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Research article

Degenerate Catalan-Daehee numbers and polynomials of order r arising from degenerate umbral calculus

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Abstract: Many mathematicians have studied degenerate versions of some special polynomials and numbers that can take into account the surrounding environment or a person's psychological burden in recent years, and they've discovered some interesting results. Furthermore, one of the most important approaches for finding the combinatorial identities for the degenerate version of special numbers and polynomials is the umbral calculus. The Catalan numbers and the Daehee numbers play important role in connecting relationship between special numbers.

In this paper, we first define the degenerate Catalan-Daehee numbers and polynomials and aim to study the relation between well-known special polynomials and degenerate Catalan-Daehee polynomials of order r as one of the generalizations of the degenerate Catalan-Daehee polynomials by using the degenerate Sheffer sequences. Some of them include the degenerate and other special polynomials and numbers such as the degenerate falling factorials, the degenerate Bernoulli polynomials and numbers of order r, the degenerate Euler polynomials and numbers of order r, the degenerate Daehee polynomials of order r, the degenerate Bell polynomials, and so on.

Keywords: Catalan numbers and polynomials; Catalan-Daehee polynomials; the degenerate Sheffer sequence; the degenerate Stirling numbers of the first kind; the degenerate Stirling numbers of the second kind

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1. Introduction

In combinatorics, the Catalan numbers are the numbers of Dyck words of length $2n$ or the numbers of different ways $n+1$ factors completely parenthesized or the numbers of non-isomorphic ordered trees with $n+1$ vertices or the numbers of monotonic lattice paths along the edges of a grid with $n \times n$

square cells, which do not pass above the diagonal or the numbers of noncrossing partitions of the set $\{1,...,n\}$ and arise in many other counting problems with real-world applications [3, 5, 7, 24, 28]. The Catalan-Daehee numbers are defined by assigning $\sqrt{1-4t}-1$ instead of t in the definition of Daehee numbers which play important role in connecting relationship between special numbers [10, 16]. Moreover, the generating function of Catalan numbers can be represented by the fermionic p-adic integral on $\mathbb{Z}_p$ of $(1-4t)^{\frac{x}{2}}$ and the generating function of Catalan-Daehee numbers can be represented by the p-adic Volkenborn integral on $\mathbb{Z}_p$ of the same function $(1-4t)^{\frac{x}{2}}$ [16, 17]. Various identities of Catalan-Daehee polynomials have been studied in [5, 16, 17, 29].

Many scholars in the field of mathematics have worked on degenerate versions of special polynomials and numbers which include the degenerate Stirling numbers of the first and second kinds, the degenerate Bernstein polynomials, the degenerate Bell numbers and polynomials, the degenerate gamma function, the degenerate gamma random variables, and so on [1, 2, 10–15, 19, 20, 30, 31]. We can find the motivation to study degenerate polynomials and numbers in the following real-world examples. Suppose the probability of a baseball player getting a hit in a match is p. We wonder if the probability that the player will succeed in the 11th trial after failing 9 times in 10 trials is still p. We can see cases where the probability is less than p because of the psychological burden that the player must succeed in the 11th trial [31].

In the 1970s, Rota and his collaborators [22–24] began to construct a rigorous foundation for the classical umbral calculus, which consisted of a symbolic technique for the manipulation of numerical and polynomial sequences. The umbral calculus has received much attention from researchers because of its numerous applications in many fields of mathematics, physics, chemistry, and engineering [4, 6, 9, 11, 13, 15, 16, 20–26, 28]. For instance, the connection between Sheffer polynomials and Riordan array and the isomorphism between the Sheffer groups and the Riordan Groups are proved [25, 26]. Recently, Kim-Kim [11] introduced the λ-Sheffer sequences and the degenerate Sheffer sequences by substituting λ-linear functionals and λ-differential operators, respectively, instead of linear functionals and differential operators.

With these points in mind, in this paper, we first define the degenerate Catalan-Daehee numbers and polynomials and degenerate Catalan-Daehee polynomials of order $r(\geq 1)$ as one of the generalizations of the degenerate Catalan-Daehee polynomials. It is difficult to study identities related to degenerate Catalan-Daehee polynomials and special polynomials using the p-adic integral on $\mathbb{Z}_p$ or other properties. Thus, we explore various interesting identities related to the degenerate Catalan-Daehee polynomials of order r and special polynomials and numbers by using degenerate Sheffer sequences. At the same time we derive the inversion formulas of these identities. Some of them include the degenerate and other special polynomials and numbers such as the degenerate falling factorials, the falling factorials, the degenerate Bernoulli polynomials and numbers of order r, the degenerate Euler polynomials and numbers of order r, the degenerate Daehee polynomials of order r, the degenerate Bell polynomials, etc.

Now, we give some definitions and properties needed in this paper.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \quad (|x \setminus \lambda| \leq 1) \quad \text{(see [10-17])}. \tag{1.1}$$

By Taylor expansion, we get

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},\qquad \text{(see [10-17])}, \tag{1.2}$$

where $(x)_{0,\lambda}=1,\ (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n\geq 1)$.

The degenerate Bernoulli polynomials and degenerate Euler polynomials of order r, respectively, are given by the generating function

$$\left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\qquad \text{(see [1,11}-\text{13])}, \tag{1.3}$$

and

$$\left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\qquad \text{(see [1, 11, 13])}. \tag{1.4}$$

We note that $B_{n,\lambda}^{(r)}=B_{n,\lambda}^{(r)}(0)$ and $E_{n,\lambda}^{(r)}=E_{n,\lambda}^{(r)}(0)$ $(n\geq 0)$, which are called the degenerate Bernoulli and degenerate Euler numbers of order r, respectively.

The degenerate Bernoulli polynomials of the second kind of order r are defined by the generating function

$$\left(\frac{t}{\log_\lambda(1+t)}\right)^r(1+t)^x=\sum_{n=0}^{\infty}b_{n,\lambda(x)}^{(r)}\frac{t^n}{n!},\qquad \text{(see [8, 11])}. \tag{1.5}$$

When $x=0$, $b_{n,\lambda}^{(r)}=b_{n,\lambda}^{(r)}(0)$, which are called the degenerate Bernoulli numbers of the second kind of order r.

The degenerate Daehee polynomials of order r are defined by the generating function

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\qquad \text{(see [6, 11])}, \tag{1.6}$$

where $\log_\lambda(1+t)=\frac{1}{\lambda}((1+t)^\lambda-1)$ and $\log_\lambda(e_\lambda(t))=e_\lambda(\log_\lambda(t))=t$.
When $x=0$, $D_{n,\lambda}^{(r)}=D_{n,\lambda}^{(r)}(0)$, which are called the degenerate Daehee numbers of order r.

The Bell polynomials are defined by the generating function

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!},\qquad \text{(see [3, 15, 19, 20])}.$$

Kim-Kim introduced the degenerate Bell polynomials given by the generating function

$$e_\lambda^x(e_\lambda(t)-1)=\sum_{l=0}^{\infty}Bel_{l,\lambda}(x)\frac{t^l}{l!},\qquad \text{(see [13])}. \tag{1.7}$$

When $x=1$, $Bel_{n,\lambda}^{(r)}=Bel_{n,\lambda}^{(r)}(1)$ are called the degenerate Bell numbers.

For $n \geq 0$, it is well known that the Stirling numbers of the first and second kind, respectively are given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \text{ and } \frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad \text{(see [1, 14])},$$

and

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \text{ and } \frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [1, 14])},$$

where $(x)_0 = 1,\ (x)_n = x(x-1)\dots(x-n+1),\ (n \geq 1)$.

