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Research article

Existence of stable standing waves for the nonlinear Schrödinger equation with mixed power-type and Choquard-type nonlinearities

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Abstract: The aim of this paper is to study the existence of stable standing waves for the following nonlinear Schrödinger type equation with mixed power-type and Choquard-type nonlinearities

$$i\partial_t\psi + \Delta\psi + \lambda|\psi|^q\psi + \frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|\psi|^p}{|x-y|^\mu|y|^\alpha}dy\right)|\psi|^{p-2}\psi = 0,$$

where $N \geq 3$, $0 < \mu < N$, $\lambda > 0$, $\alpha \geq 0$, $2\alpha + \mu \leq N$, $0 < q < \frac{4}{N}$ and $2 - \frac{2\alpha+\mu}{N} < p < \frac{2N-2\alpha-\mu}{N-2}$. We firstly obtain the best constant of a generalized Gagliardo-Nirenberg inequality, and then we prove the existence and orbital stability of standing waves in the L^2-subcritical, L^2-critical and L^2-supercritical cases by the concentration compactness principle in a systematic way.

Keywords: nonlinear Schrödinger equation; standing waves; orbital stability
Mathematics Subject Classification: 35Q55

1. Introduction

In this paper, we consider the Cauchy problem for the following nonlinear Schrödinger equation with mixed power-type and Choquard-type nonlinearities

$$\begin{cases} i\partial_t\psi + \Delta\psi + \lambda|\psi|^q\psi + \frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|\psi|^p}{|x-y|^\mu|y|^\alpha}dy\right)|\psi|^{p-2}\psi = 0, & (t,x) \in [0,T)\times\mathbb{R}^N, \\ \psi(0) = \psi_0 \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $0 < \mu < N$, $\lambda > 0$, $\alpha \geq 0$, $2\alpha + \mu \leq N$, $0 < q < \frac{4}{N}$, $2 - \frac{2\alpha+\mu}{N} < p < \frac{2N-2\alpha-\mu}{N-2}$ and $\psi(t,x) : [0,T)\times\mathbb{R}^N \to \mathbb{C}$ is the complex function with $0 < T \leq \infty$.

The Eq (1.1) has several physical origins and backgrounds, which applied in various modeling scenarios arising from phenomena in science and engineering and depended on different parameter configuration, see, e.g. [20,21]. In the mathematical case $\lambda = 0$, $\alpha = 0$ and $p = 2$, the Eq (1.1) reduces to the well-known Hartree equation, in which this type Schrödinger equations have been studied in [4, 13, 14] by considering the corresponding Cauchy problem. In the physical case $N = 3$, $\lambda = 0$, $\alpha = 0$, $p = 2$ and $\mu = 2$, it was introduced by Pekar in [33] to describe the quantum theory about the polaron at rest in mathematical physics. After then, Lions in [27] used it to describe an electron trapped in its own pole. In a way, it approximated to the Hartee-Fock theory about one component plasma. Afterwards, this equation was proposed by Penrose in [31, 32] as a model of self-gravitating matter and usually called as the Schrödinger-Newton equation.

Recently, this type of equation has been studied extensively in [2,5,9–11,19,24,29,30,35,38,39,44]. Equation (1.1) admits a class of special solutions, which are called standing waves, namely solutions of the form $\psi(t, x) = e^{i\omega t}u(x)$, where $\omega \in \mathbb{R}$ is a frequency and $u \in H^1(\mathbb{R}^N)$ is a nontrivial solution satisfying the elliptic equation

$$-\Delta u + \omega u = \lambda|u|^q u + \frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|x-y|^\mu|y|^\alpha}dy\right)|u|^{p-2}u. \tag{1.2}$$

The Eq (1.2) is variational, whose action functional is defined by

$$A_\omega(u) := E(u) + \frac{\omega}{2}\|u\|_{L^2}^2,$$

where the corresponding energy functional is defined on $H^1(\mathbb{R}^N)$ by

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx - \frac{\lambda}{q+2}\int_{\mathbb{R}^N}|u|^{q+2}dx - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy. \tag{1.3}$$

To begin with, we shall focus on the existence of ground state and recall this definition.

Definition 1.1. *We say that u_c is a ground state of* (1.2) *on $S(c)$ if it is a solution having minimal energy among all the solutions which belong to $S(c)$. Namely, if*

$$E(u_c) = inf\left\{E(u), u \in S(c), (E|_{S(c)})'(u) = 0\right\},$$

where

$$S(c) := \{u \in H^1(\mathbb{R}^N) : \|u\|_{L^2}^2 = c\}.$$

Subsequently, for the evolutional type equation (1.1), one of the most important problems is to study the stability of standing waves, which is defined as follows.

Definition 1.2. *Let u be a solution of* (1.2). *We say that the standing wave $e^{i\omega t}u(x)$ is orbitally stable if for each $\varepsilon > 0$, there is a $\delta > 0$ such that if initial data $\psi_0 \in H^1(\mathbb{R}^N)$ and $\|\psi_0 - u\|_{H^1(\mathbb{R}^N)} < \delta$, then the corresponding solution to* (1.1) *with $\psi|_{t=0} = \psi_0$ satisfies*

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R}}\|\psi(t,\cdot) - e^{i\theta}u\|_{H^1(\mathbb{R}^N)} < \varepsilon.$$

Otherwise, we say that the standing wave is unstable.

Generally, there are two major methods in the research of the orbital stability of standing waves. The first one is the Grillakis-Shatah-Strauss theory about general stability/instability criterion in [16]. As a matter of fact, if we assume certain spectral properties of the linearization of (1.2) about u_ω, the criterion means that the standing wave $e^{i\omega t}u_\omega(x)$ is orbitally stable when $\frac{\partial}{\partial\omega}\|u_\omega\|_{L^2}^2 > 0$ or unstable when $\frac{\partial}{\partial\omega}\|u_\omega\|_{L^2}^2 < 0$. Moreover, it also turns out that this method is extremely useful in the case of homogeneous nonlinearities. In this paper, however, we consider the non-homogeneous Schrödinger equation with mixed power-type and Choquard-type nonlinearities. On the one hand, it is difficult for us to verify some properties of the spectrum. On the other hand, the sign of $\frac{\partial}{\partial\omega}\|u_\omega\|_{L^2}^2$ is hard to be verified for the Eq (1.1). Therefore, this method might be hard to work, see, e.g. [25, 34].

The other is the idea introduced by Cazenave and Lions in [3], which constructs orbitally stable standing waves to (1.1) is to consider the constrained minimization problems. For this method, we know that it only makes use of the conservation laws and the compactness of any minimizing sequences. Therefore, this method is quite general and may be applied to many situations. According to the idea, we naturally obtain the stability of the set of the constrained energy minimizers, and then we recall the following definition, as introduced in [3].

Definition 1.3. *We say that the set $\mathcal{M}$ is orbitally stable if for each $\varepsilon > 0$, there is a $\delta > 0$ such that if initial data $\psi_0 \in H^1(\mathbb{R}^N)$ and $\inf_{u\in\mathcal{M}} \|\psi_0 - u\|_{H^1(\mathbb{R}^N)} < \delta$, then the corresponding solution to* (1.1) *with $\psi|_{t=0} = \psi_0$ satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{u\in\mathcal{M}} \|\psi(t,\cdot) - u\|_{H^1(\mathbb{R}^N)} < \varepsilon.$$

In view of the Definition 1.3, in order to study the stability, we require that the solution of (1.1) exists globally, at least for initial data ψ_0 sufficiently close to $\mathcal{M}$. According to the results, all solutions for the nonlinear Schrödinger equation exist globally in the L^2-subcritical case. Therefore, the stability of standing waves has been studied extensively in [2, 5, 8, 29]. In the L^2-supercritical case, however, according to the local well-posedness theory, the solution of NLS with small initial data exists globally, and the solution may blow up in finite time for some large initial data. Therefore, the existence of stable standing waves in this case is of particular interest. Meanwhile, this type of problems have been considered in [18, 19, 35] by studying the corresponding minimization problem recently.

At this point, the nonlinear Schrödinger equation have attracted much attention. When $\alpha = 0$, Li and Zhao [29] showed the existence and orbital stability of standing waves in the mass subcritical case and mass critical case. Chen and Tang [2] obtained the existence of normalized ground states. The ground states of the NLS equation with combined power-type nonlinearities was studied by Jeanjean in [19] and Soave in [35, 36]. The related content with Choquard-type nonlinearities was obtained by Feng and Chen in [9, 12]. In the case $N = 3$, $\lambda = 0$, $\alpha = 0$, $p = 2$ and $\mu = 2$, the existence and orbital stability of standing waves were proved by Cazenave and Lions in [3].

From a mathematical point of view, however, the Choquard-type equation (1.2) also stimulated a lot of interest see, e.g. [6, 7, 17, 41–43]. In the case $\lambda = 0$, Du, Gao and Yang [5] studied the existence of positive ground state in the energy subcritical and the energy critical cases, established the regularity and symmetry by the moving plane method in integral forms. Furthermore, the existence and uniqueness of positive solutions was proved by Lieb [22] and Lions [27], and the orbital stability of generalized Choquard-type equation was obtained by Wang, Sun and Lv [40].

