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Research article

Almost periodic solutions of a discrete Lotka-Volterra model via exponential dichotomy theory

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Abstract: In this paper, we consider a discrete non-autonomous Lotka-Volterra model. Under some assumptions, we prove the existence of positive almost periodic solutions. Our analysis relies on the exponential dichotomy for the difference equations and the Banach fixed point theorem. Furthermore, by constructing a Lyapunov function, the exponential convergence is proved. Finally, a numerical example illustrates the effectiveness of the results.

Keywords: almost periodic solution; difference exponential dichotomy; Banach fixed point; discrete Lotka-Volterra model; exponential convergence
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1. Introduction

In recent years, using a wide variety of methods, a considerable amount of literature has been produced regarding the stability and existence of almost periodic solutions of differential equations [1–13]. Almost periodic solutions of most differential equations are vector-valued functions defined on the set of real numbers $\mathbb{R}$, but the concept of almost periodic solution, makes more sense on any additive group except $\mathbb{R}$, in mathematical modeling, discrete time models are preferred over the continuous time models. As it is directly applicable to various fields (economics, populations dynamics, species interactions, mathematical biology, ecology etc.), the theory of difference equation is much more substantiated than the corresponding theory of differential equation (see [14–20]).

Gopalsamy and Mohamad [21] established the following criteria for the existence of globally attractive positive almost periodic solutions for discrete Lotka-Volterra systems:

$$x(n+1) = \frac{\alpha(n)x(n)}{1+\beta(n)x(n)}.$$

Xia et al. [22] studied the relationship between the global quasi-uniform asymptotic stability of a difference system and the existence of almost periodic solutions. Moreover, they applied their results to a discrete Lotka-Volterra system as follows:

$$x_i(n+1) = x_i(n)\exp\left\{r_i(n) - \sum_{j=1}^{m} a_{ij}(n)x_j(n)\right\}, \quad i = 1,\ldots,m.$$

Meng [23], Niu [24] and Xue [25] et al. obtained the results of global stability, global attraction and uniform stability of almost periodic solutions of discrete Lotka-Volterra systems by constructing Lyapunov functions. To the best of our knowledge, there is no paper employing exponential dichotomy theory (for difference system) to study the existence of almost periodic solutions of discrete Lotka-Volterra systems. Different from the above mentioned works [21–25], in this paper, by using the exponential dichotomy theory for difference equations and Banach fixed point theorem, we prove the existence and uniqueness of solutions for the discrete almost periodic Lotka-Volterra model, formulated as follows:

$$x_i(n+1) = x_i(n)\exp\left\{r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right\}, \tag{1.1}$$

where $r_i(n), a_{ij}(n), b_{ij}, \tau(n)$, $i, j = 1,\ldots,N$ are the almost periodic sequences.

The rest of this paper is organized as follows. In the second section, we introduce some preconditions for the existence of almost periodic solutions. In the third section, we prove the properties of the exponential dichotomy for the difference equationsand existence of exponential stability of almost periodic solutions for the Lotka-Volterra model. In the last section, the validity of the results is proved by numerical simulations.

2. Preliminaries

For a bounded sequence f defined on $\mathbb{Z}$, we define f^- and f^+ as follows:

$$f^- = \liminf_{n\to\infty} f(n), f^+ = \limsup_{n\to\infty} f(n).$$

Taking $Z[a,b] = \{a, a+1,\ldots,b-1,b\}$, where $a, b \in \mathbb{Z}$. Similarly, we denote $Z[a,+\infty] = \{a, a+1, a+2,\ldots\}$.

Definition 2.1. *[2–4] If for $\epsilon > 0$, there is a constant $l(\epsilon) > 0$ such that in any interval of length $l(\epsilon) > 0$ there exists a number $\bar{\tau} \in \mathbb{Z}$ such that the inequality*

$$|x(n+\bar{\tau}) - x(n)| < \epsilon,$$

is satisfied for all $n \in \mathbb{Z}$. Then a sequence $x(n)$ is called to be almost periodic sequence.