Moreover, the degenerate Stirling numbers of the first and second kind, respectively are given by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \text{ and } \frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [12, 14])}, \tag{1.8}$$

and

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \text{ and } \frac{1}{k!}\big(e_\lambda(t)-1\big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [12, 14])}. \tag{1.9}$$

For $k \geq 0$, as an extension of the notion of the degenerate Stirling numbers of the second kind, Kim et al. introduced Jindalrae-Stirling numbers of the second kind by

$$\frac{1}{k!}(e_\lambda(e_\lambda(t)-1)-1)^k = \sum_{n=k}^{\infty} S_{j,\lambda}^{(2)}(n,k)\frac{t^n}{n!}, \quad \text{(see [18])}. \tag{1.10}$$

From (1.9) and (1.10), we note that

$$S_{j,\lambda}^{(2)}(n,k) = \sum_{m=k}^{n} S_{2,\lambda}(n,m)S_{2,\lambda}(m,k). \tag{1.11}$$

Let $\mathbb{C}$ be the complex number field and let $\mathscr{F}$ be the set of all power series in the variable t over $\mathbb{C}$ with

$$\mathscr{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \;\middle|\; a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ be the vector space all linear functional on $\mathbb{P}$:

$$\mathbb{P}_n = \{\ P(x) \in \mathbb{C}[x] \mid degP(x) \leq n\},\ \ (n \geq 0).$$

Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{C}$.

Recently, Kim-Kim [11] considered λ-linear functional and λ-differential operator as follows:

For $f(t)=\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathscr{F}$ and a fixed nonzero real number λ, each λ gives rise to the linear functional $\langle f(t) \mid \cdot \rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by

$$\langle f(t) \mid (x)_{n,\lambda} \rangle_\lambda = a_n, \quad \text{for all } n \geq 0, \qquad (\text{see } [11]). \tag{1.12}$$

In particular $\langle t^k \mid (x)_{n,\lambda} \rangle_\lambda = n!\delta_{n,k}$, for all $n,\ k\ \geq 0$, where $\delta_{n,k}$ is the Kronecker's symbol.

For $\lambda=0$, we observe that the linear functional $\langle f(t) \mid \cdot \rangle_0$ agrees with the one in $\langle f(t) \mid x^n \rangle = a_k$, $(k \geq 0)$.

For each $\lambda \in \mathbb{R}$ and each nonnegative integer k, they also defined the differential operator on $\mathbb{P}$ by

$$(t^k)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & \text{if } \ k \leq n, \\ 0 & \text{if } \ k \geq n, \quad (\text{see } [11]). \end{cases} \tag{1.13}$$

and for any power series $f(t)=\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathscr{F}$, $(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}, \quad (n \geq 0)$.

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t))=0$ and such series has a multiplicative inverse $1/f(t)$ of $f(t)$. $f(t)$ is called a delta series if $o(f(t))=1$ and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t))=f(\overline{f}(t))=t$.

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively. Then there exists a unique sequence $s_{n,\lambda}(x)$ such that the orthogonality condition holds

$$\left\langle g(t)\big(f(t)\big)^k \mid s_{n,\lambda}(x) \right\rangle_\lambda = n!\delta_{n,k}, \quad (n,k \geq 0), \qquad (\text{see } [11]). \tag{1.14}$$

The sequence $s_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $(g(t),f(t))$, which is denoted by $s_{n,\lambda}(x) \sim (g(t),f(t))_\lambda$.

The sequence $s_{n,\lambda}(x) \ \sim \ (g(t),f(t))_\lambda$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)} e_\lambda^x\big(\overline{f}(t)\big) \ = \ \sum_{k=0}^{\infty} \frac{s_{k,\lambda}(x)}{k!} t^k, \quad (n,k \geq 0), \qquad (\text{see } [11]). \tag{1.15}$$

Assume that for each $\lambda \in \mathbb{R}^*$ of the set of nonzero real numbers, $s_{n,\lambda}(x)$ is λ-Sheffer for $(g_\lambda(t), f_\lambda(t))$. Assume also that $\lim_{\lambda\to 0} f_\lambda(t) = f(t)$ and $\lim_{\lambda\to 0} g_\lambda(t) = g(t)$, for some delta series $f(t)$ and an invertible series $g(t)$. Then $\lim_{\lambda\to 0} \overline{f}_\lambda(t) = \overline{f}(t)$, where is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$. Let $\lim_{\lambda\to 0} s_{k,\lambda}(x) = s_k(x)$. In this case, Kim-Kim called that the family $\{s_{n,\lambda}(x)\}_{\lambda\in\mathscr{R}-\{0\}}$ of λ-Sheffer sequences $s_{n,\lambda}$ is the degenerate (Sheffer) sequences for the Sheffer polynomial $s_n(x)$.

Let $s_{n,\lambda}(x) \sim (g(t),f(t))_\lambda$ and $r_{n,\lambda}(x) \sim (h(t),g(t))_\lambda$, $(n \geq 0)$. Then

$$\begin{aligned} & s_{n,\lambda}(x) = \sum_{k=0}^{n} z_{n,k} r_{k,\lambda}(x), \quad (n \geq 0), \\ & \quad \text{where } \ z_{n,k} = \frac{1}{k!} \left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))} \big(l(\overline{f}(t))\big)^k \mid (x)_{n,\lambda} \right\rangle_\lambda, \quad (n,k \geq 0), \qquad (\text{see } [11]). \end{aligned} \tag{1.16}$$

2. Degenerate Catalan-Daehee polynomials arising from degenerate Sheffer sequences

In this section, we define the degenerate Catalan-Daehee polynomials of order r, and derive several identities between the degenerate Catalan-Daehee polynomials of order r and some other polynomials arising from degenerate Sheffer sequences.

As is known, the Catalan numbers C_n are given by the generating function

$$\frac{1-\sqrt{1-4t}}{2t}=\frac{2}{1+\sqrt{1-4t}}=\sum_{n=0}^{\infty}C_nt^n,\qquad \text{(see [5, 16, 17])}.$$

The Catalan numbers $C_n^{(r)}$ of order r, as a generalization of Catalan numbers, are given by the generating function

$$\left(\frac{1-\sqrt{1-4t}}{2t}\right)^r=\left(\frac{2}{1+\sqrt{1-4t}}\right)^r=\sum_{n=0}^{\infty}C_n^{(r)}t^n,\qquad \text{(see [16, 17])}.$$

Kim-Kim introduced the Catalan-Daehee polynomials which are given by the generating function

$$\frac{\frac{1}{2}log(1-4t)}{\sqrt{1-4t}-1}(1-4t)^{\frac{x}{2}}=\sum_{n=0}^{\infty}\mathfrak{C}_n(x)t^n=\sum_{n=0}^{\infty}n!\mathfrak{C}_n(x)\frac{t^n}{n!},\qquad \text{(see [16, 17])}. \tag{2.1}$$

When $x=0$, $\mathfrak{C}_n:=\mathfrak{C}_n(0)$, which are called Catalan-Daehee numbers.

From (1.6) and (2.1), we note that

$$\sum_{n=0}^{\infty}\mathfrak{C}_n(x)t^n=\frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1}(1-4t)^{\frac{x}{2}}=\sum_{n=0}^{\infty}n!D_{n,\frac{1}{2}}(\frac{x}{2})(-4)^n\frac{t^n}{n!}.$$

We introduce the degenerate Catalan-Daehee polynomials $\mathfrak{C}_{n,\lambda}(x)$ which are given by the generating function

$$\left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}\right)e_\lambda^x\big(\frac{1}{2}\log_\lambda(1-4t)\big)=\sum_{n=0}^{\infty}n!\mathfrak{C}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.2}$$

When $x=0$, $\mathfrak{C}_{n,\lambda}:=\mathfrak{C}_{n,\lambda}(0)$, which are called degenerate Catalan Daehee numbers.

When $\lambda\to 0$, we note that $\mathfrak{C}_{n,\lambda}(x)=\mathfrak{C}_n(x)$.

