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Research article

Decision making algorithmic techniques based on aggregation operations and similarity measures of possibility intuitionistic fuzzy hypersoft sets

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Abstract: Soft set has limitation for the consideration of disjoint attribute-valued sets corresponding to distinct attributes whereas hypersoft set, an extension of soft set, fully addresses this scarcity by replacing the approximate function of soft sets with multi-argument approximate function. Some structures (i.e., possibility fuzzy soft set, possibility intuitionistic fuzzy soft set) exist in literature in which a possibility of each element in the universe is attached with the parameterization of fuzzy sets and intuitionistic fuzzy sets while defining fuzzy soft set and intuitionistic fuzzy soft set respectively. This study aims to generalize the existing structure (i.e., possibility intuitionistic fuzzy soft set) and to make it adequate for multi-argument approximate function. Therefore, firstly, the elementary notion of possibility intuitionistic fuzzy hypersoft set is developed and some of its elementary properties i.e., subset, null set, absolute set and complement, are discussed with numerical examples. Secondly, its set-theoretic operations i.e., union, intersection, AND, OR and relevant laws are investigated with the help of numerical examples, matrix and graphical representations. Moreover, algorithms based on AND/OR operations are proposed and are elaborated with illustrative examples. Lastly, similarity measure between two possibility intuitionistic fuzzy hypersoft sets is characterized with the help of example. This concept of similarity measure is successfully applied in decision making to judge the eligibility of a candidate for an appropriate job. The proposed similarity formulation is compared with the relevant existing models and validity of the generalization of the proposed structure is discussed.

Keywords: fuzzy set; intuitionistic fuzzy set; soft set; intuitionistic fuzzy soft set; hypersoft set

Mathematics Subject Classification: 03E72, 68T35, 90B50

1. Introduction

Fuzzy sets theory (FST) [1] and intuitionistic fuzzy set theory (IFST) [2] are considered apt mathematical modes to tackle many intricate problems involving various uncertainties, in different mathematical disciplines. The former one emphasizes on the degree of true belongingness of a certain object from the initial sample space whereas the later one accentuates on degree of true membership and degree of non-membership with condition of their dependency on each other. FST and IFST have some sort of complexities which bound them to tackle issue including uncertainty efficiently. The main cause of these hurdles is potentially, the deficiency of the parametrization tool. It requests a mathematical tool free from all such obstacles to handle such issues. This inadequacy is settled down with the introduction of soft set theory (SS-Theory) [3] which is another defined class of subsets of the universe of discourse. The researchers [4–13] examined and explored some rudimentary properties, operations, laws of SS-Theory with applications in decision making.

The existing studies based on fuzzy soft sets (FS-sets), intuitionistic fuzzy soft sets (IFS-sets), etc., are widely used in different environments to solve decision-making problems. However, under certain cases, these existing approaches fail to classify the objects based on their possibility degrees. In other words, the existing studies have treated the possibility degree of each element as one. However, in many practical applications, different persons may have their possibility degrees which differ from one related to each object. To address this issue in decision-making process, Alkhazaleh et al. [14] introduced the concept of possibility of FS-sets by assigning a possibility degree to each number of FS-set. However, in this set, there is a complete lack of degree of non-membership during the analysis. To tackle it and address more appropriately, a concept of possibility intuitionistic fuzzy soft set (PIFS-set) was introduced by Bashir et al. [15]. PIFS-set is more generalized than the existing FS-sets, IFS-sets, and other sets. In PIFS-set, a degree of possibility of each component is assigned to degrees of intuitionistic fuzzy soft numbers (IFS-Ns) during evaluating the object. In numerous daily-life problems, parametric values are additionally subdivided into disjoint attribute-values sets. The existing SS-Theory is deficient for managing such sort of attributive-valued sets. Hypersoft set theory (HS-Theory) [16] is developed to make SS-Theory in accordance with attributive-valued sets to handle real life situations. Certain rudimentary properties, operations, laws, relations and points of HS-Theory are explored in [17–19] for appropriate indulgent and further use in various fields. The utilizations of HS-Theory in decision making are premeditated in [20–24] and the gluing investigations of HS-Theory with concave, convex, complex sets and parameterization are discussed in [25–27]. Kamacı [28] made very valuable research on hybrid structures of hypersoft sets and rough sets. Recently Kamacı et al. [29] investigated some new hybridized structures i.e., n-ary fuzzy hypersoft expert sets which are the extensions of existing structures (i.e., n-ary fuzzy soft expert sets).

1.1. Research gap and motivation

The following points may lead to motivation of proposed study.

1) The possibility fuzzy soft set (PFS-set), introduced by Alkhazaleh et al., is a gluing concept of

FS-Theory and SS-theory with attachment of a possibility degree to each approximate element of fuzzy soft set. They also calculated similarity measures between two PFS-sets and applied PFS-set to decision-making problem for clinical diagnosis. In 2012, Bashir et al. [30] extended the concept of PFS-set with introduction of possibility fuzzy soft expert set (PFSE-set) to adequate it with the consideration of multi-expert opinions. In 2014, Zhang et al. [31] characterized possibility multi-fuzzy soft set (PMFS-set) by assigning a possibility grade to each approximate member of multi-fuzzy soft set. In 2017, Kalaiselvi et al. [32] presented decision-making approach in sports via PFS-set. In 2018, Ponnalagu et al. [33] investigated more on PFSE-set with applications. In 2019, Garg et al. [34] and Khalil et al. [35] discussed decision-making algorithms based on PIFS-set and possibility m-polar fuzzy soft sets (PMPFS-set) respectively.

2) In certain real world scenarios, distinct parameters are further classified into disjoint sets having sub-parametric values (Figure 1 depicts the vivid comparison of soft set model and hypersoft set model. It presents the optimal selection of a mobile with the help of suitable parameters in case of soft set and suitable sub-parametric values in case of hypersoft set). In decision-making, the jury may endure some sort of tendency and proclivity while paying no attention to such parametric categorization during the decision. Soft set theory and its hybridized models have constraints regarding the consideration of such parametric categorization. Along these lines another construction requests its place in writing for tending to such obstacle, so hypersoft set is conceptualized to handle such situations. It has made the whole decision-making process more flexible and reliable. Also, it not only fulfills the requirements of existing soft-like literature for multi-attributive approximate functions but also supports the decision makers to make decisions with deep inspection.

3) Inspiring from above literature, new notions of possibility intuitionistic fuzzy hypersoft set (PIFHS-set) are conceptualized along with some elementary essential properties, aggregation operations and generalized typical results. Moreover, decision-making algorithmic approaches based on aggregation operators and similarity measures of PIFHS-sets are proposed to solve real life problems.

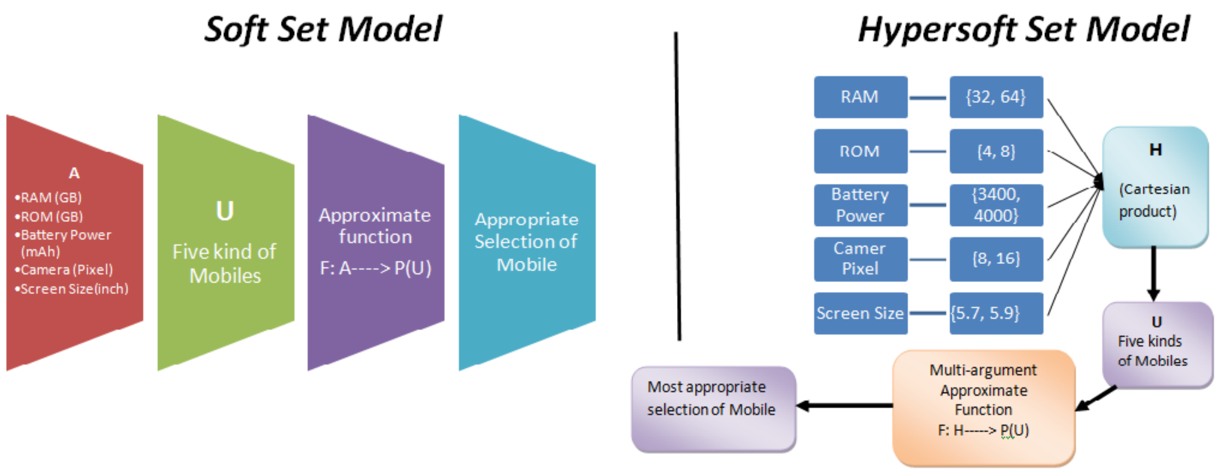

Figure 1. Comparison of soft set and hypersoft set models.

1.2. *Main contributions*

The following are the main contributions of this proposed study:

1) The existing relevant models are made adequate with the consideration of multi-argument approximate function through the development of the theory of PIFHS-sets.
2) The scenario where parameters are further partitioned into sub-parametric values in the form of sets, is tackled by using PIFHS-sets.
3) Some fundamentals like elementary properties and operations of PIFHS-sets are characterized.
4) Decision-making applications are discussed based on the proposal of aggregation operations-based algorithms for optimal product selection.
5) Similarity measures between PIFHS-sets are determined and authenticated with real-life application for recruitment process.
6) The results of proposed similarity are compared with relevant existing models.
7) The proposed structure is compared with relevant models under suitable evaluating indicators.
8) The advantageous aspects of the proposed structure are discussed.
9) The generalization of proposed structure is presented.

1.3. *Paper organization*

The remaining paper is systemized as: Section 2 presents some basic definitions and terminologies. Section 3 discusses the notions of possibility intuitionistic fuzzy hypersoft set (PIFHS-set) with properties and results. Section 4 characterizes set theoretic operations of PIFHS-set. Section 5 proposes decision making algorithms based on aggregation operations of PIFHS-set with applications. Section 6 presents similarity between two PIFHS-sets and application. Section 7 discusses the generalization and merits of proposed structure. Section 8 summarizes the paper with future directions.

2. Preliminaries

In this section, certain essential terminologies and terms like fuzzy set, intuitionistic fuzzy set, soft set, fuzzy soft set, intuitionistic fuzzy soft set, hypersoft set, fuzzy hypersoft set and intuitionistic fuzzy hypersoft set are recalled from existing literature for proper understanding of main results. Throughout the paper, $\mathcal{Z}$ will denote universe of discourse.

In 1965, Zadeh [1] initiated the concept of fuzzy sets as a generalization of classical set (crisp set) to deal with uncertain nature of data. This set employs a membership function which maps set of objects (alternatives) to unit closed interval.

Definition 2.1. (**fuzzy set**) [1] A fuzzy set $\mathcal{F}$ defined as $\mathcal{F} = \{(\hat{a}, A_{\mathcal{F}}(\hat{a}))|\hat{a} \in \mathcal{Z}\}$ such that $A_{\mathcal{F}} : \mathcal{Z} \to \mathbb{I}$ where $A_{\mathcal{F}}(\hat{a})$ denotes the belonging value of $\hat{a} \in \mathcal{F}$.

Definition 2.2. (**properties of fuzzy set**) [1] If $\mathcal{F}$ and $\mathcal{G}$ are two fuzzy sets then for all $\hat{a} \in \mathcal{Z}$, we have

(i) $\mathcal{F} \cup \mathcal{G} = \{(\hat{a}, max\{A_{\mathcal{F}}(\hat{a}), A_{\mathcal{G}}(\hat{a})\})\}$
(ii) $\mathcal{F} \cap \mathcal{G} = \{(\hat{a}, min\{A_{\mathcal{F}}(\hat{a}), A_{\mathcal{G}}(\hat{a})\})\}$
(iii) $\mathcal{F}^c = \{(\hat{a}, 1 - A_{\mathcal{F}}(\hat{a}))|\hat{a} \in \mathcal{Z}\}$

Fuzzy set emphasizes on degree of membership only for dealing with uncertain scenarios but there are many situations where non-membership degree is necessary to be considered therefore to adequate fuzzy set with such situation Atanassov [2] introduced intuitionistic fuzzy set as a generalization of fuzzy set. It provides due status to both membership and non-membership degrees of an alternative.

Definition 2.3. (**intuitionistic fuzzy set**) [2] An intuitionistic fuzzy set $\mathcal{Y}$ defined as $\mathcal{Y} = \{(\hat{b}, < A_{\mathcal{Y}}(\hat{b}), B_{\mathcal{Y}}(\hat{b}) >) | \hat{b} \in \mathcal{Z}\}$ such that $A_{\mathcal{Y}} : \mathcal{Z} \to \mathbb{I}$ and $B_{\mathcal{Y}} : \mathcal{Z} \to \mathbb{I}$, where $A_{\mathcal{Y}}(\hat{b})$ and $B_{\mathcal{Y}}(\hat{b})$ denote the belonging value and not-belonging value of $\hat{b} \in \mathcal{Y}$ with condition of $0 \leq A_{\mathcal{Y}}(\hat{b}) + B_{\mathcal{Y}}(\hat{b}) \leq 1$ and degree of hesitancy $\mathcal{H}_{\mathcal{Y}}(\hat{b}) = 1 - A_{\mathcal{Y}}(\hat{b}) - B_{\mathcal{Y}}(\hat{b})$.

Definition 2.4. (**properties of intuitionistic fuzzy set**) [2] If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are two intuitionistic fuzzy sets then for all $\hat{b} \in \mathcal{Z}$, we have

(i) $\mathcal{Y}_1 \cup \mathcal{Y}_2 = \left\{ \left(\hat{b}, < max\{A_{\mathcal{Y}_1}(\hat{b}), A_{\mathcal{Y}_2}(\hat{b})\}, min\{B_{\mathcal{Y}_1}(\hat{b}), B_{\mathcal{Y}_2}(\hat{b})\} > \right) \right\}$
(ii) $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \left\{ \left(\hat{b}, < min\{A_{\mathcal{Y}_1}(\hat{b}), A_{\mathcal{Y}_2}(\hat{b})\}, max\{B_{\mathcal{Y}_1}(\hat{b}), B_{\mathcal{Y}_2}(\hat{b})\} > \right) \right\}$
(iii) ${\mathcal{Y}_1}^c = \{(\hat{b}, < B_{\mathcal{Y}_1}(\hat{b}), A_{\mathcal{Y}_1}(\hat{b}) >)\}$

Fuzzy set and intuitionistic fuzzy set depict some kind of insufficiency regarding the consideration of parameterization tool. In order to manage this limitation, Molodtsov [3] developed soft set as a new mathematical tool to tackle uncertainties and vagueness in the data.

Definition 2.5. (**soft set**) [3] A pair $(\Psi_{\breve{\mathfrak{S}}}, \mathcal{W})$ is said to be soft set $\breve{\mathfrak{S}}$ over $\mathcal{Z}$, where $\Psi_M : \mathcal{W} \to \mathcal{P}(\mathcal{Z})$ and $\mathcal{W}$ is a subset of set of attributes $\mathcal{X}$.

Example 2.6. Lat $\mathcal{Z} = \{\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4, \hat{z}_5, \hat{z}_6\}$, $\mathcal{X} = \{\hat{p}_1, \hat{p}_2, ..., \hat{p}_9\}$ and $\mathcal{W} = \{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\}$ then approximate elements of soft set $\breve{\mathfrak{S}} = (\Psi_{\breve{\mathfrak{S}}}, \mathcal{W})$ are given as
$\Psi_{\breve{\mathfrak{S}}}(\hat{p}_1) = \{\hat{z}_1, \hat{z}_3, \hat{z}_6\}$
$\Psi_{\breve{\mathfrak{S}}}(\hat{p}_2) = \{\hat{z}_2, \hat{z}_3, \hat{z}_5\}$
$\Psi_{\breve{\mathfrak{S}}}(\hat{p}_3) = \{\hat{z}_4, \hat{z}_5, \hat{z}_6\}$
$\Psi_{\breve{\mathfrak{S}}}(\hat{p}_4) = \{\hat{z}_1, \hat{z}_2, \hat{z}_5\}$
and soft set $\breve{\mathfrak{S}}$ is stated as $\breve{\mathfrak{S}} = \{\Psi_{\breve{\mathfrak{S}}}(\hat{p}_1), \Psi_{\breve{\mathfrak{S}}}(\hat{p}_3), \Psi_{\breve{\mathfrak{S}}}(\hat{p}_3), \Psi_{\breve{\mathfrak{S}}}(\hat{p}_4)\}$ or

$$\breve{\mathfrak{S}} = \left\{ \left(\hat{p}_1, \{\hat{z}_1, \hat{z}_3, \hat{z}_6\} \right), \left(\hat{p}_2, \{\hat{z}_2, \hat{z}_3, \hat{z}_5\} \right), \left(\hat{p}_3, \{\hat{z}_4, \hat{z}_5, \hat{z}_6\} \right), \left(\hat{p}_4, \{\hat{z}_1, \hat{z}_2, \hat{z}_5\} \right) \right\}$$

Definition 2.7. (**fuzzy soft set**) [5] A pair $(\Psi_F, \mathcal{V})$ is said to be fuzzy soft set over $\mathcal{Z}$, where $\Psi_F : \mathcal{V} \to \mathcal{P}(\mathcal{F})$ and $\mathcal{P}(\mathcal{F})$ is a collection of all fuzzy subsets over $\mathcal{Z}$, $\mathcal{V} \subseteq \mathcal{X}$.

