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Research article

New Simpson type inequalities for twice differentiable functions via generalized fractional integrals

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Abstract: Fractional versions of Simpson inequalities for differentiable convex functions are extensively researched. However, Simpson type inequalities for twice differentiable functions are also investigated slightly. Hence, we establish a new identity for twice differentiable functions. Furthermore, by utilizing generalized fractional integrals, we prove several Simpson type inequalities for functions whose second derivatives in absolute value are convex.

Keywords: Simpson type inequalities; generalized fractional integrals; convex functions
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1. Introduction

It is well known that Simpson's inequality is used in several branches of mathematics in the literature. For four times continuously differentiable functions, the classical Simpson's inequality is expressed as follows:

Theorem 1. *Let $F : [a,b] \to \mathbb{R}$ denote a four times continuously differentiable mapping on (a,b), and let $\left\|F^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|F^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{F(a)+F(b)}{2}+2F\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b F(x)dx\right| \leq \frac{1}{2880}\left\|F^{(4)}\right\|_\infty (b-a)^4 .$$

The convex theory is an available way to solve a large number of problems from various branches of mathematics. Hence, many authors have researched on the results of Simpson-type for convex functions. More precisely, some inequalities of Simpson's type for s-convex functions is proved by using differentiable functions [1]. In the paper [2], it is investigated the new variants of Simpson's type inequalities based on differentiable convex mapping. For more information about Simpson type inequalities for various convex classes, we refer the reader to Refs. [3–7] and the references therein.

In the papers [8] and [9], it is extended the Simpson inequalities for differentiable functions to Riemann-Liouville fractional integrals. Thus, several paper focused on fractional Simpson and other fractional integral inequalities for various fractional integral operators [10–25]. For further information about to Simpson type inequalities, we refer the reader to Refs. [26–32] and the references therein. In the paper [33], Sarikaya et al. investigated several Simpson type inequalities for functions whose second derivatives are convex.

The first and second results on fractional Simpson inequality for twice differentiable functions were established in [34] and [35], respectively. With the help of these articles, the aim of this paper is to extend the results of given in [33] for twice differentiable functions to generalized fractional integrals. The general structure of the paper consists of four chapters including an introduction. The remaining part of the paper proceeds as follows: In Section 2, after giving a general literature survey and definition of generalized fractional integral operators, we give an equality for twice differentiable functions involving generalized fractional integrals. In Section 3, for utilizing this equality, it is considered several Simpson type inequalities for mapping whose second derivatives are convex. In the last section, some conclusions and further directions of research are discussed.

The generalized fractional integrals were introduced by Sarikaya and Ertuğral as follows:

Definition 1. *[36] Let us note that a function* $\varphi : [0,\infty) \to [0,\infty)$ *satisfies the following condition:*

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We consider the following left-sided and right-sided generalized fractional integral operators

$$_{a+}I_{\varphi}F(x) = \int_a^x \frac{\varphi(x-t)}{x-t} F(t)dt, \quad x > a \tag{1.1}$$

and

$$_{b-}I_{\varphi}F(x) = \int_x^b \frac{\varphi(t-x)}{t-x} F(t)dt, \quad x < b, \tag{1.2}$$

respectively.

The most significant feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, conformable fractional integral, etc. These important special cases of the integral operators (1.1) and (1.2) are mentioned as follows:

1) Let us consider $\varphi(t) = t$. Then, the operators (1.1) and (1.2) reduce to the Riemann integral.
2) If we choose $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\alpha > 0$, then the operators (1.1) and (1.2) reduce to the Riemann-Liouville fractional integrals $J_{a+}^{\alpha}F(x)$ and $J_{b-}^{\alpha}F(x)$, respectively. Here, Γ is Gamma function.
3) For $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators (1.1) and (1.2) reduce to the k-Riemann-Liouville fractional integrals $J_{a+,k}^{\alpha}F(x)$ and $J_{b-,k}^{\alpha}F(x)$, respectively. Here, Γ_k is k-Gamma function.

In recent years, several papers have devoted to obtain inequalities for generalized fractional integrals [37–43].