In this paper, the study of the existence and stability of standing waves for (1.1) with $\alpha > 0$ in the energy space H^1 is of particular interest, in which the time of existence only depends on the H^1-norm of initial data. Therefore, by the Gagliardo-Nirenberg inequality and the concentration compactness principle in the study of orbital stability of standing waves, see, e.g. [3, 14, 15, 17, 26, 44], we can obtain the following main results:

In the mass subcritical case, i.e., $2 - \frac{2\alpha+\mu}{N} < p < \frac{2+2N-2\alpha-\mu}{N}$ and $c > 0$, or in the mass critical case, i.e., $p = \frac{2+2N-2\alpha-\mu}{N}$ and $0 < c < \|Q_p\|_{L^2}^2$, where Q_p is a ground state to the elliptic equation

$$-\Delta Q_p + Q_p = \frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N} \frac{|Q_p|^p}{|x-y|^\mu |y|^\alpha} dy\right)|Q_p|^{p-2}Q_p, \tag{1.4}$$

it is easy for us to see that the energy functional is bounded from below on $S(c)$. In particular, for $\alpha = 0$, in view of (1.4), the Riesz potential $I_\mu : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$I_\mu(x) = \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{N-\mu}{2})\pi^{\frac{N}{2}} 2^{N-\mu}|x|^\mu} \text{ with } \Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx, \ s > 0.$$

Therefore, applying the concepts by Cazenave and Lions in [3], we consider the following constrained minimization problem

$$m(c) := \inf_{u \in S(c)} E(u). \tag{1.5}$$

However, compared with the work for the classical Schrödinger equation, there are two major difficulties in the analysis of stable standing waves. One is that the Eq (1.2) does not enjoy the scaling invariance and the space translation invariance due to the inhomogeneous nonlinearity $\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^\mu|y|^\alpha} dy\right)|u|^{p-2}u$, the other is that the nonlinear term with a convolution is difficult to handle. Therefore, the usual methods cannot work. In order to overcome these difficulties, we need to prove the boundedness of the translation sequence $\{y_n\}$, and then apply it to prove the compactness of all minimizing sequences for (1.5). Based on the result, we can obtain the existence of minimizers for the minimization problem (1.5) and the stability of standing waves.

Theorem 1.4. *Let $N \geq 3$, $0 < \mu < N$, $\lambda > 0$, $\alpha \geq 0$, $2\alpha + \mu \leq N$ and $0 < q < \frac{4}{N}$. Assume one of the following conditions hold:*

(1) $2 - \frac{2\alpha+\mu}{N} < p < \frac{2+2N-2\alpha-\mu}{N}$, $c > 0$;

(2) $p = \frac{2+2N-2\alpha-\mu}{N}$, $0 < c < \|Q_p\|_{L^2}^2$, where Q_p is the solution of Eq (1.4).

Then the set $\mathcal{M}_c := \{u \in H^1(\mathbb{R}^N) : u \in S(c),\ E(u) = m(c)\}$ is not empty and orbitally stable.

In the mass supercritical case, i.e., $\frac{2+2N-2\alpha-\mu}{N} < p < \frac{2N-2\alpha-\mu}{N-2}$, it is obvious that the energy functional is unbounded from below on $S(c)$. Indeed, if we define $u_s(x) = s^{\frac{N}{2}}u(sx)$ for $s > 0$ such that $\|u_s\|_{L^2}^2 = \|u\|_{L^2}^2 = c$, then we have

$$E(u_s) = \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda s^{\frac{Nq}{2}}}{q+2}\int_{\mathbb{R}^N} |u|^{q+2}dx - \frac{s^{Np-2N+2\alpha+\mu}}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy. \tag{1.6}$$

In view of (1.6), we can obtain that $E(u_s) \to -\infty$ as $s \to \infty$. Therefore, we cannot study the existence of stable standing waves for (1.1) by considering the global minimization problem (1.5). Applying the concepts by Jeanjean in [19], Luo and Yang in [28], we consider the following constrained local minimization problem

$$m(c) := \inf_{u \in V(c)} E(u), \tag{1.7}$$

where $V(c) := S(c) \cap B_{r_0} = \{u \in S(c) : \|\nabla u\|_{L^2}^2 < r_0\}$ for $r_0 > 0$ with $c \in (0, c_0)$, and B_{r_0} is defined by

$$B_{r_0} := \{u \in H^1(\mathbb{R}^N) : \|\nabla u\|_{L^2}^2 < r_0\}.$$

More precisely, we can obtain the property that

$$-\infty < m(c) := \inf_{u \in V(c)} E(u) < 0 \leq \inf_{u \in \partial V(c)} E(u),$$

where $\partial V(c) := \{u \in S(c) : \|\nabla u\|_{L^2}^2 = r_0\}$.

However, the energy functional of (1.3) does not keep invariant by translation due to the inhomogeneous nonlinearity $\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^\mu |y|^\alpha} dy\right)|u|^{p-2}u$. Similarly, in order to prove the compactness of all minimizing sequences for the minimization problem (1.7), we can solve it by proving the boundedness of any translation sequences. As consequence, we can obtain the existence of minimizers for the minimization problem (1.7) and the stability of standing waves.

Theorem 1.5. *Let $N \geq 3$, $0 < \mu < N$, $\lambda > 0$, $\alpha \geq 0$, $2\alpha + \mu \leq N$, $0 < q < \frac{4}{N}$ and $\frac{2+2N-2\alpha-\mu}{N} < p < \frac{2N-2\alpha-\mu}{N-2}$. Then there exists a $c_0 > 0$ with $c \in (0, c_0)$ such that the following conclusions hold:*

(1) $\emptyset \neq \mathcal{M}_c := \{u \in H^1(\mathbb{R}^N) : u \in V(c),\ E(u) = m(c)\} \subset V(c) \subset S(c)$;

(2) The set $\mathcal{M}_c$ is orbitally stable.

This paper is organized as follows. In section 2, we give some preliminaries. Next, we obtain the best constant of the Gagliardo-Nirenberg inequality (2.5). In section 3, we prove the Theorem 1.4. In section 4, we give some properties for (1.1) in the mass supercritical case. In section 5, we prove the Theorem 1.5. In section 6, we make a summary for this paper.

Notation: Throughout this paper, we use the following notation. $C > 0$ stands for a constant that may be different from line to line when it does not cause any confusion. $H^1(\mathbb{R}^N)$ is the usual Sobolev space with norm $\|u\|_{H^1} = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)dx\right)^{\frac{1}{2}}$. $L^s(\mathbb{R}^N)$ with $1 \leq s < \infty$ denotes the Lebesgue space with the norm $\|u\|_{L^s} = \left(\int_{\mathbb{R}^N} |u|^s dx\right)^{\frac{1}{s}}$. $B_R(y)$ denotes the ball in $\mathbb{R}^N$ centered at y with radius R.

2. Preliminaries

In this section, we we will collect some preliminaries, and then we obtain the best constant of the Gagliardo-Nirenberg inequality (2.6).

Lemma 2.1. *([13]) Let $N \geq 3$, $0 < \mu < N$, $\lambda > 0$, $\alpha \geq 0$, $2\alpha + \mu \leq N$, $0 < q < \frac{4}{N}$, $2 - \frac{2\alpha+\mu}{N} < p < \frac{2N-2\alpha-\mu}{N-2}$ and $\psi_0 \in H^1(\mathbb{R}^N)$. Then, there exists $T = T(\|\psi_0\|_{H^1})$ such that (1.1) admits a unique solution $\psi \in C\left([0, T), H^1\right)$. Let $[0, T^*)$ be the maximal time interval on which the solution $\psi(t)$ is well-defined, if $T^* < \infty$, then $\lim_{t \to T^*} \|\psi(t)\|_{H^1} = \infty$. Moreover, there are conservations of mass and energy,*

$$\|\psi(t)\|_{L^2}^2 = \|\psi_0\|_{L^2}^2, \quad E(\psi(t)) = E(\psi_0),$$

for all $0 \leq t < T^*$.

Next, we can establish the following Gagliardo-Nirenberg inequality related to (1.2) and the concentration compactness principle.

Lemma 2.2. *([39]) Let $N \geq 3$ and $0 < q < \frac{4}{N-2}$, then the following sharp Gagliardo-Nirenberg inequality*

$$\|u\|_{L^{q+2}}^{q+2} \leq C(q)\|u\|_{L^2}^{q+2-\frac{Nq}{2}}\|\nabla u\|_{L^2}^{\frac{Nq}{2}},$$

holds for any $u \in H^1(\mathbb{R}^N)$. The sharp constant $C(q)$ is

$$C(q) = \frac{2q+4}{4+2q-Nq}\left(\frac{4-Nq+2q)}{Nq}\right)^{\frac{Nq}{4}}\frac{1}{\|Q_q\|_{L^2}^q},$$

where Q_q is a ground state solution of the elliptic equation $-\Delta Q_q + Q_q = |Q_q|^q Q_q$.

In particular, in the L^2-critical case, i.e., $q = \frac{4}{N}$, $C(q) = \frac{q+2}{2\|Q_q\|_{L^2}^q}$.

Lemma 2.3. *([4]) Let $N \geq 3$ and $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying:*

$$\int_{\mathbb{R}^N}|u_n|^2dx = \lambda,$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ satisfying one of the three possibilities:

(i) (Compactness) There exists $\{y_{n_k}\}_{k=1}^{\infty} \subset \mathbb{R}^N$ such that $u_{n_k}(\cdot - y_{n_k}) \to u$ as $k \to \infty$ in $L^2(\mathbb{R}^N)$, namely

$$\forall\ \varepsilon > 0,\ \exists R > 0,\ \int_{B_R(y_{n_k})}|u_{n_k}(x)|^2dx \geq \lambda - \varepsilon;$$

(ii) (Vanishing)

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_{n_k}(x)|^2dx = 0 \text{ for all } R < \infty;$$

(iii) (Dichotomy) There exists $\sigma \in (0,\lambda)$ and $u_{n_k}^{(1)}$, $u_{n_k}^{(2)}$ bounded in $H^1(\mathbb{R}^N)$ such that:

$$\begin{cases} |u_{n_k}^{(1)}| + |u_{n_k}^{(2)}| \leq |u_{n_k}|, \\ \text{Supp } u_{n_k}^{(1)} \cap \text{Supp } u_{n_k}^{(2)} = \emptyset, \\ \|u_{n_k}^{(1)}\|_{H^1} + \|u_{n_k}^{(2)}\|_{H^1} \leq C\|u_{n_k}\|_{H^1}, \\ \|u_{n_k}^{(1)}\|_{L^2}^2 \to \sigma,\ \|u_{n_k}^{(2)}\|_{L^2}^2 \to \lambda - \sigma,\ \text{as } k \to \infty, \\ \liminf\limits_{k\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_{n_k}|^2 - |\nabla u_{n_k}^{(1)}|^2 - |\nabla u_{n_k}^{(2)}|^2\right)dx \geq 0, \\ \|u_{n_k} - (u_{n_k}^{(1)} + u_{n_k}^{(2)})\|_{L^s} \to 0 \text{ as } k \to 0 \text{ for all } 2 \leq s < \frac{2N}{N-2}\ (2 \leq s < \infty \text{ if } N = 1). \end{cases} \tag{2.1}$$

Lemma 2.4. *(Hardy-Littlewood-Sobolev inequality [23]) Let $N \geq 3$, $p > 1$, $r > 1$, $0 < \mu < N$, $\alpha \geq 0$, $2\alpha + \mu \leq N$, $u \in L^p(\mathbb{R}^N)$ and $v \in L^r(\mathbb{R}^N)$. Then, there exists a constant $C(\alpha,\mu,N,p,r)$, independent of u, v, such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u(x)v(y)}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy\right| \leq C(\alpha,\mu,N,p,r)\|u\|_{L^p}\|v\|_{L^r}, \tag{2.2}$$

where

$$\frac{1}{p} + \frac{1}{r} + \frac{2\alpha+\mu}{N} = 2.$$

Remark 2.5. *(1) By the Lemma 2.4, we know that* $|x|^{-\mu} * v \in L^{\frac{Ns}{N-(N-\mu)s}}(\mathbb{R}^N)$ *for* $v \in L^s(\mathbb{R}^N)$ *with* $s \in (1, \frac{N}{N-\mu})$ *and*

$$\int_{\mathbb{R}^N} \left||x|^{-\mu} * v\right|^{\frac{Ns}{N-(N-\mu)s}} dx \le C\left(\int_{\mathbb{R}^N} |v|^s dx\right)^{\frac{N}{N-(N-\mu)s}},$$

where $C > 0$ *is a constant depending only on* N, α, μ *and* s.