Definition 2.8. (**intuitionistic fuzzy soft set**) [13] A pair $(\Psi_{IF}, \mathcal{V})$ is said to be fuzzy soft set over $\mathcal{Z}$, where $\Psi_{IF} : \mathcal{V} \to \mathcal{P}(\mathcal{IF})$ and $\mathcal{P}(\mathcal{IF})$ is a collection of all intuitionistic fuzzy subsets over $\mathcal{Z}$, $\mathcal{V} \subseteq \mathcal{X}$.

In many real-world scenarios the classification of attributes into sub-attributive values in the form of sets is necessary. The existing concept of soft set is not sufficient and incompatible with such scenarios so Smarandache [16] introduced hypersoft sets to address the insufficiency of soft set and to handle the situations with multi-argument approximate function (MAAF).

Definition 2.9. (**hypersoft set**) [16] Let $\mathcal{Z} = \{\hat{z}_1, \hat{z}_2, ..., \hat{z}_n\}$ be an initial universe and $\mathcal{X} = \{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\}$ be a set of parameters. The respective attribute-valued non-overlapping sets of each element of $\mathcal{X}$ are
$Q_1 = \{\hat{q}_{11}, \hat{q}_{12}, ..., \hat{q}_{1n}\}$
$Q_2 = \{\hat{q}_{21}, \hat{q}_{22}, ..., \hat{q}_{2n}\}$
$Q_3 = \{\hat{q}_{31}, \hat{q}_{32}, ..., \hat{q}_{3n}\}$

....

....

....

$Q_n = \{\hat{q}_{n1}, \hat{q}_{n2}, ..., \hat{q}_{nn}\}$

and $Q = Q_1 \times Q_2 \times Q_3 \times .... \times Q_n = \{\hat{q}_1, \hat{q}_2, \hat{q}_3, ..., \hat{q}_r\}$ where each $\hat{q}_i$ $(i = 1, 2, ..., r)$ is a n-tuple element of Q and $r = \prod_{i=1}^{n} |Q_i|$, $| \bullet |$ denotes set cardinality then a MAAF is a mapping

$$\xi_H : \mathcal{V} \to \mathcal{P}(\mathcal{Z}) \tag{2.1}$$

and defined as

$$\xi_H(\{\hat{q}_1, \hat{q}_2, ..., \hat{q}_k\}) = \mathcal{P}(\{\hat{z}_1, \hat{z}_2, ..., \hat{z}_n\})$$

where $\mathcal{P}(\mathcal{Z})$ denotes the power set of $\mathcal{Z}$, $\mathcal{V} \subseteq Q$ with $k \leq r$. The pair $(\xi_H, \mathcal{V})$ is known as hypersoft set.

Definition 2.10. (**fuzzy and intuitionistic fuzzy hypersoft set**) [16] Fuzzy hypersoft set and intuitionistic fuzzy hypersoft set are hypersoft sets defined over fuzzy universe and intuitionistic fuzzy universes respectively i.e., in Eq (2.1) $\mathcal{P}(\mathcal{Z})$ is replaced with $\mathcal{P}(\mathcal{F})$ and $\mathcal{P}(\mathcal{IF})$.

3. Possibility intuitionistic fuzzy hypersoft set (PIFHS-set)

Consider the daily life scenario of clinical study to diagnose heart diseases in patients, doctors (decision-makers) usually prefer chest pain type, resting blood pressure, serum cholesterol etc., as diagnostic parameters. After keen analysis, it is vivid that these parameters are required to be further partitioning into their sub-parametric values i.e., chest pain type (typical angina, atypical angina, etc.), resting blood pressure (110 mmHg, 150 mmHg, 180 mmHg, etc.) and serum cholesterol (210 mg/dl, 320 mg/dl, 430 mg/dl, etc.). Patients are advised to visit medical laboratories for test reports regarding indicated parameters. As the efficiency of medical instruments in laboratories varies which leads different observations (data) for each patient. It is quite ambiguous for any decision-maker (i.e., medical practitioner) to assess the exact role of each parameter in diagnosis. Again this assessment is judged by relevant medical field specialist on the basis of standard values of each sub-attribute in accordance with WHO or some local medical body by considering a possibility degree within [0,1]. The literature has no suitable model to deal (i) sub-attribute values in the form of disjoint sets, and (ii) possibilitic data collectively. In order to meet the demand of literature, the novel model PIFHS-set is being characterized. The case (i) is addressed by considering multi-argument approximate function which considers the cartesian product of attribute-valued disjoint sets as its domain and then maps it to power set of initial universe (collection of intuitionistic fuzzy sets) and the case (ii) is tackled by assigning a possibility degree with each intuitionistic fuzzy number attached with each sub-parameter.

Now in this section definition and elementary properties of possibility intuitionistic fuzzy hypersoft set are conceptualized with appropriate examples.

Definition 3.1. (**possibility intuitionistic fuzzy hypersoft set**) The pair $(\mathfrak{F}_\mu, \mathcal{A})$ is said to be possibility intuitionistic fuzzy hypersoft set (PIFHS-set) over hypersoft universe $(\mathcal{Z}, \mathcal{A})$ if

$$\mathfrak{F}_\mu : \mathcal{A} \to (I \times I)_{\mathcal{Z}} \times I_{\mathcal{Z}}$$

defined by

$$\mathfrak{F}_\mu(\alpha) = (\mathfrak{F}(\alpha)(u), \mu(\alpha)(u)),$$

with

$$\mathfrak{F}(\alpha)(u) = <\psi_1(u), \psi_2(u)> \forall\, u \in \mathcal{Z}$$

where

(i) $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times .... \times \mathcal{A}_n$, $\mathcal{A}_i$ are disjoint attribute-valued sets corresponding to distinct attributes $a_i, i = 1, 2, ..., n$ respectively,
(ii) $\mathfrak{F} : \mathcal{A} \to (I{\times}I)_{\mathcal{Z}}$ and $\mu : \mathcal{A} \to I_{\mathcal{Z}}$, $I_{\mathcal{Z}}$ and $(I{\times}I)_{\mathcal{Z}}$ are the collections of all fuzzy and intuitionistic fuzzy subsets of $\mathcal{Z}$ respectively,
(iii) $(\mathfrak{F}(\alpha)(u)$ is the degree of membership of $u \in \mathcal{Z}$ in $\mathfrak{F}(\alpha)$,
(iv) $\mu(\alpha)(u)$ is the degree of possibility of membership of $u \in \mathcal{Z}$ in $\mathfrak{F}(\alpha)$.

so $\mathfrak{F}_\mu(\alpha_i)$ can be written as:

$$\mathfrak{F}_\mu(\alpha_i) = \left\{ \left(\frac{u_1}{\mathfrak{F}(\alpha_i)(u_1)}, \mu(\alpha_i)(u_1)\right), \left(\frac{u_2}{\mathfrak{F}(\alpha_i)(u_2)}, \mu(\alpha_i)(u_2)\right), ..., \left(\frac{u_n}{\mathfrak{F}(\alpha_i)(u_n)}, \mu(\alpha_i)(u_n)\right) \right\}.$$

For convenience, PIFHS-set is denoted by $\mathfrak{F}_\mu$ and collection of all PIFHS-sets is denoted by Ω_{pifhss}. The pictorial representations of existing structure PIFS-set (possibility intuitionistic fuzzy soft set) and proposed structure PIFHS-set are presented in Figure 2.

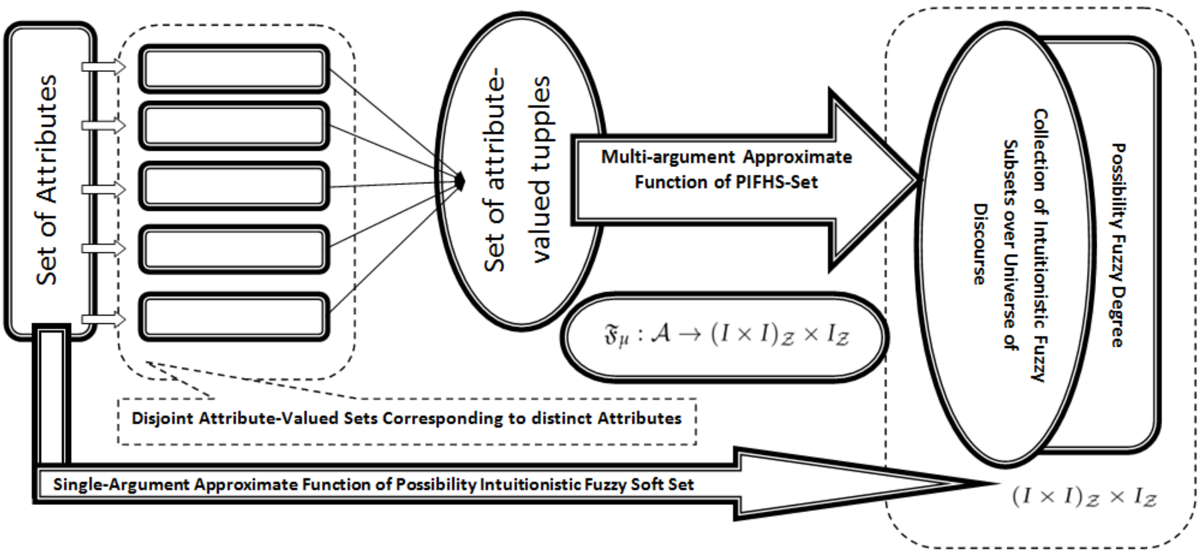

Figure 2. Pictorial representations and comparison of PIFS-set and PIFHS-set.

Example 3.2. Assume that Mr. Smith wants to purchase a mobile tablet from a mobile market. There are four kinds of mobile tablets (options) which form the set of discourse $\mathcal{Z} = \{m_1, m_2, m_3, m_4\}$. The best selection may be evaluated by observing the attributes i.e., a_1 = Camera Resolution (Mega pixels), a_2 = storage (Giga Bytes), and a_3 = Battery power (mAh). The attribute-valued sets corresponding to these attributes are:
$\mathcal{A}_1 = \{a_{11} = 8, a_{12} = 16\}$
$\mathcal{A}_2 = \{a_{21} = 32, a_{22} = 64\}$

$\mathcal{A}_3 = \{a_{31} = 3400\}$
then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$
$\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where each α_i, $i = 1, 2, 3, 4$, is a 3-tuple element.
Now

$$\mathfrak{F}_\mu(\alpha_1) = \left\{ \left(\tfrac{m_1}{<0.3,0.1>}, 0.2\right), \left(\tfrac{m_2}{<0.4,0.2>}, 0.3\right), \left(\tfrac{m_3}{<0.5,0.3>}, 0.4\right), \left(\tfrac{m_4}{<0.6,0.4>}, 0.5\right) \right\}$$

$$\mathfrak{F}_\mu(\alpha_2) = \left\{ \left(\tfrac{m_1}{<0.7,0.2>}, 0.8\right), \left(\tfrac{m_2}{<0.6,0.3>}, 0.8\right), \left(\tfrac{m_3}{<0.6,0.4>}, 0.7\right), \left(\tfrac{m_4}{<0.5,0.5>}, 0.6\right) \right\}$$

$$\mathfrak{F}_\mu(\alpha_3) = \left\{ \left(\tfrac{m_1}{<0.5,0.1>}, 0.1\right), \left(\tfrac{m_2}{<0.4,0.1>}, 0.2\right), \left(\tfrac{m_3}{<0.5,0.1>}, 0.3\right), \left(\tfrac{m_4}{<0.6,0.2>}, 0.4\right) \right\}$$

$$\mathfrak{F}_\mu(\alpha_4) = \left\{ \left(\tfrac{m_1}{<0.7,0.1>}, 0.2\right), \left(\tfrac{m_2}{<0.5,0.1>}, 0.3\right), \left(\tfrac{m_3}{<0.6,0.4>}, 0.4\right), \left(\tfrac{m_4}{<0.7,0.2>}, 0.5\right) \right\}$$

Then $\mathfrak{F}_\mu$ is a PIFHS-set over $(\mathcal{Z}, \mathcal{A})$. Its matrix representation is

$$\mathfrak{F}_\mu = \begin{pmatrix} \left((0.3, 0.1), 0.2\right) & \left((0.4, 0.2), 0.3\right) & \left((0.5, 0.3), 0.4\right) & \left((0.6, 0.4), 0.5\right) \\ \left((0.7, 0.2), 0.8\right) & \left((0.6, 0.3), 0.8\right) & \left((0.6, 0.4), 0.7\right) & \left((0.5, 0.5), 0.6\right) \\ \left((0.5, 0.1), 0.1\right) & \left((0.4, 0.1), 0.2\right) & \left((0.5, 0.1), 0.3\right) & \left((0.6, 0.2), 0.4\right) \\ \left((0.7, 0.1), 0.2\right) & \left((0.5, 0.1), 0.3\right) & \left((0.6, 0.4), 0.4\right) & \left((0.7, 0.2), 0.5\right) \end{pmatrix}.$$

Definition 3.3. (**PIFHS-subset**) Let $\mathfrak{F}_\mu, \mathfrak{G}_\eta \in \Omega_{pifhss}$, then $\mathfrak{F}_\mu$ is said to be a possibility intuitionistic fuzzy hypersoft subset (PIFHS-subset) of $\mathfrak{G}_\eta$, denoted by $\mathfrak{F}_\mu \subseteq \mathfrak{G}_\eta$, if

(i) $\mu(\alpha)$ is a intuitionistic fuzzy subset of $\eta(\alpha)$, for all $\alpha \in \mathcal{A}$
(ii) $\mathfrak{F}(\alpha)$ is a intuitionistic fuzzy subset of $\mathfrak{G}(\alpha)$, for all $\alpha \in \mathcal{A}$.

Example 3.4. Consider $\mathfrak{F}_\mu$ from Example 3.2 and let

$$\mathfrak{G}_\eta(\alpha_1) = \left\{ \left(\tfrac{m_1}{<0.4,0.2>}, 0.3\right), \left(\tfrac{m_2}{<0.5,0.3>}, 0.4\right), \left(\tfrac{m_3}{<0.6,0.4>}, 0.5\right), \left(\tfrac{m_4}{<0.6,0.4>}, 0.6\right) \right\}$$

$$\mathfrak{G}_\eta(\alpha_2) = \left\{ \left(\tfrac{m_1}{<0.8,0.2>}, 0.9\right), \left(\tfrac{m_2}{<0.6,0.4>}, 0.9\right), \left(\tfrac{m_3}{<0.6,0.4>}, 0.8\right), \left(\tfrac{m_4}{<0.5,0.5>}, 0.7\right) \right\}$$

$$\mathfrak{G}_\eta(\alpha_3) = \left\{ \left(\tfrac{m_1}{<0.6,0.2>}, 0.2\right), \left(\tfrac{m_2}{<0.5,0.2>}, 0.3\right), \left(\tfrac{m_3}{<0.6,0.2>}, 0.4\right), \left(\tfrac{m_4}{<0.7,0.3>}, 0.5\right) \right\}$$

$$\mathfrak{G}_\eta(\alpha_4) = \left\{ \left(\tfrac{m_1}{<0.8,0.2>}, 0.3\right), \left(\tfrac{m_2}{<0.7,0.2>}, 0.4\right), \left(\tfrac{m_3}{<0.6,0.4>}, 0.5\right), \left(\tfrac{m_4}{<0.7,0.3>}, 0.6\right) \right\}$$

then

$$\mathfrak{F}_\mu \subseteq \mathfrak{G}_\eta.$$

Definition 3.5. (**equality of PIFHS-sets**) Let $\mathfrak{F}_\mu, \mathfrak{G}_\eta \in \Omega_{pifhss}$, then $\mathfrak{F}_\mu$ is said to be equal to $\mathfrak{G}_\eta$ if $\mathfrak{F}_\mu \subseteq \mathfrak{G}_\eta$ and $\mathfrak{G}_\eta \subseteq \mathfrak{F}_\mu$.

