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Research article

On a boundary value problem of arbitrary orders differential inclusion with nonlocal, integral and infinite points boundary conditions

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Abstract: In this work, we are concerned with a boundary value problem of fractional orders differential inclusion with nonlocal, integral and infinite points boundary conditions. We prove some existence results for that nonlocal boundary value problem. Next, the existence of maximal and minimal solutions is proved. Finally, the sufficient condition for the uniqueness and continuous dependence of solution are studied.

Keywords: differential inclusion; boundary value problem; existence of solutions; continuous dependence
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1. Introduction

The models of the differential and integral equations have been appeared in different applications (see [1–4, 6, 7, 9, 12–20]).

Boundary value problems involving fractional differential equations arise in physical sciences and applied mathematics. In some of these problems, subsidiary conditions are imposed locally. In some other cases, nonlocal conditions are imposed. It is sometimes better to impose nonlocal conditions since the measurements needed by a nonlocal condition may be more precise than the measurement given by a local condition. Consequently, a variety of excellent results on fractional boundary value problems (abbreviated BVPs) with resonant conditions have been achieved. For instance, Bai [4] studied a type of fractional differential equations with m-points boundary conditions. The existence of nontrivial solutions was established by using coincidence degree theory. Applying the same method, Kosmatov [17] investigated the fractional order three points BVP with resonant case.

Although the study of fractional BVPs at resonance has acquired fruitful achievements, it should be noted that such problems with Riemann-Stieltjes integrals are very scarce, so it is worthy of further explorations. Riemann-Stieltjes integral has been considered as both multipoint and integral in a single framework, which is more common, see the relevant works due to Ahmad et al. [1].

The boundary value problems with nonlocal, integral and infinite points boundary conditions have been studied by some authors (see, for example [8, 10–12]).

Here, we discuss the boundary value problem of the nonlinear differential inclusions of arbitrary (fractional) orders

$$\frac{dx}{dt} \in F_1(t, x(t), I^{\gamma} f_2(t, D^{\alpha} x(t))), \quad \alpha, \gamma \in (0, 1] \quad t \in (0, 1) \tag{1.1}$$

with the nonlocal boundary condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \ a_k > 0 \ \ \tau_k \in [0, 1], \tag{1.2}$$

the integral condition

$$\int_0^1 x(s) dg(s) = x_0 \tag{1.3}$$

and the infinite point boundary condition

$$\sum_{k=1}^{\infty} a_k x(\tau_k) = x_0, \ a_k > 0 \ and \ \tau_k \in [0, 1]. \tag{1.4}$$

We study the existence of solutions $x \in C[0, 1]$ of the problems (1.1) and (1.2), and deduce the existence of solutions of the problem of (1.1) with the conditions (1.3) and (1.4). Then the existence of the maximal and minimal solutions will be proved. The sufficient condition for the uniqueness and continuous dependence of the solution will be studied.

This paper is organised as: In Section 2, we prove the existence of continuous solutions of the problems (1.1) and (1.2), and deduce the existence of solutions of the problem of (1.1) with the conditions (1.3) and (1.4). In Section 3, the existence of the maximal and minimal solutions is proved. In Section 4, the sufficient condition for the uniqueness and continuous dependence of the solution are studied. Next, in Section 5, we extend our results to the nonlocal problems (1.3) and (2.1). Finally, some existence results is proved for the nonlocal problems (1.4) and (2.1) in Section 6.

2. Main results

Consider the following assumptions:

(I) (i) The set $F_1(t, x, y)$ is nonempty, closed and convex for all $(t, x, y) \in [0, 1] \times R \times R$.
(ii) $F_1(t, x, y)$ is measurable in $t \in [0, 1]$ for every $x, y \in R$.
(iii) $F_1(t, x, y)$ is upper semicontinuous in x and y for every $t \in [0, 1]$.
(iv) There exist a bounded measurable function $a_1 : [0, 1] \longrightarrow R$ and a positive constant K_1, such that

$$\begin{aligned} \|F_1(t, x, y)\| &= \sup\{|f_1| : f_1 \in F_1(t, x, y)\} \\ &\leq |a_1(t)| + K_1(|x| + |y|). \end{aligned}$$

Remark 2.1. *From the assumptions (i)–(iv) we can deduce that (see [3, 6, 7, 13]) there exists* $f_1 \in F_1(t, x, y)$*, such that*
(v) $f_1 : [0, 1]\times R\times R \longrightarrow R$ *is measurable in t for every* $x, y \in R$ *and continuous in* x, y *for* $t \in [0, 1]$*, and there exist a bounded measurable function* $a_1 : [0, 1] \to R$ *and a positive constant* $K_1 > 0$ *such that*

$$|f_1(t, x, y)| \leq |a_1(t)| + K_1(|x| + |y|),$$

and the functional f_1 *satisfies the differential equation*

$$\frac{dx}{dt} = f_1(t, x(t), I^\gamma f_2(t, D^\alpha x(t))), \quad \alpha, \gamma \in (0, 1] \text{ and } t \in (0, 1]. \tag{2.1}$$