Lemma 2.1. *[2–4] Let $x : \mathbb{Z} \to \mathbb{R}$ and $y : \mathbb{Z} \to \mathbb{R}$. The following holds:*
(a) If $x(n)$ is an almost periodic sequence, then it is bounded.
(b) If $x(n), y(n)$ are almost periodic sequences, $x(n) + y(n)$ and $x(n)\cdot y(n)$ are almost periodic.
(c) If $x(n)$ is an almost periodic sequence, then $X(n)$ is almost periodic if and only if $X(n)$ is bounded on $\mathbb{Z}$ where

$$X(n) = \sum_{k=0}^{n-1} x(n).$$

(d) If $x(n)$ is an almost periodic sequence and $X(\cdot)$ is defined on the value field of $x(n)$, then $X \circ x$ is almost periodic.

Consider the following almost periodic difference system:

$$x(n+1) = A(n)x(n) + f(n), \tag{2.1}$$

where $A : \mathbb{Z} \to \mathbb{Z}^{m\times m}$ is an almost periodic matrix sequence and $f : \mathbb{Z} \to \mathbb{Z}^m$ is an almost periodic vector sequence. Its linear system is

$$x(n+1) = A(n)x(n). \tag{2.2}$$

Definition 2.2. *[26] Suppose that $\Phi(n)$ is a fundamental matrix of the difference system (2.2). If there exists a projection P such that $P^2 = P$ and two positive constants K, v, such that*

$$\begin{cases} \left\|\Phi(r)P\Phi^{-1}(s+1)\right\| \le K(\frac{1}{1+v})^{r-s-1}, & r \ge s, \\ \left\|\Phi(r)(I-P)\Phi^{-1}(s+1)\right\| \le K(\frac{1}{1+v})^{s+1-r}, & s \ge r, \end{cases} \tag{2.3}$$

then the difference system (2.2) is said to admit an exponential dichotomy.

Lemma 2.2. *[26] Let $r(n) > 0$ be an almost periodic sequence on $\mathbb{Z}$ and $\inf_{n\in\mathbb{Z}} r(n) > 0$, then the linear system*

$$\Delta x(n) = -r(n)x(n)$$

admits an exponential dichotomy on $\mathbb{Z}$.

Lemma 2.3. *[27] [Chapter 7, Exercise 4] Suppose that system (2.2) admits an exponential dichotomy satisfying definition 2.2. Then system (2.2) does not admit a non-trivial bounded solution. i.e If there is a bounded solution, it's only a zero solution.*

Lemma 2.4. *[27] [Chapter 7, Theorem 7.6.5] Suppose that $\Phi(n)$ is the normal fundamental matrix of the linear system (2.2). If the system (2.2) admits exponential dichotomy with the projection P and the constants K, v, M with $\|f(n)\| \le M$. Then the system (2.1) has a unique bounded solution which can be uniquely expressed by*

$$x(n) = \sum_{k=-\infty}^{n-1} \Phi(n)P\Phi^{-1}(k+1)f(k) - \sum_{k=n}^{+\infty} \Phi(n)(I-P)\Phi^{-1}(k+1)f(k).$$

3. Main results

We consider the discrete almost periodic Lotka-Volterra system as follows:

$$x_i(n+1) = x_i(n)\exp\left\{r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right\}, \tag{3.1}$$

where $r_i(n), a_{ij}(n)$, $b_{ij}(n)$ and $\tau(n)$ are positive almost periodic sequences defined on $\mathbb{Z}$. Throughout this paper, we use the notations $\frac{e^{r_i^+}}{a_{ii}^-} = B_i$ and $B = \max\{B_i\}$. Further, we always assume that

system (3.1) satisfies the following conditions:
(H_1) $r_i^- > \sum_{j\neq i}^N a_{ij}^+ B_j + \sum_{j=1}^N b_{ij}^+ B_j$, $\frac{1}{a_{ii}^+}(r_i^- - \sum_{j\neq i}^N a_{ij}^+ B_j - \sum_{j=1}^N b_{ij}^+ B_j) = D_i$.