As a generalization of the degenerate Catalan-Daehee polynomials, we also introduce degenerate Catalan-Daehee polynomials $\mathfrak{C}_{n,\lambda}^{(r)}(x)$ of order r are given by the generating function

$$\left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}\right)^r e_\lambda^x\big(\frac{1}{2}\log_\lambda(1-4t)\big)=\sum_{n=0}^{\infty}n!\mathfrak{C}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{2.3}$$

When $x=0$, $\mathfrak{C}_{n,\lambda}^{(r)}:=\mathfrak{C}_{n,\lambda}^{(r)}(0)$, which are called degenerate Catalan Daehee numbers of order r.

It easy to see that the compositional inverse of $f(t)=\frac{1}{4}(1-e_\lambda(2t))$ such that $f(\overline{f}(t))=\overline{f}(f(t))=t$ is

$$\overline{f}(t)=\frac{1}{2}\log_\lambda(1-4t). \tag{2.4}$$

From(1.15), (2.2), (2.3) and (2.4) we have

$$n!\mathfrak{C}_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t}, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda, \tag{2.5}$$

and

$$n!\mathfrak{C}_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r, \frac{1}{4}(1-e_\lambda(2t)\right)_\lambda. \tag{2.6}$$

Theorem 1. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathfrak{C}_{n,\lambda}^{(r)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\left(\binom{n}{m}(-1)^m 2^{2m-k} S_{1,\lambda}(m,k)(n-m)!\mathfrak{C}_{n-m,\lambda}^{(r)}\right)(x)_{k,\lambda}.$$

Proof. From (1.2), (1.15) and (2.6), we consider the following two Sheffer sequence as follows:

$$n!\mathfrak{C}_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r,\ \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda \quad \text{and} \quad (x)_{n,\lambda} \sim \left(1,\ t\right)_\lambda. \tag{2.7}$$

From (1.16) and (2.7), we have

$$n!\mathfrak{C}_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} z_{n,k}(1)_{k,\lambda}. \tag{2.8}$$

From (1.8) and (2.3), we obtain

$$\begin{aligned} z_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}\right)^r \left(\frac{1}{2}\log_\lambda(1-4t)\right)^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{2^k}\left\langle \left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}\right)^r \,\middle|\, \left(\frac{\log_\lambda(1+(-4t))^k}{k!}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{2^k}\sum_{m=k}^{n}\binom{n}{m}(-4)^m S_{1,\lambda}(m,k)\left\langle \left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}\right)^r \,\middle|\, (x)_{n-m,\lambda}\right\rangle_\lambda \\ &= \sum_{m=k}^{n}\binom{n}{m}(-1)^m 2^{2m-k} S_{1,\lambda}(m,k)(n-m)!\mathfrak{C}_{n-m,\lambda}^{(r)}. \end{aligned} \tag{2.9}$$

Therefore, from (2.8) and (2.9), we have the desired result. □

The next theorem gives the inversion formula of Theorem 1.

Theorem 2. *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, we have*

$$(1)_{n,\lambda} = \sum_{k=0}^{n}\frac{k!r!}{4^k}(-1)^k\left(\sum_{l=k}^{n}2^l\binom{n}{l}(n-l+r)_r S_{2,\lambda}(l,k)S_{2,\lambda}(n-l+r,r)\right)\mathfrak{C}_{k,\lambda}^{(r)}(x).$$

Proof. From (2.7), we consider the following two degenerate Sheffer sequences

$$(x)_{n,\lambda} \sim \big(1,\, t\big)_\lambda \quad \text{and} \quad n!\mathfrak{C}_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r,\, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda. \tag{2.10}$$

From (1.16) and (2.10) , we have

$$(1)_{n,\lambda} = \sum_{k=0}^{n} \widetilde{z_{n,k}}\mathfrak{C}_{k,\lambda}^{(r)}(x). \tag{2.11}$$

First, by (1.9), we observe that

$$\left(\frac{e_\lambda(t)-1}{t}\right)^r = \frac{r!}{t^r}\frac{(e_\lambda(t)-1)^r}{r!} = r!\sum_{m=0}^{\infty}(m+r)_r S_{2,\lambda}(m+r,r)\frac{t^m}{n!}. \tag{2.12}$$

Then, from (1.2), (1.9), (1.16) and (2.12) we have

$$\begin{aligned}
\widetilde{z_{n,k}} &= \frac{1}{k!}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r \big(\frac{1}{4}(1-e_\lambda(2t))\big)^k \,\middle|\, (x)_{n,\lambda} \right\rangle_\lambda \\
&= \frac{1}{4^k}(-1)^k\sum_{l=k}^{n}2^l S_{2,\lambda}(l,k)\binom{n}{l}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r \middle| (x)_{n-l,\lambda} \right\rangle_\lambda \\
&= \frac{r!}{4^k}(-1)^k\sum_{l=k}^{n}2^l S_{2,\lambda}(l,k)\binom{n}{l}(n-l+r)_r S_{2,\lambda}(n-l+r,r).
\end{aligned} \tag{2.13}$$

Therefore, from (2.11) and (2.13), we have what we want. □

Theorem 3. *For $n \in \mathbb{N}\cup\{0\}$, we have*

$$\mathfrak{C}_{n,\lambda}^{(r)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=l}^{n}\frac{n!}{m!}(-1)^m 2^{2m-l}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\mathfrak{C}_{n-m,\lambda}^{(r)}\right)(x)_k.$$

Proof. We note that

$$(x)_n \sim (1, e_\lambda(t)-1)_\lambda \tag{2.14}$$

because of $e_\lambda^x(\log(1+t)) = (1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}$.

From (2.6) and (2.14), we consider the following two degenerate Sheffer sequences.

$$n!\mathfrak{C}_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r,\, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda \quad \text{and} \quad (x)_n \sim \big(1,\, e_\lambda(t)-1\big)_\lambda. \tag{2.15}$$

From (1.8), (1.9), (1.16) and (2.15), we observe that

$$n!\mathfrak{C}_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} z_{n,k}(x)_k, \tag{2.16}$$

and

$$\begin{aligned}\frac{\left(e_{\lambda}(\frac{1}{2}\log_{\lambda}(1-4t))-1\right)^{k}}{k!}&=\sum_{l=k}^{\infty}S_{2,\lambda}(l,k)\frac{\left(\frac{1}{2}\log_{\lambda}(1-4t)\right)^{l}}{l!}\\&=\sum_{m=l}^{\infty}\sum_{l=k}^{\infty}(-1)^{m}2^{2m-l}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\frac{t^{m}}{m!}.\end{aligned}\tag{2.17}$$

From (1.16) and (2.17), we obtain

$$\begin{aligned}z_{n,k}&=\frac{1}{k!}\left\langle\left(\frac{\frac{1}{2}\log_{\lambda}(1-4t)}{e_{\lambda}(\frac{1}{2}\log_{\lambda}(1-4t))-1}\right)^{r}\left(e_{\lambda}(\frac{1}{2}\log_{\lambda}(1-4t))-1\right)^{k}\,\middle|\,(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\sum_{m=l}^{n}\sum_{l=k}^{n}(-1)^{m}2^{2m-l}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\binom{n}{m}\left\langle\left(\frac{\frac{1}{2}\log_{\lambda}(1-4t)}{e_{\lambda}(\frac{1}{2}\log_{\lambda}(1-4t))-1}\right)^{r}\,\middle|\,(x)_{n-m,\lambda}\right\rangle_{\lambda}\\&=\sum_{m=l}^{n}\sum_{l=k}^{n}(-1)^{m}2^{2m-l}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\binom{n}{m}(n-m)!\mathfrak{C}_{n-m,\lambda}^{(r)}\\&=\sum_{l=k}^{n}\sum_{m=l}^{n}\frac{n!}{m!}(-1)^{m}2^{2m-l}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\mathfrak{C}_{n-m,\lambda}^{(r)}.\end{aligned}\tag{2.18}$$

From (2.16) and (2.18), we get the desired result.

□

The next theorem is the inversion formula of Theorem 3.

