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Research article

Approximation properties of the new type generalized Bernstein-Kantorovich operators

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Abstract: In this paper, we introduce new type of generalized Kantorovich variant of α-Bernstein operators and study their approximation properties. We obtain estimates of rate of convergence involving first and second order modulus of continuity and Lipschitz function are studied for these operators. Furthermore, we establish Voronovskaya type theorem of these operators. The last section is devoted to bivariate new type α-Bernstein-Kantorovich operators and their approximation behaviors. Also, some graphical illustrations and numerical results are provided.

Keywords: α-Bernstein-Kantorovich operators; Lipschitz class; Voronovskaya type theorem
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1. Introduction

Approximation theory has an important place in application areas such as analysis and CAGD. In particular, Bernstein polynomials play an important role in approximation theory. Hence, due to the increasing interest in Bernstein polynomials, the question arises of how to construct its modifications that give better convergence results. That's why the books of Lorentz [4] and Lupaş [5] are of great importance.

For $\psi \in C(I = [0, 1])$, the classical Bernstein-Kantorovich operators are defined by

$$B_\rho(\psi;\zeta) = \sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}(\zeta) \int_{\frac{\vartheta}{\rho+1}}^{\frac{\vartheta+1}{\rho+1}} \psi(t), \quad \zeta \in I, \tag{1.1}$$

where $p_{\rho,\vartheta}(\zeta) = \binom{\rho}{\vartheta}\zeta^\vartheta(1-\zeta)^{\rho-\vartheta}$ is the Bernstein basis function. For Kantorovich-type modifications of Bernstein operators, we refer to the articles [6–15]. Recently, Chen et al. [1] introduced a generalization of the α-Bernstein operators as follows:

$$T_{\rho}^{(\alpha)}(\zeta) = \sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)}(\zeta)\psi\left(\frac{\vartheta}{\rho}\right), \quad \zeta \in I, \tag{1.2}$$

where $\underset{\rho\geq 2}{p_{\rho,\vartheta}^{(\alpha)}(\zeta)} = \left[\binom{\rho-2}{\vartheta}(1-\alpha)\zeta + \binom{\rho-2}{\vartheta-2}(1-\alpha)(1-\zeta) + \binom{\rho}{\vartheta}\alpha\zeta(1-\zeta)\right]\zeta^{i-1}(1-\zeta)^{\rho-\vartheta-1}$, $\zeta \in [0,1]$ .The authors of [1] have studied many approximation properties of α-Bernstein operators (1.2) such as rate of convergence and shape of preserving properties. After that, Mohiuddine et al. [16] introduced the Kantorovich variant of α-Bernstein operators (1.2) and examined the approximation properties. In [20], authors presented a Kantorovich variant of the operators proposed by [1] based on non-negative parameters and studied the estimate of the rate of approximation by using the modulus of smoothnes and Lipschitz type function for these operators. Very recently, Deo et al. [17] studied the direct local approximation theorem, Voronovskaya type asymtotic estimate formula and bounded variation for α-Bernstein-Kantorovich operators [16].

Mohiuddine and Özger [18] introduced Stancu variant of α-Bernstein-Kantorovich operators and studied approximation properties for these operators. Very recently, Q. B. Cai et al. [19] introduced the bivariate α-Bernstein-Kantorovich operators based q-integer and studied degree of approximation for these bivariate operators in terms of the partial moduli of continuity and Peetre's K-functional. We present the following new type generalized Kantorovich-Bernstein operators.

2. The new type generalized Bernstein-Kantorovich operators

In [2] Kantorovich variant of $T_{\rho,\alpha}(\psi;\zeta)$ (1.2) defined as follows:

$$K_{\rho,\alpha}(\psi;\zeta) = (\rho+1)\sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \int_{\frac{\vartheta}{(\rho+1)}}^{\frac{(\vartheta+1)}{(\rho+1)}} \psi(t)dt.$$

In this paper, we introduce the new type generalized Kantorovich variant of $T_{\rho}^{\alpha}(\psi,\zeta)$ as follows:

$$K_{\rho,\alpha}^{l}(\psi;\zeta) = \sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \int_{0}^{1}\dots\int_{0}^{1} \psi\left(\frac{\vartheta+t_1+\dots+t_l}{\rho+l}\right)dt_1\dots dt_l, \tag{2.1}$$

where $\psi \in [0,1]$, $l \in \mathbb{Z}^{+}$ and $\alpha \in [0,1]$.

In particular, if $l = 1$ and $\alpha = 1$, then the operator

$$K_{\rho,\alpha}(\psi;\zeta) = \sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \int_{0}^{1} \psi\left(\frac{\vartheta+t}{\rho+1}\right)dt,$$

i.e., it reduces to classical Bernstein - Kantorovich operators. In this work, $C[0,1]$ denote the space of all bounded real valued continuous function on $[0,1]$. This space is equipped with the following norm:

$$\|\psi\| = \sup_{\zeta\in[0,1]} |\psi(\zeta)|.$$

Moments of positive operators have an important place in the approximation theory. Therefore, from the definition of $K^{l}_{\rho,\alpha}(\psi;\zeta)$ and next two lemmas, we can derive the formula for moments of $K^{l}_{\rho,\alpha}(t^{m};\zeta)$.

Lemma 1. *The following formulas hold*

$$\left(\frac{\vartheta+t_1+...+t_l}{\rho+l}\right)^m=\sum_{j_0+...+j_l=m}\binom{m}{j_0,...,j_l}\frac{\vartheta^{j_0}t_1^{j_1}...t_l^{j_l}}{(\rho+l)^m}$$

and

$$\int_0^1...\int_0^1\frac{\vartheta^{j_0}t_1^{j_1}...t_l^{j_l}}{(\rho+l)^m}dt_1...dt_l=\frac{\vartheta^{j_0}}{(\rho+l)^m(j_1+1)...(j_l+1)}.$$

Lemma 2. *[1] Let $\alpha\in[0,1]$. Then moments of the operators $T^{\alpha}_{\rho}(\zeta)$ (1.2) are as follows:*

$$\begin{aligned}
&T^{(\alpha)}_{\rho}(1;\zeta)=1,\\
&T^{(\alpha)}_{\rho}(\zeta;\zeta)=\zeta,\\
&T^{(\alpha)}_{\rho}\left(\zeta^2;\zeta\right)=\zeta^2+\frac{\rho+2(1-\alpha)}{\rho^2}\zeta(1-\zeta),\\
&T^{(\alpha)}_{\rho}\left(\zeta^3;\zeta\right)=\zeta^3+\frac{3\left[\rho+2(1-\alpha)\right]}{\rho^2}\zeta^2(1-\zeta)+\frac{\rho+6(1-\alpha)}{\rho^3}\zeta(1-\zeta)(1-2\zeta),\\
&T^{(\alpha)}_{\rho}\left(\zeta^4;\zeta\right)=\zeta^4+\frac{6\left[\rho+2(1-\alpha)\right]}{\rho^2}\zeta^3(1-\zeta)+\frac{4\left[\rho+6(1-\alpha)\right]}{\rho^3}\zeta^2(1-2\zeta)(1-\zeta)\\
&\qquad+\frac{\left[12(\rho-6)(1-\alpha)+3\rho(\rho-2)\right]\zeta(1-\zeta)+\left[14(1-\alpha)+\rho\right]}{\rho^4}\zeta(1-\zeta).
\end{aligned}$$