The first result on fractional Simpson inequality for twice differentiable functions was proved by Budak et al. in [34] as follows:

Theorem 2. *Suppose* $F:[a,b]\to\mathbb{R}$ *is an twice differentiable mapping* (a,b) *so that* $F''\in L_1([a,b])$. *Suppose also the mapping* $|F''|$ *is convex on* $[a,b]$. *Then, we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}F(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}F(a)\right]\right|\\ \leq\;&\frac{(b-a)^2}{6}\Lambda(\alpha)\left[|F''(a)|+|F''(b)|\right].\end{aligned}$$

Here,

$$\Lambda(\alpha)=\frac{1}{4(\alpha+2)}\left(\alpha\left(\frac{\alpha+1}{3}\right)^{\frac{2}{\alpha}}+\frac{3}{\alpha+1}\right)-\frac{1}{8}.$$

The other version of fractional Simpson inequality for twice differentiable functions was proved in [35] as follows:

Lemma 1. *[35] Let us consider the function* $\varpi:[0,1]\to\mathbb{R}$ *by* $\varpi(t)=\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}t-t^{\alpha+1}$ *with* $\alpha>0$.

1) If $0<\alpha\leq\frac{1}{2}$, *then we have*

$$\int_0^1|\varpi(t)|\,dt=\frac{1-\alpha^2}{3(\alpha+2)}.$$

2) If $\alpha>\frac{1}{2}$, *then there exist a real number* c_α *such that* $0<c_\alpha<1$ *and we obtain the following equality*

$$\int_0^1|\varpi(t)|\,dt=2\left(\frac{(c_\alpha)^{\alpha+2}}{\alpha+2}-\frac{(1-2\alpha)c_\alpha+(\alpha+1)(c_\alpha)^2}{3}\right)+\frac{1-\alpha^2}{3(\alpha+2)}.$$

Theorem 3. *[35] Assume that* $F:[a,b]\to\mathbb{R}$ *is an absolutely continuous mapping* (a,b) *such that* $F''\in L_1([a,b])$. *Assume also that the mapping* $|F''|$ *is convex on* $[a,b]$. *Then, we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{b-}F\left(\frac{a+b}{2}\right)+J^{\alpha}_{a+}F\left(\frac{a+b}{2}\right)\right]\right| \quad (1.3)\\ \leq\;&\frac{(b-a)^2}{8(\alpha+1)}\Omega_1(\alpha)\left[|F''(a)|+|F''(b)|\right].\end{aligned}$$

Here, $\Omega_1(\alpha)$ *is defined by*

$$\Omega_1(\alpha)=\begin{cases}\frac{1-\alpha^2}{3(\alpha+2)}, & 0<\alpha\leq\frac{1}{2},\\[2ex] 2\left(\frac{(c_\alpha)^{\alpha+2}}{\alpha+2}-\frac{(1-2\alpha)c_\alpha+(\alpha+1)(c_\alpha)^2}{3}\right)+\frac{1-\alpha^2}{3(\alpha+2)}, & \alpha>\frac{1}{2}.\end{cases}$$

2. Some equalities for twice differentiable functions

In this section, we give an identity on twice differentiable functions for using the main results.

Lemma 2. *Let $F:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping (a,b) such that $F''\in L_1([a,b])$. Then, the following equality*

$$\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right] \tag{2.1}$$

$$=\frac{(b-a)^2}{8\Upsilon(1)}\int_0^1\left(\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right)\left[F''\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt$$

is valid. Here, $\Omega(t)=\int_0^t\Upsilon(s)\,ds$ and $\Upsilon(s)=\int_0^s\frac{\varphi\left(\left(\frac{b-a}{2}\right)u\right)}{u}du$.

Proof. By using integration by parts, we obtain

$$\begin{aligned}K_1 &= \int_0^1\left(\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right)F''\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\\ &= -\frac{2\left(\Omega(1)-\frac{2}{3}\Upsilon(1)\right)}{b-a}F'\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_0^1\left(\frac{2}{3}\Upsilon(1)-\Upsilon(t)\right)F'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\\ &= -\frac{2\left(\Omega(1)-\frac{2}{3}\Upsilon(1)\right)}{b-a}F'\left(\frac{a+b}{2}\right)+\frac{4\Upsilon(1)}{3(b-a)^2}F(b)+\frac{8\Upsilon(1)}{3(b-a)^2}F\left(\frac{a+b}{2}\right)\\ &\quad-\frac{4}{(b-a)^2}\int_0^1\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t}F\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt.\end{aligned} \tag{2.2}$$