Set $D = \min_{1\leq n\leq n}\{D_i\}$. We further suppose that Eq (3.3) satisfies :
(H_2) $\frac{(D+B)\sum_{j=1}^N (a_{ij}^+ + b_{ij}^+)}{D^2 r_i^-} < 1$.
(H_3) $r_i^- > 1 + \frac{(D+B)\sum_{j=1}^N (a_{ij}^+ + b_{ij}^+)}{D^2}$.
Let $u_i(t) = \frac{1}{x_i(t)}$, $\dot{u}_i(t) = \frac{-1}{x_i^2(t)}\dot{x}_i(t)$, $\dot{x}_i(t) = \dot{u}_i(t)\frac{-1}{u_i^2(t)}$, then we have,

$$\dot{u}_i(t) = -r_i(t)u_i(t) + u_i(t)\left[\sum_{j=1}^N a_{ij}(t)\frac{1}{u_j(t)} + \sum_{j=1}^N b_{ij}(t)\frac{1}{u_j(t-\tau(t))}\right]. \tag{3.2}$$

then we can get the difference equation as follow:

$$\Delta u_i(n) = -r_i(n)u_i(n) + u_i(n)\left[\sum_{j=1}^N a_{ij}(n)\frac{1}{u_j(n)} + \sum_{j=1}^N b_{ij}(n)\frac{1}{u_j(n-\tau(n))}\right]. \tag{3.3}$$

Theorem 3.1. *If $(H_1) - (H_3)$ hold, then there exists a unique positive almost periodic solution $u_i(n)$ of (3.3), i.e, system (3.3) has a unique almost periodic solution $x_i(n)$.*

Theorem 3.2. *Let $x^*(n) = \{x_1^*(n), \ldots, x_N^*(n)\}$ be a positive almost periodic solution of (3.1) in the $\mathcal{B}$. If $(H_1) - (H_3)$ hold, then the solution $x(n) = \{x_1(n), \ldots, x_N(n)\}$ of (3.1) converges exponentially to $x^*(n)$ as $n \to \infty$.*

4. Proof of main results

4.1. Existence of bounded solution

We consider the almost periodic Lotka-Volterra system as follows:

$$\dot{x}_i(t) = x_i(t)[r_i(t) - \sum_{j=1}^N a_{ij}(t)x_j(t) - \sum_{j=1}^N b_{ij}(t)x_j(t-\tau(t))], \quad i = 1, \ldots, N, \tag{4.1}$$

where $x_i(t)$ denotes the density of prey species x_i at time t; $r_i(t), a_{ij}(t)$, $b_{ij}(t)$ and $\tau(t)$ are positive continuous almost periodic functions defined on $t \in (-\infty, +\infty)$.

For the purpose of convenience, we should use piecewise constant variable differential equations to obtain the discrete model. Assume that the average growth rate changes over regular time intervals in (4.1), and then we can incorporate this aspect in (4.1) and obtain the following semi-discretization modified system (4.1):

$$\frac{\dot{x}_i(t)}{x_i(t)} = r_i([t]) - \sum_{j=1}^N a_{ij}([t])x_j([t]) - \sum_{j=1}^N b_{ij}(t)x_j([t] - \tau([t])), \tag{4.2}$$

where $t \neq \ldots -2, -1, 0, 1, 2, \ldots$ and $[t]$ denotes the integer part t, $t \in (-\infty, +\infty)$.

Integrating over $n \le t < n+1, n = \ldots, -2, -1, 0, 1, 2, \ldots$, we have

$$x_i(t) = x_i(n)\exp\left\{r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right\}(t-n).$$

Let $t \to n+1$, then lead us to

$$x_i(n+1) = x_i(n)\exp\left\{r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right\}, \tag{4.3}$$

where $r_i(n), a_{ij}(n)$ and $b_{ij}(n)$ are positive almost periodic sequences defined on $\mathbb{Z}$.

Proposition 4.1. *For every solution* $\{(x_1(n), \ldots, x_N(n))^T\}$ *of (4.3), we have*

$$x_i(n) \le B_i, \quad i = 1, \ldots, N. \tag{4.4}$$

Proof: Obviously, when $n \ge n_0$ $x_i(n) > 0$. Suppose that there exists a $l_0 \in Z[n_0, +\infty)$, such that $x_i(l_0+1) \ge x_i(l_0)$, then

$$x_i(l_0+1) = x_i(l_0)\exp\left\{r_i(l_0) - \sum_{j=1}^{N} a_{ij}(l_0)x_j(l_0) - \sum_{j=1}^{N} b_{ij}(l_0)x_j(l_0-\tau(l_0))\right\} \ge x_i(l_0),$$

then we get,

$$r_i(l_0) - \sum_{j=1}^{N} a_{ij}(l_0)x_j(l_0) - \sum_{j=1}^{N} b_{ij}(l_0)x_j(l_0-\tau(l_0)) \ge 0.$$