(II) $f_2 : [0, 1] \times R \longrightarrow R$ *is measurable in t for any* $x \in R$ *and continuous in x for* $t \in [0, 1]$*, and there exist a bounded measurable function* $a_2 : [0, 1] \to R$ *and a positive constant* $K_2 > 0$ *such that*

$$|f_2(t, x)| \leq |a_2(t)| + K_2|x|, \ \forall t \in [0, 1] \text{ and } x \in R$$

and

$$\sup_{t\in[0,1]} |a_i(t)| \leq a_i, i = 1, 2.$$

(III) $2K_1\gamma + K_1K_2\alpha < \alpha\gamma\Gamma(2 - \alpha), \ \alpha, \gamma \in (0, 1]$.

Remark 2.2. *From (I) and (v) we can deduce that every solution of (1.1) is a solution of (2.1). Now, we shall prove the following lemma.*

Lemma 2.1. *If the solution of the problems (1.2)–(2.1) exists then it can be expressed by the integral equation*

$$\begin{aligned} y(t) \;=\; & \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - \theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] \\ & + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\phi, y(\theta))d\theta)ds. \end{aligned} \tag{2.2}$$

Proof. Consider the boundary value problems (1.2)–(2.1) be satisfied. Operating by $I^{1-\alpha}$ on both sides of (2.1) we can obtain

$$D^\alpha x(t) = I^{1-\alpha}\frac{dx}{dt} = I^{1-\alpha} f_1(t, x(t), I^\gamma f_2(t, D^\alpha x(t))). \tag{2.3}$$

Taking

$$D^\alpha x(t) = y(t), \tag{2.4}$$

then we obtain

$$x(t) = x(0) + I^\alpha y(t). \tag{2.5}$$

Putting $t = \tau$ and multiplying (2.5) by $\Sigma_{k=1}^m a_k$, then we get

$$\Sigma_{k=1}^m a_k x(\tau_k) = \Sigma_{k=1}^m a_k x(0) + \Sigma_{k=1}^m a_k I^\alpha y(\tau_k), \tag{2.6}$$

$$x_0 = \Sigma_{k=1}^m a_k x(0) + \Sigma_{k=1}^m a_k I^\alpha y(\tau_k) \tag{2.7}$$

and

$$x(0) = \frac{1}{\Sigma_{k=1}^m a_k}[x_0 - \Sigma_{k=1}^m a_k I^\alpha y(\tau_k)]. \tag{2.8}$$

Then

$$x(t) = \frac{1}{\Sigma_{k=1}^m a_k}[x_0 - \Sigma_{k=1}^m a_k I^\alpha y(\tau_k)] + I^\alpha y(t). \tag{2.9}$$

Substituting (2.8) and (2.9) in (2.5), which completes the proof.

Theorem 2.1. *Let assumptions (I)–(III) be satisfied. Then the integral equation (2.2) has at least one continuous solution.*

Proof. Define a set Q_r as

$$Q_r = \{y \in C[0,1] : \|y\| \leq r\},$$

$$r = \frac{\Gamma(\gamma+1)[a_1 + K_1 A|x_0|] + k_1 a_2}{\alpha\gamma\Gamma(2-\alpha) - [2K_1\alpha + K_1K_2\gamma]}, \quad (\sum_{k=1}^m a_k)^{-1} = A,$$

and the operator F by

$$\begin{aligned} Fy(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] \\ &+ \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y(\theta))d\theta)ds. \end{aligned}$$

For $y \in Q_r$, then

$$\begin{aligned} |Fy(t)| &= |\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] \\ &+ \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y(\theta))d\theta)ds| \\ &\leq K_1 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}[\frac{|x_0|}{\sum_{k=1}^m a_k} + \frac{\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta)|d\theta}{\sum_{k=1}^m a_k} \\ &+ \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta)|d\theta + \int_0^s \frac{(s-\phi)^{\gamma-1}}{\Gamma(\gamma)}(K_2|y(\theta)| + |a_2(t)|)d\theta] + |a_1(t)|)ds \\ &\leq \frac{1}{\Gamma(2-\alpha)}[K_1A|x_0| + \frac{K_1\|y\|}{\Gamma(\alpha+1)} + K_1\frac{\|y\|}{\Gamma(\alpha+1)} + \frac{K_1K_2\|y\|}{\Gamma(\gamma+1)} + \frac{K_1a_2}{\Gamma(\gamma+1)} + a_1] \\ &\leq \frac{1}{\Gamma(2-\alpha)}[K_1A|x_0| + \frac{K_1 r}{\Gamma(\alpha+1)} + K_1\frac{r}{\Gamma(\alpha+1)} + \frac{K_1K_2 r}{\Gamma(\gamma+1)} + \frac{K_1a_2}{\Gamma(\gamma+1)} + a_1] \\ &\leq \frac{1}{\Gamma(2-\alpha)}[K_1A|x_0| + \frac{2K_1 r}{\Gamma(\alpha+1)} + \frac{K_1K_2 r}{\Gamma(\gamma+1)} + \frac{K_1a_2}{\Gamma(\gamma+1)} + a_1] \leq r. \end{aligned}$$

