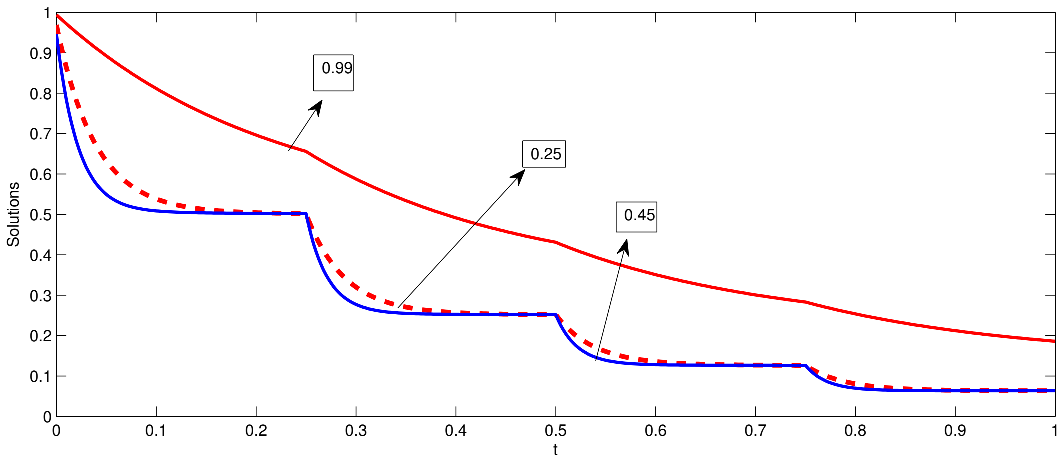

Figure 2. Graphical presentation for different fractional order and given impulsive points of Example 2.

6. Conclusions

We have in fact obtained some conditions necessary for the solution of existence, uniqueness and stability of the said implicit-impulsive FDEs with involving CFFD. we obtain this conditions using the fixed point theorem as Krasnoselskii's and Banach contraction principle. In this article, we have used Banach's contraction theorem for the uniqueness of solution and Krasnoselskii's fixed point theorem for the existence of the solution for the said problem (1.2). Also we have studied this problem for the stability of H-U and g-H-U stable. All the results have been demonstrated by a proper example. We have also presented the solution through graph by taking different fractional order and impulsive points using RKM methods.

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Conflict of interest

No conflict of interest exist.

References

1. J. T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, *Commun. Nonlinear Sci.,* **16** (2011), 1140–1153.
2. R. Metzler, K. Joseph, Boundary value problems for fractional diffusion equations, *Phys. A*, **278** (2000), 107–125.
3. K. B. Oldham, Fractional differential equations in electrochemistry, *Adv. Eng. Softw.*, **41** (2010), 9–12. https://doi.org/10.1016/j.advengsoft.2008.12.012
4. F. A. Rihan, Numerical modeling of fractional-order biological systems, *Abstr. Appl. Anal.,* **2013** (2013), 1–11. https://doi.org/10.1155/2013/816803
5. J. Sabatier, O. P. Agrawal, J. A. T. Machado, *Advances in fractional Calculus,* Dordrecht, Springer, 2007.
6. V. E. Tarasov, *Fractional dynamics: Application of fractional Calculus to dynamics of particles,* Fields and Media, Springer, Heidelberg, Higher Education Press, Beijing, 2010.
7. M. D. Ortigueira, *Fractional Calculus for scientists and engineers: Lecture notes in electrical engineering,* 84, Springer, Dordrecht, 2011.
8. J. Hristov, Derivatives with non-singular kernels from the Caputo-Fabrizio definition and beyond: Appraising analysis with emphasis on diffusion models, *Front. Fract. Calc.,* **1** (2017), 270–342.
9. M. I. Abbas, On the Hamdard and Riemann-Liouville fractional neutral functional integro-differential equations with finite delay, *J. Pseudo-Differ. Oper.,* **10** (2019), 1–10.
10. M. I. Abbas, Ulam stability of fractional impulsive differential equations with Riemann-Liouville integral boundary conditions, *J. Contemp. Math. Anal.,* **50** (2015), 209–219. https://doi.org/10.3103/S1068362315050015
11. A. Atangana, B. S. T. Alkahtani, New model of groundwater flowing within a confine aquifer: application of Caputo-Fabrizio derivative, *Arabian J. Geo.,* **9** (2016), 1–6. https://doi.org/10.1007/s12517-015-2060-8
12. A. A. Kilbas, M. Saigo, RK. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integr. Transf. Spec. F.,* (2004), 31–49. https://doi.org/10.1080/10652460310001600717
13. M. Caputo, M. Fabrizio, A new definition of fractional derivative of without singular kernel, *Prog. Fract. Differ. Appl.,* **1** (2015), 73–85 .
14. T. Abdeljawad, D. Baleanu, Monotonicity results for fractional difference operators with discrete exponential kernels, *Adv. Differ. Equ.,* **2017** (2017), 1–9. https://doi.org/10.1186/s13662-017-1126-1
15. R. A. Khan, K. Shah, Existence and uniqueness of solutions to fractional order multi-point boundary value problems, *Commun. Appl. Anal.*, **19** (2015), 515–526.
16. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.,* **27** (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222

17. S. M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.,* **222** (1998), 126–137. https://doi.org/10.1006/jmaa.1998.5916

18. S. M. Jung, Hyers-Ulam stability of linear differential equations of first order II, *Appl. Math. Lett.,* **19** (2006), 854–858. https://doi.org/10.1016/j.aml.2005.11.004

19. D. D. Bajnov, P. S. Simeonov, *Systems with impulse effect stability, theory and applications. Ellis Horwood Series in mathematics and its applications,* Halsted Press, New York, 1989.