Example 3.6. Considering data in example 3.2, we have

$$\mathfrak{F} = \begin{pmatrix} \left((0.1, 0.3), 0.8\right) & \left((0.2, 0.4), 0.7\right) & \left((0.3, 0.5), 0.6\right) & \left((0.4, 0.6), 0.5\right) \\ \left((0.2, 0.7), 0.2\right) & \left((0.3, 0.6), 0.2\right) & \left((0.4, 0.6), 0.3\right) & \left((0.5, 0.5), 0.4\right) \\ \left((0.1, 0.5), 0.9\right) & \left((0.1, 0.4), 0.8\right) & \left((0.1, 0.5), 0.7\right) & \left((0.2, 0.6), 0.6\right) \\ \left((0.1, 0.7), 0.8\right) & \left((0.1, 0.5), 0.7\right) & \left((0.4, 0.6), 0.6\right) & \left((0.2, 0.7), 0.5\right) \end{pmatrix}$$

and

$$\mathfrak{G}_\eta = \begin{pmatrix} \begin{pmatrix} (0.1, 0.3), 0.8 \end{pmatrix} & \begin{pmatrix} (0.2, 0.4), 0.7 \end{pmatrix} & \begin{pmatrix} (0.3, 0.5), 0.6 \end{pmatrix} & \begin{pmatrix} (0.4, 0.6), 0.5 \end{pmatrix} \\ \begin{pmatrix} (0.2, 0.7), 0.2 \end{pmatrix} & \begin{pmatrix} (0.3, 0.6), 0.2 \end{pmatrix} & \begin{pmatrix} (0.4, 0.6), 0.3 \end{pmatrix} & \begin{pmatrix} (0.5, 0.5), 0.4 \end{pmatrix} \\ \begin{pmatrix} (0.1, 0.5), 0.9 \end{pmatrix} & \begin{pmatrix} (0.1, 0.4), 0.8 \end{pmatrix} & \begin{pmatrix} (0.1, 0.5), 0.7 \end{pmatrix} & \begin{pmatrix} (0.2, 0.6), 0.6 \end{pmatrix} \\ \begin{pmatrix} (0.1, 0.7), 0.8 \end{pmatrix} & \begin{pmatrix} (0.1, 0.5), 0.7 \end{pmatrix} & \begin{pmatrix} (0.4, 0.6), 0.6 \end{pmatrix} & \begin{pmatrix} (0.2, 0.7), 0.5 \end{pmatrix} \end{pmatrix}$$

then $\mathfrak{F}_\mu = \mathfrak{G}_\eta$.

Definition 3.7. (**possibility null IFH-set**) A PIFHS-set $\mathfrak{F}_\mu$ is said to be a possibility null intuitionistic fuzzy hypersoft set, denoted by $\Phi_{(0)}$, if $\mathfrak{F}(\alpha) =< 0, 0 >$, and $\mu(\alpha) = 0$, for all $\alpha \in \mathcal{A}$.

Example 3.8. Considering matrix notation of $\mathfrak{F}_\mu$ given in example 3.2, we have

$$\Phi_{(0)} = \begin{pmatrix} \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} \\ \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} \\ \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} \\ \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} & \begin{pmatrix} (0.0, 0.0), 0.0 \end{pmatrix} \end{pmatrix}$$

Definition 3.9. (**possibility absolute IFH-set**) A PIFHS-set $\mathfrak{F}_\mu$ is said to be a possibility absolute intuitionistic fuzzy hypersoft set, denoted by $\mathbb{A}_{(1)}$, if $\mathfrak{F}(\alpha) =< 1, 0 >$, and $\mu(\alpha) = 1$, for all $\alpha \in \mathcal{A}$.

Example 3.10. Considering matrix notation of $\mathfrak{F}_\mu$ given in example 3.2, we have

$$\mathbb{A}_{(1)} = \begin{pmatrix} \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} \\ \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} \\ \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} \\ \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} & \begin{pmatrix} (1.0, 0.0), 1.0 \end{pmatrix} \end{pmatrix}$$

Definition 3.11. (**complement of PIFHS-set**) The complement of a PIFHS-set $\mathfrak{F}_\mu$, denoted by $\mathfrak{F}^c_\mu$, is defined by $\mathfrak{F}^c_\mu = \mathfrak{G}_\eta$ such that $\eta(\alpha) = \mu^c(\alpha)$ and $\mathfrak{G}(\alpha) = \mathfrak{F}^c(\alpha), \forall \alpha \in \mathcal{A}$, where c is a intuitionistic fuzzy complement.

Example 3.12. Considering matrix notation of $\mathfrak{F}_\mu$ given in example 3.2, we have

$$\mathfrak{F}^c_\mu = \mathfrak{G}_\eta = \begin{pmatrix} \begin{pmatrix} (0.1, 0.3), 0.8 \end{pmatrix} & \begin{pmatrix} (0.2, 0.4), 0.7 \end{pmatrix} & \begin{pmatrix} (0.3, 0.5), 0.6 \end{pmatrix} & \begin{pmatrix} (0.4, 0.6), 0.5 \end{pmatrix} \\ \begin{pmatrix} (0.2, 0.7), 0.2 \end{pmatrix} & \begin{pmatrix} (0.3, 0.6), 0.2 \end{pmatrix} & \begin{pmatrix} (0.4, 0.6), 0.3 \end{pmatrix} & \begin{pmatrix} (0.5, 0.5), 0.4 \end{pmatrix} \\ \begin{pmatrix} (0.1, 0.5), 0.9 \end{pmatrix} & \begin{pmatrix} (0.1, 0.4), 0.8 \end{pmatrix} & \begin{pmatrix} (0.1, 0.5), 0.7 \end{pmatrix} & \begin{pmatrix} (0.2, 0.6), 0.6 \end{pmatrix} \\ \begin{pmatrix} (0.1, 0.7), 0.8 \end{pmatrix} & \begin{pmatrix} (0.1, 0.5), 0.7 \end{pmatrix} & \begin{pmatrix} (0.4, 0.6), 0.6 \end{pmatrix} & \begin{pmatrix} (0.2, 0.7), 0.5 \end{pmatrix} \end{pmatrix}$$

4. Set theoretic operations of PIFHS-sets

In this section, definitions and properties of set theoretic operations like union, intersection, AND-operation and OR-operation of PFS-sets are developed and are explained with the help of suitable examples.

Definition 4.1. (**union and intersection of PIFHS-sets**) Let $\mathfrak{A}_\mu, \mathfrak{B}_\eta \in \Omega_{pifhss}$ then

(i) their union, denoted by $\mathfrak{A}_\mu \cup \mathfrak{B}_\eta$, is a PIFHS-set $\mathfrak{C}_\nu$ such that
$\mathfrak{C}(\alpha) = \coprod\{\mathfrak{A}(\alpha), \mathfrak{B}(\alpha)\}$ and
$\nu(\alpha) = max\{\mu(\alpha), \eta(\alpha)\}$
where $\coprod$ denotes intuitionistic union.

(ii) their intersection, denoted by $\mathfrak{A}_\mu \cap \mathfrak{B}_\eta$, is a PIFHS-set $\mathfrak{D}_\omega$ such that
$\mathfrak{D}(\alpha) = \prod\{\mathfrak{A}(\alpha), \mathfrak{B}(\alpha)\}$ and
$\omega(\alpha) = min\{\mu(\alpha), \eta(\alpha)\}$
where $\prod$ denotes intuitionistic intersection.

Example 4.2. Let $\mathfrak{A}_\mu, \mathfrak{B}_\eta \in \Omega_{pifhss}$ with matrix notations as $\mathfrak{A}_\mu =$

$$\begin{pmatrix} \left((0.1,0.2),0.2\right) & \left((0.2,0.3),0.3\right) & \left((0.3,0.4),0.4\right) & \left((0.4,0.5),0.5\right) \\ \left((0.5,0.5),0.8\right) & \left((0.6,0.4),0.8\right) & \left((0.7,0.3),0.7\right) & \left((0.9,0.1),0.6\right) \\ \left((0.4,0.3),0.1\right) & \left((0.6,0.4),0.2\right) & \left((0.7,0.2),0.3\right) & \left((0.4,0.1),0.4\right) \\ \left((0.6,0.2),0.2\right) & \left((0.7,0.3),0.3\right) & \left((0.5,0.2),0.4\right) & \left((0.7,0.2),0.5\right) \end{pmatrix}$$

and $\mathfrak{B}_\eta =$

$$\begin{pmatrix} \left((0.2,0.1),0.3\right) & \left((0.3,0.2),0.4\right) & \left((0.4,0.3),0.5\right) & \left((0.5,0.4),0.6\right) \\ \left((0.6,0.4),0.9\right) & \left((0.7,0.3),0.9\right) & \left((0.8,0.2),0.8\right) & \left((1.0,0.0),0.7\right) \\ \left((0.5,0.2),0.2\right) & \left((0.7,0.3),0.3\right) & \left((0.8,0.1),0.4\right) & \left((0.5,0.0),0.5\right) \\ \left((0.7,0.1),0.3\right) & \left((0.8,0.2),0.4\right) & \left((0.6,0.1),0.5\right) & \left((0.8,0.1),0.6\right) \end{pmatrix}$$

then

$$\mathfrak{C}_\nu = \mathfrak{A}_\mu \cup \mathfrak{B}_\eta = \mathfrak{B}_\eta$$

and

$$\mathfrak{D}_\omega = \mathfrak{A}_\mu \cap \mathfrak{B}_\eta = \mathfrak{A}_\mu$$

Proposition 4.3. *Let $\mathfrak{A}_\mu, \mathfrak{B}_\eta, \mathfrak{E}_\psi \in \Omega_{pifhss}$. Then the following properties hold:*

(i) $\mathfrak{A}_\mu \cup \mathfrak{B}_\eta = \mathfrak{B}_\eta \cup \mathfrak{A}_\mu$
(ii) $\mathfrak{A}_\mu \cap \mathfrak{B}_\eta = \mathfrak{B}_\eta \cap \mathfrak{A}_\mu$
(iii) $\mathfrak{A}_\mu \cup (\mathfrak{B}_\eta \cup \mathfrak{E}_\psi) = (\mathfrak{A}_\mu \cup \mathfrak{B}_\eta) \cup \mathfrak{E}_\psi$
(iv) $\mathfrak{A}_\mu \cap (\mathfrak{B}_\eta \cap \mathfrak{E}_\psi) = (\mathfrak{A}_\mu \cap \mathfrak{B}_\eta) \cap \mathfrak{E}_\psi$

Proposition 4.4. *Let $\mathfrak{E}_\psi \in \Omega_{pifhss}$. Then the following properties hold:*

(i) $\mathfrak{E}_\psi \cup \mathfrak{E}_\psi = \mathfrak{E}_\psi$
(ii) $\mathfrak{E}_\psi \cap \mathfrak{E}_\psi = \mathfrak{E}_\psi$
(iii) $\mathfrak{E}_\psi \cup \Phi_{(0)} = \mathfrak{E}_\psi$
(iv) $\mathfrak{E}_\psi \cap \Phi_{(0)} = \Phi_{(0)}$
(v) $\mathfrak{E}_\psi \cup \mathcal{A}_{(1)} = \mathcal{A}_{(1)}$
(vi) $\mathfrak{E}_\psi \cap \mathcal{A}_{(1)} = \mathfrak{E}_\psi$

Proposition 4.5. *Let $\mathfrak{A}_\mu, \mathfrak{B}_\eta, \mathfrak{E}_\psi \in \Omega_{pifhss}$. Then the following properties hold:*

(i) $\mathfrak{A}_\mu \cup (\mathfrak{B}_\eta \cap \mathfrak{E}_\psi) = (\mathfrak{A}_\mu \cup \mathfrak{B}_\eta) \cap (\mathfrak{A}_\mu \cup \mathfrak{E}_\psi)$

(ii) $\mathfrak{A}_\mu \cap (\mathfrak{B}_\eta \cup \mathfrak{E}_\psi) = (\mathfrak{A}_\mu \cap \mathfrak{B}_\eta) \cup (\mathfrak{A}_\mu \cap \mathfrak{E}_\psi)$

Proof. (i) For all $\alpha \in \mathcal{A}$,
$\lambda_{\mathfrak{A}(\alpha)\tilde{\cup}(\mathfrak{B}(\alpha)\tilde{\cap}\mathfrak{E}(\alpha))}(\alpha) = \tilde{\cup}\left\{\lambda_{\mathfrak{A}(\alpha)}(\alpha), \lambda_{(\mathfrak{B}(\alpha)\tilde{\cap}\mathfrak{E}(\alpha))}(\alpha)\right\}$
$= \tilde{\cup}\left\{\lambda_{\mathfrak{A}(\alpha)}(\alpha), \tilde{\cap}(\lambda_{\mathfrak{B}(\alpha)}(\alpha), \lambda_{\mathfrak{E}(\alpha)}(\alpha))\right\}$
$= \left\{ \begin{array}{l} < \alpha,\ max(T_{\mathfrak{A}(\alpha)}(\alpha), min(T_{\mathfrak{B}(\alpha)}(\alpha), T_{\mathfrak{E}(\alpha)}(\alpha))), \\ \quad min(F_{\mathfrak{A}(\alpha)}(\alpha), max(F_{\mathfrak{B}(\alpha)}(\alpha), F_{\mathfrak{E}(\alpha)}(\alpha))) > \end{array} \right\}$
$= \left\{ < \alpha, min\begin{pmatrix} max(T_{\mathfrak{A}(\alpha)}(\alpha), T_{\mathfrak{B}(\alpha)}(\alpha)), \\ max(T_{\mathfrak{A}(\alpha)}(\alpha), T_{\mathfrak{E}(\alpha)}(\alpha)) \end{pmatrix}, max\begin{pmatrix} min(F_{\mathfrak{A}(\alpha)}(\alpha), F_{\mathfrak{B}(\alpha)}(\alpha)), \\ min(F_{\mathfrak{A}(\alpha)}(\alpha), F_{\mathfrak{E}(\alpha)}(\alpha)) \end{pmatrix} > \right\}$
$= \tilde{\cap}(\tilde{\cup}(\lambda_{\mathfrak{A}(\alpha)\tilde{\cup}\mathfrak{B}(\alpha)}(\alpha), \lambda_{\mathfrak{A}(\alpha)\tilde{\cup}\mathfrak{E}(\alpha)}(\alpha))$
$= \lambda_{(\mathfrak{A}(\alpha)\tilde{\cup}\mathfrak{B}(\alpha))\tilde{\cap}(\mathfrak{A}(\alpha)\tilde{\cup}\mathfrak{E}(\alpha))}(\alpha)$
and
$\gamma_{\mu(\alpha)\tilde{\cup}(\eta(\alpha)\tilde{\cap}\psi(\alpha))}(\alpha)$
$= \max\left\{\gamma_{\mu(\alpha)}(\alpha), \gamma_{(\eta(\alpha)\tilde{\cap}\psi(\alpha))}(\alpha)\right\}$
$= \max\left\{\gamma_{\mu(\alpha)}(\alpha), min(\gamma_{\eta(\alpha)}(\alpha), \gamma_{\psi(\alpha)}(\alpha))\right\}$
$= \min\left\{\ \max(\gamma_{\mu(\alpha)}(\alpha), \gamma_{\eta(\alpha)}(\alpha)), \max(\gamma_{\mu(\alpha)}(\alpha), \gamma_{\psi(\alpha)}(\alpha))\ \right\}$
$= \min\left\{\gamma_{(\mu(\alpha)\tilde{\cup}\eta(\alpha))}(\alpha), \gamma_{(\mu(\alpha)\tilde{\cup}\psi(\alpha))}(\alpha)\right\}$
$= \gamma_{(\mu(\alpha)\tilde{\cup}\eta(\alpha))\tilde{\cap}_{(\mu(\alpha)\tilde{\cup}\psi(\alpha))}}(\alpha)$
(ii) can be proved in a similarly way as in (i). □

Definition 4.6. (**AND & OR operations of PIFHS-sets**) Let $(\mathcal{P}_\mu, C), (Q_\eta, \mathcal{D}) \in \Omega_{pifhss}$, then

(i) their AND-operation, denoted by $(\mathcal{P}_\mu, C) \wedge (Q_\eta, \mathcal{D})$, is a PIFHS-set $(\mathcal{R}_\nu, \mathcal{G})$ defined by

$$(\mathcal{R}_\nu, \mathcal{G}) = (\mathcal{R}_\nu, C \times \mathcal{D})$$

where $\mathcal{R}_\nu(c,d) = (\mathcal{R}(c,d)(u), \nu(c,d)(u))$, for all $(c,d) \in C \times \mathcal{D}$, such that
$\mathcal{R}(c,d) = \nabla\{\mathcal{P}(c), Q(d)\}$ and $\nu(c,d) = min\{\mu(c), \eta(d)\}$, for all $(c,d) \in C \times \mathcal{D}$ and $u \in \mathcal{Z}$. Here ∇ denotes intuitionistic fuzzy intersection.