Theorem 4. *For $n\in\mathbb{N}\cup\{0\}$ and $r\in\mathbb{N}$, we have*

$$(x)_{n}=\sum_{k=0}^{n}k!\left(\sum_{l=k}^{n}\sum_{m=l}^{n}\binom{n}{m}(-1)^{k}2^{l-2k}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)b_{n-m,\lambda}^{(r)}\right)\mathfrak{C}_{k,\lambda}^{(r)}(x),$$

where $b_{n,\lambda}^{(r)}$ are the Bernoulli numbers of the second kind of order r.

Proof. From (2.15), we consider the following two degenerate Sheffer sequences.

$$(x)_{n}\sim\left(1,\,t\right)_{\lambda}\quad\text{and}\quad n!\mathfrak{C}_{n,\lambda}^{(r)}(x)\sim\left(\left(\frac{e_{\lambda}(t)-1}{t}\right)^{r},\,\frac{1}{4}(1-e_{\lambda}(2t))\right)_{\lambda}.\tag{2.19}$$

From (1.16) and (2.19), we have

$$(x)_{n}=\sum_{k=0}^{n}\widetilde{z_{n,k}}k!\mathfrak{C}_{k,\lambda}^{(r)}(x).\tag{2.20}$$

From (1.5), (1.8), (1.9) and (1.16), we get

$$
\begin{aligned}
\widetilde{z_{n,k}} &= \frac{1}{k!}\left\langle \left(\frac{t}{\log_\lambda(1+t)}\right)^r \frac{1}{4^k}(1-e_\lambda(2\log_\lambda(1+t)))^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{(-1)^k}{4^k}\sum_{l=k}^{n} S_{2,\lambda}(l,k)2^l \left\langle \left(\frac{t}{\log_\lambda(1+t)}\right)^r \,\middle|\, \left(\frac{(\log_\lambda(1+t))^l}{l!}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{(-1)^k}{4^k}\sum_{l=k}^{n} S_{2,\lambda}(l,k)2^l \sum_{m=l}^{n} S_{1,\lambda}(m,l)\binom{n}{m}\left\langle \left(\frac{t}{\log_\lambda(1+t)}\right)^r \,\middle|\, (x)_{n-m,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\sum_{m=l}^{n}\binom{n}{m}(-1)^k 2^{l-2k} S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)b_{n-m,\lambda}^{(r)}.
\end{aligned}
\tag{2.21}
$$

Combining (2.20) and (2.21), we prove the theorem. □

Theorem 5. *For $n \in \mathbb{N}\cup\{0\}$, we have*

(1) when $r_1 = r_2$, $\quad \mathfrak{C}_{n,\lambda}^{(r_1)}(x) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^m 2^{2n-k}S_{1,\lambda}(n,k)B_{k,\lambda}^{(r_2)}(x),$

(2) when $r_1 \neq r_2$, $\quad \mathfrak{C}_{n,\lambda}^{(r_1)}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}(-1)^l 2^{2l-k}(n-l)!S_{1,\lambda}(l,k)\mathfrak{C}_{n-k,\lambda}^{(r_1-r_2)}B_{k,\lambda}^{(r_2)}(x),$

where $B_{k,\lambda}^{(r)}(x)$ are the Bernoulli polynomials of order r.

Proof. From (1.3),(1.15) and (2.6), we consider two degenerate Sheffer sequences as follows:

$$
n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1},\ \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda \quad \text{and} \quad B_{n,\lambda}^{(r_2)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_2},\ t\right)_\lambda. \tag{2.22}
$$

From (1.16) and (2.22), we have

$$
n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x) = \sum_{k=0}^{n} z_{n,k}B_{k,\lambda}^{(r_2)}(x), \tag{2.23}
$$

From (1.8) and (1.16), we have
when $r_1 = r_2$,

$$
\begin{aligned}
z_{n,k} &= \frac{1}{k!}\left\langle \frac{1}{2^k}(\log_\lambda(1-4t))^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{2^k}S_{1,\lambda}(n,k)(-4)^n = (-1)^m 2^{2n-k}S_{1,\lambda}(n,k),
\end{aligned}
\tag{2.24}
$$

when $r_1 > r_2$,

$$
\begin{aligned}
z_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}{\frac{1}{2}\log_\lambda(1-4t)}\right)^{r_2-r_1}\frac{1}{2^k}\big(\log_\lambda(1-4t)\big)^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{2^k}\sum_{l=k}^{n} S_{1,\lambda}(l,k)(-4)^l\binom{n}{l}\left\langle \left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}\right)^{r_1-r_2} \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}(-1)^l 2^{2l-k}\binom{n}{l}S_{1,\lambda}(l,k)(n-l)!\mathfrak{C}_{n-k,\lambda}^{(r_1-r_2)},
\end{aligned}
\tag{2.25}
$$

and when $r_1 < r_2$,

$$
\begin{aligned}
z_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}{\frac{1}{2}\log_\lambda(1-4t)}\right)^{r_2-r_1}\frac{1}{2^k}\big(\log_\lambda(1-4t)\big)^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}(-1)^l 2^{2l-k}\binom{n}{l}S_{1,\lambda}(l,k)\left\langle \left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda(\frac{1}{2}\log_\lambda(1-4t))-1}\right)^{r_1-r_2} \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}(-1)^l 2^{2l-k}\binom{n}{l}S_{1,\lambda}(l,k)(n-l)!\mathfrak{C}_{n-k,\lambda}^{(r_1-r_2)}.
\end{aligned} \tag{2.26}
$$

Therefore, from (2.23), (2.24), (2.25) and (2.26), we have the desired result. □

The following theorem gives the inversion formula of Theorem 5.

Theorem 6. *For $n \in \mathbb{N}\cup\{0\}$, we have*

(1) when $r_1 = r_2$, $\quad B_{n,\lambda}^{(r_2)}(x) = \sum_{k=0}^{n} k!\Big((-1)^k 2^{n-2k} S_{2,\lambda}(n,k)\Big)\mathfrak{C}_{k,\lambda}^{(r_1)}(x),$

(2) when $r_1 \neq r_2$, $\quad B_{n,\lambda}^{(r_2)}(x) = \sum_{k=0}^{n}(-1)^k k!\Big(\sum_{l=k}^{n}\binom{n}{l}2^{l-2k}S_{2,\lambda}(l,k)B_{n-l,\lambda}^{(r_2-r_1)}\Big)\mathfrak{C}_{k,\lambda}^{(r_1)}(x).$

Proof. From (2.22), we consider the two degenerate Sheffer sequences as follows:

$$
B_{n,\lambda}^{(r_2)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_2},\, t\right)_\lambda \quad \text{and} \quad n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1},\, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda. \tag{2.27}
$$

From (1.16) and (2.27), we have

$$
B_{n,\lambda}^{(r_2)}(x) = \sum_{k=0}^{n}\widetilde{z_{n,k}}k!\mathfrak{C}_{k,\lambda}^{(r_1)}(x). \tag{2.28}
$$