Thus, the class of functions $\{Fy\}$ is uniformly bounded on Q_r and $F : Q_r \to Q_r$. Let $y \in Q_r$ and $t_1, t_2 \in [0,1]$ such that $|t_2 - t_1| < \delta$, then

$$
\begin{aligned}
&|Fy(t_2) - Fy(t_1)| \\
= &\ |\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}(f_1(s, x(s), I^\gamma f_2(s, y(s)))ds - \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}(f_1(s, x(s), I^\gamma f_2(s, y(s)))ds| \\
\leq &\ |\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}(f_1(s, x(s), I^\gamma f_2(s, y(s)))ds - \int_0^{t_1} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}(f_1(s, x(s), I^\gamma f_2(s, y(s)))ds \\
&+ \int_0^{t_1} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}(f_1(s, x(s), I^\gamma f_2(s, y(s)))ds - \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}(f_1(s, x(s), I^\gamma f_2(s, y(s)))ds| \\
\leq &\ \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}|(f_1(s, x(s), I^\gamma f_2(s, y(s)))|ds \\
&+ \int_0^{t_1} \frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}|(f_1(s, x(s), I^\gamma f_2(s, y(s)))ds| \\
\leq &\ \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}|(f_1(s, x(s), I^\gamma f_2(s, y(s)))|ds \\
&+ \int_0^{t_1} \frac{(t_2-s)^{\alpha} - (t_1-s)^{\alpha}}{\Gamma(1-\alpha)(t_2-s)^{\alpha}(t_1-s)^{\alpha}}.|(f_1(s, x(s), I^\gamma f_2(s, y(s)))|ds.
\end{aligned}
$$

Thus, the class of functions $\{Fy\}$ is equicontinuous on Q_r and $\{Fy\}$ is compact operator by the Arzela-Ascoli Theorem [5].

Now we prove that F is continuous operator. Let $y_n \subset Q_r$ be convergent sequence such that $y_n \to y$, then

$$
\begin{aligned}
Fy_n(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_n(\theta)d\theta] \\
&\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_n(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y_n(\theta)))ds.
\end{aligned}
$$

Using Lebesgue dominated convergence Theorem [5] and assumptions (iv)–(II) we have

$$
\begin{aligned}
\lim_{n\to\infty} Fy_n(t) &= \lim_{n\to\infty} \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_n(\theta)d\theta] \\
&\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_n(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y_n(\theta)))ds \\
&= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} y_n(\theta)d\theta] \\
&\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} y_n(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, \lim_{n\to\infty} y_n(\theta)))ds \\
&= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] \\
&\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y(\theta)))ds \ = \ Fy(t).
\end{aligned}
$$

Then $F : Q_r \to Q_r$ is continuous, and by Schauder Fixed Point Theorem [5] there exists at least one solution $y \in C[0,1]$ of (2.2). Now

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}[x(0) + I^{\alpha} y(t)]\\ &= \frac{d}{dt} I^{\alpha} I^{1-\alpha} f_1(t, x(t), I^{\gamma} f_2(t, y(t)))\\ &= \frac{d}{dt} I f_1(t, x(t), I^{\gamma} f_2(t, y(t)))\\ &= f_1(t, x(t), I^{\gamma} f_2(t, y(t))).\end{aligned}$$

Putting $t = \tau$ and using (2.9), we obtain

$$x(t) = \frac{1}{\Sigma_{k=1}^{m} a_k}[x_0 - \Sigma_{k=1}^{m} a_k I^{\alpha} y(\tau_k)] + I^{\alpha} y(t),$$

$$\Sigma_{k=1}^{m} a_k x(\tau_k) = \Sigma_{k=1}^{m} a_k \frac{1}{\Sigma_{k=1}^{m} a_k}[x_0 - \Sigma_{k=1}^{m} a_k I^{\alpha} y(\tau_k)] + \Sigma_{k=1}^{m} a_k I^{\alpha} y(\tau_k),$$

then

$$\Sigma_{k=1}^{m} a_k x(\tau_k) = x_0,\ a_k > 0 \ and\ \tau_k \in [0,1].$$

This proves the equivalence between the problems (1.2)–(2.1) and the integral equation (2.2). Then there exists at least one solution $y \in C[0,1]$ of the problems (1.2)–(2.1).