Lemma 3. *For $\alpha\in[0,1]$, $l\in\mathbb{Z}^+$ and $\rho\in\mathbb{N}$, we have*

$$K^{l}_{\rho,\alpha}(t^m;\zeta)=\sum_{j_0+...+j_l=m}\binom{m}{j_0,...,j_l}\frac{\rho^{j_0}}{(\rho+l)^m(j_1+1)...(j_l+1)}T^{(\alpha)}_{\rho}(t^{j_0};\zeta),\tag{2.2}$$

where

$$T^{(\alpha)}_{\rho}(\psi;\zeta)=\sum_{\vartheta=0}^{\rho}\psi\left(\frac{\vartheta}{\rho}\right)p^{(\alpha)}_{\rho,\vartheta}(\zeta),\quad \text{(see [1]).}$$

Proof. It follows from (2.1) that

$$\begin{aligned}
K^{l}_{\rho,\alpha}(t^m;\zeta)&=\sum_{\vartheta=0}^{\rho}p^{(\alpha)}_{\rho,\vartheta}(\zeta)\int_0^1...\int_0^1\left(\frac{\vartheta+t_1+...+t_l}{\rho+l}\right)^m dt_1...dt_l\\
&=\sum_{\vartheta=0}^{\rho}p^{(\alpha)}_{\rho,\vartheta}(\zeta)\sum_{j_0+...+j_l=m}\binom{m}{j_0,...,j_l}\int_0^1\int_0^1...\int_0^1\frac{\vartheta^{j_0}t_1^{j_1}...t_l^{j_l}}{(\rho+l)^m}dt_1...dt_l\\
&=\sum_{\vartheta=0}^{\rho}p^{(\alpha)}_{\rho,\vartheta}(\zeta)\sum_{j_0+...+j_l=m}\binom{m}{j_0,...,j_l}\frac{\vartheta^{j_0}}{(\rho+l)^m(j_1+1)...(j_l+1)}
\end{aligned}$$

$$= \sum_{j_0+...+j_l=m} \binom{m}{j_0, ..., j_l} \frac{\rho^{j_0}}{(\rho+l)^m (j_1+1)...(j_l+1)} \sum_{\vartheta=0}^{\rho} \frac{\vartheta^{j_0}}{\rho^{j_0}} p_{\rho,\vartheta}^{(\alpha)}(\zeta)$$
$$= \sum_{j_0+...+j_l=m} \binom{m}{j_0, ..., j_l} \frac{\rho^{j_0}}{(\rho+l)^m (j_1+1)...(j_l+1)} T_\rho^{(\alpha)}(t^{j_0}; \zeta).$$

□

Using formula (2.2), we can calculate $K_{\rho,\alpha}^l(t^j;\zeta)$ for $j = 0, 1, 2$.

Lemma 4. *Let* $\alpha \in [0,1]$, $l \in \mathbb{Z}^+$ *and* $\rho \in \mathbb{N}$, *We have*

$$(i)\ K_{\rho,\alpha}^l(1;\zeta) = 1,$$
$$(ii)\ K_{\rho,\alpha}^l(t;\zeta) = \frac{l}{2(\rho+l)} + \frac{\rho}{(\rho+l)}\zeta,$$
$$(iii)\ K_{\rho,\alpha}^l(t^2;\zeta) = \frac{3l^2+l}{12(\rho+l)^2} + \frac{(\rho(l+1)+2(1-\alpha))}{(\rho+l)^2}\zeta + \left(\frac{\rho^2-\rho-2(1-\alpha)}{(\rho+l)^2}\right)\zeta^2.$$

Proof. We give the proof for only $K_{\rho,\alpha}^l(t^2;\zeta)$. Using (2.2), we get

$$K_{\rho,\alpha}^l(t^2;\zeta) = \sum_{j_0+...+j_l=2} \binom{2}{j_0, ..., j_l} \frac{\rho^{j_0}}{(\rho+l)^m (j_1+1)...(j_l+1)} T_\rho^{(\alpha)}(t^{j_0}; \zeta)$$
$$= \binom{l}{2}\frac{1}{2(\rho+l)^2} + \binom{l}{1}\frac{1}{3(\rho+l)^2} + \binom{l}{1}\frac{\rho}{(\rho+l)^2}\tau + \frac{\rho^2}{(\rho+l)^2}\left(\zeta^2 + \frac{\rho+2(1-\alpha)}{\rho^2}\zeta(1-\zeta)\right)$$
$$= \frac{3l^2+l}{12(\rho+l)^2} + \frac{(\rho(l+1)+2(1-\alpha))}{(\rho+l)^2}\zeta + \left(\frac{\rho^2-(\rho+2(1-\alpha))}{(\rho+l)^2}\right)\zeta^2.$$

□

The linearity property of $K_{\rho,\alpha}^l(\psi;\zeta)$ allows us to obtain the next lemma.

Lemma 5. *Let* $\alpha \in [0,1]$, $\rho \in \mathbb{N}$ *and* $l \in \mathbb{Z}^+$. *For every* $\zeta \in [0,1]$ *there holds*

$$K_{\rho,\alpha}^l(t-\zeta;\zeta) = \frac{l(1-2\zeta)}{2(\rho+l)},$$
$$K_{\rho,\alpha}^l((t-\zeta)^2;\zeta) = \frac{3l^2+l}{12(\rho+l)^2} + \frac{\left(\rho+2-2\alpha-l^2\right)}{(\rho+l)^2}\zeta + \frac{l^2-\rho+2\alpha-2}{(\rho+l)^2}\zeta^2 = \mu_{\rho,\alpha}^2$$

Proof. Using the linearity property of $K_{\rho,\alpha}^l(t;\zeta)$ and Lemma 4, we can prove all the above equalites with the same method. Thus, we give proof for only $K_{\rho,\alpha}^l((t-\zeta)^2;\zeta)$.

$$K_{\rho,\alpha}^l((t-\zeta)^2;\zeta) = K_{\rho,\alpha}^l(t^2;\zeta) - 2\zeta K_{\rho,\alpha}^l(t;\zeta) + \zeta^2 K_{\rho,\alpha}^l(1;\zeta)$$
$$= \frac{3l^2+l}{12(\rho+l)^2} + \frac{(\rho(l+1)+2(1-\alpha))}{(\rho+l)^2}\zeta + \left(\frac{\rho^2-(\rho+2(1-\alpha))}{(\rho+l)^2}\right)\zeta^2 - 2\zeta\left(\frac{l}{2(\rho+l)} + \frac{\rho}{(\rho+l)}\zeta\right) + \zeta^2$$
$$= \frac{3l^2+l}{12(\rho+l)^2} + \frac{\rho+2-2\alpha-l^2}{(\rho+l)^2}\zeta + \frac{l^2-\rho+2\alpha-2}{(\rho+l)^2}\zeta^2.$$

□

Lemma 6. *Let* $\alpha \in [0,1]$, $\rho \in \mathbb{N}$ *and* $l \in \mathbb{Z}^{+}$. *For every* $\zeta \in [0,1]$ *there hold*

$$\lim_{\rho\to\infty}(\rho+l)K_{\rho,\alpha}^{l}((t-\zeta);\zeta) = \frac{l(1-2\zeta)}{2},$$
$$\lim_{\rho\to\infty}(\rho+l)K_{\rho,\alpha}^{l}((t-\zeta)^2;\zeta) = \zeta-\zeta^2.$$

Proof. By Lemma 5, we have

$$\begin{aligned}\lim_{\rho\to\infty}(\rho+l)K_{\rho,\alpha}^{l}((t-\zeta);\zeta) &= \lim_{\rho\to\infty}(\rho+l)\frac{l(1-2\zeta)}{2(\rho+l)}\\ &= \frac{l(1-2\zeta)}{2}\end{aligned}$$

and

$$\begin{aligned}\lim_{\rho\to\infty}(\rho+l)K_{\rho,\alpha}^{l}((t-\zeta)^2;\zeta) &= \lim_{\rho\to\infty}(\rho+l)\frac{3l^2+l}{12(\rho+l)^2}+\frac{\left(\rho+2-2\alpha-l^2\right)}{(\rho+l)^2}\zeta+\frac{\left(l^2-\rho+2\alpha-2\right)}{(\rho+l)^2}\zeta^2\\ &= \zeta-\zeta^2.\end{aligned}$$

□

In the next theorem, we examined Korovkin type approximation theorem for $K_{\rho,\alpha}^{l}(\psi;\zeta)$.