(2) By the Lemma 2.4 and the Sobolev embedding theorem, we can obtain that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha} dxdy \le C\left(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{2N-2\alpha-\mu}} dx\right)^{2-\frac{2\alpha+\mu}{N}} \le C\|u\|^{2p}_{H^1(\mathbb{R}^N)}, \tag{2.3}$$

for $p \in [2 - \frac{2\alpha+\mu}{N}, \frac{2N-2\alpha-\mu}{N-2}]$ *if* $N \ge 3$ *and* $p \in [2 - \frac{2\alpha+\mu}{N}, +\infty]$ *if* $N = 1, 2$, *where* $C > 0$ *is a constant depending only on* N, α, μ *and* p.

By applying the idea of M.Weinstein [37], the best constant for the generalized Gagliardo-Nirenberg inequality (2.6) can be obtained by considering the existence of the following functional

$$J_{\alpha,\mu,p}(u) = \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{Np-2N+2\alpha+\mu}{2}} \left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{2p-Np+2N-2\alpha-\mu}{2}}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha} dxdy}. \tag{2.4}$$

More precisely, we can obtain the following theorem.

Theorem 2.6. *Let* $N \ge 3$, $0 < \mu < N$, $\alpha \ge 0$, $2\alpha + \mu \le N$ *and* $2 - \frac{2\alpha+\mu}{N} < p < \frac{2N-2\alpha-\mu}{N-2}$, *then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha} dxdy \le C_{\alpha,\mu,p}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{Np-2N+2\alpha+\mu}{2}} \left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{2p-Np+2N-2\alpha-\mu}{2}}. \tag{2.5}$$

The best constant in the generalized Gagliardo-Nirenberg inequality is defined by

$$C_{\alpha,\mu,p} = \frac{2p}{2p-Np+2N-2\alpha-\mu}\left(\frac{2p-Np+2N-2\alpha-\mu}{Np-2N+2\alpha+\mu}\right)^{\frac{Np-2N+2\alpha+\mu}{2}} \|Q_p\|^{2-2p}_{L^2},$$

where Q_p *is a ground state solution of the elliptic equation* (1.4).

In particular, in the L^2*-critical case, i.e.,* $p = \frac{2+2N-2\alpha-\mu}{N}$, $C_{\alpha,\mu,p} = p\|Q_p\|^{2-2p}_{L^2}$.

Proof. To start with, by Lemma 2.4 and applying the interpolation inequality and Sobolev imbedding, we can obtain that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha} dxdy &\le C\|u\|^{2p}_{L^{\frac{2Np}{2N-2\alpha-\mu}}} \\ &\le C_{\alpha,\mu,p}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{Np-2N+2\alpha+\mu}{2}} \left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{2p-Np+2N-2\alpha-\mu}{2}}.\end{aligned} \tag{2.6}$$

Based on the above results, the functional (2.4) is well-defined. Thus, we consider a minimizing sequence $\{u_n\}$ and the following variational problem

$$J := inf\left\{J_{\alpha,\mu,p}(u),\ u \in H^1(\mathbb{R}^N) \setminus \{0\}\right\}. \tag{2.7}$$

By the Gagliardo-Nirenberg inequality, we have $J > 0$. Similarly, we set a minimizing sequence $\{v_n\}_{n=1}^{\infty}$, which is defined by $v_n(x) = \mu_n u_n(\lambda_n x)$ with

$$\lambda_n = \frac{\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}} \text{ and } \mu_n = \frac{\|u_n\|_{L^2}^{\frac{N-2}{2}}}{\|\nabla u_n\|_{L^2}^{\frac{N}{2}}},$$

so that $\|v_n\|_{L^2} = \|\nabla v_n\|_{L^2} = 1$ and $J(v_n) = J(u_n) \to J > 0$ as $n \to \infty$.

By the Schwarz symmetrization properties, namely

$$\int_{\mathbb{R}^N} |v^*|^p dx = \int_{\mathbb{R}^N} |v|^p dx \text{ and } \int_{\mathbb{R}^N} |\nabla v^*|^2 dx \le \int_{\mathbb{R}^N} |\nabla v|^2 dx,$$

we may assume that v_n is spherically symmetric and satisfies $\|v_n^*\|_{H^1} \le \|v_n\|_{H^1}$. Consequently, there exists a subsequence, which we still denote by $\{v_n\}_{n=1}^{\infty}$, and $v \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n \to v$ in $L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N)$. Since $\|v\|_{L^2} = \lim_{n\to\infty} \|v_n\|_{L^2}$, it implies that

$$\|v\|_{L^2} = \|\nabla v\|_{L^2} = 1 \text{ and } J(v) = J.$$

On the basis of the standard variational principle, if $w \in H^1(\mathbb{R}^N)$, we have

$$\frac{dJ_{\alpha,\mu,p}(v + tw)}{dt}\Big|_{t=0} = 0.$$

Then, we can obtain that v satisfies the following elliptic equation

$$-\frac{Np - 2N + 2\alpha + \mu}{2}\Delta v + \frac{2p - Np + 2N - 2\alpha - \mu}{2} v = pJ\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N} \frac{|v|^p}{|x-y|^\mu |y|^\alpha} dy\right)|v|^{p-2}v.$$

Now, we define $v(x) = au(bx)$ with $b = \left(\frac{2p-Np+2N-2\alpha-\mu}{Np-2N+2\alpha+\mu}\right)^{\frac{1}{2}}$ and $a = \left(\frac{(2p-Np+2N-2\alpha-\mu)^{\frac{N-\mu}{2}+1}}{2pJ(Np-2N+2\alpha+\mu)^{\frac{N-\mu}{2}}}\right)^{\frac{1}{2p-2}}$, so that u is a solution of (1.4) and $J(u) = J(v) = J$.

Then, we can establish the following Pohozaev identity (see Lemma 3.1 in [5]) related to (1.4). Multiplying (1.4) by Q_p and by $x \cdot \nabla Q_p$, and integrating by parts, we have

$$\int_{\mathbb{R}^N} |\nabla Q_p|^2 dx + \int_{\mathbb{R}^N} |Q_p|^2 dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Q_p|^p |Q_p|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy, \tag{2.8}$$

and

$$(N-2)\int_{\mathbb{R}^N} |\nabla Q_p|^2 dx + N\int_{\mathbb{R}^N} |Q_p|^2 dx = \frac{2N - 2\alpha - \mu}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Q_p|^p |Q_p|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy. \tag{2.9}$$

From these identities, we can get the following relations

$$\|\nabla Q_p\|_{L^2}^2 = \frac{Np - 2N + 2\alpha + \mu}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Q_p|^p |Q_p|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy = \frac{2N - Np - 2\alpha - \mu}{Np - 2p - 2N + 2\alpha + \mu}\|Q_p\|_{L^2}^2, \tag{2.10}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Q_p|^p |Q_p|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy = \frac{2p}{2p - Np + 2N - 2\alpha - \mu}\|Q_p\|_{L^2}^2. \tag{2.11}$$

Having said all of above, we derive the best constant

$$C_{\alpha,\mu,p} = \frac{1}{J} = \frac{2p}{2p - Np + 2N - 2\alpha - \mu}\left(\frac{2p - Np + 2N - 2\alpha - \mu}{Np - 2N + 2\alpha + \mu}\right)^{\frac{Np-2N+2\alpha+\mu}{2}} \|Q_p\|_{L^2}^{2-2p}.$$

□

Theorem 2.7. *Assume $N \geq 3$, $0 < \mu < N$, $\alpha \geq 0$, $2\alpha+\mu \leq N$ and $u \in H^1(\mathbb{R}^N)$ is a ground state solution of the elliptic equation* (1.4)*.*

If $p \geq \frac{2N-2\alpha-\mu}{N-2}$ or $p \leq 2 - \frac{2\alpha+\mu}{N}$, then the equation has no nontrivial solution.

Proof. Once we have the Theorem 2.6, combined with (2.8) and (2.9), we can obtain that

$$\left((N-2) - \frac{2N-2\alpha-\mu}{p}\right)\int_{\mathbb{R}^N} |\nabla u|^2 dx + \left(N - \frac{2N-2\alpha-\mu}{p}\right)\int_{\mathbb{R}^N} |u|^2 dx = 0.$$

If $p \geq \frac{2N-2\alpha-\mu}{N-2}$ or $p \leq 2 - \frac{2\alpha+\mu}{N}$, then $u \equiv 0$. The conclusion was arrived. □

3. Proof of Theorem 1.4

In this section, we prove the Theorem 1.4 in seven steps.

Step 1. We prove that the minimization problem (1.5) is well-defined and every minimization sequence of (1.5) is bounded in $H^1(\mathbb{R}^N)$. By the definition of $E(u)$ and applying the Lemma 2.2 and the Young inequality, see, e.g. [29, 39], we have

$$E(u) \geq \left(\frac{1}{2} - \varepsilon\right)\|\nabla u\|_{L^2}^2 - \delta_1(\varepsilon, \|u\|_{L^2}) - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha} dxdy. \tag{3.1}$$

for any $\varepsilon > 0$ and $u \in S(c)$.