3. Maximal and minimal solutions

Here, we shall study the maximal and minimal solutions for the problems (1.2) and (2.1). Let $y(t)$ be any solution of (2.2), let $u(t)$ be a solution of (2.2), then $u(t)$ is said to be a maximal solution of (2.2) if it satisfies the inequality

$$y(t) \le u(t), \quad t \in [0,1].$$

A minimal solution can be defined by similar way by reversing the above inequality.

Lemma 3.1. *Let the assumptions of Theorem 2.1 be satisfied. Assume that $x(t)$ and $y(t)$ are two continuous functions on* $[0,1]$ *satisfying*

$$\begin{aligned}y(t) &\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta) d\theta]\\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta) d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y(\theta))) ds \ \ t \in [0,1],\\ x(t) &\ge \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} x(\theta) d\theta]\\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} x(\theta) d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, x(\theta))) ds \ \ t \in [0,1],\end{aligned}$$

where one of them is strict.

Let functions f_1 and f_2 be monotonic nondecreasing in y, then

$$y(t) < x(t), \quad t > 0. \tag{3.1}$$

Proof. Let the conclusion (3.1) be not true, then there exists t_1 with

$$y(t_1) < x(t_1), \quad t_1 > 0 \quad \text{ and } y(t) < x(t), \quad 0 < t < t_1.$$

Since f_1 and f_2 are monotonic functions in y, then we have

$$\begin{aligned} y(t_1) \quad &\leq \quad \int_0^{t_1} \frac{(t_1 - s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - \theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y(\theta)))ds \\ &< \quad \int_0^{t_1} \frac{(t - s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - \theta)^{\alpha-1}}{\Gamma(\alpha)} x(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} x(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, x(\theta)))ds \\ &< \quad x(t_1), \ t_1 \in [0, 1]. \end{aligned}$$

This contradicts the fact that $y(t_1) = x(t_1)$, then $y(t) < x(t)$. This completes the proof.

For the existence of the continuous maximal and minimal solutions for (2.1), we have the following theorem.

Theorem 3.1. *Let the assumptions of Theorem 2.1 be hold. Moreover, if f_1 and f_2 are monotonic nondecreasing functions in y for each $t \in [0, 1]$, then Eq (2.1) has maximal and minimal solutions.*

Proof. First, we should demonstrate the existence of the maximal solution of (2.1). Let $\epsilon > 0$ be given. Now consider the integral equation

$$\begin{aligned} y_\epsilon(t) \quad &= \quad \int_0^{t} \frac{(t - s)^{-\alpha}}{\Gamma(1-\alpha)} f_{1,\epsilon}\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - \theta)^{\alpha-1}}{\Gamma(\alpha)} y_\epsilon(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_\epsilon(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_{2,\epsilon}(\theta, y_\epsilon(\theta))d\theta\Big)ds, \quad t \in [0, 1], \end{aligned}$$

where

$$f_{1,\epsilon}(s, x_\epsilon(s), y_\epsilon(s)) = f_1(s, x_\epsilon(s), y_\epsilon(s)) + \epsilon,$$
$$f_{2,\epsilon}(s, x_\epsilon(s)) = f_2(s, x_\epsilon(s)) + \epsilon.$$

For $\epsilon_2 > \epsilon_1$, we have

$$\begin{aligned} y_{\epsilon_2}(t) \quad &= \quad \int_0^{t} \frac{(t - s)^{-\alpha}}{\Gamma(1-\alpha)} f_{1,\epsilon_2}\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - \theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_2}(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_2}(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_{2,\epsilon_2}(\theta, y_{\epsilon_2}(\theta))d\theta\Big)ds, \quad t \in [0, 1], \end{aligned}$$

$$y_{\epsilon_2}(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\Big[f_1\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_2}(\theta)d\theta] + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_2}(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}(f_2(\theta, y_{\epsilon_2}(\theta)) + \epsilon_2)d\theta\Big) + \epsilon_2\Big]ds,\ t \in [0,1].$$

Also

$$\begin{aligned} y_{\epsilon_1}(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_{1,\epsilon_1}\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_1}(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_1}(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_{2,\epsilon_1}(\theta, y_{\epsilon_1}(\theta))d\theta\Big)ds, \\ y_{\epsilon_1}(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\Big[f_1\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\phi)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_1}(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_1}(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}(f_2(\theta, y_{\epsilon_1}(\theta)) + \epsilon_1)d\theta\Big) + \epsilon_1\Big]ds \\ &> \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\Big[f_1\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_2}(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_2}(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}(f_2(\theta, y_{\epsilon_2}(\tau)) + \epsilon_2)d\theta\Big) + \epsilon_2\Big]ds. \end{aligned}$$