Theorem 7. *Let* $\alpha \in [0,1]$, $\rho \in \mathbb{N}$ *and* $l \in \mathbb{Z}^{+}$. *For each* $\psi \in C[0,1]$, *we have* $K_{\rho,\alpha}^{l}(\psi;\zeta) \rightrightarrows \psi$ *on* $[0,1]$, *where the symbol* $\rightrightarrows$ *denotes the uniform convergence.*

Proof. By the Korovkin's Theorem it is sufficient to show that

$$\lim_{\rho\to\infty}\left\|K_{\rho,\alpha}^{l}(t^m;\zeta)-\zeta^m\right\|_{C[0,1]} = 0, \quad m=0,1,2.$$

By Lemma 4 (i), (ii) and (iii), it is clear that

$$\lim_{\rho\to\infty}\left\|K_{\rho,\alpha}^{l}(1;\zeta)-1\right\|_{C[0,1]} = 0$$

and

$$\left|K_{\rho,\alpha}^{l}(t;\zeta)-\zeta\right| = \left|\frac{l(1-2\zeta)}{2(\rho+l)}\right|$$

which yields

$$\lim_{\rho\to\infty}\left\|K_{\rho,\alpha}^{l}(t;\zeta)-\zeta\right\|_{C[0,1]} = 0.$$

Similarly

$$\left|K_{\rho,\alpha}^{l}\left(t^2;\zeta\right)-\zeta^2\right| = \frac{3l^2+l}{12(\rho+l)^2}+\frac{\left(\rho+2-2\alpha-l^2\right)}{(\rho+l)^2}\zeta+\frac{\left(l^2-\rho+2\alpha-2\right)}{(\rho+l)^2}\zeta^2$$

which concludes

$$\lim_{\rho\to\infty}\left\|K_{\rho,\alpha}^{l}\left(t^2;\zeta\right)-\zeta^2\right\|_{C[0,1]} = 0$$

Thus the proof is completed. □

3. Local approximation

For $\psi \in C[0,1]$ and $\delta > 0$,first and second order modulus of smoothness for ψ defined as

$$w(\psi,\delta) = \sup_{0<h\leq\delta} \sup_{\zeta,\zeta+h\in[0,1]} |\psi(\zeta+h) - \psi(\zeta)|,$$

and

$$\omega_2(\psi;\delta) = \sup_{0<h\leq\delta} \sup_{\zeta,\zeta+h\in[0,1]} |\psi(\zeta-h) - 2\psi(\zeta) + \psi(\zeta+h)|.$$

Recall that the Peetre's K-functional is defined by

$$K_2(\psi;\delta) = \inf_{g\in C^2[0,1]} \left\{ \|\psi - g\| + \delta \left\|g''\right\| \right\} \qquad \delta > 0,$$

where $C^2[0,1] := \left\{ g \in C[0,1] : g', g'' \in C[0,1] \right\}$.

Then, we know that (Theorem 2.4 in [21]),

$$K_2(\psi;\delta) \leq L\omega_2(\psi;\delta), \tag{3.1}$$

where L absolute constant.

Lemma 8. *Let* $\psi \in C[0,1]$. *Consider the operators*

$${}^*K^l_{\rho,\alpha}(\psi;\zeta) = K^l_{\rho,\alpha}(\psi;\zeta) + \psi(\zeta) - \psi\left(\frac{l}{2(\rho+l)} + \frac{\rho}{\rho+l}\zeta\right) \tag{3.2}$$

Then, for all $g \in C^2[0,1]$, *we have*

$$\begin{aligned} \left|{}^*K^l_{\rho,\alpha}(g;\tau) - g(\tau)\right| \leq & \left[\frac{3l^2 + l}{12(\rho+l)^2} + \frac{\left(\rho + 2 - 2\alpha - l^2\right)}{(\rho+l)^2}\zeta + \frac{l^2 - \rho + 2\alpha - 2}{(\rho+l)^2}\zeta^2 \right. \\ & \left. + \left(\frac{l}{2(\rho+l)} + \frac{\rho}{\rho+l}\zeta - \zeta \right)^2 \right] \left\|g''\right\|. \end{aligned} \tag{3.3}$$

Proof. From (3.2) we have

$$\begin{aligned} {}^*K^l_{\rho,\alpha}((t-\zeta);\zeta) &= K^l_{\rho,\alpha}((t-\zeta);\zeta) - \left[\frac{l}{2(\rho+l)} + \frac{\rho}{\rho+l}\zeta - \zeta \right] \\ &= K^l_{\rho,\alpha}(t;\zeta) - \zeta K^l_{\rho,\alpha}(1;\zeta) - \left(\frac{l}{2(\rho+l)} + \frac{\rho}{\rho+l}\zeta \right) + \zeta = 0. \end{aligned} \tag{3.4}$$

Let $\zeta \in [0,1]$ and $g \in C^2[0,1]$. Using the Taylor's formula,

$$g(t) - g(\zeta) = (t-\zeta)g'(\zeta) + \int_\zeta^t (t-u)g''(u)du, \tag{3.5}$$

Applying ${}^{*}K^{l}_{\rho,\alpha}$ to both sides of the (3.5) and using (3.4), we have

$$\begin{aligned}
{}^{*}K^{l}_{\rho,\alpha}(g;\zeta)-g(\zeta) &= {}^{*}K^{l}_{\rho,\alpha}\left((t-\zeta)g^{'}(\zeta);\zeta\right)+{}^{*}K^{l}_{\rho,\alpha}\left(\int_{\zeta}^{t}(t-u)g^{''}(u)du;\zeta\right)\\
&= g^{'}(\zeta){}^{*}K^{l}_{\rho,\alpha}\left((t-\zeta);\zeta\right)+K^{l}_{\rho,\alpha}\left(\int_{\zeta}^{t}(t-u)g^{''}(u)du;\zeta\right)\\
&\quad-\int_{\zeta}^{\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta}\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-u\right)g^{''}(u)du\\
&= K^{l}_{\rho,\alpha}\left(\int_{\zeta}^{t}(t-u)g^{''}(u)du;\zeta\right)-\int_{\zeta}^{\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta}\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-u\right)g^{''}(u)du.
\end{aligned}$$

On the other hand, since

$$\int_{\zeta}^{t}|t-u|\left|g^{''}(u)\right|du\leq\left\|g^{''}\right\|\int_{\zeta}^{t}|t-u|\,du\leq(t-\zeta)^{2}\left\|g^{''}\right\|$$

and

$$\begin{aligned}
&\left|\int_{\zeta}^{\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta}\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-u\right)g^{''}(u)du\right|\\
&\leq\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-\zeta\right)^{2}\left\|g^{''}\right\|,
\end{aligned}$$

we conclude that

$$\begin{aligned}
\left|{}^{*}K^{l}_{\rho,\alpha}(g;\zeta)-g(\zeta)\right| &= \left|K^{l}_{\rho,\alpha}\left(\int_{\zeta}^{t}(t-u)g^{''}(u)du;\zeta\right)-\int_{\zeta}^{\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta}\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-u\right)g^{''}(u)du\right|\\
&\leq\left\|g^{''}\right\|K^{l}_{\rho,\alpha}\left((t-\zeta)^{2};\zeta\right)+\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-\zeta\right)^{2}\left\|g^{''}\right\|.
\end{aligned}$$