With help of the Eq (2.2) and using the change of the variable $x=\frac{1+t}{2}b+\frac{1-t}{2}a$ for $t\in[0,1]$, it can be rewritten as follows

$$\begin{aligned}K_1 &= -\frac{2\left(\Omega(1)-\frac{2}{3}\Upsilon(1)\right)}{b-a}F'\left(\frac{a+b}{2}\right)+\frac{4\Upsilon(1)}{3(b-a)^2}F(b)+\frac{8\Upsilon(1)}{3(b-a)^2}F\left(\frac{a+b}{2}\right)\\ &\quad-\frac{4}{(b-a)^2}\,{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right).\end{aligned} \tag{2.3}$$

Similarly, we get

$$K_2 = \int_0^1\left(\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right)F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \tag{2.4}$$

$$= \frac{2\left(\Omega(1)-\frac{2}{3}\Upsilon(1)\right)}{b-a}F'\left(\frac{a+b}{2}\right)+\frac{2}{b-a}\int_0^1\left(\frac{2}{3}\Upsilon(1)-\Upsilon(t)\right)F'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$= \frac{2\left(\Omega(1)-\frac{2}{3}\Upsilon(1)\right)}{b-a}F'\left(\frac{a+b}{2}\right)+\frac{4\Upsilon(1)}{3(b-a)^2}F(a)+\frac{8\Upsilon(1)}{3(b-a)^2}F\left(\frac{a+b}{2}\right)$$

$$-\frac{4}{(b-a)^2}\int_0^1\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t}F\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$= \frac{2\left(\Omega(1)-\frac{2}{3}\Upsilon(1)\right)}{b-a}F'\left(\frac{a+b}{2}\right)+\frac{4\Upsilon(1)}{3(b-a)^2}F(a)+\frac{8\Upsilon(1)}{3(b-a)^2}F\left(\frac{a+b}{2}\right)$$

$$-\frac{4}{(b-a)^2}\,{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right).$$

From Eqs (2.3) and (2.4), we have

$$K_1+K_2 \tag{2.5}$$

$$= \frac{4\Upsilon(1)}{3(b-a)^2}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{4}{(b-a)^2}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right].$$

Multiplying the both sides of (2.5) by $\frac{(b-a)^2}{8\Upsilon(1)}$, we obtain Eq (2.1). This ends the proof of Lemma 2. □

3. New Simpson's type inequalities for twice differentiable functions

In this section, we establish several Simpson type inequalities for mapping whose second derivatives are convex.

Theorem 4. *Let us consider that the assumptions of Lemma 2 are valid. Let us also consider that the mapping $|F''|$ is convex on $[a,b]$. Then, we get the following inequality*

$$\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right]\right| \tag{3.1}$$

$$\leq \frac{(b-a)^2}{8\Upsilon(1)}\Psi_1^{\varphi}\left[|F''(a)|+|F''(b)|\right],$$

where Ψ_1^{φ} is defined by

$$\Psi_1^{\varphi}=\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|dt.$$

Proof. By taking modulus in Lemma 2, we have

$$\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right]\right| \tag{3.2}$$

$$\leq \frac{(b-a)^2}{8\Upsilon(1)}\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|$$

$$\times\left[\left|F''\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt.$$

By using convexity of $|F''|$, we obtain

$$\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{8\Upsilon(1)}\left[\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|\left[\left(\frac{1+t}{2}\right)|F''(b)|+\left(\frac{1-t}{2}\right)|F''(a)|\right]\right.$$

$$\left.+\left(\frac{1+t}{2}\right)|F''(a)|+\left(\frac{1-t}{2}\right)|F''(b)|\right]dt$$

$$= \frac{(b-a)^2}{8\Upsilon(1)}\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|dt\left[|F''(a)|+|F''(b)|\right]$$

$$= \frac{(b-a)^2}{8\Upsilon(1)}\Psi_1^{\varphi}\left[|F''(a)|+|F''(b)|\right].$$

This finishes the proof of Theorem 4. □

Remark 1. *If we choose $\varphi(t)=t$ in Theorem 4, then Theorem 4 reduces to [33, Theorem 2.2].*

Remark 2. *Let us consider $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 4. Then, the inequality (3.1) reduces to the inequality (1.3).*