In the case $2 - \frac{2\alpha+\mu}{N} < p < \frac{2+2N-2\alpha-\mu}{N}$, we have $0 < Np - 2N + 2\alpha + \mu < 2$, which implies that

$$\begin{aligned} E(u) &\geq \left(\frac{1}{2} - \varepsilon\right)\|\nabla u\|_{L^2}^2 - \delta_1(\varepsilon, \|u\|_{L^2}) - \varepsilon\|\nabla u\|_{L^2}^2 - \delta_2(\varepsilon, \|u\|_{L^2}) \\ &= \left(\frac{1}{2} - 2\varepsilon\right)\|\nabla u\|_{L^2}^2 - \delta_3(\varepsilon, \|u\|_{L^2}) \geq -\delta_3(\varepsilon, \|u\|_{L^2}) > -\infty. \end{aligned}$$

In the case $p = \frac{2+2N-2\alpha-\mu}{N}$, we have $Np - 2N + 2\alpha + \mu = 2$, $2p - Np + 2N - 2\alpha - \mu \leq \frac{4}{N}$, $\frac{1}{2p} \leq \frac{N}{2N+4}$ and $C_{\alpha,\mu,p} \leq \frac{N+2}{N}\|Q_p\|_{L^2}^{-\frac{4}{N}}$. By (3.1) and $\|u\|_{L^2} < \|Q_p\|_{L^2}$, it follows from the Theorem 2.6 that

$$\begin{aligned} E(u) &\geq \left(\frac{1}{2} - \varepsilon\right)\|\nabla u\|_{L^2}^2 - \delta_1(\varepsilon, \|u\|_{L^2}) - \frac{1}{2}\left(\frac{\|u\|_{L^2}}{\|Q_p\|_{L^2}}\right)^{\frac{4}{N}}\|\nabla u\|_{L^2}^2 \\ &= \frac{1}{2}\left(1 - \left(\frac{\|u\|_{L^2}}{\|Q_p\|_{L^2}}\right)^{\frac{4}{N}} - 2\varepsilon\right)\|\nabla u\|_{L^2}^2 - \delta_1(\varepsilon, \|u\|_{L^2}) \geq -\delta_1(\varepsilon, \|u\|_{L^2}) > -\infty. \end{aligned}$$

Therefore, $E(u)$ has a lower bound and the variational problem (1.5) is well-defined. Moreover, it is easy for us to see that every minimization sequence of (1.5) is bounded in $H^1(\mathbb{R}^N)$.

Step 2. We do the scaling transform of the energy functional (1.3) for $s > 0$ sufficiently small. Based on the above analysis, in the case $2 - \frac{2\alpha+\mu}{N} < p < \frac{2+2N-2\alpha-\mu}{N}$ or $p = \frac{2+2N-2\alpha-\mu}{N}$, in view of (1.6), we can find an $s > 0$ sufficiently small such that $E(u_s) < 0$.

Next, we choose $\{u_n\}_{n=1}^{\infty} \subset S(c)$ be a minimizing sequence bounded in $H^1(\mathbb{R}^N)$ satisfying

$$\|u_n\|_{L^2}^2 = c, \quad \lim_{n\to\infty} E(u_n) = m(c).$$

Then, there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that one of the three possibilities in Lemma 2.3 holds.

Step 3. We prove that the vanishing case in Lemma 2.3 does not occur. If not, by Lion's lemma, we have $u_{n_k} \to 0$ in $L^s(\mathbb{R}^N)$ as $k \to \infty$ for all $s \in (2, \frac{2N}{N-2})$. Hence,

$$\int_{\mathbb{R}^N} |u_{n_k}|^{q+2} dx \to 0 \text{ and } \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{n_k}|^p |u_{n_k}|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy \to 0,$$

and thus,

$$\lim_{k\to\infty} E(u_{n_k}) = \lim_{k\to\infty} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx \geq 0,$$

which contradicts $m(c) < 0$. Hence, the vanishing does not occur.

Step 4. We prove that the dichotomy case in Lemma 2.3 does not occur. To begin with, we recall that

$$m(\theta\eta) \leq \theta m(\eta) \text{ for } \eta \in (0, c) \text{ and } \theta \in (1, \frac{c}{\eta}). \tag{3.2}$$

Indeed, by choosing $\{u_n\}_{n=1}^{\infty} \subset S(\eta)$ satisfying $\lim_{n\to\infty} E(u_n) = m(\eta)$, we can obtain that $\|\sqrt{\theta}u_n\|_{L^2}^2 = \theta\|u_n\|_{L^2}^2 = \theta\eta$ and

$$\begin{aligned} m(\theta\eta) &\leq \liminf_{n\to\infty} \frac{\theta}{2}\|\nabla u_n\|_{L^2}^2 - \frac{\lambda\theta^{\frac{q+2}{2}}}{q+2}\|u_n\|_{L^{q+2}}^{q+2} - \frac{\theta^p}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n|^p|u_n|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy \\ &< \liminf_{n\to\infty} \theta\left(\frac{1}{2}\|\nabla u_n\|_{L^2}^2 - \frac{\lambda}{q+2}\|u_n\|_{L^{q+2}}^{q+2} - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n|^p|u_n|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy\right) \\ &= \theta m(\eta). \end{aligned} \tag{3.3}$$

Hence, we can obtain that

$$m(c) < m(\eta) + m(c-\eta) \leq m(\eta) + m^\infty(c-\eta) \text{ for any } \eta \in (0, c), \tag{3.4}$$

where $m(0) = 0$, $m^\infty(c-\eta) = \inf_{S(c-\eta)} E^\infty(u)$ and $E^\infty(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{\lambda}{q+2}\|u\|_{L^{q+2}}^{q+2}$.

Afterwards, we suppose by contradiction that (iii) in Lemma 2.3 holds. Thus, there exist $\{u_{n_k}^{(1)}\}$ and $\{u_{n_k}^{(2)}\}$ such that

$$d_{n_k} = \text{dist}\{\text{Supp } u_{n_k}^{(1)}, \text{ Supp } u_{n_k}^{(2)}\} \to \infty,$$

and

$$\int_{\mathbb{R}^n} |u_{n_k}^{(1)}|^2 dx \to \sigma, \quad \int_{\mathbb{R}^n} |u_{n_k}^{(2)}|^2 dx \to c - \sigma,$$

as $k \to \infty$. Similar to the proof of the Brézis-Lieb Lemma [1], we know that

$$|u_n - u|^p - |u_n|^p \to |u|^p,$$

in $L^s(\mathbb{R}^N)$ for $s \in [2, \frac{2N}{N-2})$ as $n \to \infty$, which implies that

$$\int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dy - \int_{\mathbb{R}^N} \frac{|u_n - u|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dy \to \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dy,$$

in $L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ as $n \to \infty$. Hence, by some basic calculation we can obtain that

$$\int_{\mathbb{R}^N} |u|^{q+2} dx - \int_{\mathbb{R}^N} |u_{n_k}^{(1)}|^{q+2} dx = \int_{\mathbb{R}^N} |u_{n_k}^{(2)}|^{q+2} dx + 2\int_{\mathbb{R}^N} |u_{n_k}^{(1)}|^{q+2} |u_{n_k}^{(2)}|^{q+2} dx,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|x|^{-\mu} * (\frac{1}{|x|^\alpha}|u_n|^p)\right)(\frac{1}{|x|^\alpha}|u_n|^p) dx - \int_{\mathbb{R}^N} \left(|x|^{-\mu} * (\frac{1}{|x|^\alpha}|u_{n_k}^{(1)}|^p)\right)(\frac{1}{|x|^\alpha}|u_{n_k}^{(1)}|^p) dx \\
&= \int_{\mathbb{R}^N} \left(|x|^{-\mu} * (\frac{1}{|x|^\alpha}|u_{n_k}^{(2)}|^p)\right)(\frac{1}{|x|^\alpha}|u_{n_k}^{(2)}|^p) dx + 2\int_{\mathbb{R}^N} \left(|x|^{-\mu} * (\frac{1}{|x|^\alpha}|u_{n_k}^{(2)}|^p)\right)(\frac{1}{|x|^\alpha}|u_{n_k}^{(1)}|^p) dx.
\end{aligned}$$

By the Lemma 2.3, we know that Supp $u_{n_k}^{(1)} \cap$ Supp $u_{n_k}^{(2)} = \emptyset$, then

$$\int_{\mathbb{R}^N} |u_{n_k}^{(1)}|^{q+2} |u_{n_k}^{(2)}|^{q+2} dx \to 0 \text{ and } \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{n_k}^{(1)}|^p |u_{n_k}^{(2)}|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy \to 0, \text{ as } k \to \infty.$$

We consequently obtain that

$$\begin{aligned}
m(c) &= \lim_{k\to\infty} \left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx - \frac{\lambda}{q+2}\int_{\mathbb{R}^N} |u_{n_k}|^{q+2} dx - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{n_k}|^p |u_{n_k}|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy\right) \\
&\geq \liminf_{k\to\infty} \left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_{n_k}^{(1)}|^2 dx - \frac{\lambda}{q+2}\int_{\mathbb{R}^N} |u_{n_k}^{(1)}|^{q+2} dx - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{n_k}^{(1)}|^p |u_{n_k}^{(1)}|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy\right) + o_k(1) \\
&\quad + \liminf_{k\to\infty} \left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_{n_k}^{(2)}|^2 dx - \frac{\lambda}{q+2}\int_{\mathbb{R}^N} |u_{n_k}^{(2)}|^{q+2} dx - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{n_k}^{(2)}|^p |u_{n_k}^{(2)}|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy\right) \\
&\geq m(\sigma) + m^\infty(c - \sigma) + o_k(1).
\end{aligned}$$

Letting $k \to \infty$, we have $m(c) \geq m(\sigma) + m^\infty(c - \sigma)$, which is a contradiction with (3.4). Hence, the dichotomy does not occur.