Using Lemma 5, we get

$$\begin{aligned}
\left|{}^{*}K^{l}_{\rho,\alpha}(g;\zeta)-g(\zeta)\right| &\leq\left[\frac{3l^{2}+l}{12(\rho+l)^{2}}+\frac{\left(\rho+2-2\alpha-l^{2}\right)}{(\rho+l)^{2}}\zeta+\frac{l^{2}-\rho+2\alpha-2}{(\rho+l)^{2}}\zeta^{2}\right.\\
&\quad\left.+\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-\zeta\right)^{2}\right]\left\|g^{''}\right\|.
\end{aligned}$$

□

Theorem 9. *Let $\rho \in \mathbb{N}$, $\alpha \in [0,1]$ and $l \in \mathbb{Z}^{+}$. Then, for every $\psi \in C[0,1]$, there exists a constant $M > 0$ such that*

$$\left|K^{l}_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| \leq M\omega_2\left(\psi;\ \sqrt{\delta^{l}_{\rho,\alpha}(\zeta)}\right)+\omega\left(\psi;\beta^{l}_{\rho,\alpha}(\zeta)\right),$$

where

$$\begin{aligned}\delta^{l}_{\rho,\alpha}(\zeta)=&\left[\frac{3l^2+l}{12(\rho+l)^2}+\frac{\left(\rho+2-2\alpha-l^2\right)}{(\rho+l)^2}\zeta+\frac{l^2-\rho+2\alpha-2}{(\rho+l)^2}\zeta^2\right.\\&\left.+\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-\zeta\right)^2\right]\left\|g''\right\|\end{aligned}$$

and

$$\beta^{l}_{\rho,\alpha}(\zeta)=\left|\frac{l}{2(\rho+l)}+\left(\frac{\rho}{\rho+l}-1\right)\zeta\right|.$$

Proof. It follows from Lemma 8, that

$$\begin{aligned}\left|K^{l}_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| &\leq \left|{}^{*}K^{l}_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right|+\left|\psi(\zeta)-\psi\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta\right)\right|\\&\leq \left|{}^{*}K^{l}_{\rho,\alpha}(\psi-g;\zeta)-(\psi-g)(\zeta)\right|\\&\quad+\left|\psi(\zeta)-\psi\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta\right)\right|+\left|{}^{*}K^{l}_{\rho,\alpha}(g;\zeta)-g(\zeta)\right|\\&\leq \left|{}^{*}K^{l}_{\rho,\alpha}(\psi-g;\zeta)\right|+\left|(\psi-g)(\zeta)\right|\\&\quad+\left|\psi(\zeta)-\psi\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta\right)\right|+\left|{}^{*}K^{l}_{\rho,\alpha}(g;\zeta)-g(\zeta)\right|.\end{aligned}$$

Now, considering the boundedness of the ${}^{*}K^{l}_{\rho,\alpha}$ and inequality (3.3), we get

$$\begin{aligned}\left|K^{l}_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| &\leq 4\left\|\psi-g\right\|+\left|\psi(\zeta)-\psi\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta\right)\right|\\&\quad+\left[\frac{3l^2+l}{12(\rho+l)^2}+\frac{\left(\rho+2-2\alpha-l^2\right)}{(\rho+l)^2}\zeta+\frac{l^2-\rho+2\alpha-2}{(\rho+l)^2}\zeta^2\right.\\&\quad\left.+\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-\zeta\right)^2\right]\left\|g''\right\|\\&\leq 4\left\|\psi-g\right\|+\omega\left(\psi;\left|\left(\frac{l}{2(\rho+l)}+\left(\frac{\rho}{\rho+l}-1\right)\zeta\right)^2\right|\right)+\delta^{l}_{\rho,\alpha}(\zeta)\left\|g''\right\|.\end{aligned}$$

Now, taking infimum on the right side over all $g \in C^2[0,1]$ and using (3.1), we get the following result

$$\begin{aligned}\left|K^{l}_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| &\leq 4K_2\left(\psi;\delta^{l}_{\rho,\alpha}(\zeta)\right)+\omega\left(\psi;\beta_{\rho}(\zeta)\right)\\&\leq M\omega_2\left(\psi;\ \sqrt{\delta^{l}_{\rho,\alpha}(\zeta)}\right)+\omega\left(\psi;\beta^{l}_{\rho,\alpha}(\zeta)\right).\end{aligned}$$

□

Let us consider the Lipschitz-type with two parameters [22]. For $\beta_1 \geq 0, \beta_2 > 0$, we define

$$\text{Lip}_M^{\beta_1,\beta_2}(\eta) = \left\{ \psi \in C[0,1] : |\psi(t) - \psi(\zeta)| \leq M \frac{|t-\zeta|^\eta}{(t+\beta_1\zeta^2+\beta_2\zeta)^{\frac{\eta}{2}}}; t \in [0,1], \zeta \in (0,1] \right\},$$

where $0 < \eta \leq 1$.

Theorem 10. *Let $\psi \in Lip_M^{\beta_1,\beta_2}(\eta)$. Then for all $\zeta \in (0,1]$, we have*

$$\left| K_{\rho,\alpha}^l(\psi;\zeta) - \psi(\zeta) \right| \leq M \left(\frac{K_{\rho,\alpha}^l\left((t-\zeta)^2;\zeta\right)}{\beta_1\zeta^2+\beta_2\zeta} \right)^{\frac{\eta}{2}}.$$

Proof. Let we prove theorem for the case $0 < \eta \leq 1$, applying Holder's inequality with $p = \frac{2}{\eta}, q = \frac{2}{2-\eta}$,

$$\begin{aligned}
\left| K_{\rho,\alpha}^l(\psi;\zeta) - \psi(\zeta) \right| &\leq \sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \int_0^1 \cdots \int_0^1 \left| \psi\left(\frac{\vartheta+t_1+...+t_l}{\rho+l}\right) - \psi(\zeta) \right| dt_1...dt_l \\
&\leq \sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \left(\int_0^1 \cdots \int_0^1 \left| \psi\left(\frac{\vartheta+t_1+...+t_l}{\rho+l}\right) - \psi(\zeta) \right|^{\frac{2}{\eta}} dt_1...dt_l \right)^{\frac{\eta}{2}} \\
&\leq \left\{ \sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \int_0^1 \cdots \int_0^1 \left| \psi\left(\frac{\vartheta+t_1+...+t_l}{\rho+l}\right) - \psi(\zeta) \right|^{\frac{2}{\eta}} dt_1...dt_l \right\}^{\frac{\eta}{2}} \left\{ \sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \right\}^{\frac{2-\eta}{2}} \\
&= \left(\sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \int_0^1 \cdots \int_0^1 \left| \psi\left(\frac{\vartheta+t_1+...+t_l}{\rho+l}\right) - \psi(\zeta) \right|^{\frac{2}{\eta}} dt_1...dt_l \right)^{\frac{\eta}{2}} \\
&\leq M \left(\sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \int_0^1 \cdots \int_0^1 \frac{\left(\frac{\vartheta+t_1+...+t_l}{\rho+l} - \zeta\right)^2}{\left(\frac{\vartheta+t_1+...+t_l}{\rho+l} + \beta_1\zeta^2 + \beta_2\zeta\right)} dt_1...dt_l \right)^{\frac{\eta}{2}} \\
&\leq \frac{M}{(\beta_1\zeta^2+\beta_2 x)^{\frac{\eta}{2}}} \left(\sum_{\vartheta=0}^{\rho} p_{\rho,\vartheta}^{(\alpha)} \int_0^1 \cdots \int_0^1 \left(\frac{\vartheta+t_1+...+t_l}{\rho+l} - \zeta\right)^2 dt_1...dt_l \right)^{\frac{\eta}{2}} \\
&= \frac{M}{(\beta_1 x^2+\beta_2 x)^{\frac{\eta}{2}}} \left(K_{\rho,\alpha}^l\left((t-x)^2;x\right)\right)^{\frac{\eta}{2}}
\end{aligned}$$

□

Theorem 11. *Let $\psi \in C[0,1]$, $\alpha \in [0,1]$ and $l \in \mathbb{Z}^+$. Then the inequality*

$$\left| K_{\rho,\alpha}^l(\psi;\zeta) - \psi(\zeta) \right| \leq 2w(\psi, \delta_{\rho,\alpha}^l(\zeta))$$

takes place, where $\delta_{\rho,\alpha}^l(\zeta) = \sqrt{K_{\rho,\alpha}^l(t-\zeta)^2}$.