(ii) their OR-operation, denoted by $(\mathcal{P}_\mu, C) \vee (Q_\eta, \mathcal{D})$, is a PIFHS-set $(\mathcal{S}_\kappa, \mathcal{H})$ defined by

$$(\mathcal{S}_\kappa, \mathcal{H}) = (\mathcal{S}_\kappa, C \times \mathcal{D})$$

where $\mathcal{S}_\kappa(c,d) = (\mathcal{S}(c,d)(u), \nu(c,d)(u))$, for all $(c,d) \in C \times \mathcal{D}$, such that
$\mathcal{S}(c,d) = \triangle\{\mathcal{P}(c), Q(d)\}$ and $\kappa(c,d) = max\{\mu(c), \eta(d)\}$, for all $(c,d) \in C \times \mathcal{D}$ and $u \in \mathcal{Z}$. Here $\triangle$ denotes intuitionistic fuzzy intersection.

Example 4.7. Let $\mathfrak{A}_\mu, \mathfrak{B}_\eta \in \Omega_{pifhss}$ with matrix notations as

$$\mathfrak{A}_\mu = \begin{pmatrix} \left((0.1,0.2),0.2\right) & \left((0.2,0.3),0.3\right) & \left((0.3,0.4),0.4\right) & \left((0.4,0.5),0.5\right) \\ \left((0.5,0.5),0.8\right) & \left((0.6,0.4),0.8\right) & \left((0.7,0.3),0.7\right) & \left((0.9,0.1),0.6\right) \\ \left((0.4,0.3),0.1\right) & \left((0.6,0.4),0.2\right) & \left((0.7,0.2),0.3\right) & \left((0.4,0.1),0.4\right) \\ \left((0.6,0.2),0.2\right) & \left((0.7,0.3),0.3\right) & \left((0.5,0.2),0.4\right) & \left((0.7,0.2),0.5\right) \end{pmatrix}$$

and

$$\mathfrak{B}_\eta = \begin{pmatrix} \begin{pmatrix} (0.2,0.1),0.3 \\ (0.6,0.4),0.9 \\ (0.5,0.2),0.2 \\ (0.7,0.1),0.3 \end{pmatrix} & \begin{pmatrix} (0.3,0.2),0.4 \\ (0.7,0.3),0.9 \\ (0.7,0.3),0.3 \\ (0.8,0.2),0.4 \end{pmatrix} & \begin{pmatrix} (0.4,0.3),0.5 \\ (0.8,0.2),0.8 \\ (0.8,0.1),0.4 \\ (0.6,0.1),0.5 \end{pmatrix} & \begin{pmatrix} (0.5,0.4),0.6 \\ (1.0,0.0),0.7 \\ (0.5,0.0),0.5 \\ (0.8,0.1),0.6 \end{pmatrix} \end{pmatrix},$$

we have

$$(\mathcal{R}_\nu, \mathcal{G}) = \begin{pmatrix}
((0.1,0.2),0.2) & ((0.2,0.3),0.3) & ((0.3,0.4),0.4) & ((0.4,0.5),0.5) \\
((0.1,0.4),0.2) & ((0.2,0.3),0.3) & ((0.3,0.4),0.4) & ((0.4,0.5),0.5) \\
((0.1,0.2),0.2) & ((0.2,0.3),0.3) & ((0.3,0.4),0.4) & ((0.4,0.5),0.5) \\
((0.1,0.2),0.2) & ((0.2,0.3),0.3) & ((0.3,0.4),0.4) & ((0.4,0.5),0.5) \\
((0.2,0.5),0.3) & ((0.3,0.4),0.4) & ((0.4,0.3),0.5) & ((0.5,0.4),0.6) \\
((0.5,0.5),0.8) & ((0.6,0.4),0.8) & ((0.7,0.3),0.7) & ((0.9,0.1),0.6) \\
((0.5,0.5),0.2) & ((0.6,0.4),0.3) & ((0.7,0.3),0.4) & ((0.5,0.1),0.5) \\
((0.5,0.5),0.3) & ((0.6,0.4),0.4) & ((0.6,0.3),0.5) & ((0.8,0.1),0.6) \\
((0.2,0.3),0.1) & ((0.3,0.4),0.2) & ((0.4,0.3),0.3) & ((0.4,0.4),0.4) \\
((0.4,0.4),0.1) & ((0.6,0.4),0.2) & ((0.7,0.2),0.3) & ((0.4,0.1),0.4) \\
((0.4,0.3),0.1) & ((0.6,0.4),0.2) & ((0.7,0.2),0.3) & ((0.4,0.1),0.4) \\
((0.4,0.3),0.1) & ((0.6,0.4),0.2) & ((0.6,0.2),0.3) & ((0.4,0.1),0.4) \\
((0.2,0.2),0.2) & ((0.3,0.3),0.3) & ((0.4,0.3),0.4) & ((0.5,0.4),0.5) \\
((0.6,0.4),0.2) & ((0.7,0.3),0.3) & ((0.5,0.2),0.4) & ((0.7,0.2),0.5) \\
((0.5,0.2),0.2) & ((0.7,0.3),0.3) & ((0.5,0.2),0.4) & ((0.5,0.2),0.5) \\
((0.6,0.2),0.2) & ((0.7,0.3),0.3) & ((0.5,0.2),0.4) & ((0.7,0.2),0.5)
\end{pmatrix}$$

and

$$(\mathcal{S}_\kappa, \mathcal{H}) = \begin{pmatrix}
((0.2,0.1),0.3) & ((0.3,0.2),0.4) & ((0.4,0.3),0.5) & ((0.5,0.4),0.6) \\
((0.6,0.2),0.9) & ((0.7,0.3),0.9) & ((0.8,0.2),0.8) & ((1.0,0.0),0.7) \\
((0.5,0.2),0.2) & ((0.7,0.3),0.3) & ((0.8,0.1),0.4) & ((0.5,0.0),0.5) \\
((0.7,0.1),0.3) & ((0.8,0.2),0.4) & ((0.6,0.1),0.5) & ((0.8,0.1),0.6) \\
((0.5,0.1),0.8) & ((0.6,0.2),0.8) & ((0.7,0.3),0.7) & ((0.9,0.1),0.6) \\
((0.6,0.4),0.9) & ((0.7,0.3),0.9) & ((0.8,0.2),0.8) & ((1.0,0.0),0.7) \\
((0.5,0.2),0.8) & ((0.7,0.3),0.8) & ((0.8,0.1),0.7) & ((0.9,0.0),0.6) \\
((0.7,0.1),0.8) & ((0.8,0.2),0.8) & ((0.7,0.1),0.7) & ((0.9,0.1),0.6) \\
((0.4,0.1),0.3) & ((0.6,0.2),0.4) & ((0.7,0.2),0.5) & ((0.5,0.1),0.6) \\
((0.6,0.3),0.9) & ((0.7,0.3),0.9) & ((0.8,0.2),0.8) & ((1.0,0.0),0.7) \\
((0.5,0.2),0.2) & ((0.7,0.3),0.3) & ((0.8,0.1),0.4) & ((0.5,0.0),0.5) \\
((0.7,0.1),0.3) & ((0.8,0.2),0.4) & ((0.7,0.1),0.5) & ((0.8,0.1),0.6) \\
((0.6,0.1),0.3) & ((0.7,0.2),0.4) & ((0.5,0.2),0.5) & ((0.7,0.2),0.6) \\
((0.6,0.2),0.9) & ((0.7,0.3),0.9) & ((0.8,0.2),0.8) & ((1.0,0.0),0.7) \\
((0.6,0.2),0.2) & ((0.7,0.3),0.3) & ((0.8,0.1),0.4) & ((0.7,0.0),0.5) \\
((0.7,0.1),0.3) & ((0.8,0.2),0.4) & ((0.6,0.1),0.5) & ((0.8,0.1),0.6)
\end{pmatrix}$$

5. Application of AND-operation and OR-operation of PIFHS-sets

In this section, two algorithms are proposed for decision making to have right decision regarding best selection of certain material/product using AND-operation and OR-operation of PIFHS-sets. Moreover, proposed algorithms are explained with illustrated examples.

Example 5.1. Suppose John wants to buy a washing machine from the market. There are five kinds of washing machines w_1, w_2, w_3 that form the universe of discourse $\mathcal{Z} = \{w_1, w_2, w_3, w_4, w_5\}$. Such selection is made on the basis of three parameters y_1 = power (watts), y_2 = voltage, y_3 = capacity in kg, which depict their performances. The attribute-valued sets corresponding to these parameters are $\ddot{\mathcal{Y}}_1, \ddot{\mathcal{Y}}_2, \ddot{\mathcal{Y}}_3$, such that $\ddot{\mathcal{Y}}_1 = \{y_{11} = 400, y_{12} = 500\}$, $\ddot{\mathcal{Y}}_2 = \{y_{21} = 220, y_{22} = 240\}$, $\ddot{\mathcal{Y}}_3 = \{y_{31} = 10\}$, therefore, $\mathcal{P} = \ddot{\mathcal{Y}}_1 \times \ddot{\mathcal{Y}}_2 \times \ddot{\mathcal{Y}}_3 = \{p_1, p_2, p_3, p_4\}$ where each p_i is a 3-tuple element of $\mathcal{P}$.

Now we propose two algorithms (i.e., one for AND-operation and other for OR-operation) of PIFHS-sets to have right selection.

Algorithm 1: optimal product selection based on AND-operation of PIFHS-sets

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Step 1 Construct PIFHS-sets $\mathcal{W}_\zeta, \mathcal{L}_\xi$ according to Experts.

Step 2 Calculate AND-operation $\mathcal{V}_\delta$ of PIFHS-sets constructed in step 1.

Step 3 Present $\mathcal{V}_\delta$ in matrix notation with reduced fuzzy numerical grades $\rho_{rf} = |T_\rho(p) - F_\rho(p)|$ of $\rho(p) =< T_\rho(p), F_\rho(p) >$.

Step 4 Mark the highest numerical grade ρ_{rf} in each row of matrix.

Step 5 Calculate score $\mathbb{S}$ = Sum of the products of ρ_{rf_i} with the corresponding possibility σ_i.

Step 6 Decision = Max$\{\mathbb{S}\}$.

The flow chart of Algorithm 1 is presented in Figure 3.

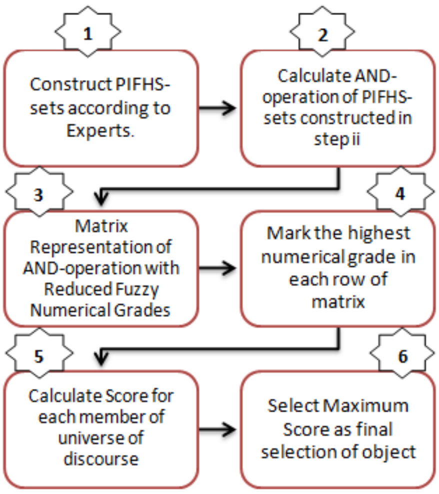

Figure 3. AND-algorithm.

Step 1 Consider we have two experts whose observations $\mathcal{W}_{\zeta}$ and $\mathcal{L}_{\xi}$ are as follows:

$$\mathcal{W}_{\zeta}(p_1) = \left\{ \left(\frac{w_1}{<0.51,0.31>}, 0.2\right), \left(\frac{w_2}{<0.31,0.21>}, 0.3\right), \left(\frac{w_3}{<0.32,0.22>}, 0.4\right), \left(\frac{w_4}{<0.41,0.31>}, 0.5\right), \left(\frac{w_5}{<0.51,0.41>}, 0.6\right) \right\},$$

$$\mathcal{W}_{\zeta}(p_2) = \left\{ \left(\frac{w_1}{<0.32,0.22>}, 0.9\right), \left(\frac{w_2}{<0.42,0.32>}, 0.8\right), \left(\frac{w_3}{<0.52,0.42>}, 0.7\right), \left(\frac{w_4}{<0.62,0.32>}, 0.6\right), \left(\frac{w_5}{<0.52,0.42>}, 0.5\right) \right\},$$

$$\mathcal{W}_{\zeta}(p_3) = \left\{ \left(\frac{w_1}{<0.43,0.33>}, 0.8\right), \left(\frac{w_2}{<0.53,0.43>}, 0.7\right), \left(\frac{w_3}{<0.63,0.33>}, 0.6\right), \left(\frac{w_4}{<0.73,0.13}, 0.7\right), \left(\frac{w_5}{<0.83,0.13>}, 0.8\right) \right\},$$

$$\mathcal{W}_{\zeta}(p_4) = \left\{ \left(\frac{w_1}{<0.54,0.44>}, 0.4\right), \left(\frac{w_2}{<0.64,0.14>}, 0.5\right), \left(\frac{w_3}{<0.74,0.24>}, 0.6\right), \left(\frac{w_4}{<0.34,0.24>}, 0.7\right), \left(\frac{w_5}{<0.54,0.14>}, 0.8\right) \right\},$$

$$\mathcal{L}_{\xi}(p_1) = \left\{ \left(\frac{w_1}{<0.29,0.19>}, 0.1\right), \left(\frac{w_2}{<0.28,0.18>}, 0.2\right), \left(\frac{w_3}{<0.27,0.17>}, 0.3\right), \left(\frac{w_4}{<0.26,0.16>}, 0.4\right), \left(\frac{w_5}{<0.25,0.15>}, 0.5\right) \right\},$$

$$\mathcal{L}_{\xi}(p_2) = \left\{ \left(\frac{w_1}{<0.38,0.28>}, 0.3\right), \left(\frac{w_2}{<0.37,0.27>}, 0.4\right), \left(\frac{w_3}{<0.35,0.25>}, 0.5\right), \left(\frac{w_4}{<0.35,0.15>}, 0.6\right), \left(\frac{w_5}{<0.36,0.26>}, 0.7\right) \right\},$$

$$\mathcal{L}_{\xi}(p_3) = \left\{ \left(\frac{w_1}{<0.44,0.34>}, 0.9\right), \left(\frac{w_2}{<0.45,0.35>}, 0.8\right), \left(\frac{w_3}{<0.46,0.36>}, 0.7\right), \left(\frac{w_4}{<0.47,0.37>}, 0.6\right), \left(\frac{w_5}{<0.48,0.38>}, 0.5\right) \right\},$$

$$\mathcal{L}_{\xi}(p_4) = \left\{ \left(\frac{w_1}{<0.54,0.44>}, 0.1\right), \left(\frac{w_2}{<0.55,0.45>}, 0.2\right), \left(\frac{w_3}{<0.56,0.36>}, 0.3\right), \left(\frac{w_4}{<0.57,0.37>}, 0.4\right), \left(\frac{w_5}{<0.58,0.38>}, 0.5\right) \right\},$$