Applying Lemma 3.1, we obtain

$$y_{\epsilon_2} < y_{\epsilon_1},\ t \in [0,1].$$

As shown before, the family of function $y_\epsilon(t)$ is equi-continuous and uniformly bounded, then by Arzela Theorem, there exist decreasing sequence ϵ_n, such that $\epsilon_n \to 0$ as $n \to \infty$, and $u(t) = \lim_{n\to\infty} y_{\epsilon_n}(t)$ exists uniformly in $[0,1]$ and denote this limit by $u(t)$. From the continuity of the functions, $f_{2,\epsilon_n}(t, y_{\epsilon_n}(t))$, we get $f_{2,\epsilon_n}(t, y_{\epsilon_n}(t)) \longrightarrow f_2(t, y(t))$ as $n \to \infty$ and

$$u(t) = \lim_{n\to\infty} y_{\epsilon_n}(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\Big[f_1\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_n}(\theta)d\theta] + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_{\epsilon_n}(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}(f_2(\theta, y_{\epsilon_n}(\theta)) + \epsilon_n)\Big) + \epsilon_n\Big]ds,\ \ t \in [0,1].$$

Now we prove that $u(t)$ is the maximal solution of (2.1). To do this, let $y(t)$ be any solution of (2.1), then

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y(\theta))d\theta\Big)ds, \\ y_\epsilon(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\Big[f_1\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_\epsilon(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_\epsilon(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}(f_2(\theta, y_\epsilon(\theta)) + \epsilon)d\theta\Big) + \epsilon\Big]ds \end{aligned}$$

and

$$
\begin{aligned}
y_\epsilon(t) \quad > \quad & \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_\epsilon(\theta)d\theta] \\
& + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_\epsilon(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y_\epsilon(\theta))\Big)ds.
\end{aligned}
$$

Applying Lemma 3.1, we obtain

$$y(t) < y_\epsilon(t), t \in [0,1].$$

From the uniqueness of the maximal solution it clear that $y_\epsilon(t)$ tends to $u(t)$ uniformly in $[0,1]$ as $\epsilon \to 0$.

By a similar way as done above, we can prove the existence of the minimal solution.

4. Uniqueness of the solution

Here, we study the sufficient condition for the uniqueness of the solution $y \in C[0,1]$ of problems (1.2) and (2.1). Consider the following assumptions:

(I^*) (i) The set $F_1(t,x,y)$ is nonempty, closed and convex for all $(t,x,y) \in [0,1] \times R \times R$.
(ii) $F_1(t,x,y)$ is measurable in $t \in [0,1]$ for every $x, y \in R$.
(iii) F_1 satisfies the Lipschitz condition with a positive constant K_1 such that

$$H(F_1(t,x_1,y_1), F_1(t,x_2,y_2)| \leq K_1(|x_1 - x_2| + |y_1 - y_2|),$$

where $H(A,B)$ is the Hausdorff metric between the two subsets $A, B \in [0,1] \times E$.

Remark 4.1. *From this assumptions we can deduce that there exists a function $f_1 \in F_1(t,x,y)$, such that (iv) $f_1 : [0,1] \times R \times R \to R$ is measurable in $t \in [0,1]$ for every $x, y \in R$ and satisfies Lipschitz condition with a positive constant K_1 such that (see [3, 7])*

$$|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| \leq K_1(|x_1 - x_2| + |y_1 - y_2|).$$

(II^*) *$f_2 : [0,1] \times R \longrightarrow R$ is measurable in $t \in [0,T]$ and satisfies Lipschitz condition with positive constant K_2, such that*

$$|f_2(t,x) - f_2(t,y)| \leq K_2|x - y|.$$

From the assumption (I^*), we have

$$|f_1(t,x,y)| - |f_1(t,0,0)| \leq |f_1(t,x,y) - f_1(t,0,0)| \leq K_1(|x| + |y|).$$

Then

$$
\begin{aligned}
|f_1(t,x,y)| &\leq K_1(|x| + |y|) + |f_1(t,0,0)| \\
&\leq K_1(|x| + |y|) + |\, a_1(t) \,|,
\end{aligned}
$$

where $|a_1(t)| = \sup_{t \in I} |f_1(t,0,0)|$.

From the assumption (II^*), we have

$$|f_2(t,y)| - |f_2(t,0)| \leq |f_2(t,y) - f_2(t,0)| \leq K_2|y|.$$

Then

$$\begin{aligned} |f_2(t,x)| &\leq K_2|x| + |f_2(t,0)| \\ &\leq K_2|x| + \mid a_2(t) \mid, \end{aligned}$$

where $|a_2(t)| = \sup_{t\in I}|f_1(t,0)|$.