Applying the concentration compactness principle of the Lemma 2.3, there exists a sequence $\{y_{n_k}\}_{k=1}^\infty$ such that

$$\int_{B_{R(\varepsilon)}(y_{n_k})} |u_{n_k}(x)|^2 dx \geq \lambda - \varepsilon. \tag{3.5}$$

If we denote $\tilde{u}_{n_k}(\cdot) = u_{n_k}(\cdot + y_{n_k})$, then there exists $\tilde{u}$ satisfying $\int_{\mathbb{R}^N} |\tilde{u}(x)|^2 dx = \lambda$, namely

$$\tilde{u}_{n_k} \rightharpoonup \tilde{u} \text{ in } H^1(\mathbb{R}^N) \text{ and } \tilde{u}_{n_k} \to \tilde{u} \text{ in } L^s(\mathbb{R}^N) \text{ for all } s \in [2, \frac{2N}{N-2}).$$

Step 5. We prove that the compactness case in Lemma 2.3 will occur. We firstly prove that the sequence $\{y_{n_k}\}_{k=1}^{\infty}$ is bounded. Indeed, if it was not true, then up to a subsequence, we assume that $|y_{n_k}| \to \infty$ as $n \to \infty$. We consequently deduce that

$$\int_{\mathbb{R}^N}\left(|x|^{-\mu} * (\frac{1}{|x|^{\alpha}}|u_{n_k}|^p)\right)\frac{1}{|x|^{\alpha}}|u_{n_k}|^p dx = \int_{\mathbb{R}^N}\left(|x+y_{n_k}|^{-\mu} * (\frac{1}{|x+y_{n_k}|^{\alpha}}|\tilde{u}_{n_k}|^p)\right)\frac{1}{|x+y_{n_k}|^{\alpha}}|\tilde{u}_{n_k}|^p dx \to 0,$$

as $k \to \infty$, which yields $m(c) \geq m^{\infty}(c)$. In fact, by the definition of $m^{\infty}(c)$, we know that $m^{\infty}(c)$ is attained by a nontrivial function v_c, which yields

$$m^{\infty}(c) = \lim_{k\to\infty} E^{\infty}(\tilde{u}_{n_k}) \geq \frac{1}{2}\|\nabla\tilde{u}\|_{L^2}^2 - \frac{\lambda}{q+2}\|\tilde{u}\|_{L^{q+2}}^{q+2}.$$

We can see that $\tilde{u}$ is a minimizer of $m^{\infty}(c)$, and then we can obtain

$$m(c) < m^{\infty}(c) - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_c|^p|v_c|^p}{|x|^{\alpha}|x-y|^{\mu}|y|^{\alpha}}dxdy < m^{\infty}(c).$$

This yields $m(c) + \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_c|^p|v_c|^p}{|x|^{\alpha}|x-y|^{\mu}|y|^{\alpha}}dxdy < m^{\infty}(c)$, which is a contradiction with $m(c) \geq m^{\infty}(c)$. Accordingly, $\{y_{n_k}\}_{k=1}^{\infty}$ is bounded, and up to subsequence, we assume that $\lim_{k\to\infty} y_{n_k} = y_0$. We consequently deduce that

$$\|u_{n_k}(x) - \tilde{u}(x-y_0)\|_{L^s} \leq \|u_{n_k}(x+y_{n_k}) - \tilde{u}(x)\|_{L^s} + \|\tilde{u}_{n_k}(x-y_{n_k}+y_0) - \tilde{u}(x)\|_{L^s} \to 0,$$

which implies $u_{n_k} \to \tilde{u}(x-y_0)$ in $L^s(\mathbb{R}^N)$ for all $s \in [2, \frac{2N}{N-2})$, namely $u(x) = \tilde{u}(x-y_0)$ and

$$m(c) = \liminf_{k\to\infty} E(u_{n_k}) \geq E(u) \geq m(c).$$

Therefore, $E(u) = m(c)$ and $u_{n_k} \to u$ in $H^1(\mathbb{R}^n)$ as $k \to \infty$. This completes the proof.

Step 6. We prove that the Cauchy problem (1.1) admits a global solution $\psi(t)$ with $\psi(0) = \psi_0$ if $2 - \frac{2\alpha+\mu}{N} < p < \frac{2+2N-2\alpha-\mu}{N}$ and $\psi_0 \in H^1(\mathbb{R}^N)$ or $p = \frac{2+2N-2\alpha-\mu}{N}$ and $\|\psi_0\|_{L^2} < \|Q_p\|_{L^2}$.

Indeed, by Lemma 2.1, we know that it suffices to bound $\|\nabla\psi(t)\|_{L^2}$ in the existence time. By Lemma 2.2, Theorem 2.6, the conversation law and the Young inequality, we have

$$\begin{aligned}\|\nabla\psi(t)\|_{L^2}^2 &= 2E(\psi(t)) + \frac{2\lambda}{q+2}\|\psi(t)\|_{L^{q+2}}^{q+2} + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi(t)|^p|\psi(t)|^p}{|x|^{\alpha}|x-y|^{\mu}|y|^{\alpha}}dxdy \\ &\leq 2E(\psi(0)) + 2\varepsilon\|\nabla\psi(t)\|_{L^2}^2 + 2\delta_1(\varepsilon, \|\psi(t)\|_{L^2}) \\ &\quad + \frac{C_{\alpha,\mu,p}}{p}\|\nabla\psi(t)\|_{L^2}^{Np-2N+2\alpha+\mu}\|\psi(t)\|_{L^2}^{2p-Np+2N-2\alpha-\mu}. \end{aligned} \tag{3.6}$$

Similar to the step 1, in the case $2 - \frac{2\alpha+\mu}{N} < p < \frac{2+2N-2\alpha-\mu}{N}$, we have

$$\|\nabla\psi(t)\|_{L^2}^2 \leq 2E(\psi(0)) + 2\varepsilon\|\nabla\psi(t)\|_{L^2}^2 + 2\delta_1(\varepsilon, \|\psi(t)\|_{L^2}) + 2\varepsilon\|\nabla\psi(t)\|_{L^2}^2 + 2\delta_2(\varepsilon, \|\psi(t)\|_{L^2}^2).$$

In the case $p = \frac{2+2N-2\alpha-\mu}{N}$, we have

$$\|\nabla\psi(t)\|_{L^2}^2 \leq 2E(\psi(0)) + 2\varepsilon\|\nabla\psi(t)\|_{L^2}^2 + 2\delta_1(\varepsilon, \|\psi(t)\|_{L^2}) + \left(\frac{\|\psi(t)\|_{L^2}}{\|Q_p\|_{L^2}}\right)^{\frac{4}{N}}\|\nabla\psi(t)\|_{L^2}^2.$$

The above argument implies the boundedness of $\|\nabla\psi(t)\|_{L^2}^2$ since $\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2} < \|Q_p\|_{L^2}$. Then we come to the conclusion.

Step 7. We prove that the set $\mathcal{M}_c$ is orbitally stable. We firstly observed that the solution ψ of (1.1) exists globally, then argue by contradiction that there exist constant $\varepsilon_0 > 0$ and a sequence $\{\psi_{0,n}\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^N)$ such that

$$\inf_{u\in\mathcal{M}_c} \|\psi_{0,n} - u\|_{H^1(\mathbb{R}^N)} < \frac{1}{n} \tag{3.7}$$

and there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ such that the corresponding solution sequence $\psi_n(t_n)$ of (1.1) satisfies

$$\sup_{t_n\in\mathbb{R}} \inf_{u\in\mathcal{M}_c} \|\psi_n(t_n) - u\|_{H^1(\mathbb{R}^N)} \geq \varepsilon_0. \tag{3.8}$$

Subsequently, we claim that there exists $v \in \mathcal{M}_c$ satisfies $\lim\limits_{n\to\infty} \|\psi_{0,n} - v\|_{H^1(\mathbb{R}^N)} = 0$. Indeed, in view of (3.7), there exists $\{v_n\}_{n=1}^{\infty} \subset S(c)$ be a minimizing sequence such that

$$\|\psi_{0,n} - v_n\|_{H^1(\mathbb{R}^N)} < \frac{2}{n}, \tag{3.9}$$

Due to $\{v_n\}_{n=1}^{\infty} \subset \mathcal{M}_c$ be a minimizing sequence, by the argument above, there exists $v \in \mathcal{M}_c$ satisfies

$$\lim_{n\to\infty} \|v_n - v\|_{H^1(\mathbb{R}^N)} = 0. \tag{3.10}$$

Thus, the claim follows from (3.9) and (3.10) immediately. Consequently,

$$\lim_{n\to\infty} \|\psi_{0,n}\|_{L^2}^2 = \|v\|_{L^2}^2 = c, \ \lim_{n\to\infty} E(\psi_{0,n}) = E(v) = m(c).$$

By the conservation of mass and energy, we have

$$\lim_{n\to\infty} \|\psi_n(t_n)\|_{L^2}^2 = c, \ \lim_{n\to\infty} E(\psi_n(t_n)) = E(v) = m(c).$$

Similarly, by the argument above, we can see that $\{\psi_n(t_n)\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N)$. Hence,

$$\begin{aligned} E(\tilde{\psi}_n) = {} & \frac{c}{\|\psi_n(t_n)\|_{L^2}^2} E(\psi_n(t_n)) + \left[\left(\frac{\sqrt{c}}{\|\psi_n(t_n)\|_{L^2}}\right)^2 - \left(\frac{\sqrt{c}}{\|\psi_n(t_n)\|_{L^2}}\right)^{q+2}\right] \frac{\lambda}{q+2} \|\psi_n(t_n)\|_{L^{q+2}}^{q+2} \\ & + \left[\left(\frac{\sqrt{c}}{\|\psi_n(t_n)\|_{L^2}}\right)^2 - \left(\frac{\sqrt{c}}{\|\psi_n(t_n)\|_{L^2}}\right)^{2p}\right] \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi_n(t_n)|^p |\psi_n(t_n)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy, \end{aligned}$$

for $\tilde{\psi}_n = \frac{\sqrt{c}\cdot\psi_n(t_n)}{\|\psi_n(t_n)\|_{L^2}}$ and $\|\tilde{\psi}_n\|_{L^2}^2 = c$. From the above results, we have

$$\lim_{n\to\infty} E(\tilde{\psi}_n) = \lim_{n\to\infty} E(\psi_n(t_n)) = m(c).$$

Hence, $\{\tilde{\psi}_n(t_n)\}_{n=1}^{\infty}$ is a minimizing sequence of (1.5). By the analysis above, there exists $\tilde{v} \in \mathcal{M}_c$ satisfies

$$\tilde{\psi}_n \to \tilde{v} \text{ in } H^1(\mathbb{R}^N). \tag{3.11}$$

By the definition of $\tilde{\psi}_n$, we know

$$\tilde{\psi}_n - \psi_n(t_n) \to 0 \text{ in } H^1(\mathbb{R}^N). \tag{3.12}$$

We consequently obtain that $\psi_n(t_n) \to \tilde{v}$ in $H^1(\mathbb{R}^N)$, and which contradicts (3.8). This completes the proof.