Step 2 Now $\mathcal{W}_{\zeta} \wedge \mathcal{L}_{\xi} = \mathcal{V}_{\delta}$ where

$$\mathcal{V}_{\delta}(p_1, p_1) = \left\{ \left(\frac{w_1}{<0.29,0.31>}, 0.1\right), \left(\frac{w_2}{<0.28,0.21>}, 0.2\right), \left(\frac{w_3}{<0.27,0.22>}, 0.3\right), \left(\frac{w_4}{<0.26,0.31>}, 0.4\right), \left(\frac{w_5}{<0.25,0.41>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_1, p_2) = \left\{ \left(\frac{w_1}{<0.38,0.31>}, 0.2\right), \left(\frac{w_2}{<0.31,0.27>}, 0.3\right), \left(\frac{w_3}{<0.32,0.25>}, 0.4\right), \left(\frac{w_4}{<0.35,0.31>}, 0.5\right), \left(\frac{w_5}{<0.36,0.41>}, 0.6\right) \right\},$$

$$\mathcal{V}_{\delta}(p_1, p_3) = \left\{ \left(\frac{w_1}{<0.44,0.34>}, 0.2\right), \left(\frac{w_2}{<0.31,0.35>}, 0.3\right), \left(\frac{w_3}{<0.32,0.36>}, 0.4\right), \left(\frac{w_4}{<0.41,0.37>}, 0.5\right), \left(\frac{w_5}{<0.48,0.41>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_1, p_4) = \left\{ \left(\frac{w_1}{<0.51,0.44>}, 0.1\right), \left(\frac{w_2}{<0.31,0.45>}, 0.2\right), \left(\frac{w_3}{<0.32,0.36>}, 0.3\right), \left(\frac{w_4}{<0.41,0.37>}, 0.4\right), \left(\frac{w_5}{<0.51,0.41>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_2, p_1) = \left\{ \left(\frac{w_1}{<0.29,0.22>}, 0.1\right), \left(\frac{w_2}{<0.28,0.32>}, 0.2\right), \left(\frac{w_3}{<0.27,0.42>}, 0.3\right), \left(\frac{w_4}{<0.26,0.32>}, 0.4\right), \left(\frac{w_5}{<0.25,0.42>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_2, p_2) = \left\{ \left(\frac{w_1}{<0.32,0.28>}, 0.3\right), \left(\frac{w_2}{<0.37,0.32>}, 0.4\right), \left(\frac{w_3}{<0.35,0.42>}, 0.5\right), \left(\frac{w_4}{<0.35,0.32>}, 0.6\right), \left(\frac{w_5}{<0.36,0.42>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_2, p_3) = \left\{ \left(\frac{w_1}{<0.32,0.34>}, 0.9\right), \left(\frac{w_2}{<0.42,0.35>}, 0.8\right), \left(\frac{w_3}{<0.46,0.42>}, 0.7\right), \left(\frac{w_4}{<0.47,0.37>}, 0.6\right), \left(\frac{w_5}{<0.48,0.42>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_2, p_4) = \left\{ \left(\frac{w_1}{<0.32,0.44>}, 0.1\right), \left(\frac{w_2}{<0.42,0.45>}, 0.2\right), \left(\frac{w_3}{<0.52,0.42>}, 0.3\right), \left(\frac{w_4}{<0.57,0.37>}, 0.4\right), \left(\frac{w_5}{<0.52,0.42>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_3, p_1) = \left\{ \left(\frac{w_1}{<0.29,0.33>}, 0.1\right), \left(\frac{w_2}{<0.28,0.43>}, 0.2\right), \left(\frac{w_3}{<0.27,0.33>}, 0.3\right), \left(\frac{w_4}{<0.26,0.16>}, 0.4\right), \left(\frac{w_5}{<0.25,0.15>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_3, p_2) = \left\{ \left(\frac{w_1}{<0.38,0.33>}, 0.3\right), \left(\frac{w_2}{<0.37,0.43>}, 0.4\right), \left(\frac{w_3}{<0.35,0.33>}, 0.5\right), \left(\frac{w_4}{<0.35,0.15>}, 0.6\right), \left(\frac{w_5}{<0.36,0.26>}, 0.7\right) \right\},$$

$$\mathcal{V}_{\delta}(p_3, p_3) = \left\{ \left(\frac{w_1}{<0.43,0.34>}, 0.8\right), \left(\frac{w_2}{<0.45,0.43>}, 0.7\right), \left(\frac{w_3}{<0.46,0.36>}, 0.6\right), \left(\frac{w_4}{<0.47,0.37>}, 0.6\right), \left(\frac{w_5}{<0.48,0.38>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_3, p_4) = \left\{ \left(\frac{w_1}{<0.43,0.44>}, 0.1\right), \left(\frac{w_2}{<0.53,0.45>}, 0.2\right), \left(\frac{w_3}{<0.56,0.36>}, 0.3\right), \left(\frac{w_4}{<0.57,0.37>}, 0.4\right), \left(\frac{w_5}{<0.58,0.38>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_4, p_1) = \left\{ \left(\frac{w_1}{<0.29,0.44>}, 0.1\right), \left(\frac{w_2}{<0.28,0.18>}, 0.2\right), \left(\frac{w_3}{<0.27,0.24>}, 0.3\right), \left(\frac{w_4}{<0.26,0.24>}, 0.4\right), \left(\frac{w_5}{<0.25,0.15>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_4, p_2) = \left\{ \left(\frac{w_1}{<0.38,0.44>}, 0.3\right), \left(\frac{w_2}{<0.37,0.27>}, 0.4\right), \left(\frac{w_3}{<0.35,0.25>}, 0.5\right), \left(\frac{w_4}{<0.35,0.24>}, 0.6\right), \left(\frac{w_5}{<0.36,0.26>}, 0.7\right) \right\},$$

$$\mathcal{V}_{\delta}(p_4, p_3) = \left\{ \left(\frac{w_1}{<0.44,0.44>}, 0.4\right), \left(\frac{w_2}{<0.45,0.35>}, 0.5\right), \left(\frac{w_3}{<0.46,0.36>}, 0.6\right), \left(\frac{w_4}{<0.34,0.37>}, 0.6\right), \left(\frac{w_5}{<0.48,0.38>}, 0.5\right) \right\},$$

$$\mathcal{V}_{\delta}(p_4, p_4) = \left\{ \left(\frac{w_1}{<0.54,0.44>}, 0.1\right), \left(\frac{w_2}{<0.55,0.45>}, 0.2\right), \left(\frac{w_3}{<0.56,0.36>}, 0.3\right), \left(\frac{w_4}{<0.34,0.37>}, 0.4\right), \left(\frac{w_5}{<0.54,0.38>}, 0.5\right) \right\},$$

Step 3 Matrix notation of $\mathcal{V}_\delta$ is given as

$$\mathcal{V}_\delta = \begin{pmatrix} (0.02, 0.1) & (0.07, 0.2) & (0.05, 0.3) & (0.05, 0.4) & (\mathbf{0.16}, 0.5) \\ (\mathbf{0.07}, 0.2) & (0.04, 0.3) & (\mathbf{0.07}, 0.4) & (0.04, 0.5) & (0.05, 0.6) \\ (\mathbf{0.10}, 0.2) & (0.04, 0.3) & (0.04, 0.4) & (0.04, 0.5) & (0.07, 0.5) \\ (0.07, 0.1) & (\mathbf{0.14}, 0.2) & (0.04, 0.3) & (0.04, 0.4) & (0.10, 0.5) \\ (0.07, 0.1) & (0.04, 0.2) & (0.15, 0.3) & (0.06, 0.4) & (\mathbf{0.17}, 0.5) \\ (0.04, 0.3) & (0.05, 0.4) & (\mathbf{0.07}, 0.5) & (0.03, 0.6) & (0.06, 0.5) \\ (0.02, 0.9) & (0.07, 0.8) & (0.04, 0.7) & (\mathbf{0.10}, 0.6) & (0.06, 0.5) \\ (0.12, 0.1) & (0.03, 0.2) & (0.10, 0.3) & (\mathbf{0.20}, 0.4) & (0.10, 0.5) \\ (0.04, 0.1) & (\mathbf{0.15}, 0.2) & (0.06, 0.3) & (0.10, 0.4) & (0.10, 0.5) \\ (0.05, 0.3) & (0.06, 0.4) & (0.02, 0.5) & (\mathbf{0.20}, 0.6) & (0.10, 0.7) \\ (0.09, 0.8) & (0.02, 0.7) & (\mathbf{0.10}, 0.6) & (\mathbf{0.10}, 0.6) & (\mathbf{0.10}, 0.5) \\ (0.01, 0.1) & (0.08, 0.2) & (\mathbf{0.20}, 0.3) & (\mathbf{0.20}, 0.4) & (\mathbf{0.20}, 0.5) \\ (\mathbf{0.15}, 0.1) & (0.10, 0.2) & (0.03, 0.3) & (0.02, 0.4) & (0.10, 0.5) \\ (0.06, 0.3) & (0.10, 0.4) & (0.10, 0.5) & (\mathbf{0.11}, 0.6) & (0.10, 0.7) \\ (0.00, 0.4) & (\mathbf{0.10}, 0.5) & (\mathbf{0.10}, 0.6) & (0.03, 0.6) & (\mathbf{0.10}, 0.5) \\ (0.10, 0.1) & (0.10, 0.2) & (\mathbf{0.20}, 0.3) & (0.03, 0.4) & (0.16, 0.5) \end{pmatrix}$$

Step 4 Reduced fuzzy numerical grades ρ_{rf} (see Figure 4) and possibilities σ are given in Table 1. The value of ρ_{rf} and σ against the pairs $(p_i, p_i), i = 1, 2, 3$ are considered as 0 as both the parameters are the same.

$Score(w_1) = \mathbb{S}(w_1) = (0.07 \times 0.2) + (0.10 \times 0.2) + (0.15 \times 0.1) = 0.049$
$Score(w_2) = \mathbb{S}(w_2) = (0.14 \times 0.2) + (0.10 \times 0.5) = 0.078$
$Score(w_3) = \mathbb{S}(w_3) = (0.07 \times 0.4) + 0 + (0.15 \times 0.2) + 0 + (0.20 \times 0.3) + (0.10 \times 0.6) + 0 = 0.178$
$Score(w_4) = \mathbb{S}(w_4) = (0.10 \times 0.6) + (0.20 \times 0.4) + (0.20 \times 0.6) + 0 + (0.20 \times 0.4) + (0.11 \times 0.6) = 0.406$
$Score(w_5) = \mathbb{S}(w_5) = 0 + (0.17 \times 0.5) + 0 + (0.20 \times 0.5) + (0.10 \times 0.5) = 0.235$

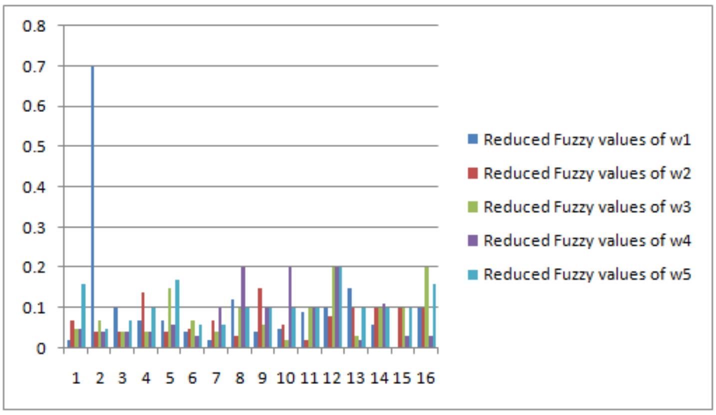

Figure 4. Reduced fuzzy values in $\mathcal{V}_\delta$ under AND-algorithm.

Table 1. AND-operation based grade table.

$\mathcal{V}_\delta$	w_i	Highest ρ_{rf}	σ_i
(p_1, p_1)	w_5	0	0
(p_1, p_2)	w_1, w_3	0.07, 0.07	0.2,0.4
(p_1, p_3)	w_1	0.10	0.2
(p_1, p_4)	w_2	0.14	0.2
(p_2, p_1)	w_5	0.17	0.5
(p_2, p_2)	w_3	0	0
(p_2, p_3)	w_4	0.10	0.6
(p_2, p_4)	w_4	0.20	0.4
(p_3, p_1)	w_3	0.15	0.2
(p_3, p_2)	w_4	0.20	0.6
(p_3, p_3)	w_3, w_4, w_5	0	0
(p_3, p_4)	w_3, w_4, w_5	0.20, 0.20, 0.20	0.3, 0.4, 0.5
(p_4, p_1)	w_1	0.15	0.1
(p_4, p_2)	w_4	0.11	0.6
(p_4, p_3)	w_2, w_3, w_5	0.10, 0.10, 0.10	0.5, 0.6, 0.5
(p_4, p_4)	w_3	0	0

Step 5 and Step 6 Score of w_4 is maximum (as depicted in Figure 5), therefore it is selected.

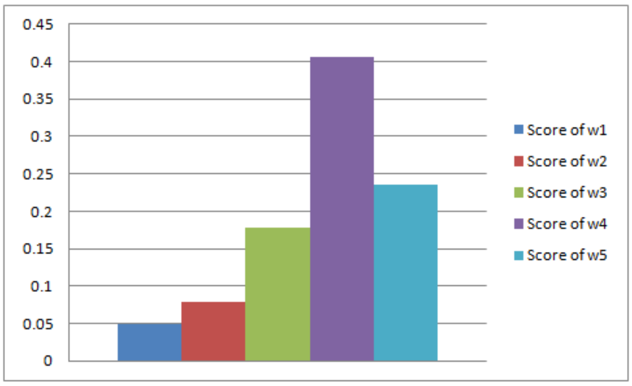

Figure 5. Score values under AND-algorithm.

Example 5.2. Considering the Example 5.1 for purchasing of dryer with same sub-attributive sets, an alternate algorithm is proposed which is based on OR-operation of PIFHS-sets.

Algorithm 2: optimal product selection based on OR-operation of PIFHS-sets

Step 1 Construct PIFHS-sets $\mathcal{W}_\zeta, \mathcal{L}_\xi$ according to Experts.

Step 2 Calculate OR-operation $\mathcal{X}_\varrho$ of PIFHS-sets constructed in step 1.

Step 3 Present $\mathcal{X}_\varrho$ in matrix notation with reduced fuzzy numerical grades $\rho_{rf} = |T_\rho(p) - F_\rho(p)|$ of $\rho(p) =< T_\rho(p), F_\rho(p) >$.

Step 4 Mark the highest numerical grade ρ_{rf} in each row of matrix.

Step 5 Calculate score $\mathbb{S}$ = Sum of the products of ρ_{rf_i} with the corresponding possibility σ_i.

Step 6 Decision = Max{$\mathbb{S}$}.

The flow chart of Algorithm 2 is presented in Figure 6.

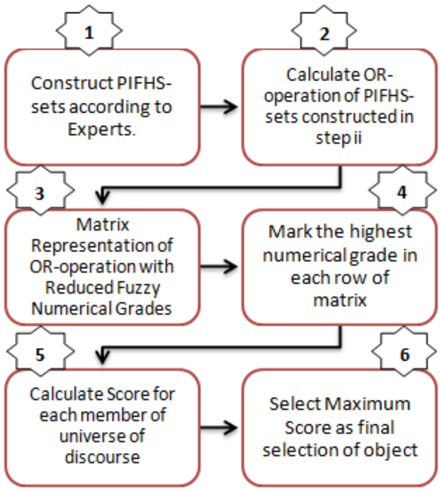

Figure 6. OR-algorithm.