It follows that

$$x_i(l_0) \le \frac{r_i(l_0) - \sum_{j\ne i}^{N} a_{ij}(l_0)x_j(l_0) - \sum_{j=1}^{N} b_{ij}(l_0)x_j(l_0-\tau(l_0))}{a_{ii}(l_0)} \le \frac{r_i(l_0)}{a_{ii}(l_0)} \le \frac{r_i^+}{a_{ii}^-}.$$

By the fact that $\max\limits_{x\in\mathbb{R}}(x\exp(a-bx)) = \frac{\exp(a-1)}{b}$, for $a, b > 0$. Hence,

$$\begin{aligned} x_i(l_0+1) &= x_i(l_0)\exp\left\{r_i(l_0) - \sum_{j=1}^{N} a_{ij}(l_0)x_j(l_0) - \sum_{j=1}^{N} b_{ij}(l_0)x_j(l_0-\tau(l_0))\right\} \\ &\le x_i(l_0)\exp\{r_i(l_0) - a_{ii}(l_0)x_i(l_0)\} \\ &\le x_i(l_0)\exp\{r_i^+ - a_{ii}^- x_i(l_0)\} \\ &\le \frac{\exp(r_i^+ - 1)}{a_{ii}^-} \le \frac{\exp(r_i^+)}{a_{ii}^-} = B_i. \end{aligned}$$

We claim that $x_i(n) \le B_i$, for $n \ge l_0$. By way of contradiction, we assume that there exists a $q_0 \ge l_0$, such that $x_i(q_0) > B_i$, then, $q_0 \ge l_0 + 2$. Let $x_i(\tilde{q}_0) > B_i$, where $\tilde{q}_0 \ge l_0 + 2$ is the smallest integer, then, $x_i(\tilde{q}_0 - 1) < x_i(\tilde{q}_0)$. The above argument produces that $x_i(\tilde{q}_0) \le B_i$, which is contradictory. The proof of our claim is complete.

Now, considering that $x_i(n+1) < x_i(n)$ for all $n \in Z[n_0, +\infty)$. Then $\lim_{n\to+\infty} x_i(n)$ exists, we denote $\lim_{n\to+\infty} x_i(n) = \bar{x}_i$. We claim that $\bar{x}_i \leq \frac{e^{r_i^+}}{a_{ii}^-}$. Let us assume that $\bar{x}_i > \frac{e^{r_i^+}}{a_{ii}^-}$. Taking the limit of both sides in system (3.1), gives us

$$\bar{x}_i = \bar{x}_i \exp\left\{\lim_{n\to+\infty}\left(r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right)\right\}.$$

Hence,

$$\lim_{n\to+\infty}\left(r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right) = 0.$$

However,

$$\begin{aligned}&\lim_{n\to+\infty}\left(r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right)\\ \leq\ &\lim_{n\to+\infty}(r_i(n) - a_{ii}(n)x_i(n)) \leq r_i^+ - a_{ii}^-\bar{x}_i < r_i^+ - e^{r_i^+} < 0,\end{aligned}$$

which is a contradiction. The proof is complete.

Proposition 4.2. *Assume that* $(H_1), (H_2)$ *and* (H_3) *hold. For every solution* $\{(x_1(n), \ldots, x_N(n))^T\}$ *of (3.1), we have*

$$x_i(n) \geq D_i, \quad i = 1, \ldots, N. \tag{4.5}$$

Proof: By proposition 4.1, there exists a $n^* \in \mathbb{N}$, such that

$$B_i - \varepsilon \leq x_i(n) \leq B_i + \varepsilon, \quad n \geq n^*.$$

Suppose that there exists a $k_0 \geq n^*$, such that $x_i(k_0+1) \leq x_i(k_0)$. Note that

$$\begin{aligned}x_i(n+1) =& x_i(n)\exp\left\{r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right\}\\ \geq& x_i(n)\exp\left\{r_i(n) - a_{ii}(n)x_i(n) - \sum_{j\neq i}^{N} a_{ij}(n)B_j - \sum_{j=1}^{N} b_{ij}(n)B_j\right\}\\ \geq& x_i(n)\exp\left\{r_i^- - a_{ii}^+x_i(n) - \sum_{j\neq i}^{N} a_{ij}^+B_j - \sum_{j=1}^{N} b_{ij}^+x_jB_j\right\}.\end{aligned}$$