And from (1.3), (1.5) and (1.16), we get
when $r_1 = r_2$,

$$
\begin{aligned}
\widetilde{z_{n,k}} &= \frac{1}{k!}\left\langle \frac{(-1)^k}{4^k}(e_\lambda(2t)-1)^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= (-1)^k 2^{-2k}\left\langle \sum_{l=k}^{\infty}S_{2,\lambda}(l,k)2^l\frac{t^l}{l!} \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda = (-1)^k 2^{n-2k}S_{2,\lambda}(n,k),
\end{aligned} \tag{2.29}
$$

when $r_1 > r_2$,

$$
\begin{aligned}
\widetilde{z_{n,k}} &= \frac{1}{k!}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^{r_1-r_2}\frac{1}{4^k}(1-e_\lambda(2t))^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= (-1)^k 2^{-2k}\sum_{l=k}^{n}S_{2,\lambda}(l,k)2^l\binom{n}{l}\left\langle \left(\frac{t}{e_\lambda(t)-1}\right)^{r_2-r_1} \,\middle|\, (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= (-1)^k\sum_{l=k}^{n}2^{l-2k}\binom{n}{l}S_{2,\lambda}(l,k)B_{n-l,\lambda}^{(r_2-r_1)},
\end{aligned} \tag{2.30}
$$

and when $r_1 < r_2$,

$$\begin{aligned}\widetilde{z_{n,k}} &= \frac{1}{k!}\left\langle \left(\frac{t}{e_\lambda(t)-1}\right)^{r_2-r_1}\frac{1}{4^k}(1-e_\lambda(2t))^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= (-1)^k\sum_{l=k}^{n}2^{l-2k}\binom{n}{l}S_{2,\lambda}(l,k)B_{n-l,\lambda}^{(r_2-r_1)}.\end{aligned} \tag{2.31}$$

From (2.28), (2.29), (2.30) and (2.31), we arrive at the desired result. □

Theorem 7. *For $n \in \mathbb{N}\cup\{0\}$ and $s \in \mathbb{N}$, we have*

$$\begin{aligned}\mathfrak{C}_{n,\lambda}^{(r_1)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\Bigg(&\frac{1}{2^{k+1}}\sum_{l=k}^{n}S_{1,\lambda}(l,k)(-4)^l\binom{n}{l}\sum_{m=0}^{n-l}m!\mathfrak{C}_{m,\lambda}^{(r_1)}\binom{n-l}{m} \\ &\times\sum_{j=i}^{n-l-m}\sum_{i=0}^{r_2}(-1)^{n-l-m}2^{r_2-i-j-2(n-l-m)}S_{2,\lambda}(j,i)S_{1,\lambda}(n-l-m,j)\Bigg)E_{k,\lambda}^{(r_2)}(x),\end{aligned}$$

where $E_{n,\lambda}^{(r)}(x)$ are the degenerate Euler polynomials of order r.

Proof. From (1.4), (1.15) and (2.6), we consider the following two degenerate Sheffer sequences

$$n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1}, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda \quad \text{and} \quad E_{n,\lambda}^{(r_2)}(x) \sim \left(\left(\frac{e_\lambda(t)+1}{2}\right)^{r_2}, t\right)_\lambda. \tag{2.32}$$

From (1.16) and (2.32), we give

$$n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x) = \sum_{k=0}^{n}z_{n,k}E_{k,\lambda}^{(r_2)}(x). \tag{2.33}$$

Observe that

$$\begin{aligned}\left(e_\lambda(\tfrac{1}{2}\log(1-4t))+1\right)^r &= \left(e_\lambda(\tfrac{1}{2}\log(1-4t))-1+2\right)^r \\ &= \sum_{i=0}^{r}\binom{r}{i}\left(e_\lambda(\tfrac{1}{2}\log(1-4t))-1\right)^i 2^{r-i} \\ &= \sum_{i=0}^{r}\binom{r}{i}2^{r-i}i!\sum_{j=i}^{\infty}S_{2,\lambda}(j,i)\frac{(\frac{1}{2})^j(\log(1-4t))^j}{j!} \\ &= \sum_{i=0}^{r}\binom{r}{i}2^{r-i-j}i!\sum_{j=i}^{\infty}S_{2,\lambda}(j,i)\sum_{d=j}^{\infty}S_{1,\lambda}(d,j)(-4)^d\frac{t^d}{d!} \\ &= \sum_{d=i}^{\infty}\sum_{j=i}^{d}\sum_{i=0}^{r}(-1)^d 2^{r-i-j+2d}S_{2,\lambda}(j,i)S_{1,\lambda}(d,j)\frac{t^d}{d!}.\end{aligned} \tag{2.34}$$

From (1.2), (1.8), (1.16), (2.3) and (2.34), we obtain

$$
\begin{aligned}
z_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)+1}{2}\right)^{r_2}\left(\frac{\frac{1}{2}(\log_\lambda(1-4t))}{e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)-1}\right)^{r_1} \right.\\
&\qquad\qquad \left. \times \frac{1}{2^k}\left(\log_\lambda(1-4t)\right)^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{2^{k+1}}\sum_{l=k}^{n} S_{1,\lambda}(l,k)(-4)^l\binom{n}{l}\sum_{m=0}^{n-l} m!\mathfrak{C}^{(r_1)}_{m,\lambda}\binom{n-l}{m} \\
&\qquad\qquad \times \left\langle \left(\frac{e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)+1}{2}\right)^{r_2} \,\middle|\, (x)_{n-l-m,\lambda}\right\rangle_\lambda \\
&= \frac{1}{2^{k+1}}\sum_{l=k}^{n} S_{1,\lambda}(l,k)(-4)^l\binom{n}{l}\sum_{m=0}^{n-l} m!\mathfrak{C}^{(r_1)}_{m,\lambda}\binom{n-l}{m} \\
&\qquad \times \sum_{j=i}^{n-l-m}\sum_{i=0}^{r_2}(-1)^{n-l-m}2^{r_2-i-j+2(n-l-m)}S_{2,\lambda}(j,i)S_{1,\lambda}(n-l-m,j).
\end{aligned} \tag{2.35}
$$

Therefore, from (2.33) and (2.35), we arrive at the desired result. □

Theorem 8. *For $n\in\mathbb{N}\cup\{0\}$ and $s\in\mathbb{N}$, we have*

$$
E^{(r_2)}_{n,\lambda}(x) = \sum_{k=0}^{n} k!\left(2^{-2k}(-1)^k\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}\frac{2^l(1)_{n-l-m+1,\lambda}}{n-l-m+1} \times S_{2,\lambda}(l,k)E^{(r_2)}_{m,\lambda}\right)\mathfrak{C}^{(r_1)}_{k,\lambda}(x),
$$

where $E^{(r)}_{n,\lambda}(x)$ are the degenerate Euler polynomials of order r.

Proof. From (1.4), (1.15) and (2.5), we consider two degenerate Sheffer sequences as follows:

$$
E^{(r_2)}_{n,\lambda}(x) \sim \left(\left(\frac{e_\lambda(t)+1}{2}\right)^{r_2},\, t\right)_\lambda \quad \text{and} \quad n!\mathfrak{C}^{(r_1)}_{n,\lambda}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1},\, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda. \tag{2.36}
$$

From (1.16) and (2.36), we have

$$
E^{(r_2)}_{n,\lambda}(x) = \sum_{k=0}^{n}\widetilde{z_{n,k}}k!\mathfrak{C}^{(r_1)}_{k,\lambda}(x), \tag{2.37}
$$

and from (1.2), (1.4), (1.9) and (1.16), we get

$$
\begin{aligned}
\widetilde{z_{n,k}} &= \left\langle \left(\frac{2}{e_\lambda(t)+1}\right)^{r_2}\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1}\frac{1}{4^k}\frac{(1-e_\lambda(2t))^k}{k!} \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda. \\
&= 2^{-2k}(-1)^k\sum_{l=k}^{n} S_{2,\lambda}(l,k)2^l\binom{n}{l}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^{r_1} \,\middle|\, \left(\frac{2}{e_\lambda(t)+1}\right)^{r_2}_\lambda (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= 2^{-2k}(-1)^k\sum_{l=k}^{n}\binom{n}{l}2^lS_{2,\lambda}(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}E^{(r_2)}_{m,\lambda}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^{r_1} \,\middle|\, (x)_{n-l-m,\lambda}\right\rangle_\lambda \\
&= 2^{-2k}(-1)^k\sum_{l=k}^{n}\binom{n}{l}2^lS_{2,\lambda}(l,k)\sum_{m=0}^{n-l}\binom{n-l}{m}E^{(r_2)}_{m,\lambda}\frac{(1)_{n-l-m+1,\lambda}}{n-l-m+1}.
\end{aligned} \tag{2.38}
$$

From (2.37) and (2.38), we deduce the desired result. □

Theorem 9. *For $n \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{N}$, we have*

$$\mathfrak{C}^{(r_1)}_{n-m,\lambda}(x) = \frac{1}{n!}\sum_{k=0}^{n}\left(\sum_{m=l}^{n}\sum_{l=k}^{m}\binom{n}{m}(-1)^m 2^{2m-l}(n-m)!S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\mathfrak{C}^{(r_1+r_2)}_{n-m,\lambda}\right)b^{(r_2)}_{k,\lambda}(x),$$

where $b^{(r)}_{n,\lambda}(x)$ are degenerate Bernoulli polynomials of the second kind of order r.