Theorem 4.1. *Let the assumptions* (I^*) *and* (II^*) *be satisfied. Then the solution of the problems (1.2) and (2.1) is unique.*

Proof. *Let* $y_1(t)$ *and* $y_2(t)$ *be solutions of the problems (1.2) and (2.1), then*

$$\begin{aligned} |y_1(t) - y_2(t)| &= |\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_1(\theta)d\theta] \\ &+ \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_1(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y_1(\theta))d\theta)ds \\ &- \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_2(\theta)d\theta] \\ &+ \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y_2(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y_2(\theta))d\theta)ds| \\ &\leq \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}[K_1 \frac{\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)}}{\sum_{k=1}^m a_k} \|y_1 - y_2\| d\theta]ds \\ &+ K_1 \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} \|y_1 - y_2\| d\theta + K_1 \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} |f_2(\theta, y_1(\theta)) - f_2(\theta, y_2(\theta))| d\theta]ds \\ &\leq \frac{1}{\Gamma(2-\alpha)}[\frac{K_1 \sum_{k=1}^m a_k}{\sum_{k=1}^m a_k \Gamma(1+\alpha)} \|y_1 - y_2\| + \frac{K_1}{\Gamma(1+\alpha)} \|y_1 - y_2\| + \frac{K_1 K_2}{\Gamma(1+\gamma)} \|y_1 - y_2\|] \\ &\leq \frac{2K_1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \|y_1 - y_2\| + \frac{K_1 K_2}{\Gamma(1+\gamma)\Gamma(2-\alpha)} \|y_1 - y_2\| \\ &\leq [\frac{2K_1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} + \frac{K_1 K_2}{\Gamma(1+\gamma)\Gamma(2-\alpha)}] \|y_1 - y_2\|. \end{aligned}$$

Then

$$\|y_1 - y_2\|[\alpha\gamma\Gamma(2-\alpha) - (2K_1\alpha + K_1K_2\gamma)] < 0.$$

Since $(\alpha\gamma\Gamma(2-\alpha) - (2K_1\alpha + K_1K_2\gamma)) < 1$, *then* $y_1(t) = y_2(t)$ *and the solution of (1.2) and (2.1) is unique.*

4.1. Continuous dependence of the solution

Definition 4.1. *The unique solution of the problems (1.2) and (2.1) depends continuously on initial data* x_0, *if* $\epsilon > 0$, $\exists\delta > 0$, *such that*

$$|x_0 - x_0^*| \leq \delta \quad \Rightarrow \|y - y^*\| \leq \epsilon,$$

where y^* is the unique solution of the integral equation

$$\begin{aligned} y^*(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0^* - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y^*(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y^*(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y^*(\theta))d\theta)ds|. \end{aligned}$$

Theorem 4.2. *Let the assumptions (I^*) and (II^*) be satisfied, then the unique solution of (1.2) and (2.1) depends continuously on x_0.*

Proof. Let $y(t)$ and $y^*(t)$ be the solutions of problems (1.2) and (2.1), then

$$\begin{aligned} |y(t) - y^*(t)| &= |\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y(\theta))d\theta)ds \\ &\quad - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0^* - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y^*(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y^*(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y^*(\theta))d\theta)ds| \\ &\quad + K_1 \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} |y(\theta) - y^*(\theta)|d\theta + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\theta, y(\theta)) - f_2(\theta, y^*(\theta))|d\theta]ds \\ &\leq \frac{1}{\Gamma(2-\alpha)}[\frac{K_1\|x_0 - x_0^*\|}{\sum_{k=1}^m a_k} + \frac{\sum_{k=1}^m a_k K_1 \|y - y^*\|}{\sum_{k=1}^m a_k \Gamma(\alpha+1)} + \frac{K_1\|y - y^*\|}{\Gamma(\alpha+1)} + \frac{K_1K_2\|y - y^*\|}{\Gamma(\gamma+1)}] \\ &\leq \frac{K_1 A\delta}{\Gamma(2-\alpha)} + [\frac{2K_1}{\Gamma(\alpha+1)\Gamma(2-\alpha)} + \frac{K_1K_2}{\Gamma(\gamma+1)\Gamma(2-\alpha)}]\|y - y^*\|, \end{aligned}$$

then we obtain

$$|y(t) - y^*(t)| \leq (AK_1\delta)(\alpha\gamma\Gamma(2-\alpha) - (2K_1\alpha + K_1K_2\gamma))^{-1} \leq \epsilon$$

and

$$\|y(t) - y^*(t)\| \leq \epsilon.$$

Definition 4.2. *The unique solution of the problems (1.2) and (2.1) depends continuously on initial data a_k, if $\epsilon > 0$, $\exists\delta > 0$, such that*

$$\sum_{k=1}^m |a_k - a_k^*| \leq \delta \quad \Rightarrow \|y - y^*\| \leq \epsilon,$$

where y^* is the unique solution of the integral equation

$$\begin{aligned} y^*(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k^* \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y^*(\theta)d\theta] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y^*(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y^*(\theta))d\theta)ds|. \end{aligned}$$

Theorem 4.3. *Let the assumptions* (I^*) *and* (II^*) *be satisfied, then the unique solution of problems (1.2) and (2.1) depends continuously on* a_k.