Proof. It is known that

$$|\psi(t)-\psi(\zeta)| \leq w(\psi,\delta)\left(\frac{(t-\zeta)^2}{\delta^2}+1\right), \text{ for any } \delta>0,$$

So, we have

$$\begin{aligned}\left|K^l_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| &\leq K^l_{\rho,\alpha}(|\psi(t)-\psi(\zeta)|;\zeta)\\ &\leq w(\psi,\delta)\left(\frac{1}{\delta^2}K^l_{\rho,\alpha}\left((t-\zeta)^2;\zeta\right)+1\right)\end{aligned}$$

Choosing $\delta=\delta_\rho(\zeta)=\sqrt{K^l_{\rho,\alpha}(t-\zeta)^2}$, we have

$$\left|K^l_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| \leq 2w(\psi,\delta^l_{\rho,\alpha}(\zeta))$$

□

Let $C^k[I]$ denote space of k-times continuously differentiable function on I.

Theorem 12. *For any $\psi\in C^1[0,1]$ and $\zeta\in[0,1]$, we have*

$$\left|K^l_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| \leq \left|\frac{l(1-2\zeta)}{2(\rho+l)}\right|\left|\psi^{'}(\zeta)\right| + 2\sqrt{K^l_{\rho,\alpha}((t-\zeta)^2;\zeta)}w(\psi^{'},\sqrt{K^l_{\rho,\alpha}((t-\zeta)^2;\zeta)}) \tag{3.6}$$

Proof. Let $\psi\in C^1[0,1]$. For any $t\in[0,1]$, $\zeta\in[0,1]$, we have

$$\psi(t)-\psi(\zeta)=\psi^{'}(\zeta)(t-\zeta)+\int_\zeta^t(\psi^{'}(u)-\psi^{'}(\zeta))du.$$

Using $K^l_{\rho,\alpha}(.;\zeta)$ on both sides of the above equation, we have

$$K^l_{\rho,\alpha}(\psi(t)-\psi(\zeta);\zeta)=\psi^{'}(\zeta)K^l_{\rho,\alpha}((t-\zeta);\zeta)+K^l_{\rho,\alpha}\left(\int_\zeta^t(\psi^{'}(u)-\psi^{'}(\zeta))du;\zeta\right).$$

Using the property of modulus of continuity $|\psi(t)-\psi(\zeta)|\leq w(\psi,\delta)\left(\frac{|t-\zeta|}{\delta}+1\right)$, $\delta>0$, we obtain

$$\left|\int_\zeta^t(\psi^{'}(u)-\psi^{'}(\zeta))du\right| \leq w(\psi^{'},\delta)\left(\frac{(t-\zeta)^2}{\delta}+|t-\zeta|\right),$$

it follows that

$$\left|K^l_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| \leq \left|\psi^{'}(\zeta)\right|\left|\left(K^l_{\rho,\alpha}(t-\zeta);\zeta\right)\right| + w(\psi^{'},\delta)\left\{\frac{1}{\delta}K^l_{\rho,\alpha}((t-\zeta)^2;\zeta)+K^l_{\rho,\alpha}(|t-\zeta|;\zeta)\right\}$$

From Cauchy-Schwarz inequality, we have

$$\left|K^l_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| \leq \left|\psi^{'}(\zeta)\right|\left|\left(K^l_{\rho,\alpha}(t-\zeta);\zeta\right)\right| + w(\psi^{'},\delta)\left\{\frac{1}{\delta}\sqrt{K^l_{\rho,\alpha}((t-\zeta)^2;\zeta)}+1\right\}\sqrt{K^l_{\rho,\alpha}((t-\zeta)^2;\zeta)}$$

Now, taking $\delta=\sqrt{K^l_{\rho,\alpha}((t-\zeta)^2;\zeta)}$, we obtain (3.6). □

In the next section, we state the direct global approximation theorem for operators $K^l_{\rho,\alpha}(\psi;\zeta)$.

4. Global approximation

Let $AC\,[0,1]$ denote the absolutely continuous on $[0,1]$. For $\psi \in C\,[0,1]$, the first and second order Ditzian-Totik moduli of smoothness are defined by

$$w_\theta(\psi,\delta) = \sup_{0<h\le\delta}\ \sup_{\zeta,\zeta+h\theta(\zeta)\in[0,1]} |\psi(\zeta + h\theta(\zeta)) - \psi(\zeta)|$$

and

$$w_{2,\phi}(\psi,\delta) = \sup_{0<h\le\delta}\ \sup_{\zeta,\zeta+h\phi(\zeta)\in[0,1]} |\psi(\zeta - h\phi(\zeta)) - 2\psi(\zeta) + \psi(\zeta + h\phi(\zeta))|,$$

respectively.

Moreover, the second-order modified K-functional for $\psi \in C\,[0,1]$ is defined by

$$K_{2,\phi}(\psi,\delta) = \inf\left\{\|\psi - g\| + \delta\left\|\phi^2 g''\right\| + \delta^2\left\|g''\right\| : g \in W^2(\phi)\right\},$$

where $\delta > 0, \phi(x) = \sqrt{x(1-x)}$ $(x \in [0,1])$and

$$W^2(\phi) = \left\{g \in C\,[0,1] : g' \in AC\,[0,1]\,, \phi^2 g'' \in C[0,1]\right\}.$$

It is well-known [23] that, for any $\delta > 0$,

$$K_{2,\phi}(\psi,\delta^2) \le D w_{2,\phi}(\psi,\delta), \tag{4.1}$$

holds for some absolute constant $D > 0$.

Theorem 13. *Let $\rho \in \mathbb{N}$, $\alpha \in [0,1]$ and $l \in \mathbb{Z}^+$. Then, for every $\psi \in C\,[0,1]$ and $\zeta \in [0,1]$, there exist an absolute $C > 0$ such that*

$$\left\|K^l_{\rho,\alpha}(\psi;\zeta) - \psi(\zeta)\right\| \le C w_{2,\phi}\left(\psi, \frac{1}{\sqrt{\rho+l}}\right) + w_{\theta_l}\left(\psi, \frac{l}{\rho+l}\right), \tag{4.2}$$

where $\theta_l = l\,(1+2x)$.