Step 1 This step is same as in step 1 of algorithm 1.
Step 2 Now $\mathcal{W}_\zeta \vee \mathcal{L}_\xi = \mathcal{X}_\varrho$ where

$$\mathcal{X}_\varrho(p_1,p_1) = \left\{ \left(\tfrac{d_1}{<0.51,0.19>}, 0.2\right), \left(\tfrac{d_2}{<0.31,0.18>}, 0.3\right), \left(\tfrac{d_3}{<0.32,0.17>}, 0.4\right), \left(\tfrac{d_4}{<0.41,0.16>}, 0.5\right), \left(\tfrac{d_5}{<0.51,0.15>}, 0.6\right) \right\},$$

$$\mathcal{X}_\varrho(p_1,p_2) = \left\{ \left(\tfrac{d_1}{<0.51,0.28>}, 0.3\right), \left(\tfrac{d_2}{<0.37,0.21>}, 0.4\right), \left(\tfrac{d_3}{<0.35,0.22>}, 0.5\right), \left(\tfrac{d_4}{<0.41,0.15>}, 0.6\right), \left(\tfrac{d_5}{<0.51,0.26>}, 0.7\right) \right\},$$

$$\mathcal{X}_\varrho(p_1,p_3) = \left\{ \left(\tfrac{d_1}{<0.51,0.31>}, 0.9\right), \left(\tfrac{d_2}{<0.45,0.21>}, 0.8\right), \left(\tfrac{d_3}{<0.46,0.22>}, 0.7\right), \left(\tfrac{d_4}{<0.47,0.31>}, 0.6\right), \left(\tfrac{d_5}{<0.51,0.38>}, 0.6\right) \right\},$$

$$\mathcal{X}_\varrho(p_1,p_4) = \left\{ \left(\tfrac{d_1}{<0.54,0.31>}, 0.2\right), \left(\tfrac{d_2}{<0.55,0.21>}, 0.3\right), \left(\tfrac{d_3}{<0.56,0.22>}, 0.4\right), \left(\tfrac{d_4}{<0.57,0.31>}, 0.5\right), \left(\tfrac{d_5}{<0.58,0.38>}, 0.6\right) \right\},$$

$$\mathcal{X}_\varrho(p_2,p_1) = \left\{ \left(\tfrac{d_1}{<0.32,0.19>}, 0.9\right), \left(\tfrac{d_2}{<0.42,0.18>}, 0.8\right), \left(\tfrac{d_3}{<0.52,0.17>}, 0.7\right), \left(\tfrac{d_4}{<0.62,0.16>}, 0.6\right), \left(\tfrac{d_5}{<0.52,0.15>}, 0.5\right) \right\},$$

$$\mathcal{X}_\varrho(p_2,p_2) = \left\{ \left(\tfrac{d_1}{<0.38,0.22>}, 0.9\right), \left(\tfrac{d_2}{<0.42,0.27>}, 0.8\right), \left(\tfrac{d_3}{<0.52,0.25>}, 0.7\right), \left(\tfrac{d_4}{<0.62,0.15>}, 0.6\right), \left(\tfrac{d_5}{<0.52,0.26>}, 0.7\right) \right\},$$

$$\mathcal{X}_\varrho(p_2,p_3) = \left\{ \left(\tfrac{d_1}{<0.44,0.22>}, 0.9\right), \left(\tfrac{d_2}{<0.45,0.32>}, 0.8\right), \left(\tfrac{d_3}{<0.52,0.36>}, 0.7\right), \left(\tfrac{d_4}{<0.62,0.32>}, 0.6\right), \left(\tfrac{d_5}{<0.52,0.38>}, 0.5\right) \right\},$$

$$\mathcal{X}_\varrho(p_2,p_4) = \left\{ \left(\tfrac{d_1}{<0.52,0.22>}, 0.9\right), \left(\tfrac{d_2}{<0.55,0.32>}, 0.8\right), \left(\tfrac{d_3}{<0.56,0.36>}, 0.7\right), \left(\tfrac{d_4}{<0.62,0.32>}, 0.6\right), \left(\tfrac{d_5}{<0.58,0.38>}, 0.5\right) \right\},$$

$$\mathcal{X}_\varrho(p_3,p_1) = \left\{ \left(\tfrac{d_1}{<0.43,0.19>}, 0.8\right), \left(\tfrac{d_2}{<0.53,0.18>}, 0.7\right), \left(\tfrac{d_3}{<0.63,0.17>}, 0.6\right), \left(\tfrac{d_4}{<0.73,0.13>}, 0.7\right), \left(\tfrac{d_5}{<0.83,0.13>}, 0.8\right) \right\},$$

$$\mathcal{X}_\varrho(p_3,p_2) = \left\{ \left(\tfrac{d_1}{<0.43,0.28>}, 0.8\right), \left(\tfrac{d_2}{<0.53,0.27>}, 0.7\right), \left(\tfrac{d_3}{<0.63,0.25>}, 0.6\right), \left(\tfrac{d_4}{<0.73,0.13>}, 0.7\right), \left(\tfrac{d_5}{<0.83,0.13>}, 0.8\right) \right\},$$

$$\mathcal{X}_{\varrho}(p_3,p_3)=\left\{\left(\frac{d_1}{<0.44,0.33>},0.9\right),\left(\frac{d_2}{<0.53,0.35>},0.8\right),\left(\frac{d_3}{<0.63,0.33>},0.7\right),\left(\frac{d_4}{<0.73,0.13>},0.7\right),\left(\frac{d_5}{<0.83,0.13>},0.8\right)\right\},$$

$$\mathcal{X}_{\varrho}(p_3,p_4)=\left\{\left(\frac{d_1}{<0.54,0.33>},0.8\right),\left(\frac{d_2}{<0.55,0.43>},0.7\right),\left(\frac{d_3}{<0.63,0.33>},0.6\right),\left(\frac{d_4}{<0.73,0.13>},0.7\right),\left(\frac{d_5}{<0.83,0.13>},0.8\right)\right\},$$

$$\mathcal{X}_{\varrho}(p_4,p_1)=\left\{\left(\frac{d_1}{<0.54,0.19>},0.4\right),\left(\frac{d_2}{<0.64,0.14>},0.5\right),\left(\frac{d_3}{<0.74,0.17>},0.6\right),\left(\frac{d_4}{<0.34,0.16>},0.7\right),\left(\frac{d_5}{<0.54,0.14>},0.8\right)\right\},$$

$$\mathcal{X}_{\varrho}(p_4,p_2)=\left\{\left(\frac{d_1}{<0.54,0.28>},0.4\right),\left(\frac{d_2}{<0.64,0.14>},0.5\right),\left(\frac{d_3}{<0.74,0.24>},0.6\right),\left(\frac{d_4}{<0.35,0.15>},0.7\right),\left(\frac{d_5}{<0.54,0.14>},0.8\right)\right\},$$

$$\mathcal{X}_{\varrho}(p_4,p_3)=\left\{\left(\frac{d_1}{<0.54,0.34>},0.9\right),\left(\frac{d_2}{<0.64,0.14>},0.8\right),\left(\frac{d_3}{<0.74,0.24>},0.7\right),\left(\frac{d_4}{<0.47,0.24>},0.7\right),\left(\frac{d_5}{<0.54,0.14>},0.8\right)\right\},$$

$$\mathcal{X}_{\varrho}(p_4,p_4)=\left\{\left(\frac{d_1}{<0.54,0.44>},0.4\right),\left(\frac{d_2}{<0.64,0.14>},0.5\right),\left(\frac{d_3}{<0.74,0.24>},0.6\right),\left(\frac{d_4}{<0.57,0.24>},0.7\right),\left(\frac{d_5}{<0.58,0.14>},0.8\right)\right\},$$

Step 3 Matrix notation of $\mathcal{X}_{\varrho}$ is given as

$$\mathcal{V}_{\delta}=\begin{pmatrix}
(0.32,0.2) & (0.13,0.3) & (0.15,0.4) & (0.25,0.5) & (\mathbf{0.36},0.6)\\
(0.23,0.3) & (0.16,0.4) & (0.13,0.5) & (\mathbf{0.26},0.6) & (0.25,0.7)\\
(0.20,0.9) & (\mathbf{0.24},0.8) & (\mathbf{0.24},0.7) & (0.16,0.6) & (0.13,0.6)\\
(0.23,0.2) & (\mathbf{0.34},0.3) & (\mathbf{0.34},0.4) & (0.26,0.5) & (0.20,0.6)\\
(0.13,0.9) & (0.24,0.8) & (0.35,0.7) & (\mathbf{0.46},0.6) & (0.37,0.5)\\
(0.16,0.9) & (0.15,0.8) & (0.27,0.7) & (\mathbf{0.47},0.6) & (0.26,0.7)\\
(0.22,0.9) & (0.13,0.8) & (0.16,0.7) & (\mathbf{0.30},0.6) & (0.14,0.5)\\
(\mathbf{0.30},0.9) & (0.23,0.8) & (0.20,0.7) & (\mathbf{0.30},0.6) & (0.20,0.5)\\
(0.24,0.8) & (0.35,0.7) & (0.46,0.6) & (0.60,0.7) & (\mathbf{0.70},0.8)\\
(0.15,0.8) & (0.26,0.7) & (0.38,0.6) & (0.60,0.7) & (\mathbf{0.70},0.8)\\
(0.11,0.9) & (0.18,0.8) & (0.30,0.7) & (0.60,0.7) & (\mathbf{0.70},0.8)\\
(0.21,0.8) & (0.12,0.7) & (0.30,0.6) & (0.60,0.7) & (\mathbf{0.70},0.8)\\
(0.35,0.4) & (0.50,0.5) & (\mathbf{0.57},0.6) & (0.18,0.7) & (0.40,0.8)\\
(0.26,0.4) & (\mathbf{0.50},0.5) & (\mathbf{0.50},0.6) & (0.20,0.7) & (0.40,0.8)\\
(0.20,0.9) & (\mathbf{0.50},0.8) & (\mathbf{0.50},0.7) & (0.23,0.7) & (0.40,0.8)\\
(0.10,0.4) & (\mathbf{0.50},0.5) & (\mathbf{0.50},0.6) & (0.33,0.7) & (0.44,0.8)
\end{pmatrix}$$

Step 4 Reduced fuzzy numerical grades ρ_{rf} (see Figure 7) and possibilities σ are given in Table 2. The value of ρ and σ against the pairs $(p_i,p_i), i=1,2,3$ are considered as 0 as both the parameters are the same.
$Score(d_1)=\mathbb{S}(d_1)=(0.30\times 0.9)=0.27$
$Score(d_2)=\mathbb{S}(d_2)=(0.24\times 0.8)+(0.34\times 0.3)+(0.50\times 0.5)+(0.50\times 0.8)+0=0.944$
$Score(d_3)=\mathbb{S}(d_3)=(0.24\times 0.7)+(0.34\times 0.4)+(0.57\times 0.6)+(0.50\times 0.6)+(0.50\times 0.7)+0=1.296$
$Score(d_4)=\mathbb{S}(d_4)=(0.26\times 0.6)+(0.46\times 0.6)+0+(0.30\times 0.6)+(0.30\times 0.6)=0.792$
$Score(d_5)=\mathbb{S}(d_5)=0+(0.70\times 0.8)+(0.70\times 0.8)+0+(0.70\times 0.8)=1.68$
Step 5 and Step 6 As score of d_5 is maximum (as depicted in Figure 8), therefore it is selected.

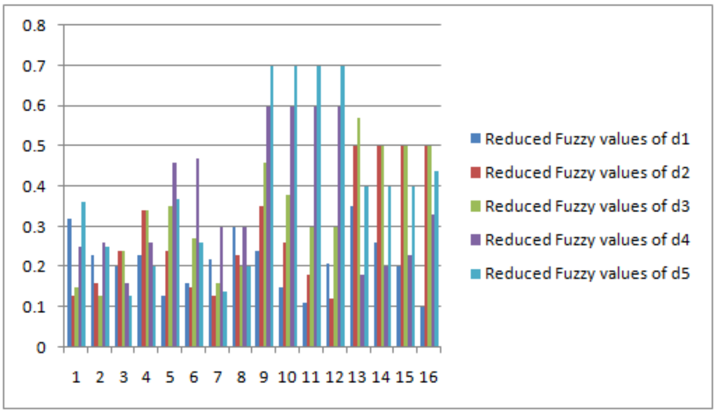

Figure 7. Reduced fuzzy values in $\mathcal{V}_\delta$ under OR-algorithm.

Table 2. OR-operation based grade table.

$\mathcal{V}_\delta$	d_i	Highest ρ_{rf}	σ_i
(p_1, p_1)	d_5	0	0
(p_1, p_2)	d_4	0.26	0.6
(p_1, p_3)	d_2, d_3	0.24, 0.24	0.8, 0.7
(p_1, p_4)	d_2, d_3	0.34, 0.34	0.3, 0.4
(p_2, p_1)	d_4	0.46	0.6
(p_2, p_2)	d_4	0	0
(p_2, p_3)	d_4	0.30	0.6
(p_2, p_4)	d_1, d_4	0.30, 0.30	0.9, 0.6
(p_3, p_1)	d_5	0.70	0.8
(p_3, p_2)	d_5	0.70	0.8
(p_3, p_3)	d_5	0	0
(p_3, p_4)	d_5	0.70	0.8
(p_4, p_1)	d_3	0.57	0.6
(p_4, p_2)	d_2, d_3	0.50, 0.50	0.5, 0.6
(p_4, p_3)	d_2, d_3	0.50, 0.50	0.8, 0.7
(p_4, p_4)	d_2, d_3	0	0

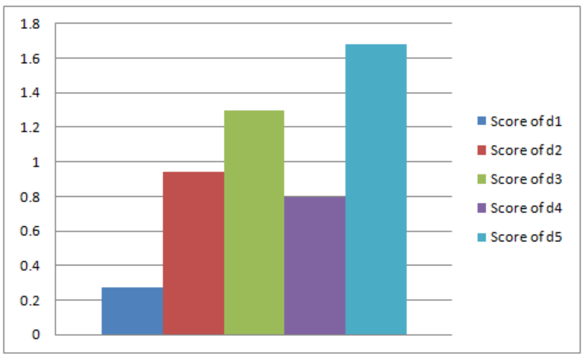

Figure 8. Score values under OR-algorithm.

The comparison of ranking of Dryers by both methods has been presented in Table 3 and Figure 9. It can be observed that mean score of AND-operation is 0.1892 and of OR-operation is 0.9964. Since $0.1892 \,\&\, 0.9964 \in [0, 1]$ but the value 0.1892 is more reliable and precise as compared to 0.9964 therefore it can be concluded that OR-operation is not consistent and requires some weight to be applied.

Table 3. Comparison between the score values of AND and OR-operations.

Aggregation Operation	d_1	d_2	d_3	d_4	d_5	Mean Score	Ranking
AND-Operation	0.049	0.078	0.178	0.406	0.235	0.1892	$d_4 > d_5 > d_3 > d_2 > d_1$
OR-Operation	0.270	0.944	1.296	0.792	1.68	0.9964	$d_5 > d_3 > d_2 > d_4 > d_1$

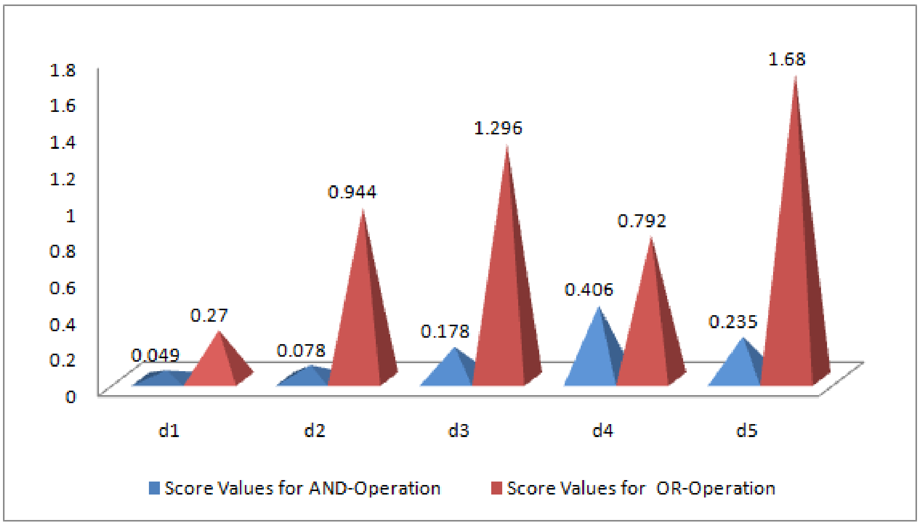

Figure 9. Comparison analysis of score values for AND-operation and OR-operation.

6. Similarity measures between PIFHS-sets

Similarity measures have broad applications in many fields like pattern recognition, image processing, region extraction, coding theory, etc. We are frequently intrigued to know whether two examples or pictures are alike or nearly alike or possibly how much they are alike. Many researchers have discussed the similarity measures between fuzzy sets, fuzzy numbers, and vague sets. Majumdar and Samanta [38–40] have employed the concept of similarity measures to soft sets, fuzzy soft sets, and generalised fuzzy soft sets. In this section, we introduce a measure of similarity between two PFSSs with partial modification in similarity methods adopted by Majumdar and Samanta [38–40]. The set theoretic approach has been employed in this regard because it is very popular and easier for computations.