Corollary 1. *If we assign $\varphi(t)=\frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$ in Theorem 4, then there exist a real number c_{α}^{k} so that $0<c_{\alpha}^{k}<1$ and the following inequality holds:*

$$\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J_{b-,k}^{\alpha}F\left(\frac{a+b}{2}\right)+J_{a+,k}^{\alpha}F\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{k(b-a)^2}{8(\alpha+k)}\Theta_1(\alpha,k)\left[|F''(a)|+|F''(b)|\right].$$

Here, $\Theta_1(\alpha,k)$ is defined by

$$\Theta_1(\alpha,k)=\begin{cases}\frac{k^2-\alpha^2}{3k(\alpha+2k)}, & 0<\frac{\alpha}{k}\leq\frac{1}{2},\\[2ex] 2\left(\frac{k\left(c_{\alpha}^{k}\right)^{\frac{\alpha+2k}{k}}}{\alpha+2k}-\frac{(k-2\alpha)c_{\alpha}^{k}+(\alpha+k)\left(c_{\alpha}^{k}\right)^2}{3k}\right)+\frac{k^2-\alpha^2}{3k(\alpha+2k)}, & \frac{\alpha}{k}>\frac{1}{2}.\end{cases} \tag{3.3}$$

Theorem 5. *Let us note that the assumptions of Lemma 2 hold. If the mapping $|F''|^q$, $q>1$ is convex on $[a,b]$, then we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;&\frac{(b-a)^2}{8\Upsilon(1)}\Psi_{\varphi}(p)\left[|F''(a)|^q+|F''(b)|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\Psi_{\varphi}(p)$ *is defined by*

$$\Psi_{\varphi}(p)=\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|^p dt\right)^{\frac{1}{p}}.$$

Proof. By using the Hölder inequality in inequality (3.2), we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;&\frac{(b-a)^2}{8\Upsilon(1)}\left\{\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|F''\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

With the help of the convexity of $|F''|^q$, we get

$$\begin{aligned}&\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;&\frac{(b-a)^2}{8\Upsilon(1)}\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|^p dt\right)^{\frac{1}{p}}\\ &\times\left[\left(\int_0^1\left[\left(\frac{1+t}{2}\right)|F''(b)|^q+\left(\frac{1-t}{2}\right)|F''(a)|^q\right]dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1\left[\left(\frac{1+t}{2}\right)|F''(a)|^q+\left(\frac{1-t}{2}\right)|F''(b)|^q\right]dt\right)^{\frac{1}{q}}\right]\\ =\;&\frac{(b-a)^2}{8\Upsilon(1)}\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|^p dt\right)^{\frac{1}{p}}\end{aligned}$$

$$\times\left[\left(\frac{3\left|F''(b)\right|^{q}+\left|F''(a)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|F''(b)\right|^{q}+3\left|F''(a)\right|^{q}}{4}\right)^{\frac{1}{q}}\right].$$

This completes the proof of Theorem 5. □

Remark 3. *Consider $\varphi(t)=t$ in Theorem 5. Then, Theorem 5 reduces to [35, Corollary 1].*

Remark 4. *If it is chosen $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 5, then Theorem 5 reduces to [35, Theorem 4].*

Corollary 2. *Let us consider $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 5. Then, we have*

$$\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{b-,k}F\left(\frac{a+b}{2}\right)+J^{\alpha}_{a+,k}F\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{8}\Psi_k(\alpha,p)\left[\left|F''(a)\right|^{q}+\left|F''(b)\right|^{q}\right]^{\frac{1}{q}},$$

where

$$\Psi_k(\alpha,p)=\left(\int_0^1\left|\frac{k}{\alpha+k}-\frac{k}{\alpha+k}t^{\frac{\alpha+k}{k}}-\frac{2}{3}(1-t)\right|^{p}dt\right)^{\frac{1}{p}}.$$

Theorem 6. *Let us note that the assumptions of Lemma 2 hold. If the mapping $|F''|^{q}$, $q\geq 1$ is convex on $[a,b]$, then we have the following inequality*

$$\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{8\Upsilon(1)}\left(\Psi_1^{\varphi}\right)^{1-\frac{1}{q}}\left\{\left(\frac{\left(\Psi_1^{\varphi}+\Psi_2^{\varphi}\right)\left|F''(b)\right|^{q}+\left(\Psi_1^{\varphi}-\Psi_2^{\varphi}\right)\left|F''(a)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left(\Psi_1^{\varphi}+\Psi_2^{\varphi}\right)\left|F''(a)\right|^{q}+\left(\Psi_1^{\varphi}-\Psi_2^{\varphi}\right)\left|F''(b)\right|^{q}}{2}\right)^{\frac{1}{q}}\right\}.$$