When $n = k_0$, we have

$$x_i(k_0) \geq x_i(k_0+1) = x_i(k_0)\exp\left\{r_i^- - a_{ii}^+x_i(k_0) - \sum_{j\neq i}^{N} a_{ij}^+B_j - \sum_{j=1}^{N} b_{ij}^+B_j\right\}.$$

Then we obtain,

$$r_i^- - a_{ii}^+x_i(k_0) - \sum_{j\neq i}^{N} a_{ij}^+B_j - \sum_{j=1}^{N} b_{ij}^+B_j \leq 0.$$

Thus,

$$x_i(k_0) \geq \frac{1}{a_{ii}^+}(r_i^- - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j).$$

Then,

$$x_i(k_0+1) \geq \frac{1}{a_{ii}^+}(r_i^- - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j)\exp\left\{r_i^- - \sum_{j=1}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j\right\} \triangleq \tilde{x}_i.$$

We claim that $x_i(n) \geq \tilde{x}_i$, for $n \geq k_0$. Suppose that there exists a $p_0 \geq k_0$, such that $x_i(p_0) < \tilde{x}_i$, then $p_0 \geq k_0 + 2$. Let $x_i(\tilde{p}_0) < \tilde{x}_i$, where $\tilde{p}_0 \geq k_0 + 2$ is the smallest integer, then, $x_i(\tilde{p}_0 - 1) \geq x_i(\tilde{p}_0)$. The above argument produces that $x_i(\tilde{p}_0) \geq \tilde{x}_i$, which is a contradiction. The proof of our claim is complete.

Next, considering that $x_i(n+1) > x_i(n)$ for all $n \in Z[n_0, +\infty)$. Then $\lim_{n\to+\infty} x_i(n)$ exists, we denoted $\lim_{n\to+\infty} x_i(n) = \underline{x}_i$. we state that $\underline{x}_i \geq \frac{1}{a_{ii}^+}(r_i^- - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j) \geq D_i$. We write $\underline{x}_i < \frac{1}{a_{ii}^+}(r_i^- - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j)$. Taking the limit of both sides in system (3.1), we obtain

$$\underline{x}_i = \underline{x}_i \exp\left\{\lim_{n\to+\infty}\left(r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right)\right\}.$$

Hence, we get

$$\begin{aligned}
0 &= \lim_{n\to+\infty}\left(r_i(n) - \sum_{j=1}^{N} a_{ij}(n)x_j(n) - \sum_{j=1}^{N} b_{ij}(n)x_j(n-\tau(n))\right) \\
&\geq \lim_{n\to+\infty} r_i^- - a_{ii}^+ x_i(n) - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j \\
&\geq r_i^- - a_{ii}^+ \underline{x}_i - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j \\
&> r_i^- - \left(r_i^- - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j\right) - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j = 0,
\end{aligned}$$

which is contradictory. Since

$$\begin{aligned}
\tilde{x}_i &= \frac{1}{a_{ii}^+}(r_i^- - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j)\exp\left\{r_i^- - \sum_{j=1}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j\right\} \\
&> \frac{1}{a_{ii}^+}(r_i^- - \sum_{j\neq i}^{N} a_{ij}^+ B_j - \sum_{j=1}^{N} b_{ij}^+ B_j) = D_i,
\end{aligned}$$

then for $n \geq n_0$, we have $x_i(n) \geq D_i$.

Proof of Theorem 2.1: Let $\mathcal{B} = \{\varphi(n) = (\varphi_1(n), \varphi_2(n), \ldots, \varphi_N(n)) | \varphi_i(n) : \mathbb{Z} \to \mathbb{R}$ is almost periodic sequence$\}$. For any $\varphi \in \mathcal{B}$, we consider equation

$$\Delta u_i(n) = -r_i(n)u_i(n) + \left[\sum_{j=1}^{N} a_{ij}(n)\frac{\varphi_i(n)}{\varphi_j(n)} + \sum_{j=1}^{N} b_{ij}(n)\frac{\varphi_i(n)}{\varphi_j(n-\tau(n))}\right]. \tag{4.6}$$

Let $u^{\varphi}(n) = (u_1^{\varphi}(n), u_2^{\varphi}(n), \ldots, u_N^{\varphi}(n))^T$ be a solution of (4.6).