Proof. Let $y(t)$ and $y^*(t)$ be the solutions of problems (1.2) and (2.1) and $(\sum_{k=1}^m a_k^*)^{-1} = A^*$,

$$\begin{aligned}
|y(t) - y^*(t)| &= |\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] \\
&+ \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y(\theta))d\theta)ds \\
&- \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^m a_k^*}[x_0 - \sum_{k=1}^m a_k^* \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y^*(\theta)d\theta] \\
&+ \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y^*(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\theta, y^*(\theta))d\theta)ds| \\
&\leq \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}[\frac{|x_0|(\sum_{k=1}^m a_k^* - \sum_{k=1}^m a_k)}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*} \\
&+ \frac{\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta)|d\theta}{\sum_{k=1}^m a_k} - \frac{\sum_{k=1}^m a_k^* \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y^*(\theta)|d\theta}{\sum_{k=1}^m a_k^*} \\
&+ K_1 \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta) - y^*(\theta)|d\theta + K_1K_2 \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}|y(\phi) - y^*(\theta)|d\theta]ds \\
&\leq \frac{1}{\Gamma(2-\alpha)}[\frac{K_1|x_0| \sum_{k=1}^m |a_k^* - a_k|}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*} \\
&+ \frac{K_1 \sum_{k=1}^m a_k(\sum_{k=1}^m a_k^* \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta)|d\theta - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta)|d\theta)}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*} \\
&+ \frac{K_1 \sum_{k=1}^m a_k^*(\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta)|d\theta - \sum_{k=1}^m a_k^* \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta)|d\theta)}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*} \\
&+ K_1 \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}|y(\theta) - y^*(\theta)|d\theta + K_1K_2 \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}|y(\phi) - y^*(\theta)|d\theta]ds \\
&\leq \frac{1}{\Gamma(2-\alpha)}[\frac{K_1|x_0|\delta}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*} + \frac{K_1(\sum_{k=1}^m a_k\delta\frac{r}{\Gamma(\alpha+1)} + \sum_{k=1}^m a_k^*\delta\frac{r}{\Gamma(\alpha+1)})}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*} \\
&+ \frac{K_1\|y - y^*\|}{\Gamma(\alpha+1)} + \frac{K_1K_2\|y - y^*\|}{\Gamma(\gamma+1)} \\
&\leq \frac{K_1\delta|x_0|}{\Gamma(2-\alpha)\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*} + \frac{K_1(\sum_{k=1}^m a_k\delta\frac{r}{\Gamma(\alpha+1)} + \sum_{k=1}^m a_k^*\delta\frac{r}{\Gamma(\alpha+1)})}{\Gamma(2-\alpha)\Gamma(\alpha+1)\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*} \\
&+ \frac{K_1\|y - y^*\|}{\Gamma(2-\alpha)\Gamma(\alpha+1)} + \frac{K_1K_2\|y - y^*\|}{\Gamma(\gamma+1)\Gamma(2-\alpha)},
\end{aligned}$$

then we obtain

$$|y(t) - y^*(t)| \leq (AA^*K_1\delta[|x_0| + r(\sum_{k=1}^m a_k + \sum_{k=1}^m a_k^*)]).\ (\alpha\gamma\Gamma(2-\alpha) - (2K_1\alpha + K_1K_2\gamma))^{-1} \leq \epsilon$$

and

$$\|y(t) - y^*(t)\| \leq \epsilon.$$

5. Riemann-Stieltjes integral condition

Let $y \in C[0,1]$ be the solution of the nonlocal boundary value problems (1.2) and (2.1). Let $a_k = (g(t_k) - g(t_{k-1}))$, g is increasing function, $\tau_k \in (t_{k-1} - t_k)$, $0 = t_0 < t_1 < t_2, ... < t_m = 1$, then, as $m \longrightarrow \infty$ the nonlocal condition (1.2) will be

$$\sum_{k=1}^{m} (g(t_k) - g(t_{k-1}) y(\tau_k) = x_0.$$

As the limit $m \longrightarrow \infty$, we obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} (g(t_k) - g(t_{k-1}) y(\tau_k) = \int_0^1 y(s) dg(s) = x_0.$$

Theorem 5.1. *Let the assumptions (I)–(III) be satisfied. If $\sum_{k=1}^{m} a_k$ be convergent, then the nonlocal boundary value problems of (1.3) and (2.1) have at least one solution given by*

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{g(1)-g(0)}[x_0 - \int_0^1 \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta) d\theta dg(s)] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta) d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\phi, y(\theta)) d\theta) ds. \end{aligned}$$

Proof. As $m \to \infty$, the solution of the nonlocal boundary value problem (2.1) will be

$$\begin{aligned} \lim_{m\to\infty} y(t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, (g(1)-g(0))^{-1} x_0 \\ &\quad -(g(1)-g(0))^{-1}. \lim_{m\to\infty} \sum_{k=1}^{m} [g(t_k) - g(t_{k-1}] \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta) d\theta \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta) d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\phi, y(\theta)) d\theta) ds \\ &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{g(1)-g(0)}[x_0 - \int_0^1 \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta) d\theta dg(s)] \\ &\quad + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta) d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\phi, y(\theta)) d\theta) ds. \end{aligned}$$