4. The supercritical case

By the definition of $E(u)$ and applying the Lemma 2.2 and Theorem 2.6, we have

$$\begin{aligned} E(u) &\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{\lambda C_1}{q+2}\|\nabla u\|_{L^2}^{\frac{Nq}{2}}\|u\|_{L^2}^{q+2-\frac{Nq}{2}} - \frac{C_2}{2p}\|\nabla u\|_{L^2}^{Np-2N+2\alpha+\mu}\|u\|_{L^2}^{2p-Np+2N-2\alpha-\mu} \\ &= \|\nabla u\|_{L^2}^2 f\left(\|u\|_{L^2}^2, \|\nabla u\|_{L^2}^2\right), \end{aligned} \tag{4.1}$$

where $C_1 = C(q)$, $C_2 = C_{\alpha,\mu,p}$. First of all, we define the following function of two variables, namely

$$f(c,r) = \frac{1}{2} - \frac{\lambda C_1}{q+2}\|\nabla u\|_{L^2}^{\frac{Nq}{2}-2}\|u\|_{L^2}^{q+2-\frac{Nq}{2}} - \frac{C_2}{2p}\|\nabla u\|_{L^2}^{Np-2N+2\alpha+\mu-2}\|u\|_{L^2}^{2p-Np+2N-2\alpha-\mu}.$$

Now, according to the configurations of parameters above, we note that

$$\beta_1 = \frac{Nq}{2} - 2,\ \beta_2 = q+2-\frac{Nq}{2},\ \beta_3 = Np-2N+2\alpha+\mu-2,\ \beta_4 = 2p-Np+2N-2\alpha-\mu,$$

And then, substitute the notation into the function, we have

$$g_c(r) := f(c,r) = \frac{1}{2} - \frac{\lambda C_1}{q+2}r^{\frac{\beta_1}{2}}c^{\frac{\beta_2}{2}} - \frac{C_2}{2p}r^{\frac{\beta_3}{2}}c^{\frac{\beta_4}{2}} \text{ for } (c,r) \in (0,\infty)\times(0,\infty).$$

In the L^2-supercritical case, however, we notice that if $N \geq 3$, $0 < \mu < N$, $\alpha \geq 0$, $2\alpha+\mu \leq N$, $0 < q < \frac{4}{N}$ and $\frac{2+2N-2\alpha-\mu}{N} < p < \frac{2N-2\alpha-\mu}{N-2}$, then

$$\beta_1 \in (-2,0),\ \beta_2 \in (\frac{4}{N},2),\ \beta_3 \in (0,\frac{4}{N-2}),\ \beta_4 \in (0,\frac{4}{N}).$$

Lemma 4.1. *The function $g_c(r)$ has a unique global maximum and the maximum value satisfies*

$$\begin{cases} \max\limits_{r>0} g_c(r) > 0 \ \ if \ \ c < c_0, \\ \max\limits_{r>0} g_c(r) = 0 \ \ if \ \ c = c_0, \\ \max\limits_{r>0} g_c(r) < 0 \ \ if \ \ c > c_0, \end{cases}$$

where

$$c_0 := \left(\frac{1}{2(\mathcal{A}+\mathcal{B})}\right)^{\frac{N}{2}} > 0, \tag{4.2}$$

with

$$\mathcal{A} = \frac{\lambda C_1}{q+2}\left(-\frac{\beta_3}{\beta_1}\frac{(q+2)C_2}{2p\lambda C_1}\right)^{\frac{\beta_1}{\beta_1-\beta_3}} > 0,\ \ \mathcal{B} = \frac{C_2}{2p}\left(-\frac{\beta_3}{\beta_1}\frac{(q+2)C_2}{2p\lambda C_1}\right)^{\frac{\beta_3}{\beta_1-\beta_3}} > 0. \tag{4.3}$$

Proof. By the definition of $g_c(r)$, we can obtain by some calculation that

$${g_c}'(r) = -\frac{\beta_1}{2}\frac{\lambda C_1}{q+2}r^{\frac{\beta_1}{2}-1}c^{\frac{\beta_2}{2}} - \frac{\beta_3}{2}\frac{C_2}{2p}r^{\frac{\beta_3}{2}-1}c^{\frac{\beta_4}{2}}.$$

Hence, there has unique solution of the equation $g_c'(r) = 0$, namely

$$r_c = \left(-\frac{\beta_3}{\beta_1}\frac{(q+2)C_2}{2p\lambda C_1}\right)^{\frac{2}{\beta_1-\beta_3}} c^{\frac{\beta_4-\beta_2}{\beta_1-\beta_3}}. \tag{4.4}$$

Moreover, considering in the analysis of $g_c(r)$ we know that $g_c(r) \to -\infty$ as $r \to 0$ and $g_c(r) \to -\infty$ as $r \to +\infty$. Therefore, we can deduce that r_c is the unique global maximum point of $g_c(r)$, namely

$$\begin{aligned}\max_{r>0} g_c(r) &= \frac{1}{2} - \frac{\lambda C_1}{q+2}\left(-\frac{\beta_3}{\beta_1}\frac{(q+2)C_2}{2p\lambda C_1}\right)^{\frac{\beta_1}{\beta_1-\beta_3}} c^{\frac{\beta_1\beta_4-\beta_1\beta_2}{2(\beta_1-\beta_3)}} c^{\frac{\beta_2}{2}} \\ &\quad - \frac{C_2}{2p}\left(-\frac{\beta_3}{\beta_1}\frac{(q+2)C_2}{2p\lambda C_1}\right)^{\frac{\beta_3}{\beta_1-\beta_3}} c^{\frac{\beta_3\beta_4-\beta_2\beta_3}{2(\beta_1-\beta_3)}} c^{\frac{\beta_4}{2}} \\ &= \frac{1}{2} - (\mathcal{A}+\mathcal{B})c^{\frac{\beta_1\beta_4-\beta_2\beta_3}{2(\beta_1-\beta_3)}}.\end{aligned}$$

In view of (4.2), we can obtain that $\max_{r>0} g_{c_0}(r_{c_0}) = 0$, and hence the lemma follows. □

Lemma 4.2. *Let $f(c_1, r_1) \geq 0$ for $(c_1, r_1) \in (0,\infty) \times (0,\infty)$. Then for any $c_2 \in (0, c_1]$, we have that*

$$f(c_2, r_2) \geq 0 \ \text{ if } \ r_2 \in [\frac{c_2}{c_1} r_1, r_1].$$

Proof. It is shown that $c \mapsto f(\cdot, r)$ is a non-increasing function, and then we have

$$f(c_2, r_1) \geq f(c_1, r_1) \geq 0. \tag{4.5}$$

By some basic calculations, $\beta_1 + \beta_2 = q > 0$, and taking into account we have

$$f(c_2, \frac{c_2}{c_1} r_1) \geq f(c_1, r_1) \geq 0. \tag{4.6}$$

Moreover, we observe that if $g_{c_2}(r') \geq 0$ and $g_{c_2}(r'') \geq 0$ then

$$f(c_2, r) = g_{c_2}(r) \geq 0 \ \text{ for any } r \in [r', r'']. \tag{4.7}$$

Indeed, there exists a local minimum point on (r', r'') when $g_{c_2}(r) < 0$ for $r \in [r', r'']$, and which contradicts to the fact that $g_{c_2}(r)$ has unique critical point with global maximum (see Lemma 4.1). By (4.5) and (4.6), we can choose $r' = \frac{c_2}{c_1} r_1$ and $r'' = r_1$, and hence the lemma follows. □

By the Lemmas 4.1 and 4.2, we can obtain that $f(c_0, r_0) = 0$ and $f(c, r_0) \geq 0$ for all $c \in (0, c_0)$ and $r_0 := r_{c_0} > 0$. According to the above results, we have the following lemma.

Lemma 4.3. *The map $c \in (0, c_0) \mapsto m(c)$ is continuous.*

Proof. Firstly, we know that the sequence $\{c_n\} \subset (0, c_0)$ satisfies $c_n \to c$. By the definition of $m(c_n)$, we can obtain that there exists $u_n \in V(c_n)$ such that $E(u_n) < 0$ and

$$E(u_n) \leq m(c_n) + \varepsilon \ \text{ for all } \ \varepsilon > 0 \text{ small enough.} \tag{4.8}$$

Next, we denote $z_n := \sqrt{\frac{c}{c_n}}u_n \in V(c) \subset S(c)$, then

$$\|\nabla z_n\|_{L^2}^2 \leq \|\nabla u_n\|_{L^2}^2 < r_0 \ \text{ for } \ c_n \geq c.$$

Instead, by Lemma 4.2, then we have $f(c_n, r) \geq 0$ for $r \in [\frac{c_n}{c} r_0, r_0]$ and $c_n < c$. Therefore, in view of (4.1) and (4.8), we have $f(c_n, \|\nabla u_n\|_{L^2}^2) < 0$ and

$$\|\nabla z_n\|_{L^2}^2 < \frac{c}{c_n}\frac{c_n}{c} r_0 = r_0 \ \text{ with } \ \|\nabla u_n\|_{L^2}^2 < \frac{c_n}{c} r_0.$$