Definition 6.1. (**similarity measures b/w PIFHS-sets**) Similarity measure between two PIFHS-sets $\mathcal{W}_\zeta$ and $\mathcal{L}_\xi$, denoted by $\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi)$, is defined as follows:

$$\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi) = \mathfrak{M}(\mathcal{W}(\varepsilon), \mathcal{L}(\varepsilon)) \times \mathfrak{M}(\zeta(\varepsilon), \xi(\varepsilon)) \tag{6.1}$$

Such that

$$\mathfrak{M}(\mathcal{W}(\varepsilon), \mathcal{L}(\varepsilon)) = max\ \mathfrak{M}_i(\mathcal{W}(\varepsilon), \mathcal{L}(\varepsilon)) \tag{6.2}$$

and

$$\mathfrak{M}(\zeta(\varepsilon), \xi(\varepsilon)) = \max\ \mathfrak{M}_i(\zeta(\varepsilon), \xi(\varepsilon)) \tag{6.3}$$

where

$$\mathfrak{M}_i(\mathcal{W}(\varepsilon), \mathcal{L}(\varepsilon)) = 1 - \frac{\sum_{j=1}^{n} \left|\mathcal{W}_{ij}(\varepsilon) - \mathcal{L}_{ij}(\varepsilon)\right|}{\sum_{j=1}^{n} \left|\mathcal{W}_{ij}(\varepsilon) + \mathcal{L}_{ij}(\varepsilon)\right|} \tag{6.4}$$

here $\mathcal{W}_{ij}(\varepsilon)$ and $\mathcal{L}_{ij}(\varepsilon)$ are reduced fuzzy numerical grades with $\mathcal{W}_{ij}(\varepsilon) = |T_{\mathcal{W}_{ij}}(\varepsilon) - F_{\mathcal{W}_{ij}}(\varepsilon)|$ and $\mathcal{L}_{ij}(\varepsilon) = |T_{\mathcal{L}_{ij}}(\varepsilon) - F_{\mathcal{L}_{ij}}(\varepsilon)|$ and

$$\mathfrak{M}_i(\zeta(\varepsilon), \xi(\varepsilon)) = 1 - \frac{\sum_{j=1}^{n} \left|\zeta_{ij}(\varepsilon) - \xi_{ij}(\varepsilon)\right|}{\sum_{j=1}^{n} \left|\zeta_{ij}(\varepsilon) + \xi_{ij}(\varepsilon)\right|} \tag{6.5}$$

Definition 6.2. (**Radical similarity b/w PIFHS-sets**) Let $\mathcal{W}_\zeta, \mathcal{L}_\xi \in \Omega_{pifhss}$ then $\mathcal{W}_\zeta$ and $\mathcal{L}_\xi$ are said to be radically similar if $\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi) \geq \frac{1}{2}$.

Proposition 6.3. *Let $\mathcal{W}_\zeta, \mathcal{L}_\xi, \mathcal{H}_\lambda \in \Omega_{pifhss}$ then the following holds:*

(i) $\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi) = \mathfrak{S}(\mathcal{L}_\xi, \mathcal{W}_\zeta)$.
(ii) $0 \leq \mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi) \leq 1$.
(iii) If $\mathcal{W}_\zeta = \mathcal{L}_\xi$ *then* $\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi) = 1$.
(iv) If $\mathcal{W}_\zeta \subseteq \mathcal{L}_\xi \subseteq \mathcal{H}_\lambda$ *then* $\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{H}_\lambda) \leq \mathfrak{S}(\mathcal{L}_\xi, \mathcal{H}_\lambda)$.
(v) $\mathcal{W}_\zeta \cap \mathcal{L}_\xi = \emptyset$ *then* $\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi) = 0$.

Proof. The proofs of (i)–(v) are straightforward and follow from Definition 6.1. □

Example 6.4. Consider PIFHS-sets from Example 5.1, we have

$$\mathfrak{M}_1(\zeta(p_1), \xi(p_1)) = 1 - \frac{\sum_{j=1}^{5} \left|\zeta_{1j}(p_1) - \xi_{1j}(p_1)\right|}{\sum_{j=1}^{5} \left|\zeta_{1j}(p_1) + \xi_{1j}(p_1)\right|}$$

$$= 1 - \frac{\left\{ \begin{array}{l} |(0.2 - 0.1)| + |(0.3 - 0.2)| + |(0.4 - 0.3)| \\ + |(0.5 - 0.4)| + |(0.6 - 0.5)| \end{array} \right\}}{\left\{ \begin{array}{l} |(0.2 + 0.1)| + |(0.3 + 0.2)| + |(0.4 + 0.3)| \\ + |(0.5 + 0.4)| + |(0.6 + 0.5)| \end{array} \right\}}$$

$= 0.86$.
Similarly
$\mathfrak{M}_2(\zeta(p_2), \xi(p_2)) = 0.77$,
$\mathfrak{M}_3(\zeta(p_3), \xi(p_3)) = 0.90$,
$\mathfrak{M}_4(\zeta(p_4), \xi(p_4)) = 0.67$, therefore
$\mathfrak{M}(\zeta(p), \xi(p)) = 0.90$.
Now

$$\mathfrak{M}_1(\mathcal{W}(p_1), \mathcal{L}(p_1)) = 1 - \frac{\sum_{j=1}^{5} \left|\mathcal{W}_{1j}(p_1) - \mathcal{L}_{1j}(p_1)\right|}{\sum_{j=1}^{5} \left|\mathcal{W}_{1j}(p_1) + \mathcal{L}_{1j}(p_1)\right|}$$

$$= 1 - \frac{\left\{\begin{array}{l} |(0.20-0.10)|+|(0.10-0.10)|+|(0.10-0.10)| \\ +|(0.10-0.10)|+|(0.10-0.10)| \end{array}\right\}}{\left\{\begin{array}{l} |(0.20+0.10)|+|(0.10+0.10)|+|(0.10+0.10)| \\ +|(0.10+0.10)|+|(0.10+0.10)| \end{array}\right\}} = 0.909.$$

Similarly
$\mathfrak{M}_2(\mathcal{W}(p_2), \mathcal{L}(p_2)) = 0.9231$,
$\mathfrak{M}_3(\mathcal{W}(p_3), \mathcal{L}(p_3)) = 0.4348$,
$\mathfrak{M}_4(\mathcal{W}(p_4), \mathcal{L}(p_4)) = 0.6667$, therefore
$\mathfrak{M}(\mathcal{W}(p), \mathcal{L}(p)) = 0.9231$.
Hence, the similarity between the two PIFHS-sets $\mathcal{W}_\zeta$ and $\mathcal{L}_\xi$ is given by $\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi) = 0.90 \times 0.9231 \cong 0.831$ which means $\mathcal{W}_\zeta$ and $\mathcal{L}_\xi$ are radically similar.

6.1. Application of similarity between PIFHS-sets in recruitment pattern recognition

In this subsection, we will try to estimate the possibility that a candidate having some qualification and experience, is suitable for a job in a company or not. For this we first propose an algorithm based on the concept of similarity measures b/w PIFHS-sets. The methodology is that a model PIFHS-set for Standard recruitment is constructed and ordinary PIFHS-sets are designed for the candidate. Then we calculate the similarity measure of this set with model PIFHS-set to observe pattern recognition. If PIFHs-set of a candidate is significantly similar to model PIFHS-set then that candidate is recommended for selection. The input variables are sub-parameters with PIFHS-numbers as approximate elements with hypothetical observation.

Algorithm 3: recruitment pattern recognition based on similarity measures b/w PIFHS-sets

---

Step 1 Construct PIFHS-set $\mathcal{W}_\zeta$ according to Company Experts Team.

Step 2 Construct PIFHS-set $\mathcal{L}_\xi$ according to External Experts.

Step 3 Represent PIFHS-sets $\mathcal{W}_\zeta$ & $\mathcal{L}_\xi$ in matrix notations having reduced fuzzy numerical grades as entries.

Step 4 Determine $\mathfrak{M}_i(\mathcal{W}(\varepsilon), \mathcal{L}(\varepsilon))$ and $\mathfrak{M}_i(\zeta(\varepsilon), \xi(\varepsilon))$ by using Equations 6.4 and 6.5.

Step 5 Determine $\mathfrak{M}(\mathcal{W}(\varepsilon), \mathcal{L}(\varepsilon))$ and $\mathfrak{M}(\zeta(\varepsilon), \xi(\varepsilon))$ by using Equations 6.2 and 6.3.

Step 6 Calculate $\mathfrak{S}(\mathcal{W}_\zeta, \mathcal{L}_\xi)$ according to Equation 6.1.

Step 7 Check the nature of similarity in accordance with Definition 6.2.

The pictorial representation of algorithm 3 is given in Figure 10.

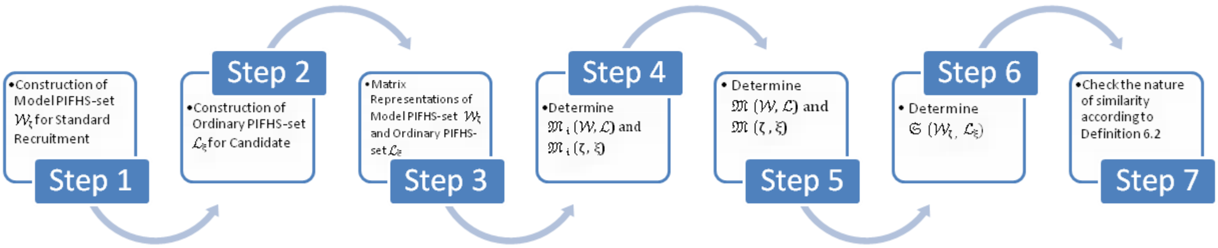

Figure 10. Pictorial representation of algorithm 3.

Example 6.5. Let there are only two elements "recommended= z_1" and "not-recommended= z_2", i.e., $\mathcal{Z} = \{z_1, z_2\}$. The selection of candidate is evaluated on the basis of certain distinct parameters. Let these parameters are p_1 = qualification, p_2 = age and p_3 = experience. These parameters are further partitioned into disjoint attribute-valued sets that are given as
$\mathcal{P}_1 = \{p_{11} = BachelorDegree, p_{12} = MasterDegree\}$
$\mathcal{P}_2 = \{p_{21} = 25years, p_{22} = 30years\}$
$\mathcal{P}_2 = \{p_{21} = 5years, p_{22} = 10years\}$
then
$Q = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3 = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8\}$ where each q_i is a 3-tupple element of Q. Let $Q_1 = \{q_3, q_4, q_7, q_8\}$ is a subset of Q.
Step 1–Step 3 Consider a model PIFHS-set for Standard recruitment is $\mathcal{W}_\zeta$ which is constructed by experts team deputed for general recruitment in Company.

$$\mathcal{W}_\zeta(q_3) = \left\{ \left(\frac{z_1}{<1,0>}, 1\right), \left(\frac{z_2}{<0,1>}, 1\right) \right\},$$
$$\mathcal{W}_\zeta(q_4) = \left\{ \left(\frac{z_1}{<0,1>}, 1\right), \left(\frac{z_2}{<1,0>}, 1\right) \right\},$$
$$\mathcal{W}_\zeta(q_7) = \left\{ \left(\frac{z_1}{<0,1>}, 1\right), \left(\frac{z_2}{<0,1>}, 1\right) \right\},$$
$$\mathcal{W}_\zeta(q_8) = \left\{ \left(\frac{z_1}{<1,0>}, 1\right), \left(\frac{z_2}{<0,1>}, 1\right) \right\},$$

and its matrix representation is given as

$$\mathcal{W}_\zeta = \begin{pmatrix} (<1,0>,1) & (<0,1>,1) \\ (<0,1>,1) & (<1,0>,1) \\ (<0,1>,1) & (<0,1>,1) \\ (<1,0>,1) & (<1,0>,1) \end{pmatrix}$$

Now we present matrix representation of $\mathcal{W}_\zeta$ with reduced fuzzy numerical grades as

$$\begin{pmatrix} (1,1) & (1,1) \\ (1,1) & (1,1) \\ (1,1) & (1,1) \\ (1,1) & (1,1) \end{pmatrix}$$

and $\mathcal{L}_\xi$ is a PIFHS-set for the candidate which is constructed by an expert outside the experts team of the company.

$$\mathcal{L}_\xi(q_3) = \left\{ \left(\frac{z_1}{<0.4,0.6>}, 0.3\right), \left(\frac{z_2}{<0.1,0.8>}, 0.4\right) \right\},$$

$$\mathcal{L}_{\xi}(q_4) = \left\{ \left(\frac{z_1}{<0.3,0.7>}, 0.4\right), \left(\frac{z_2}{<0.2,0.5>}, 0.5\right) \right\},$$

$$\mathcal{L}_{\xi}(q_7) = \left\{ \left(\frac{z_1}{<0.2,0.5>}, 0.7\right), \left(\frac{z_2}{<0.3,0.6>}, 0.8\right) \right\},$$

$$\mathcal{L}_{\xi}(q_8) = \left\{ \left(\frac{z_1}{<0.6,0.4>}, 0.8\right), \left(\frac{z_2}{<0.5,0.2>}, 0.9\right) \right\},$$

and its matrix representation is given as

$$\mathcal{L}_{\xi} = \begin{pmatrix} \left(<0.4,0.6>, 0.3 \right) & \left(<0.1,0.8>, 0.4 \right) \\ \left(<0.3,0.7>, 0.4 \right) & \left(<0.2,0.5>, 0.5 \right) \\ \left(<0.2,0.5>, 0.7 \right) & \left(<0.3,0.6>, 0.8 \right) \\ \left(<0.6,0.4>, 0.8 \right) & \left(<0.5,0.2>, 0.9 \right) \end{pmatrix}$$

Now we present matrix representation of $\mathcal{L}_{\xi}$ with reduced fuzzy numerical grades as

$$\mathcal{L}_{\xi} = \begin{pmatrix} \left(0.2,0.3 \right) & \left(0.7,0.4 \right) \\ \left(0.4,0.4 \right) & \left(0.3,0.5 \right) \\ \left(0.3,0.7 \right) & \left(0.3,0.8 \right) \\ \left(0.2,0.8 \right) & \left(0.3,0.9 \right) \end{pmatrix}$$

Step 4 and Step 5 Now we calculate similarity between $\mathcal{W}_{\zeta}$ and $\mathcal{L}_{\xi}$ according to Definition 6.1

$$\mathfrak{M}_1(\zeta(q_3), \xi(q_3)) = 1 - \frac{|(1-0.3)| + |(1-0.4)|}{|(1+0.3)| + |(1+0.4)|}$$

$= 0.5185$.
Similarly
$\mathfrak{M}_2(\zeta(q_4), \xi(q_4)) = 0.6207$,
$\mathfrak{M}_3(\zeta(q_7), \xi(q_7)) = 0.8571$,
$\mathfrak{M}_4(\zeta(q_8), \xi(q_8)) = 0.9189$, therefore
$\mathfrak{M}(\zeta(q), \xi(q)) = 0.9189$.
Now

$$\mathfrak{M}_1(\mathcal{W}(q_3), \mathcal{L}(q_3)) = 1 - \frac{|(1-0.2)| + |(1-0.7)|}{|(1+0.2)| + |(1+0.7)|}$$

$= 0.6207$.
Similarly
$\mathfrak{M}_2(\mathcal{W}(q_2), \mathcal{L}(q_2)) = 0.5185$,
$\mathfrak{M}_3(\mathcal{W}(q_3), \mathcal{L}(q_3)) = 0.4615$,
$\mathfrak{M}_4(\mathcal{W}(q_4), \mathcal{L}(q_4)) = 0.4000$, therefore
$\mathfrak{M}(\mathcal{W}(q), \mathcal{L}(q)) = 0.6207$.
Step 6 and Step 7 Hence, the similarity between the two PIFHS-sets $\mathcal{W}_{\zeta}$ and $\mathcal{L}_{\xi}$ is given by $\mathfrak{S}(\mathcal{W}_{\zeta}, \mathcal{L}_{\xi}) = 0.9189 \times 0.6207 \cong 0.5704 > \frac{1}{2}$ which means $\mathcal{W}_{\zeta}$ and $\mathcal{L}_{\xi}$ are radically similar. Therefore candidate is recommended for the job in the company.

6.2. Comparison analysis

In literature, certain decision-making algorithmic approaches have already been discussed by [14, 15, 30–34] that based on fuzzy-like and intuitionistic fuzzy-like soft set theories. Decision making is badly affected due to omission of some features. For example, in recruitment process, it is insufficient to consider qualification, experience, age as only attributes because candidates may have different qualifications, experiences and ages so it is much appropriate to further classify these attributes into their disjoint attributive sets as we have done in Example 6.5. All existing decision making models have used single set of attributes but in proposed model, distinct attributes are further partitioned into disjoint attributive sets. The consideration of such sets will make the decision making process more reliable and trust-worthy. We present a comparison analysis of our proposed structure with the relevant existing structures [14, 15] in Table 4.