Proof. From (1.5), (1.15) and (2.6), we consider the following two degenerate Sheffer sequences

$$n!\mathfrak{C}^{(r_1)}_{n,\lambda}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1}, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda \text{ and } b^{(r_2)}_{n,\lambda}(x) \sim \left(\left(\frac{t}{e_\lambda(t)-1}\right)^{r_2}, e_\lambda(t)-1\right)_\lambda. \tag{2.39}$$

From (1.16) and (2.39), we have

$$n!\mathfrak{C}^{(r_1)}_{n,\lambda}(x) = \sum_{k=0}^{n} z_{n,k} b^{(r_2)}_{k,\lambda}(x). \tag{2.40}$$

From (1.8), (1.16) and (2.3), we derive

$$\begin{aligned}
z_{n,k} &= \left\langle \left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)-1}\right)^{r_1}\left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)-1}\right)^{r_2}\right.\\
&\qquad\left.\left|\ \left(\frac{1}{k!}\left(e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)-1\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda\\
&= \sum_{m=l}^{n}\sum_{l=k}^{m}(-1)^m 2^{2m-l}S_{2,\lambda}(l,k)S_{1,\lambda}(m,l)\binom{n}{m}\left\langle\left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)-1}\right)^{r_1+r_2}\ \middle|\ (x)_{n-m,\lambda}\right\rangle_\lambda\\
&= \sum_{m=l}^{n}\sum_{l=k}^{m}(-1)^m 2^{2m-l}\binom{n}{m}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)(n-m)!\mathfrak{C}^{(r_1+r_2)}_{n-m,\lambda}.
\end{aligned} \tag{2.41}$$

Therefore, from (2.40) and (2.41), we have the desired result. □

The next theorem is the inversion formula of Theorem 9.

Theorem 10. *For $n \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{N}$, we have*

$$b^{(r_2)}_{n,\lambda}(x) = \sum_{k=0}^{n} k!\left(\sum_{m=k}^{n}\sum_{l=k}^{m}2^{l-2k}\binom{n}{m}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)b^{(r_1+r_2)}_{n,\lambda}\right)\mathfrak{C}^{(r_1)}_{k,\lambda}(x),$$

where $b^{(r)}_{n,\lambda}(x)$ are degenerate Bernoulli polynomials of the second kind of order r.

Proof. From (1.8) and (1.9), we observe that

$$\begin{aligned}\frac{(e_\lambda(2\log_\lambda(1+t))-1)^k}{k!}&=\sum_{l=k}^{\infty}S_{2,\lambda}(l,k)\frac{2^l(\log_\lambda(1+t))^l}{l!}\\&=\sum_{l=k}^{\infty}S_{2,\lambda}(l,k)2^l\sum_{m=l}^{\infty}S_{1,\lambda}(m,l)\frac{t^m}{m!}=\sum_{m=k}^{\infty}\sum_{l=k}^{m}2^lS_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\frac{t^m}{m!}.\end{aligned}\tag{2.42}$$

From (2.39), we consider the following two degenerate Sheffer sequences

$$b_{n,\lambda}^{(r_2)}(x)\sim\left(\left(\frac{t}{e_\lambda(t)-1}\right)^{r_2},\, e_\lambda(t)-1\right)_\lambda\quad\text{and}\quad n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x)\sim\left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1},\,\frac{1}{4}(1-e_\lambda(2t))\right)_\lambda.\tag{2.43}$$

We have

$$b_{n,\lambda}^{(r_2)}(x)=\sum_{k=0}^{n}\widetilde{z_{n,k}}k!\mathfrak{C}_{k,\lambda}^{(r_1)}(x).\tag{2.44}$$

From (1.5), (1.8), (1.9), (1.16) and (2.42),

$$\begin{aligned}\widetilde{z_{n,k}}&=\frac{1}{k!}\left\langle\left(\frac{t}{\log_\lambda(1+t)}\right)^{r_1+r_2}\frac{1}{4^k}(1-e_\lambda(2\log_\lambda(1+t)))^k\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda.\\&=2^{-2k}\sum_{m=k}^{n}\sum_{l=k}^{m}2^lS_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\binom{n}{m}\left\langle\left(\frac{t}{\log_\lambda(1+t)}\right)^{r_1+r_2}\middle|(x)_{n-m,\lambda}\right\rangle_\lambda\\&=\sum_{m=k}^{n}\sum_{l=k}^{m}2^{l-2k}\binom{n}{m}S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)b_{n,\lambda}^{(r_1+r_2)}.\end{aligned}\tag{2.45}$$

Thus, from (2.44) and (2.45), we get the desired result. □

Theorem 11. *For $n\in\mathbb{N}\cup\{0\}$ and $s\in\mathbb{N}$, we have*

$$\mathfrak{C}_{n,\lambda}^{(r_1)}(x)=\frac{1}{n!}\sum_{k=0}^{n}\sum_{l=k}^{n}2^lS_{1,\lambda}(n,l)S_{2,\lambda}(l,k)D_{n,\lambda}^{(r_2)}(x),$$

where $D_{n,\lambda}^{(r)}(x)$ are degenerate Daehee polynomials of order r.

Proof. From (1.6), (1.15) and (2.6), we consider the following two degenerate Sheffer sequences

$$n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x)\sim\left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1},\,\frac{1}{4}(1-e_\lambda(2t))\right)_\lambda\quad\text{and}\quad D_{n,\lambda}^{(r_2)}(x)\sim\left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_2},\, e_\lambda(t)-1\right)_\lambda.\tag{2.46}$$

From (1.16) and (2.44), we have

$$n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x)=\sum_{k=0}^{n}z_{n,k}D_{k,\lambda}^{(r_2)}(x),\tag{2.47}$$

and from (1.8), (1.9) and (1.16), we get

$$\begin{aligned} z_{n,k} &= \frac{1}{k!}\left\langle \left(e_\lambda\big(\frac{1}{2}\log_\lambda(1-4t)\big)-1\right)^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{m=k}^{n}\sum_{l=k}^{m} 2^l S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\frac{1}{m!}\left\langle t^m \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda = \sum_{l=k}^{n} 2^l S_{1,\lambda}(n,l)S_{2,\lambda}(l,k). \end{aligned} \tag{2.48}$$

From (2.47) and (2.48), we have the desired result. □

The next theorem represents the inversion formula of Theorem 11.

Theorem 12. *For $n \in \mathbb{N}\cup\{0\}$ and $s \in \mathbb{N}$, we have*

$$D_{n,\lambda}^{(r_2)}(x) = \sum_{k=0}^{n} k!\left(\sum_{l=k}^{n} 2^{l-2k}(-1)^k k! S_{1,\lambda}(n,l)S_{2,\lambda}(l,k)\right)\mathfrak{C}_{k,\lambda}^{(r_1)}(x),$$

where $D_{n,\lambda}^{(r)}(x)$ are degenerate Daehee polynomials of order r.