As mentioned above, by the definition of z_n, we can obtain that

$$\begin{aligned} E(z_n) - E(u_n) = & \frac{1}{2}(\frac{c}{c_n} - 1)\|\nabla u_n\|_{L^2}^2 - \frac{\lambda}{q+2}\left[(\frac{c}{c_n})^{\frac{q+2}{2}} - 1\right]\|u_n\|_{L^{q+2}}^{q+2} \\ & - \frac{1}{2p}\left[(\frac{c}{c_n})^p - 1\right]\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy, \end{aligned}$$

and then, we can write it as

$$m(c) \leq E(u_n) + [E(z_n) - E(u_n)].$$

At this point, by the definition of $V(c)$, we can obtain that $\|\nabla u_n\|_{L^2}^2 < r_0$ for $u \in V(c)$. Moreover, we also know that $\|u_n\|_{L^{q+2}}^{q+2}$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u|^p|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy$ are uniformly bounded, then

$$m(c) \leq E(u_n) + o_n(1) \text{ as } n \to \infty. \tag{4.9}$$

In view of (4.8) and (4.9), we have $m(c) \leq m(c_n) + \varepsilon + o_n(1)$, then there exists $u \in V(c)$ such that $E(u) < 0$ and

$$E(u) \leq m(c) + \varepsilon \text{ for all } \varepsilon > 0 \text{ small enough}.$$

Similar to the argument above, we denote $u_n := \sqrt{\frac{c_n}{c}}u \in V(c_n) \subseteq S(c_n)$, by the fact that $c_n \to c$ and $E(u_n) \to E(u)$ for $u_n \in V(c_n)$, then

$$m(c_n) \leq E(u) + [E(u_n) - E(u)] \leq m(c) + \varepsilon + o_n(1). \tag{4.10}$$

Therefore, we conclude that $m(c_n) \to m(c)$ for all $\varepsilon > 0$ small enough, and hence the lemma follows. $\square$

Lemma 4.4. *Let $\{v_n\}_{n=1}^\infty \subset B_{r_0}$ be such that $\|v_n\|_{L^{q+2}} \to 0$. Then, there exist a constant $\gamma_0 > 0$ such that*

$$E(v_n) \geq \gamma_0\|\nabla v_n\|_{L^2}^2 + o_n(1).$$

Proof. As a matter of fact, by the Theorem 2.6 we obtain that

$$\begin{aligned} E(v_n) & \geq \frac{1}{2}\|\nabla v_n\|_{L^2}^2 - \frac{C_2}{2p}\|\nabla v\|_{L^2}^{Np-2N+2\alpha+\mu}\|v\|_{L^2}^{2p-Np+2N-2\alpha-\mu} + o_n(1) \\ & \geq \|\nabla v_n\|_{L^2}^2\left(\frac{1}{2} - \frac{C_2}{2p} r_0^{\frac{\beta_3}{2}} c_0^{\frac{\beta_4}{2}}\right) + o_n(1). \end{aligned}$$

Hence, by the fact that $f(c_0, r_0) = 0$, we have that

$$\gamma_0 := \left(\frac{1}{2} - \frac{C_2}{2p} r_0^{\frac{\beta_3}{2}} c_0^{\frac{\beta_4}{2}}\right) = \frac{\lambda C_1}{q+2} r_0^{\frac{\beta_1}{2}} c_0^{\frac{\beta_2}{2}} > 0.$$

$\square$

5. Proof of Theorem 1.5

In this section, we prove the Theorem 1.5 in seven steps.

Step 1. We prove that the minimization problem (1.7) is well-defined. First of all, we have $\|\nabla u\|_{L^2}^2 = r_0$ for all $u \in \partial V(c)$. Then, in view of (4.1), we can get

$$E(u) \geq \|\nabla u\|_{L^2}^2 f(\|u\|_{L^2}^2, \|\nabla u\|_{L^2}^2) = r_0 f(c, r_0) \geq r_0 f(c_0, r_0) = 0,$$

Similarly, in view of (1.6), we can get

$$\phi_u(s) := E(u_s) < 0 \text{ for all } s > 0 \text{ small enough.}$$

As mentioned above, we obtain that

$$-\infty < m(c) := \inf_{u \in V(c)} E(u) < 0 \leq \inf_{u \in \partial V(c)} E(u). \tag{5.1}$$

Therefore, $E(u)$ has a lower bound and the variational problem (1.7) is well-defined.

Step 2. We prove that the ground state is local minimizer of $E(u)$ contained in $V(c)$ when $m(c)$ is reached. Firstly, we assume that u is a critical point of $E(u)$, its restriction $u \in S(c)$ belong to the set

$$Q_c := \{u \in S(c) : Q(u) = 0\},$$

where

$$Q(u) = \|\nabla u\|_{L^2}^2 - \frac{\lambda N q}{2(q+2)} \|u\|_{L^{q+2}}^{q+2} - \frac{Np - 2N + 2\alpha + \mu}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u|^p |u|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dx dy.$$

Moreover, by some basic calculations we have the derivative of ϕ_v, namely

$$\phi_v'(s) = \frac{d}{ds} E(v_s) = \frac{1}{s} Q(v_s). \tag{5.2}$$

Similarly, we observe the fact that if $\|\nabla v\|_{L^2} = 1$ with $v \in S(c)$ so that $u = v_s$ with $u \in S(c)$ for $s \in (0, \infty)$.

As a matter of fact, the ground states is contained in the set Q_c. In view of (5.2), if $w \in Q_c$ and $v \in S(c)$ satisfies $\|\nabla v\|_{L^2} = 1$, so that $w = v_{s_0}$, $E(w) = E(v_{s_0})$ and $\frac{d}{ds} E(v_s)(s_0) = 0$ for $s_0 \in (0, \infty)$. Just by the properties of derivatives, $s_0 \in (0, \infty)$ is a zero of the function $\phi_v'(s)$.

By the definition of $\partial V(c)$, however, when $v_s \in \partial V(c)$ we can easily acquire that $\phi_v(s) = E(v_s) \geq 0$ and

$$\phi_v(s) \to 0^-, \ \ \|\nabla v_s\|_{L^2} \to 0 \ \text{ as } \ s \to 0.$$

Hence, $s_1 > 0$ is the first zero of $\frac{d}{ds} E(v_s)$, and it is the local minima satisfying $E(v_{s_1}) < 0$ for $v_{s_1} \in V(c)$.

On the other hand, when $v_s \in \partial V(c)$ we also have $E(v_{s_1}) < 0$, $E(v_s) \geq 0$ and

$$E(v_{s_1}) \to -\infty \ \text{ as } \ s \to +\infty$$

Hence, $s_2 > s_1$ is the second zero of $E(v_s)$, and it is the local maxima satisfying $E(v_{s_2}) \geq 0$ and $m(c) \leq E(v_{s_1}) < E(v_{s_2})$. In particular, v_{s_2} cannot be a ground state if $m(c)$ is reached.

To sum up, ϕ_v' has at most two zeros, that is equal to the function $s \mapsto \frac{\phi_u'(s)}{s}$ has at most two zeros, which yields that $s_0 = s_1$ and $\omega = v_{s_0} = v_{s_1} \in V(c)$. According to the basic calculations, we obtain that

$$h(s) := \frac{\phi_u'(s)}{s} = \|\nabla u\|_{L^2}^2 - \frac{\lambda Nq}{2(q+2)} s^{\beta_1} \|u\|_{L^{q+2}}^{q+2} - \frac{Np - 2N + 2\alpha + \mu}{2p} s^{\beta_3} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u|^p |u|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy,$$

$$h'(s) = -\beta_1 \frac{\lambda Nq}{2(q+2)} s^{\beta_1 - 1} \|u\|_{L^{q+2}}^{q+2} - \beta_3 \frac{Np - 2N + 2\alpha + \mu}{2p} s^{\beta_3 - 1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u|^p |u|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy.$$

From what has been discussed above, we know that $\beta_1 < 0$ and $\beta_3 > 0$, then $h'(s) = 0$ has a unique solution, and $h(s) = 0$ has indeed at most two zeros. Moreover, the solutions were local minimizer contained in $V(c)$.

Step 3. We prove that the vanishing case does not occur. If not, we assume that

$$\int_{B_R(y_n)} |u_n|^2 dx \geq \gamma_1 > 0 \text{ for } R > 0. \tag{5.3}$$

First of all, let $\{u_n\}_{n=1}^\infty \subset B_{r_0}$ is bounded in $H^1(\mathbb{R}^N)$ be such that $\|u_n\|_{L^2}^2 \to c$ and $E(u_n) \to m(c)$ for all $c \in (0, c_0)$. By Lions' lemma, we deduce that $\|u_n\|_{L^{q+2}} \to 0$ as $n \to \infty$. At this point, by the Lemma 4.4, we have that $E(u_n) \geq o_n(1)$, which is a contradiction with $m(c) < 0$.

Step 4. We prove that the dichotomy case does not occur. Indeed, similar to (3.2), we have

$$\begin{aligned} m(c) &= \frac{c-\eta}{c} m(c) + \frac{\eta}{c} m(c) \\ &= \frac{c-\eta}{c} m\left(\frac{c}{c-\eta}(c-\eta)\right) + \frac{\eta}{c} m\left(\frac{c}{\eta}\eta\right) \\ &\leq m(c-\eta) + m(\eta), \end{aligned} \tag{5.4}$$

with a strict inequality when $m(\eta)$ is reached. But in the mass supercritical case, in view of (5.1), we can obtain that there exists $u \in V(\eta)$ satisfies

$$E(u) < 0 \ \ \textit{and} \ \ E(u) \leq m(\eta) + \varepsilon \ \text{ for all } \varepsilon > 0. \tag{5.5}$$

By the Lemma 4.2, we have $f(\eta, r) \geq 0$ for $r \in \left[\frac{\eta}{c} r_0, r_0\right]$. Therefore, in view of (4.1) and (5.5), we have

$$\|\nabla u\|_{L^2}^2 < \frac{\eta}{c} r_0. \tag{5.6}$$

Similar to (3.3), we denote $v = \sqrt{\theta} u$ such that $\|v\|_{L^2}^2 = \theta \|u\|_{L^2}^2 = \theta\eta$ and $\|\nabla v\|_{L^2}^2 = \theta \|\nabla u\|_{L^2}^2 < r_0$. Thus, for $v \in V(\theta\eta)$, we can obtain that $m(\theta\eta) \leq \theta\left(m(\eta) + \varepsilon\right)$, i.e., $m(\theta\eta) \leq \theta m(\eta)$. In particular, if $m(\eta)$ is reached, then the strict inequality follows.