Proof. If we use the operators ${}^*K^l_{\rho,\alpha}$ given by (3.2), then for a given $g \in W^2(\phi)$, we obtain that

$$\begin{aligned}\left|{}^*K^l_{\rho,\alpha}(g;\zeta) - g(\zeta)\right| \ &\le\ K^l_{\rho,\alpha}\left(\left|\int_\zeta^t |t-u|\,g''(u)du\right|;\zeta\right)\\ &\quad + \left|\int_\zeta^{\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta} \left|\frac{l}{2(\rho+l)} + \frac{\rho\zeta}{\rho+l} - u\right|\left|g''(u)\right|du\right|.\end{aligned}$$

Let $\lambda_\rho(\zeta) = \zeta(1-\zeta) + \frac{l}{(\rho+l)}$. Taking $u = \beta\zeta + (1-\beta)\,t, \beta \in [0,1]$,and also using concavity λ_ρ, we have

$$\frac{|t-u|}{\lambda_\rho(u)} = \frac{\beta\,|\zeta - t|}{\lambda_\rho(u)} \le \frac{\beta\,|\zeta - t|}{\lambda_\rho(t) + \beta\left(\lambda_\rho(\zeta) - \lambda_\rho(t)\right)} \le \frac{|\zeta - t|}{\lambda_\rho(\zeta)}.$$

Using the last inequality, we observe that

$$\begin{aligned}\left|\int_{\zeta}^{t}|t-u|\,g^{''}(u)du\right| &= \left|\int_{\zeta}^{t}\frac{|t-u|}{\lambda_\rho(u)}g^{''}(u)\lambda_\rho(u)du\right| \\ &\leq \frac{\left\|\lambda_\rho g^{''}\right\|}{\lambda_\rho(\zeta)}(t-u)^2. \end{aligned} \tag{4.3}$$

Then we get from (4.3) that

$$\begin{aligned}\left|{}^{*}K^{l}_{\rho,\alpha}(g;\zeta)-g(\zeta)\right| &\leq \frac{1}{\lambda_\rho(\zeta)}K^{l}_{\rho,\alpha}\left((t-\zeta)^2;\zeta\right)\left\|\lambda_\rho g^{''}\right\| \\ &\quad+\frac{1}{\lambda_\rho(\zeta)}\left(\frac{l\,(1-2x)}{2\,(\rho+l)}\right)^2\left\|\lambda_\rho g^{''}\right\| \\ &\leq \frac{2l^2\left\|\lambda_\rho g^{''}\right\|}{(\rho+l)} \\ &\leq \frac{2l^2\left\|\lambda_\rho g^{''}\right\|}{\rho+l}\left(\left\|\phi^2 g^{''}\right\|+\frac{1}{\rho+l}\left\|g^{''}\right\|\right).\end{aligned}$$

On the other hand, since the operators ${}^{*}K^{l}_{\rho,\alpha}(g;\zeta)$ are uniformly bounded, we get

$$\begin{aligned}\left|K^{l}_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right| &\leq \left|{}^{*}K^{l}_{\rho,\alpha}(\psi-g;\zeta)\right|+\left|{}^{*}K^{l}_{\rho,\alpha}(g;\zeta)-g(\zeta)\right|+|\psi(\zeta)-g(\zeta)| \\ &\quad+\left|\psi\left(\frac{l}{2(\rho+l)}+\frac{\rho\zeta}{\rho+l}\right)-\psi(\zeta)\right| \\ &\leq 4l^2\left[\|\psi-g\|+\left(\frac{1}{\rho+l}\left\|\phi g^{''}\right\|+\frac{1}{(\rho+l)^2}\left\|g^{''}\right\|\right)\right] \\ &\quad+\left|\psi\left(\frac{l}{2(\rho+l)}+\frac{\rho\zeta}{\rho+l}\right)-\psi(\zeta)\right|.\end{aligned}$$

Taking infimum on the right hand side of the above inequality over all $g\in W^2(\phi)$, we obtain

$$\left|K^{l}_{\rho,\alpha}(\psi;\zeta)-\psi(\zeta)\right|\leq 4l^2K_{2,\phi}\left(\psi,\frac{1}{\rho+l}\right)+\left|\psi\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta\right)-\psi(\zeta)\right|.$$

Now, using the function $\theta_{l}(\zeta)=l+2xl$, we can also get

$$\begin{aligned}\left|\psi\left(\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta\right)-\psi(\zeta)\right| &= \left|\psi\left(\zeta+\theta_{l}(\zeta)\frac{\frac{l}{2(\rho+l)}+\frac{\rho}{\rho+l}\zeta-\zeta}{\theta_{l}(\zeta)}\right)-\psi(\zeta)\right| \\ &\leq \sup_{t\in I_l(\zeta)}\left|\psi\left(t+\theta_{l}(t)\frac{\frac{l}{2}-xl}{(\rho+l)\theta_{l}(\zeta)}-\psi(t)\right)\right| \\ &\leq w_{\theta_{l}}\left(\psi;\frac{\left|\frac{l}{2}-xl\right|}{(\rho+l)\theta_{l}(\zeta)}\right) \\ &\leq w_{\theta_{l}}\left(\psi;\frac{l}{(\rho+l)}\right),\end{aligned}$$

where $I_l(\zeta)=\left\{t\in[0,1]:t+\theta_{l}(t)\frac{\frac{l}{2}-xl}{(\rho+l)}\in[0,1]\right\}$. Finally, using (4.1), we get desired result. □

5. Voronovskaya type results

Here, we Voronovskaya type result for the $K_{\rho,\alpha}^{l}(\psi;\zeta)$ operators.

Theorem 14. *Let* $\psi \in C[0,1]$. *If* ψ'' *exist at a point* $\zeta \in [0,1]$, *then we have*

$$\lim_{\rho\to\infty}(\rho+l)\left[K_{\rho,\alpha}^{l}(\psi;\zeta)-\psi(\zeta)\right]=\left(\frac{l}{2}-\zeta l\right)\psi'(\zeta)+\frac{1}{2}\zeta(1-\zeta)\psi''(\zeta),$$

where $p \in \mathbb{N}, l \in \mathbb{Z}^{+}$ *and* $\alpha \in [0,1]$.

Proof. For $\zeta \in [0,1]$, the Taylor's formula ψ is given by

$$\psi(t)=\psi(\zeta)+\psi'(\zeta)(t-\zeta)+\frac{1}{2}\psi''(\zeta)(t-\zeta)^{2}+r(t,\zeta)(t-\zeta)^{2}, \tag{5.1}$$

Here $r(t,\zeta)$ is Peano form of remainder and $r(.,\zeta) \in C[0,1]$ and $\lim_{t\to\zeta} r(t,\zeta)=0$. Applying the operator $K_{\rho,\alpha}^{l}$ to (5.1), we get

$$K_{\rho,\alpha}^{l}(\psi;\zeta)-\psi(\zeta)=\psi'(\zeta)K_{\rho,\alpha}^{l}((t-\zeta);\zeta)+\frac{1}{2}\psi''(\zeta)K_{\rho,\alpha}^{l}\left((t-\zeta)^{2};\zeta\right)+K_{\rho,\alpha}^{l}\left(r(t,\zeta)(t-\zeta)^{2};\zeta\right).$$

Using Cauchy-Schwarz inequality in the last term, we have

$$K_{\rho,\alpha}^{l}\left(r(t,\zeta)(t-\zeta)^{2};\zeta\right)\leq\sqrt{K_{\rho,\alpha}^{l}\left(r^{2}(t,\zeta);\zeta\right)}\sqrt{K_{\rho,\alpha}^{l}\left((t-\zeta)^{4};\zeta\right)}.$$

Observe that $r^{2}(t,\zeta)=0$ and $r^{2}(.,\zeta)\in C[0,1]$.

Hence from Theorem 7,

$$\lim_{\rho\to\infty}K_{\rho,\alpha}^{\gamma}\left(r^{2}(t,\zeta);\zeta\right)=r^{2}(\zeta,\zeta)=0$$

uniformly for $\zeta \in [0,1]$.