Proof. By (2.46), we consider the following two degenerate Sheffer sequences

$$D_{n,\lambda}^{(r_2)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_2},\ e_\lambda(t)-1\right)_\lambda \text{ and } n!\mathfrak{C}_{n,\lambda}^{(r_1)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^{r_1},\ \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda, \tag{2.49}$$

and from (1.16) and (2.49), we get

$$D_{n,\lambda}^{(r_2)}(x) = \sum_{k=0}^{n} \widetilde{z_{n,k}} k! \mathfrak{C}_{k,\lambda}^{(r_1)}(x). \tag{2.50}$$

From (1.8), (1.9),(1.16) and (2.42), we have

$$\begin{aligned} \widetilde{z_{n,k}} &= \frac{1}{k!}\left\langle \frac{1}{4^k}(1-e_\lambda(2\log_\lambda(1+t)))^k \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{(-1)^k}{4^k}\sum_{m=k}^{n}\sum_{l=k}^{m} 2^l S_{1,\lambda}(m,l)S_{2,\lambda}(l,k)\frac{1}{m!}\left\langle t^m \,\middle|\, (x)_{n,\lambda}\right\rangle_\lambda = (-1)^k\sum_{l=k}^{n} 2^{l-2k} S_{1,\lambda}(n,l)S_{2,\lambda}(l,k). \end{aligned} \tag{2.51}$$

From (2.50) and (2.51), we obtain the desired result. □

Theorem 13. *For $n \in \mathbb{N}\cup\{0\}$ and $s \in \mathbb{N}$, we have*

$$\begin{aligned} \mathfrak{C}_{n,\lambda}^{(r)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\Bigg(\sum_{m=k}^{n}\sum_{l=k}^{m}(-1)^m 2^{2m-l}\binom{n}{m}(n-m)! \\ S_{1,\lambda}(m,l)S_{1,\lambda}(l,k)\mathfrak{C}_{n-m,\lambda}^{(r)}\Bigg)Bel_{k,\lambda}(x), \end{aligned}$$

where $Bel_{n,\lambda}(x)$ are degenerate Bell polynomials.

Proof. From (1.8) and (2.3), we observe that

$$\begin{aligned}\frac{\log_\lambda(1+\frac{1}{2}\log_\lambda(1-4t))^k}{k!} &= \sum_{l=k}^{\infty} S_{1,\lambda}(l,k)\frac{\left(\frac{1}{2}\right)^l\left(\log_\lambda(1-4t)\right)^l}{l!} \\ &= \sum_{m=k}^{\infty}\sum_{l=k}^{m}(-1)^m 2^{2m-l} S_{1,\lambda}(l,k)S_{1,\lambda}(m,l)\frac{t^m}{m!}.\end{aligned} \tag{2.52}$$

From (1.7) and (1.15), we consider the following two degenerate Sheffer sequences as follows:

$$n!\mathfrak{C}_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda \quad \text{and} \quad Bel_{k,\lambda}(x) \sim \left(1, \log_\lambda(1+t)\right)_\lambda. \tag{2.53}$$

From (1.16) and (2.53), we have

$$n!\mathfrak{C}_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} z_{n,k} Bel_{k,\lambda}(x). \tag{2.54}$$

From (1.16), (2.3) and (2.52), we get

$$\begin{aligned} z_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)-1}\right)^r \left(\log_\lambda(1+\frac{1}{2}(\log_\lambda(1-4t)))^k \;\middle|\; (x)_{n,\lambda}\right\rangle_\lambda \\ &= \sum_{m=k}^{n}\sum_{l=k}^{m}(-1)^m 2^{2m-l} S_{1,\lambda}(l,k)S_{1,\lambda}(m,l)\binom{n}{m}\left\langle \left(\frac{\frac{1}{2}\log_\lambda(1-4t)}{e_\lambda\left(\frac{1}{2}\log_\lambda(1-4t)\right)-1}\right)^r \;\middle|\; (x)_{n-m,\lambda}\right\rangle_\lambda \\ &= \sum_{m=k}^{n}\sum_{l=k}^{m}(-1)^m 2^{2m-l}\binom{n}{m}(n-m)! S_{1,\lambda}(m,l)S_{1,\lambda}(l,k)\mathfrak{C}_{n-m,\lambda}^{(r)}. \end{aligned} \tag{2.55}$$

Thus, from (2.54) and (2.55), we have the desired result. □

The next theorem is the inversion formula of Theorem 13.

Theorem 14. *For $n \in \mathbb{N}\cup\{0\}$ and $s \in \mathbb{N}$, we have*

$$\begin{aligned} Bel_{n,\lambda}(x) = \sum_{k=0}^{n} k!r!\Bigg(&\sum_{m=k}^{n}\sum_{l=k}^{m}\sum_{d=0}^{n-m}\sum_{j=r}^{d+r}\binom{n}{m}\binom{n-m}{d}(-1)^k 2^{l-2k}(d+r)_r \\ &\times S_{2,\lambda}(m,l)S_{2,\lambda}(l,k)S_{2,\lambda}(d+r,j)S_{2,\lambda}(j,r)B_{n-m-d,\lambda}^{(r)}\Bigg)\mathfrak{C}_{k,\lambda}^{(r)}(x), \end{aligned}$$

where $Bel_{n,\lambda}(x)$ are degenerate Bell polynomials.

Proof. From (2.53), we consider two degenerate Sheffer sequences as follows:

$$Bel_{k,\lambda}(x) \sim \left(1, \log_\lambda(1+t)\right)_\lambda \quad \text{and} \quad n!\mathfrak{C}_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r, \frac{1}{4}(1-e_\lambda(2t))\right)_\lambda. \tag{2.56}$$

From (1.16) and (2.56), we have

$$Bel_{n,\lambda}(x)=\sum_{k=0}^{n}\widetilde{z_{n,k}}k!\mathfrak{C}_{k,\lambda}^{(r)}(x). \tag{2.57}$$

First, from (1.2), (1.9) and (1.10), we have two identities as follows:

$$\begin{aligned}\frac{(e_\lambda(2(e_\lambda(t)-1))-1)^k}{k!}&=(-1)^k\sum_{l=k}^{\infty}S_{2,\lambda}(l,k)\frac{2^l(e_\lambda(t)-1)^l}{l!}\\&=\sum_{m=k}^{\infty}\sum_{l=k}^{m}S_{2,\lambda}(l,k)\;2^lS_{2,\lambda}(m,k)\frac{t^m}{m!},\end{aligned} \tag{2.58}$$

and

$$\begin{aligned}\left(\frac{e_\lambda(e_\lambda(t)-1)}{t}\right)^r&=\frac{r!}{t^r}\frac{1}{r!}\big(e_\lambda(e_\lambda(t)-1)-1\big)^r=r!\frac{1}{t^r}\sum_{d=r}^{\infty}S_{J,\lambda}(d,r)\frac{t^d}{d!}\\&=r!\sum_{d=0}^{\infty}S_{J,\lambda}(d+r,r)\frac{t^d}{(d+r)!}=r!\sum_{l=0}^{\infty}(d+r)_rS_{J,\lambda}(d+r,r)\frac{t^d}{d!},\end{aligned} \tag{2.59}$$

where $S_{J,\lambda}(n,r)$ are the Jindalrae-Stirling numbers of the second kind [18].

From (1.11), (1.16), (2.58) and (2.59), we observe that

$$\begin{aligned}\widetilde{z_{n,k}}&=\frac{1}{k!}\left\langle\left(\frac{e_\lambda(e_\lambda(t)-1)-1}{e_\lambda(t)-1}\right)^r\frac{1}{4^k}(1-e_\lambda(2(e_\lambda(t)-1)))^k\;\middle|\;(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{m=k}^{n}\sum_{l=k}^{m}(-1)^k2^{l-2k}\binom{n}{m}S_{2,\lambda}(m,l)S_{2,\lambda}(l,k)\\&\qquad\times\left\langle\left(\frac{t}{e_\lambda(t)-1}\right)^r\;\middle|\;\left(\left(\frac{e_\lambda(e_\lambda(t)-1)-1}{t}\right)^r\right)_\lambda(x)_{n-m,\lambda}\right\rangle_\lambda\\&=\sum_{m=k}^{n}\sum_{l=k}^{m}(-1)^k2^{l-2k}\binom{n}{m}S_{2,\lambda}(m,l)S_{2,\lambda}(l,k)r!\sum_{d=0}^{n-m}(d+r)_r\\&\qquad\times\sum_{j=r}^{d+r}S_{2,\lambda}(d+r,j)S_{2,\lambda}(j,r)\binom{n-m}{d}B_{n-m-d,\lambda}^{(r)}.\end{aligned} \tag{2.60}$$

From (2.57) and (2.60), we get the desired result.