Step 5. We prove that the compactness case will occur. By a similar argument above, using the Lemma 2.3 and Step 5 of the proof of Theorem 1.4, we know that the sequence $\{y_n\} \subset \mathbb{R}^N$ is bounded, and up to a sequence, we assume that $y_n \to y_0$ as $n \to \infty$. We consequently deduce that

$$u_n(x - y_n) \rightharpoonup u_c \neq 0 \ \text{ in } H^1(\mathbb{R}^N).$$

First of all, we denote $w_n(x) := u_n(x - y_n) - u_c(x)$, we need to prove that the compactness holds, i.e.,

$$w_n(x) \to 0 \text{ in } H^1(\mathbb{R}^N).$$

Again, by the definition of u_n and the analysis of w_n, we can obtain that

$$\|w_n\|_{L^2}^2 = \|u_n\|_{L^2}^2 - \|u_c\|_{L^2}^2 + o_n(1) = c - \|u_c\|_{L^2}^2 + o_n(1). \tag{5.7}$$

As mentioned, we can obtain that

$$\|\nabla w_n\|_{L^2}^2 = \|\nabla u_n\|_{L^2}^2 - \|\nabla u_c\|_{L^2}^2 + o_n(1). \tag{5.8}$$

For this reason, in view of (5.7) and (5.8), we notice that any term in E fulfills the splitting properties of Brézis-Lieb [1]. Consequently,

$$E(w_n) = E(u_n(x - y_n)) - E(u_c) + o_n(1),$$

By using the fact that $\{y_n\}$ is bounded and the translation invariance holds, we have

$$E(u_n) = E(w_n) + E(u_c) + o_n(1). \tag{5.9}$$

On the one hand, in order to prove the compactness holds, we firstly prove that $\|w_n\|_{L^2}^2 \to 0$. In view of (5.7), if we note $c_1 := \|u_c\|_{L^2}^2 > 0$ so that the conclusion arrived when $c_1 = c$. Instead, if we argue by contradiction with $c_1 < c$, by the analysis of (5.7) and (5.8), we have

$$\|w_n\|_{L^2}^2 = c - c_1 + o_n(1) \le c, \quad \|\nabla w_n\|_{L^2}^2 \le \|\nabla u_n\|_{L^2}^2 < r_0.$$

While in the mass supercritical case, by the definition of w_n, we have

$$w_n \in V(\|w_n\|_{L^2}^2), \;\; E(w_n) \ge m(\|w_n\|_{L^2}^2).$$

Recalling $E(u_n) \to m(c)$ and in view of (5.9), then

$$m(c) = E(w_n) + E(u_c) + o_n(1) \ge m(\|w_n\|_{L^2}^2) + E(u_c) + o_n(1).$$

In context, by Lemma 4.3 we know that the map $c \in (0, c_0) \mapsto m(c)$ is continuous. Thus, in view of (5.7), we can deduce that $u_c \in V(c_1)$ and

$$m(c) \ge m(c - c_1) + E(u_c), \tag{5.10}$$

which implies $E(u_c) \ge m(c_1)$. For one thing, in view of (5.4) and (5.10), if $E(u_c) > m(c_1)$ then

$$m(c) > m(c - c_1) + m(c_1) \ge m(c - c_1 + c_1) = m(c).$$

It is impossible to $m(c) > m(c)$. By a process of elimination, we only have another thing that $E(u_c) = m(c_1)$, namely u_c is a local minimizer on $V(c_1)$. Similar to the argument above, if (5.4) with the strict inequality, then

$$m(c) \ge m(c - c_1) + m(c_1) > m(c - c_1 + c_1) = m(c).$$

It is impossible to $m(c) > m(c)$. Consequently, we conclude that $\|u_c\|_{L^2}^2 = c$ and $\|w_n\|_{L^2}^2 \to 0$.

On the other hand, we next prove that $\|\nabla w_n\|_{L^2}^2 \to 0$. With all that said, in view of (5.8), we can deduce that $\{w_n\}_{n=1}^{\infty} \subset B_{r_0}$ is bounded in $H^1(\mathbb{R}^N)$. By the Gagliardo-Nirenberg inequality of Lemma 2.2 and Theorem 2.6, we can obtain that $\|w_n\|_{L^{q+2}}^{q+2} \to 0$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n|^p|w_n|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy \to 0$. Consequently, by the Lemma 4.4, we have

$$E(w_n) \geq \gamma_0\|\nabla w_n\|_{L^2}^2 + o_n(1) \text{ where } \gamma_0 > 0. \tag{5.11}$$

At the end of the part, due to $u_n \rightharpoonup u_c$ in $H^1(\mathbb{R}^N)$ with $u_c \in V(c)$, in view of (5.9), we consequently deduce that $E(u_c) \geq m(c)$ and $E(w_n) \leq o_n(1)$, namely $\|\nabla w_n\|_{L^2}^2 \to 0$.

Above all, $w_n \to 0$ in $H^1(\mathbb{R}^N)$ and we come to the conclusion.

Step 6. We prove that the Cauchy problem (1.1) admits a global solution $\psi(t)$ with $\psi(0,x) = \psi_0$ if $\frac{2+2N-2\alpha-\mu}{N} < p < \frac{2N-2\alpha-\mu}{N-2}$ and $\psi_0 \in H^1(\mathbb{R}^N)$. Firstly, we denote the right hand of (5.1) by A. Since the energy $E(u)$ is the continuous function with respect to $u \in H^1(\mathbb{R}^N)$, we deduce from $E(u) = m(c) < A$ that there is a $\delta > 0$ such that $\|\psi_0 - u\|_{H^1} < \delta$ for $\psi_0 \in H^1(\mathbb{R}^N)$, and we have $E(\psi_0) < A$.

Next, we prove this by contradiction. If not, there is a $\psi_0 \in H^1(\mathbb{R}^N)$ such that $\|\psi_0 - u\|_{H^1} < \delta$ and the corresponding solution $\psi(t)$ blows up in finite time. By continuity, there is a $T_1 > 0$ such that $\|\nabla\psi(T_1)\|_{L^2}^2 > r_0$. We now consider the initial data $\tilde{\psi}_0 = \frac{\sqrt{c}\cdot\psi_0}{\|\psi_0\|_{L^2}}$. When $\delta > 0$ sufficiently small, we have

$$\tilde{\psi}_0 \in S(c) \text{ and } E(\tilde{\psi}_0) < A.$$

When $c \leq \|\psi_0\|_{L^2}^2$, we have $\|\nabla\tilde{\psi}_0\|_{L^2}^2 \leq \|\nabla\psi_0\|_{L^2}^2 < r_0$. When $c > \|\psi_0\|_{L^2}^2$, due to $0 < c < c_0$, we have $\|\nabla\tilde{\psi}_0\|_{L^2}^2 < r_0$. This implies that $\tilde{\psi}_0 \in V(c)$. Since the solution of (1.1) depends continuously on the initial data and $\|\nabla\psi(T_1)\|_{L^2}^2 > r_0$, there is a $T_2 > 0$ such that $\|\nabla\tilde{\psi}(T_2)\|_{L^2}^2 > r_0$, where $\tilde{\psi}(t)$ is the solution of (1.1) with initial data $\tilde{\psi}_0$. Consequently, we deduce from the continuity that there is a $T_3 > 0$ such that $\|\nabla\tilde{\psi}(T_3)\|_{L^2}^2 = r_0$. This implies that $\tilde{\psi}(T_3) \in \partial V(c)$. It follows that

$$A > E(\tilde{\psi}_0) = E(\tilde{\psi}(T_3)) \geq \inf_{u\in\partial V(c)} E(u) = A,$$

which is a contradiction.

Step 7. We prove that the set $\mathcal{M}_c$ is orbitally stable. We argue by contradiction, i.e., we assume that there is $\varepsilon_0 > 0$, a sequence of initial data $\{\psi_{0,n}\} \subset H^1(\mathbb{R}^N)$ and a sequence $\{t_n\} \subset \mathbb{R}$ satisfy the maximal solution $\psi_n(t)$ with $\psi_n(0) = \psi_{0,n}$ such that

$$\lim_{n\to\infty}\inf_{u\in\mathcal{M}_c}\|\psi_{0,n} - u\|_{H^1} = 0, \quad \inf_{u\in\mathcal{M}_c}\|\psi_n(t_n) - u\|_{H^1} \geq \varepsilon_0. \tag{5.12}$$

Similar to the argument of (3.10), there is a $v \in \mathcal{M}_c$ such that $\lim_{n\to\infty}\|\psi_{0,n} - v\|_{H^1} = 0$. Next, due to $v \in V(c)$, we have $\tilde{\psi}_n = \frac{\sqrt{c}\cdot\psi_n(t_n)}{\|\psi_n(t_n)\|_{L^2}} \in V(c)$ and

$$\lim_{n\to\infty} E(\tilde{\psi}_n) = \lim_{n\to\infty} E(\psi_n(t_n)) = \lim_{n\to\infty} E(\psi_{0,n}) = E(v) = m(c),$$

which implies that $\{\tilde{\psi}_n\}$ is a minimizing sequence for (1.7). Thanks to the compactness of all minimizing sequence of (1.7), there is a $\tilde{u} \in \mathcal{M}_c$ satisfies $\tilde{\psi}_n \to \tilde{u}$ in $H^1(\mathbb{R}^N)$. Moreover, by the definition of $\tilde{\psi}_n$, it follows that $\tilde{\psi}_n \to \psi_n(t_n)$ in $H^1(\mathbb{R}^N)$. Consequently, we have $\psi_n(t_n) \to \tilde{u}$ in $H^1(\mathbb{R}^N)$, which contradicts to (5.12). This completes the proof.

6. Conclusions

In this work, we study the stability of set of energy minimizers in the mass subcritical, mass critical and mass supercritical cases. Due to appearance of the inhomogeneous nonlinearity $\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|x-y|^\mu|y|^\alpha}dy\right)|u|^{p-2}u$, the non-vanishing of any minimizing sequence is hard to exclude. By a rather delicate analysis, we can overcome this difficulty by proving the boundedness of any translation sequence. To the best of our knowledge, there are no any results about instability or strong instability. However, for its mathematical interest, these problems will be the object of a future investigation.

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Conflict of interest

The author declares no conflicts of interest.

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