6. Infinite-point boundary condition

Theorem 6.1. *Let the assumptions (I)–(III) be satisfied, then the nonlocal boundary value problems of (1.4) and (2.1) have at least one solution given by*

$$y(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^{\infty} a_k}[x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\phi, y(\theta))d\theta)ds.$$

Proof. Let the assumptions of Theorem 2.1 be satisfied. Let $\sum_{k=1}^{m} a_k$ be convergent, then take the limit to (1.4), we have

$$\lim_{m\to\infty} y(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \lim_{m\to m} \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \lim_{m\to\infty} \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\phi, y(\theta))d\theta)ds.$$

Now

$$|a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta| \le |a_k| \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} |y(\theta)|d\theta \le \frac{|a_k|||y||}{\Gamma(\alpha+1)} \le \frac{|a_k|r}{\Gamma(\alpha+1)}$$

and by the comparison test ($\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta$) is convergent,

$$y(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \frac{1}{\sum_{k=1}^{\infty} a_k}[x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} \frac{(\tau_k-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta] + \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} y(\theta)d\theta, \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} f_2(\phi, y(\theta))d\theta)ds.$$

Furthermore, from (2.9) we have

$$\sum_{k=1}^{\infty} a_k x(\tau_k) = \sum_{k=1}^{\infty} a_k [\frac{1}{\sum_{k=1}^{\infty} a_k}(x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds)] + \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds)] = x_0.$$

Example 6.1. Consider the following nonlinear integro-differential equation

$$\frac{dx}{dt} = t^4 e^{-t} + \frac{x(t)}{\sqrt{t+2}} + \frac{1}{3} I^{\gamma}(\cos(5t+1) + \frac{1}{9}[t^5 \sin D^{\frac{1}{3}} x(t) + e^{-3t} x(t)]) \tag{6.1}$$

with boundary condition

$$\sum_{k=1}^{m} \left[\frac{1}{k} - \frac{1}{k+1}\right] x(\tau_k) = x_0, \ a_k > 0 \ \ \tau_k \in [0,1]. \tag{6.2}$$

Let

$$f_1(t, x(t), I^{\gamma} f_2(t, D^{\alpha} x(t))) = t^4 e^{-t} + \frac{x(t)}{\sqrt{t+2}} + \frac{1}{3} I^{\gamma}(\cos(5t+1) + \frac{1}{9}[t^5 \sin D^{\frac{1}{3}} x(t) + e^{-3t} x(t)]),$$

then

$$|f_1(t, x(t), I^\gamma f_2(t, D^\alpha x(t))) = |t^4 e^{-t} + \frac{x(t)}{\sqrt{2t+4}} + \frac{1}{4}(\frac{4}{3} I^\gamma (\cos(5t+1) + \frac{1}{9}[t^5 \sin D^{\frac{1}{3}} x(t) + e^{-3t} x(t)]))|$$

and also

$$|I^\gamma f_2(t, D^\alpha x(t))| \le \frac{1}{4}(\frac{4}{3} I^\gamma |\cos(5t+1) + \frac{1}{3}[t^5 \sin D^{\frac{1}{3}} x(t) + e^{-3t} x(t)]|.$$

It is clear that the assumptions (I) and (II) of Theorem 2.1 are satisfied with $a_1(t) = t^4 e^{-t} \in L^1[0,1]$, $a_2(t) = \frac{4}{3} I^\gamma |(\cos(5t+1)| \in L^1[0,1]$, and let $\alpha = \frac{1}{3}$, $\gamma = \frac{2}{3}$, then $2K_1\gamma + K_1K_2\alpha = \frac{1}{2} < \alpha\gamma\Gamma(2-\alpha) = \frac{1}{2}\Gamma(2-\alpha)$. Therefore, by applying Theorem 2.1, the nonlocal problems (6.1) and (6.2) has a continuous solution.

7. Conclusions

In this paper, we have studied a boundary value problem of fractional order differential inclusion with nonlocal, integral and infinite points boundary conditions. We have prove some existence results for that a single nonlocal boundary value problem, in of proving some existence results for a boundary value problem of fractional order differential inclusion with nonlocal, integral and infinite points boundary conditions. Next, we have proved the existence of maximal and minimal solutions. Then we have established the sufficient conditions for the uniqueness of solutions and continuous dependence of solution on some initial data and on the coefficients a_k are studied. Finally, we have proved the existence of a nonlocal boundary value problem with Riemann-Stieltjes integral condition and with infinite-point boundary condition. An example is given to illustrate our results.

Conflict of interest

The authors declare no conflict of interest.

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