Here, Ψ_2^{φ} is defined by in Theorem 4 and Ψ_2^{φ} is defined by

$$\Psi_2^{\varphi}=\int_0^1 t\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|dt.$$

Proof. By applying power-mean inequality in (3.2), we obtain

$$\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{2\Upsilon(1)}\left[{}_{a+}I_{\varphi}F\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}F\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{8\Upsilon(1)}\left[\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|\left|F''\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|dt\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|\left|F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $|F''|^q$ is convex, we have

$$\int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|\left|F''\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt$$

$$\leq \int_0^1\left|\Omega(1)-\Omega(t)-\frac{2}{3}\Upsilon(1)(1-t)\right|\left[\frac{1+t}{2}|F''(b)|^q+\frac{1-t}{2}|F''(a)|^q\right]dt$$

$$= \frac{\left(\Psi_1^{\varphi}+\Psi_2^{\varphi}\right)|F''(b)|^q+\left(\Psi_1^{\varphi}-\Psi_2^{\varphi}\right)|F''(a)|^q}{2}$$

and similarly

$$\int_0^1\left|\Omega(1)-\frac{2}{3}\Upsilon(1)+\frac{2}{3}\Upsilon(1)t-\Omega(t)\right|\left|F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt$$

$$\leq \frac{\left(\Psi_1^{\varphi}+\Psi_2^{\varphi}\right)|F''(a)|^q+\left(\Psi_1^{\varphi}-\Psi_2^{\varphi}\right)|F''(b)|^q}{2}.$$

Then, we obtain the desired result of Theorem 6. □

Remark 5. *If we take $\varphi(t)=t$ in Theorem 6, then Theorem 6 reduces to [33, Theorem 2.5].*

Remark 6. *Let us consider $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 6. Then, Theorem 6 reduces to [35, Theorem 5].*

Corollary 3. *If we choose $\varphi(t)=\frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$ in Theorem 6, then there exist a real number σ_{α}^{k} so that $0<\sigma_{\alpha}^{k}<1$ and we have the inequality*

$$\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J_{b-,k}^{\alpha}F\left(\frac{a+b}{2}\right)+J_{a+,k}^{\alpha}F\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{8(\alpha+1)} (\Theta_1(\alpha,k))^{1-\frac{1}{q}} \left\{ \left(\frac{(\Theta_1(\alpha,k)+\Theta_2(\alpha,k)) |F''(b)|^q + (\Theta_1(\alpha,k)-\Theta_2(\alpha,k)) |F''(a)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{(\Theta_1(\alpha,k)+\Theta_2(\alpha,k)) |F''(a)|^q + (\Theta_1(\alpha,k)-\Theta_2(\alpha,k)) |F''(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.$$

Here, $\Theta_1(\alpha,k)$ *is defined as in (3.3) and* $\Theta_2(\alpha,k)$ *is defined by*

$$\Theta_2(\alpha,k) = \begin{cases} \frac{3k^2+\alpha k-2\alpha^2}{18k(\alpha+3k)}, & 0<\alpha\leq\frac{1}{2}, \\ 2\left(\frac{k\left(\sigma_\alpha^k\right)^{\frac{\alpha+3k}{k}}}{\alpha+3k} - \frac{3(k-2\alpha)\left(\sigma_\alpha^k\right)^2+4(\alpha+k)\left(\sigma_\alpha^k\right)^3}{18k} \right) + \frac{3k^2+\alpha k-2\alpha^2}{18k(\alpha+3k)}, & \alpha>\frac{1}{2}. \end{cases}$$

4. Conclusions

Fractional versions of Simpson inequalities for differentiable convex functions are investigated extensively. On the other hand, Simpson type inequalities for twice differentiable functions are also considered slightly. Hence, Simpson type inequality for twice differentiable functions by generalized fractional integrals are established in this paper. Furthermore, we prove that our results generalize the inequalities obtained by Sarikaya et al. [33] and Hezenci et al. [35]. In the future studies, authors can try to generalize our results by utilizing different kind of convex function classes or other type fractional integral operators.

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Conflicts of interest

The authors declare that they have no competing interests.

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