Table 4. Comparison analysis.

Authors	Structure	Nature of Similarity/Value	Reduced Fuzzy Grade Method	Focus on Sub-attributes
Alkhazaleh et al. [14]	Possibility FS-set	Radical/0.74	Not Applicable	Ignored
Bashir et al. [15]	Possibility IFS-set	Non-Radical/0.26	Reduced Fuzzy Grade = $\frac{T+F}{2}$	Ignored
Majumdar & Samanta [38]	Generalised FS-set	Radical/0.60	Not Applicable	Ignored
Majumdar & Samanta [39]	Fuzzy soft set	Radical/0.71	Not Applicable	Ignored
Majumdar & Samanta [40]	Soft set	Radical/0.5	Not Applicable	Ignored
Proposed Structure	Possibility IFHS-set	Radical/0.831	Reduced Fuzzy Grade = $\lvert T - F\rvert$	Considered

7. Discussion

Here we discuss that our proposed structure i.e., possibility intuitionistic fuzzy hypersoft set is the most generalized and flexible structure as

(i) It reduces to possibility fuzzy hypersoft set (PFHS-set) if non-membership degree is ignored.
(ii) If membership and non-membership degrees are ignored and only disjoint attribute-valued sets corresponding to distinct attributes are considered with possibility degree, then it becomes possibility hypersoft set (PHS-set).
(iii) It reduces to fuzzy hypersoft set (FHS-set) if possibility and non-membership degrees are ignored.
(iv) If possibility degree along with membership and non-membership degrees, is omitted then it reduces to hypersoft set (HS-set).
(v) If only single set of parameters is considered then it becomes possibility intuitionistic fuzzy soft set (PIFS-set).
(vi) It reduces to possibility fuzzy soft set (PFS-set) if non-membership degree is ignored and only

single set of parameters is considered.

(vii) It reduces to possibility soft set (PS-set) if membership and non-membership degrees are ignored and only single set of parameters is considered.

(viii) It reduces to fuzzy soft set (FS-set) if possibility and non-membership degrees are ignored and only single set of parameters is considered.

(ix) If possibility, membership and non-membership degrees are ignored and only single set of parameters is considered then it becomes soft set (S-set).

The Figure 11 depicts the generalization of proposed structure.

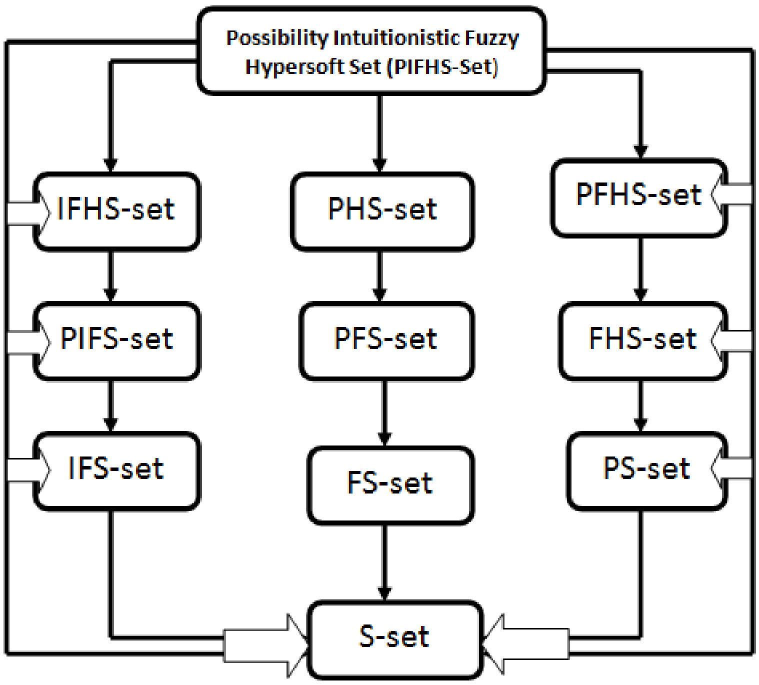

Figure 11. Generalization of proposed structure i.e., PIFHS-set.

7.1. Merits of proposed structure

In this subsection, some merits of the proposed structure (i.e., PIFHS-set) are highlighted, which are given below:

(i) The introduced approach took the significance of the idea of possibility alongside the IFHS-set to deal with current decision-making issues. The considered possibility degree mirrors the possibility of the existence of the level of acknowledgment and excusal; along these lines, this association has tremendous potential in the genuine depiction inside the space of computational incursions.

(ii) As the proposed structure emphasizes on in-depth study of attributes (i.e., further partitioning of attributes) rather than focussing on attributes merely therefore it makes the decision-making process better, flexible and more reliable.

(iii) It contains all the characteristics and properties of the existing structures i.e., PFHS-set, PHS-set, FHS-set, HS-set, PIFS-set, PFS-set, PS-set, FS-set and S-set so it is not unreasonable to call it the generalized form of all these structures.

The advantage of the proposed structure can easily be judged from the Table 5. The comparison is evaluated with features: MG (Membership Grade), NMG (Non Membership Grade), DOP (Degree of Possibility), SAAF (Single Argument Approximate function) and MAAF (Multi Argument Approximate function).

Table 5. Comparison with existing models under appropriate features.

Authors	Structure	MD	NMD	DOP	SAAF	MAAF
Maji et al. [5]	Fuzzy soft sets	✓	×	×	✓	×
Maji et al. [13]	Intuitionistic FS-sets	✓	✓	×	✓	×
Alkhazaleh et al. [14]	Possibility fuzzy soft set	✓	×	✓	✓	×
Bashir et al. [15]	Possibility intuitionistic FS-set	✓	✓	✓	✓	×
Smarandache [16]	Hypersoft set	×	×	×	✓	✓
Bashir et al. [30]	Possibility fuzzy soft expert set	✓	×	✓	✓	×
Zhang et al. [31]	Possibility multi-fuzzy soft se	✓	×	✓	✓	×
Kalaiselvi et al. [32]	Possibility fuzzy soft set	✓	×	✓	✓	×
Ponnalagu et al. [33]	Possibility fuzzy soft expert set	✓	×	✓	✓	×
Garg et al. [34]	Possibility intuitionistic FS-set	✓	✓	✓	✓	×
Khalil et al. [35]	Possibility m-polar fuzzy soft set	✓	×	✓	✓	×
Debnath [36]	Fuzzy hypersoft set	✓	×	×	✓	✓
Jafar et al. [37]	Intuitionistic fuzzy HS-set	✓	✓	×	✓	✓
Proposed Structure	Possibility intuitionistic FHS-set	✓	✓	✓	✓	✓

8. Conclusions

The key features of this work can be summarized as follows:

(i) The novel notion of possibility intuitionistic fuzzy hypersoft set (PIFHS-set) is characterized and some elementary properties i.e., PIFHS-subset, possibility null IFHS-set, possibility absolute IFHS-set, and complement of PIFHS-set are discussed with illustrated numerical examples.

(ii) The set theoretic operations of PIFHS-sets i.e., union, intersection, AND, OR, are characterized with the help of elaborated examples. Their fundamental laws and properties are also discussed.

(iii) Two decision-making algorithms based on AND and OR operations are proposed and explained with the help of daily life problems.

(iv) Similarity between PIFHS-sets is formulated and successfully applied to real world decision making problem. A comparison of similarity for proposed model is made with some existing models.

(v) Authors have carve out a conceptual framework for a generalized model i.e., PIFHS-set to deal decision-making real life problems by considering hypothetical data. The authors are devoted to discuss some case studies based on PIFHS-set by using real data. Moreover, other types of similarity measures, entropies and aggregation operators will be investigated for studying multi-criteria decision-making problems.

(vi) The proposed study may also be extended for developing the following hybridized models:

- Possibility Interval-valued Intuitionistic fuzzy hypersoft set

- Possibility Neutrosophic hypersoft set
- Possibility Interval-valued Neutrosophic hypersoft set
- Possibility Interval-valued fuzzy hypersoft set
- Possibility fuzzy hypersoft expert set etc.

with applications in decision-making.

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Conflict of interest

The authors declare no conflict of interest.

References

1. L. A. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338–353. doi: 10.1016/S0019-9958(65)90241-X.
2. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1986), 87–96. doi: 10.1016/S0165-0114(86)80034-3.
3. D. Molodtsov, Soft set theory – First results, *Comput. Math. Appl.*, **37** (1999), 19–31. doi: 10.1016/S0898-1221(99)00056-5.
4. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, *Comput. Math. Appl.*, **45** (2003), 555–562. doi: 10.1016/S0898-1221(03)00016-6.
5. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *Journal of Fuzzy Mathematics*, **9** (2001), 589–602.
6. D. Pei, D. Miao, From soft set to information system, In: *2005 IEEE International Conference on Granular Computing*, 2005, 617–621. doi: 10.1109/GRC.2005.1547365.
7. M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Sabir, On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547–1553. doi: 10.1016/j.camwa.2008.11.009.
8. K. V. Babitha, J. J. Sunil, Soft set relations and functions, *Comput. Math. Appl.*, **60** (2010), 1840–1849. doi: 10.1016/j.camwa.2010.07.014.
9. K. V. Babitha, J. J. Sunil, Transitive closure and ordering in soft set, *Comput. Math. Appl.*, **61** (2011), 2235–2239. doi: 10.1016/j.camwa.2011.07.010.
10. A. Sezgin, A. O. Atagün, On operations of soft sets, *Comput. Math. Appl.*, **61** (2011), 1457–1467. doi: 10.1016/j.camwa.2011.01.018.
11. X. Ge, S. Yang, Investigations on some operations of soft sets, *World Academy of Science, Engineering and Technology*, **51** (2011), 1112–1115.
12. F. Li, Notes on soft set operations, *ARPN Journal of Systems and Softwares*, **1** (2011), 205–208.

13. P. K. Maji, R. Biswas, A. R. Roy, Intuitionistic fuzzy soft sets, *The Journal of Fuzzy Mathematics*, **9** (2001), 677–692.

14. S. Alkhazaleh, A. R. Salleh, N. Hassan, Possibility fuzzy soft set, *Advances in Decision Sciences*, **2011** (2011), 479756. doi: 10.1155/2011/479756.

15. M. Bashir, A. R. Salleh, S. Alkhazaleh, Possibility intuitionistic fuzzy soft set, *Advances in Decision Sciences*, **2012** (2012), 404325. doi: 10.1155/2012/404325.

16. F. Smarandache, Extension of soft set of hypersoft set, and then to plithogenic hypersoft set, *Neutrosophic Sets Syst.*, **22** (2018), 168–170. doi: 10.5281/zenodo.2838716.

17. M. Saeed, M. Ahsan, M. K. Siddique, M. R. Ahmad, A study of the fundamentals of hypersoft set theory, *International Journal of Scientific and Engineering Research*, **11** (2020), 320–329.

18. M. Saeed, A. U. Rahman, M. Ahsan, F. Smarandache, An inclusive study on fundamentals of hypersoft set, In: *Theory and application of hypersoft set*, Brussel: Pons Publishing House, 2021, 1–23.

19. F. Abbas, G. Murtaza, F. Smarandache, Basic operations on hypersoft sets and hypersoft points, *Neutrosophic Sets Syst.*, **35** (2020), 407–421. doi: 10.5281/zenodo.3951694.

20. M. Saqlain, N. Jafar, S. Moin, M. Saeed, S. Broumi, Single and multi-valued neutrosophic hypersoft set and tangent similarity measure of single valued neutrosophic hypersoft sets, *Neutrosophic Sets Syst.*, **32** (2020), 317–329. doi: 10.5281/zenodo.3723165.

21. M. Saqlain, S. Moin, N. Jafar, M. Saeed, F. Smarandache, Aggregate operators of neutrosophic hypersoft sets, *Neutrosophic Sets Syst.*, **32** (2020), 235–247. doi: 10.5281/zenodo.3723155.

22. M. Saeed, M. Ahsan, A. U. Rahman, M. H. Saeed, A. Mehmood, An application of neutrosophic hypersoft mapping to diagnose brain tumor and propose appropriate treatment, *J. Intell. Fuzzy Syst.*, **41** (2021), 1677–1699. doi: 10.3233/JIFS-210482.

23. M. Ihsan, A. U. Rahman, M. Saeed, Hypersoft expert set with application in decision making for recruitment process, *Neutrosophic Sets Syst.*, **42** (2021), 191–207. doi: 10.5281/zenodo.4711524.

24. R. M. Zulqarnain, X. L. Xin, B. Ali, S. Broumi, S. Abdal, M. I. Ahamad, Decision-making ppproach based on correlation coefficient with its properties under interval-valued neutrosophic hypersoft set environment, *Neutrosophic Sets Syst.*, **40** (2021), 12–28. doi: 10.5281/zenodo.4549309.

25. A. U. Rahman, M. Saeed, F. Smarandache, M. R. Ahmad, Development of hybrids of hypersoft set with complex fuzzy set, complex intuitionistic fuzzy set and complex neutrosophic set, *Neutrosophic Sets Syst.*, **38** (2020), 335–354. doi: 10.5281/zenodo.4300520.

26. A. U. Rahman, M. Saeed, F. Smarandache, Convex and concave hypersoft sets with some properties, *Neutrosophic Sets Syst.*, **38** (2020), 497–508. doi: 10.5281/zenodo.4300580.

27. A. U. Rahman, M. Saeed, A. Dhital, Decision making application based on neutrosophic parameterized hypersoft set theory, *Neutrosophic Sets Syst.*, **41** (2021), 1–14. doi: 10.5281/zenodo.4625665.

28. H. Kamacı, On hybrid structures of hypersoft sets and rough sets, *International Journal of Modern Science and Technology*, **6** (2021), 69–82.

29. H. Kamacı, M. Saqlain, n-ary Fuzzy hypersoft expert sets, *Neutrosophic Sets Syst.*, **43** (2021), 180–211. doi: 10.5281/zenodo.4914849.

30. M. Bashir, A. R. Salleh, Possibility fuzzy soft expert set, *Open Journal of Applied Sciences*, **12** (2012), 208–211. doi: 10.4236/ojapps.2012.24B047.

31. H. D. Zhang, L. Shu, Possibility multi-fuzzy soft set and its application in decision making, *J. Intell. Fuzzy Syst.*, **27** (2014), 2115–2125. doi: 10.3233/IFS-141176.

32. S. Kalaiselvi, V. Seenivasan, Decision making in sports via possibility fuzzy soft set, *International Journal of Engineering Technology and Applied Science*, **3** (2017), 1–5.

33. K. Ponnalagu, P. Mounika, A study on possibility fuzzy soft expert set, *International Journal of Engineering, Science and Mathematics*, **7** (2018), 256–262.

34. H. Garg, R. Arora, Algorithms based on COPRAS and aggregation operators with new information measures for possibility intuitionistic fuzzy soft decision-making, *Math. Probl. Eng.*, **2020** (2020), 1563768. doi: 10.1155/2020/1563768.

35. A. M. Khalil, S. G. Li, H. X. Li, S. Q. Ma, Possibility m-polar fuzzy soft sets and its application in decision-making problems, *J. Intell. Fuzzy Syst.*, **37** (2019), 929–940. doi: 10.3233/JIFS-181769.

36. S. Debnath, Fuzzy hypersoft sets and its weightage operator for decision making, *Journal of Fuzzy Extension and Applications*, **2** (2021), 163–170. doi: 10.22105/jfea.2021.275132.1083.

37. M. N. Jafar, M. Saeed, M. Haseeb, A. Habib, Matrix theory for intuitionistic fuzzy hypersoft sets and its application in multi-attributive decision-making problems, In: *Theory and application of hypersoft set*, Brussel: Pons Publishing House, 2021, 65–84. doi: 10.5281/zenodo.4787683.

38. P. Majumdar, S. K. Samanta, Generalised fuzzy soft sets, *Comput. Math. Appl.*, **59** (2010), 1425–1432. doi: 10.1016/j.camwa.2009.12.006.

39. P. Majumdar, S. K. Samanta, On similarity measure of fuzzy soft sets, *Int. J. Advance. Soft Comput. Appl.*, **3** (2011), 1–8.

40. P. Majumdar, S. K. Samanta, Similarity measure of soft sets, *New Math. Nat. Comput.*, **4** (2008), 1–12. doi: 10.1142/S1793005708000908.