□

3. Conclusions

In this paper, we introduced the degenerate Catalan-Daehee numbers and polynomials of order r ($r\geq 1$). It was shown that the degenerate Catalan-Daehee polynomials of order r were expressed based on the degenerate falling factorials, the falling factorials, the degenerate Bernoulli polynomials of order r, the Euler polynomials (of order r), the degenerate Bernoulli polynomials of the second kind of order r, the degenerate Deahee polynomials of order r, and the degenerate Bell polynomials. We also obtained inverse formula for each of them.

It is difficult to single out where and why these formulas play an important role, but we do not doubt that they will be helpful to researchers in need of these identities. Further research would be related with the degenerate versions of some special combinatorial numbers and polynomials and then contribution in mathematics and physics applications.

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Conflict of interest

The authors declare no conflict of interest.

References

1. L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, *Utilitas Math.*, **15** (1979), 51–88.
2. L. Carlitz, Weighted Stirling numbers of the first and second kind, *Fibonacci Quart.*, **18** (1980), 147–162.
3. L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions.* Revised and enlarged edition. D. Reidel Publishing Co., Dordrecht, 1974. xi+343 pp. ISBN: 90-277-0441-4 05-02. doi: 10.1007/978-94-010-2196-8.
4. R. Dere, Y. Simsek, Applications of umbral algebra to some special polynomials, *Adv. Stud. Contemp. Math.*, **22** (2012), 433–438.
5. D. V. Dolgy, G. W. Jang, D. S. Kim, T. Kim, Explicit expressions for Catalan-Daehee numbers, *Proc. Jangjeon Math. Soc.*, **20** (2017), 1–9. doi: 10.23001/pjms2017.20.1.1.
6. T. Ernst, Examples of a q-umbral calculus, *Adv. Stud. Contemp. Math. (Kyungshang),* **16** (2008), 1–22.
7. I. Kucukoglu, B. Simsek, Y. Simsek, New classes of Catalan-type numbers and polynomials with their applications related to p-adic integrals and computational algorithms, *Turk. J. Math.,* **44** (2020), 2337–2355. doi:10.3906/mat-2008-24.
8. D. S. Kim, T. Kim, H. I. Kwon, T. Mansour, Nonlinear differential equation for Korobov numbers, *Adv. Stud. Contemp. Math. (Kyungshang),* **26** (2016), 733–740.
9. D. S. Kim, T. Kim, Some identities of Bernoulli and Euler polynomials arising from umbral calculus, *Adv. Stud. Contemp. Math. (Kyungshang),* **23** (2013), 159–171. doi:10.1186/1687-6180-2013-159.
10. D. S. Kim, T. Kim, Some identities of degenerate Daehee numbers arising from certain differential equations, *Jo. Nonlinear Sci. and Appl.,* **10** (2017), 744–751. doi:10.22436/jnsa.010.02.35.
11. D. S. Kim, T. Kim, Degenerate Sheffer sequence and λ-Sheffer sequence, *J. Math. Anal. Appl.,* **23** (2020), 124521. doi:10.1016/j.jmaa.2020.124521.

12. D. S. Kim, T. Kim, A note on a new type of degenerate Bernoulli numbers, *Russ. J. Math. Phys.,* **27** (2020), 227–235. doi:10.1134/S1061920820020090.

13. H. K. Kim, Degenerate Lah-Bell polynomials arising from degenerate Sheffer sequences, *Adv. Difference Equ.,* **2020** (2020), 16. doi:10.1186/s13662-020-03152-4.

14. T. Kim, A note on degenerate Stirling polynomials of the second kind, *Proc. Jangjeon Math. Soc.,* **20** (2017), 319–331. doi:10.17777/pjms2017.20.3.319.

15. T. Kim, D. S. Kim, Some identities of extended degenerate *r*-central Bell polynomials arising from umbral calculus, *Rev. R. Acad. Cienc. Exactas. Nat. Ser. A Mat. RACSAM*, **114** (2020), 19. doi:10.1007/s13398-019-00732-2.

16. T. Kim, D. S. Kim, Some identities of Catalan-Daehee polynomials arising from umbral calculus, *Appl. Comput. Math.*, **16** (2017), 177–189.

17. T. Kim, D. S. Kim, Differential equations associated with Catalan-Darhee numbers and their applications, *RACSAM Rev. R. Acad. Cienc. Exactas Fs. Mat. Ser. A Mat.*, (2016). doi:10.1007/s13398-016-0349-4.

18. T. Kim, D. S. Kim, L. C. Jang, H. Lee, Jindalrae and Gaenari numbers and polynomials in connection with Jindalrae-Stirling numbers, *Adv. Difference Equ.,* **2020** (2020), 19. doi:10.1186/s13662-020-02701-1.

19. T. Kim, D. S. Kim, H. Lee, J. Kwon, A note no some identities of new type degenerate Bell polynomials, *Mathematics,* **7** (2019), 1086, 12. doi:10.3390/math7111086.

20. T. Kim, D. S. Kim, H. Y. Kim, H. Lee, L. C. Jang, Degenerate Bell polynomials associated with umbral calculus, *J. Inequalities Appl.,***2020** (2020), 15. doi:10.1186/s13660-020-02494-7.

21. A. K. Kwaśniewski, On ψ-umbral extensions of Stirling numbers and Dobinski-like formulas, *Adv. Stud. Contemp. Math. (Kyungshang),* **12** (2006), 73–100.

22. S. Roman, G. Rota, The umbral calculus, *Adv. Math.*, **27** (1978), 95–188. doi:10.1016/0001-8708(78)90087-7.

23. G. Rota, D. Kahaner, A. Odlyzko, On the foundations of combinatorial theory VIII: Finite operator calculus, *J. Math. Anal. Appl.,* **42** (1973), 684-760. doi:10.1016/0022-247X(73)90172-8.

24. R. Mullin, G. C. Rota, *On the foundations of combinatorial theory III. Theory of binomial enumeration,* In: B. Harris, Ed., Graph Theory and its Applications, Academic Press, 1970, 167–213.

25. L. W. Shapiro, S. Getu, W. J. Woan, L. Woodson, The Riordan group, *Discrete Appl. Math.,* **34** (1991), 229–239. doi:10.1016/0166-218X(91)90088-E.

26. T. X. He, L. C. Hsu, P. J. S. Shiue, The Sheffer group and the Riordan group, *Discrete Appl. Math.,* **155** (2007), 1895–1909. doi:10.1016/j.dam.2007.04.006.

27. R. P. Stanley, *Enumerative Combinatorics.*, Vol.2, Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999, 581.

28. Y. Simsek, Special Numbers and Polynomials Including Their Generating Functions in Umbral Analysis Methods, *Axioms,* **7** (2018), 22. doi:10.3390/axioms7020022.

29. W. Zang, L. Chen, On the Catran numbers and some of theire identities, *Symmetry*, **2019** (2019), 62. doi:10.3390/sym11010062.

30. D. S. Kim, T. Kim, Degenerate Bernstein polynomials. *RACSAM* **113** (2019), 2913–2920. doi:10.1007/s13398-018-0594-9.

31. T. Kim, D. S. Kim, J. Kwon, H. Lee, A note on degenerate gamma random variables, *Revista Edu.*, **388** (2020), 39–44.