20. M. Benchohra, J. Henderson, S. Ntouyas, *Impulsive diferential equations and inclusions: Contemporary mathematics and its applications,* Hindawi Publishing Corporation, New York, 2006. https://doi.org/10.1155/9789775945501

21. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of impulsive differential equations,* World Scientific, Singapore, 1989. https://doi.org/10.1142/0906

22. A. Atangana, D. Baleanu, Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer, *J. Eng. Mech.,* **143** (2017), D4016005. https://doi.org/10.1061/(ASCE)EM.1943-7889.0001091

23. T. A. Burton, T. Furumochi, Krasnoselskii's fixed point theorem and stability, *Nonlinear Anal.-Theor.,* **49** (2002), 445–54. https://doi.org/10.1016/S0362-546X(01)00111-0

24. J. E. Prussing, L. J. Wellnitz, W. G. Heckathorn, Optimal impulsive time-fixed direct-ascent interception, *J. Guid. Control Dynam.,* **12** (1989), 487–494. https://doi.org/10.2514/3.20436

25. X. Liu, K. Rohlf, Impulsive control of a Lotka-Volterra system, *J. Math. Cont. Inf.,* **15** (1998), 269–284. https://doi.org/10.1093/imamci/15.3.269

26. T. Yang, L. Chua, Impulsive stabilization for control and synchronization of chaotic systems: Theory and application to secure communication, *IEEE T. Circuits-I*, **44** (1997), 976–988. https://doi.org/10.1109/81.633887

27. J. Losada, J. J. Nieto, Properties of a new fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.,* **1** (2015), 87–92.

28. K. Liu, J. Wang, Y. Zhou, D. O'Regan, Hyers-Ulam stability and existence of solutions for fractional differential equations with Mittag-Leffler kernel, *Chaos, Soliton. Fract.,* **132** (2020),109534. https://doi.org/10.1016/j.chaos.2019.109534

29. J. Sheng, W. Jiang, D. Pang, S. Wang, Controllability of nonlinear fractional dynamical systems with a Mittag-Leffler kernel, *Mathematics,* **8** (2020), 2139. https://doi.org/10.3390/math8122139

30. D. Aimene, D. Baleanu, D. Seba, Controllability of semilinear impulsive Atangana-Baleanu fractional differential equations with delay, *Chaos, Soliton. Fract.,* **128** (2019), 51–57. https://doi.org/10.1016/j.chaos.2019.07.027

31. D. Kumar, J. Singh, D. Baleanu, On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law, *Math. Method. Appl. Sci.,* **43** (2020), 443–457. https://doi.org/10.1002/mma.5903

32. A. Atangana, Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties, *Physica A: Stat. Mech. Appl.,* **505** (2018), 688–706. https://doi.org/10.1016/j.physa.2018.03.056

33. A. Atangana, J. F. Gomez-Aguilar, Existence and data dependence theorems for solutions of an ABC-fractional order impulsive system, *Chaos Soliton. Fract.,* **131** (2020), 109477. https://doi.org/10.1016/j.chaos.2019.109477

34. Eiman, K. Shah, M. Sarwar, D. Baleanu, Study on Krasnoselskii's fixed point theorem for Caputo-Fabrizio fractional differential equations, *Adv. Differ. Equ.,* **2020** (2020), 1–9. https://doi.org/10.1186/s13662-020-02624-x

35. K. M. Owolabi, A. Shikonogo, Fractal fractional operator method on HER2+ and breast cancer dynamics, *Appl. Comput. Math.,* **7** (2021), 1–19. https://doi.org/10.1007/s40819-021-01030-5

36. K. M. Owolabi, Analysis and numerical simulation of cross-reaction systems with the Caputo-Fabrizio and Riezs operators, *Numer. Meth. Part. D. E.,* **2021** (2021), 1–23.

37. E. J. Moore, S. Sirisubtawee, S. Koonprasert, A Caputo-Fabrizio fractional differential equation model for HIV/AIDS with treatment compartment, *Adv. Differ. Equ.*, **2019** (2019), 200. https://doi.org/10.1186/s13662-019-2138-9

38. D. Baleanu, S. S. Sajjadi, A. Jajarmi, Z. Defterli, On a nonlinear dynomical system with both chaotic and nonchaotic behaviors: A new fractional analysis and control, *Adv. Differ. Equ.*, **2021** (2021), 234. https://doi.org/10.1186/s13662-021-03393-x

39. D. Baleanu, S. S. Sajjadi, J. H. Asad, A. Jajarmi, E. Estiri, Hyperchaotic behaviors, optimal control and synchronization of a nonautonomous cardiac conduction System, *Adv. Differ. Equ.*, **2021** (2021), 175. https://doi.org/10.1186/s13662-021-03320-0

40. D. Baleanu, S. Zibaei, M. Namjoo, A. Jajarmi, A nonstandard finite difference scheme for the modeling and nonidentical synchronization of a noval fractional chaotic system, *Adv. Differ. Equ.,* **2021** (2021), 308. https://doi.org/10.1186/s13662-021-03454-1

41. M. M. Meerschaert, A. B. David, H. P. Scheffler, B. Baeumer, Stochastic solution of space-time fractional diffusion equations, *Phys. Rev. E,* **65** (2002), 041103. https://doi.org/10.1103/PhysRevE.65.041103

42. R. Schumer, A. B. David, M. M. Meerschaert, B. Baeumer, Fractal mobile/immobile solute transport, *Water Resour. Res.,* **39** (2003), 1296.

43. X. Zheng, H. Wang, H. Fu, Well-posedness of fractional differential equations with variable-order Caputo-Fabrizio derivative, *Chaos Soliton. Fract.,* **138** (2020), 109966. https://doi.org/10.1016/j.chaos.2020.109966