Therefore

$$\lim_{\rho\to\infty}(\rho+l)\left[K_{\rho,\alpha}^{l}(\psi;\zeta)-\psi(\zeta)\right]=\psi'(\zeta)\lim_{\rho\to\infty}(\rho+l)\,K_{\rho,\alpha}^{l}((t-\zeta);\zeta)+\frac{1}{2}\psi''(\zeta)\lim_{\rho\to\infty}(\rho+l)\,K_{\rho,\alpha}^{l}\left((t-\zeta)^{2};\zeta\right).$$

From Lemma 6,

$$\lim_{\rho\to\infty}(\rho+l)\left[K_{\rho,\alpha}^{l}(\psi;\zeta)-\psi(\zeta)\right]=\left(\frac{l}{2}-\zeta l\right)\psi'(\zeta)+\frac{1}{2}\zeta(1-\zeta)\psi''(\zeta).$$

□

Now, we show graphical analysis for the convergence of operators $K_{\rho,\alpha}^{l}(\psi;\zeta)$ to the function $\psi(\zeta)=1+\zeta\sin(10\zeta)$.

In Figure 1, we show the approximation to this function ψ by the operators $K_{\rho,0.9}^{2}(\psi;\zeta)$ for $\rho=20,50,100$ respectively.

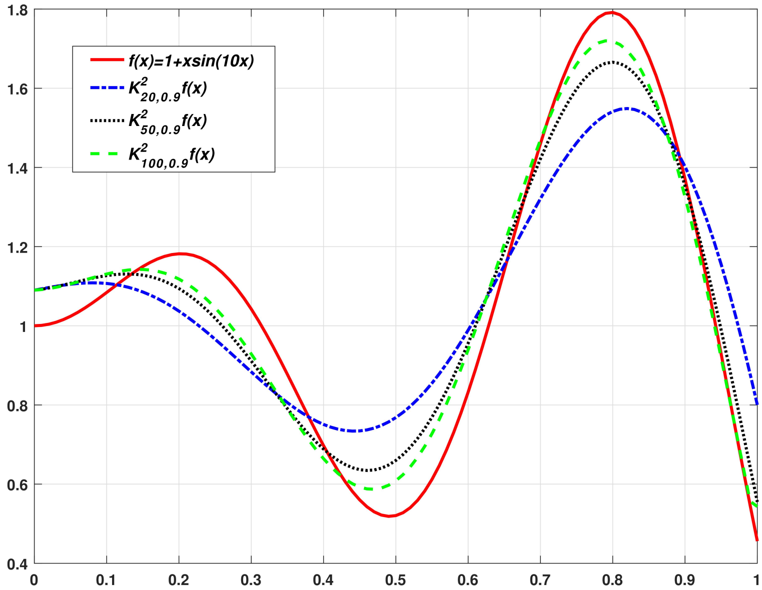

Figure 1. Approximation to ψ by $K^l_{\rho,\alpha}$ for $l = 2, \psi(x) = 1 + x\sin(10x)$ and $\rho = 20, 50, 100$.

Morever, in Table 1, we compute the error of approximation $K^2_{\rho,0.9}(\psi;\zeta)$ of our $\psi(\zeta) = 1+\zeta\sin(10\zeta)$ for $\rho = 20, 50, 100$.

Table 1. Error of approximation.

ζ	$\left\|K^l_{20,0.9}(\psi;\zeta) - \psi(\zeta)\right\|$	$\left\|K^l_{50,0.9}(\psi;\zeta) - \psi(\zeta)\right\|$	$\left\|K^l_{100,0.9}(\psi;\zeta) - \psi(\zeta)\right\|$
0.15	0.064216061	0.021762009	0.007340463
0.20	0.14487018	0.087839632	0.063907778
0.25	0.183644366	0.130465015	0.104166221
0.30	0.158749953	0.131315495	0.113558559
0.35	0.071337791	0.086468662	0.087398881
0.40	0.053489031	0.008709032	0.032989989
0.45	0.174356459	0.076385157	0.032542776
0.50	0.247626587	0.139305019	0.088014747
0.55	0.243166628	0.157342571	0.115543775
0.60	0.156507953	0.123329353	0.106886498
0.65	0.012905048	0.049061689	0.066417709
0.70	0.138822143	0.038181349	0.009556201
0.75	0.241568052	0.105238164	0.042950102
0.80	0.250138778	0.125723189	0.072927239
0.85	0.150111243	0.091301023	0.07198989

6. Bivariate α- Bernstein-Kantorovich operators

In this section, we introduce the bivariate extension of the operators (2.1). The bivariate extension of the $K^{l}_{\rho,\alpha}(\psi;\zeta)$ (2.1) can be defined by

$$K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(\psi;\zeta,\gamma)=\sum_{\vartheta_1=0}^{\rho_1}\sum_{\vartheta_2=0}^{\rho_2}p^{(\alpha_1)}_{\rho_1,\vartheta_1}p^{(\alpha_2)}_{\rho_2,\vartheta_2}\int_0^1\cdots\int_0^1\left(\int_0^1\cdots\int_0^1\psi\left(\frac{\vartheta_1+t_1+\ldots+t_{l_1}}{\rho_1+l_1},\frac{\vartheta_2+t_1+\ldots+t_{l_2}}{\rho_1+l_2}\right)dt_1\ldots dt_{l_1}\right)dt_1\ldots dt_{l_2}$$

where $(\zeta.\gamma)\in I^2=[0,1]\times[0,1]$, $\alpha_1,\alpha_2\in[0,1]$ and $l_1,l_2\in\mathbb{Z}^+$.

The bivariate α-Bernstein-Kantorovich operators can be rewritten as

$$K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(\cdot;\zeta,\gamma)=K^{l_1}_{\rho_1,\alpha_1}(\cdot;\zeta)\times K^{l_1}_{\rho_2,\alpha_2}(\cdot;\gamma)$$

Lemma 15. *Let $e_{ij}(\zeta,\gamma)=\zeta^i\gamma^j$, $0\le i+j\le 2$. For $(\zeta.\gamma)\in I^2=[0,1]\times[0,1]$, $l_1,l_2\in\mathbb{Z}^+$ and $\alpha_1,\alpha_2\in[0,1]$, we have*

$$\begin{aligned}
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(e_{00};\zeta,\gamma) &= 1,\\
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(e_{10};\zeta,\gamma) &= \frac{l_1}{2(\rho_1+l_1)}+\frac{\rho_1}{(\rho_1+l_1)}\zeta,\\
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(e_{10};\zeta,\gamma) &= \frac{l_2}{2(\rho_2+l_2)}+\frac{\rho_2}{(\rho_2+l_2)}\gamma\\
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(e_{20};\zeta,\gamma) &= \frac{3l_1^2+l_1}{12(\rho_1+l_1)^2}+\frac{(\rho_1(l_1+1)+2(1-\alpha_1))}{(\rho_1+l_1)^2}\zeta+\left(\frac{\rho_1^2-\rho_1-2(1-\alpha_1)}{(\rho_1+l_1)^2}\right)\zeta^2\\
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(e_{02};\zeta,\gamma) &= \frac{3l_2^2+l_2}{12(\rho_2+l_2)^2}+\frac{(\rho_2(l_2+1)+2(1-\alpha_2))}{(\rho_2+l_2)^2}\gamma+\left(\frac{\rho_2^2-\rho_2-2(1-\alpha_2)}{(\rho_2+l_2)^2}\right)\gamma^2
\end{aligned}$$

Remark 16. *According to above Lemma 15, we get*

$$\begin{aligned}
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(e_{10}-\zeta;\zeta,\gamma) &= \frac{l_1(1-2\zeta)}{2(\rho_1+l_1)}\\
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}(e_{01}-\gamma;\zeta,\gamma) &= \frac{l_2(1-2\gamma)}{2(\rho_2+l_2)}\\
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}((e_{10}-\zeta)^2;\zeta,\gamma) &= \frac{3l_1^2+l_1}{12(\rho_1+l_1)^2}+\frac{\left(\rho_1+2-2\alpha_1-l_1^2\right)}{(\rho_1+l_1)^2}\zeta+\frac{l_1^2-\rho_1+2\alpha_1-2}{(\rho_1+l_1)^2}\zeta^2=\delta_{\rho_1,\alpha_1}(\zeta)\\
K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}((e_{01}-\gamma)^2;\zeta,\gamma) &= \frac{3l_2^2+l_2}{12(\rho_2+l_2)^2}+\frac{\left(\rho_2+2-2\alpha_2-l_2^2\right)}{(\rho_2+l_2)^2}\gamma+\frac{l_2^2-\rho_2+2\alpha_2-2}{(\rho_2+l_2)^2}\gamma^2=\delta_{\rho_2,\alpha_2}(\gamma)
\end{aligned}$$

In the next theorem, we obtain the uniform convergence of the bivariate α-Bernstein-Kantorovich operators to the bivariate functions defined on $I^2=[0,1]\times[0,1]$.

Theorem 17. *Let $C(I^2)$ be the space of continuous bivariate function on $I^2=[0,1]\times[0,1]$. Then for any $\psi\in C(I^2)$, we have*

$$\lim_{\rho_1,\rho_2\to\infty}\left\|K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}\psi-\psi\right\|=0.$$

Proof. Using Lemma 16, we get

$$\begin{aligned}\left\|K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}e_{00}-e_{00}\right\| &= 0, \left\|K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}e_{10}-e_{10}\right\| \to 0\\ \left\|K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}e_{01}-e_{01}\right\| &\to 0, \left\|K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}\left(e_{20}+e_{02}\right)-\left(e_{20}+e_{02}\right)\right\| \to 0 \text{ as } \rho_1,\rho_2\to\infty.\end{aligned}$$

Hence, by Volkov's theorem [3], we deduce

$$\lim_{\rho_1,\rho_2\to\infty}\left\|K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}\psi-\psi\right\|=0.$$

□

We shall use the following modulus of continuity for bivariate real functions:

$$w\left(f;\delta_n,\delta_m\right)=\sup\left\{\left|f(t,s)-f(x,y)\right|:(t,s),(x,y)\in I^2,|t-x|\le\delta_n,|s-y|\le\delta_m\right\}.$$

Theorem 18. *Let $\psi\in C(I^2)$. Then for all $(\zeta,\gamma)\in I^2$, the inequality*

$$\left|K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}\left(\psi;\zeta,\gamma\right)-\psi\left(\zeta,\gamma\right)\right|\le 4w\left(\psi;\delta_{\rho_1,\alpha_1}(\zeta),\delta_{\rho_2,\alpha_2}(\gamma)\right)$$

holds, where $\delta_{\rho_1,\alpha_1}(\zeta),\delta_{\rho_2,\alpha_2}(\gamma)$ are as in Remark 16.

Proof. By the linearity and positivity properties of the $K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}$, we can write

$$\begin{aligned}\left|K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}\left(\psi;\zeta,\gamma\right)-\psi\left(\zeta,\gamma\right)\right| &\le K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}\left(\left|\psi(t,s)-\psi\left(\zeta,\gamma\right)\right|;\zeta,\gamma\right)\\ &\le w\left(\psi;\delta_1,\delta_2\right)\left[K^{l_1}_{\rho_1,\alpha_1}\left(1;\zeta\right)+\frac{1}{\delta_1}K^{l_1}_{\rho_1,\alpha_1}\left(\left|t-\zeta\right|;\zeta\right)\right]\\ &\quad\times\left[K^{l_1}_{\rho_2,\alpha_2}\left(1;\gamma\right)+\frac{1}{\delta_2}K^{l_1}_{\rho_2,\alpha_2}\left(\left|s-\gamma\right|;\gamma\right)\right].\end{aligned}$$

Applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}K^{l_1}_{\rho_1,\alpha_1}\left(\left|t-\zeta\right|;\zeta\right) &\le K^{l_1}_{\rho_1,\alpha_1}\left(\left(t-\zeta\right)^2;\zeta\right)^{\frac{1}{2}},\\ K^{l_1}_{\rho_2,\alpha_2}\left(\left|s-\gamma\right|;\gamma\right) &\le K^{l_1}_{\rho_2,\alpha_2}\left(\left(s-\gamma\right)^2;\gamma\right)^{\frac{1}{2}}.\end{aligned}$$

Choosing $\delta_1=\delta_{\rho_1,\alpha_1}(\zeta)$ and $\delta_2=\delta_{\rho_2,\alpha_2}(\gamma)$, we have desired result. □

Finally, in Figures 2 and 3, we show graphical analysis for the convergence of operators $K^{l_1,l_2}_{\rho_1,\rho_2,\alpha_1,\alpha_2}\left(\psi;\zeta,\gamma\right)$ to the function $\psi(\zeta)=\cos(2\pi\zeta)+\sin(3\pi\gamma)$.

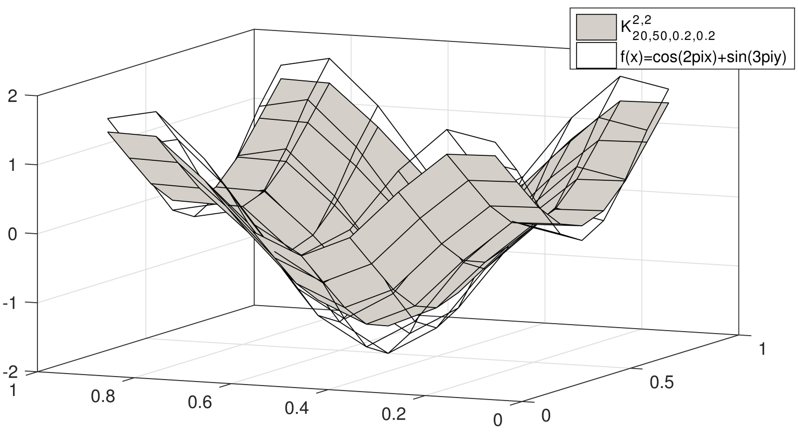

Figure 2. The result of choice $n_1 = 20, n_2 = 50, \alpha_1 = \alpha_2 = 0.2, l_1 = l_2 = 2$.

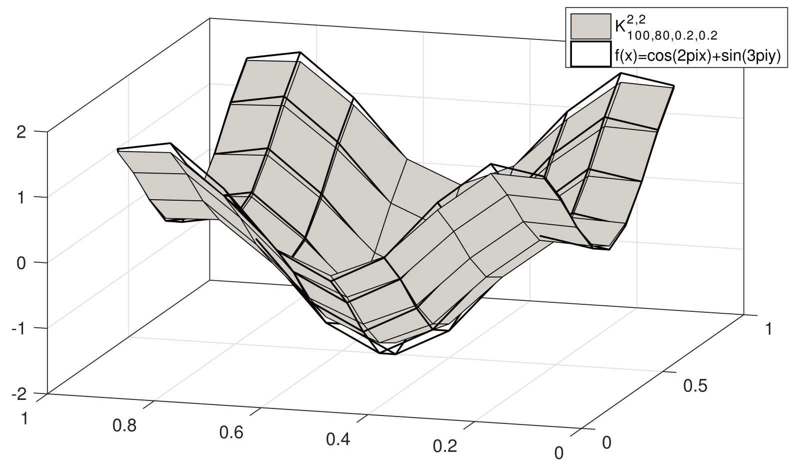

Figure 3. The result of choice $n_1 = 100, n_2 = 80, \alpha_1 = \alpha_2 = 0.2, l_1 = l_2 = 2$.

7. Conclusions

In this paper, we introduced new type of generalized Kantorovich variant of α-Bernstein operators. We obtained estimates of rate of convergence involving first and second order modulus of continuity. Furthermore, we established Voronovskaya type theorem for these operators. Also, some graphical illustrations and numerical results are provided.

Conflict of interest

The author declares no conflict of interest.

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