Since $r_i^- > 0$, then $\Delta u_i(n) = -r_i(n)u_i(n)$ admits an exponential dichotomy on $\mathbb{Z}$. By Lemma 2.4, we have a bounded solution of the form

$$u_i^{\varphi}(n) = \sum_{m=-\infty}^{n-1} \Phi_i(n)\Phi_i^{-1}(m+1)f_i(m).$$

In view of $\Delta u_i(n) = -r_i(n)u_i(n) \Longrightarrow u_i(n+1) = (1 - r_i(n))u_i(n)$, we have

$$\Phi_i(n) = (1 - r_i(n-1))(1 - r_i(n-2))\ldots(1 - r_i(m+1))\Phi_i(m+1).$$

We can get

$$u_i^{\varphi}(n) \;=\; \sum_{m=-\infty}^{n-1} \prod_{s=m+1}^{n-1} (1 - r_i(s)) \left[\sum_{j=1}^{N} a_{ij}(m)\frac{\varphi_i(m)}{\varphi_j(m)} + \sum_{j=1}^{N} b_{ij}(m)\frac{\varphi_i(m)}{\varphi_j(m-\tau(m))} \right].$$

According to Lemma 2.1, and using the almost periodicity of $\prod_{s=m+1}^{n-1}(1 - r_i(s))$, we deduce that u_i^{φ} is also almost periodic.

We define a mapping $\mathfrak{T} : \mathbf{B} \to \mathbf{B}$ by setting

$$\mathfrak{T}(\varphi(n)) = u_i^{\varphi}(n), \quad \varphi \in \mathbf{B},$$

with the norm $\|\mathfrak{T}\| = \sup_{n\in\mathbb{Z}} |\mathfrak{T}|$.

Let $\varphi(n) = (\varphi_1(n), \varphi_2(n), \ldots, \varphi_N(n)), \psi(n) = (\psi_1(n), \psi_2(n), \ldots, \psi_N(n)) \in \mathbf{B}$, we have

$$\begin{aligned}
&\|\mathfrak{T}_i(\varphi(n)) - \mathfrak{T}_i(\psi(n))\| = \sup_{n\in\mathbb{Z}} |\mathfrak{T}_i(\varphi(n)) - \mathfrak{T}_i(\psi(n))| \\
&= \sup_{n\in\mathbb{Z}} \Bigg| \sum_{m=-\infty}^{n-1} \prod_{s=m+1}^{n-1} (1 - r_i(s)) \Bigg[\sum_{j=1}^{N} a_{ij}(m) \left(\frac{\varphi_i(m)}{\varphi_j(m)} - \frac{\psi_i(m)}{\psi_j(m)} \right) \\
&+ \sum_{j=1}^{N} b_{ij}(m) \left(\frac{\varphi_i(m)}{\varphi_j(m-\tau(m))} - \frac{\psi_i(m)}{\psi_j(m-\tau(m))} \right) \Bigg] \Bigg|.
\end{aligned}$$

We observe that

$$\begin{aligned}
\frac{\varphi_i(m)}{\varphi_j(m)} - \frac{\psi_i(m)}{\psi_j(m)} &= \frac{\varphi_i(m) - \psi_i(m)}{\varphi_j(m)} + \frac{\psi_i(m)}{\varphi_j(m)} - \frac{\psi_i(m)}{\psi_j(m)} \\
&\le \left(\frac{\varphi_i(m) - \psi_i(m)}{D} \right) + \frac{\psi_i(m)\left(\psi_j(m) - \varphi_j(m)\right)}{\varphi_j(m)\psi_j(m)} \\
&\le \frac{\varphi_i(m) - \psi_i(m)}{D} + \frac{B\left[\left(\psi_j(m) - \varphi_j(m)\right)\right]}{D^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\varphi_i(m)}{\varphi_j(m-\tau(m))}-\frac{\psi_i(m)}{\psi_j(m-\tau(m))}\\
=&\frac{\varphi_i(m)-\psi_i(m)}{\varphi_j(m-\tau(m))}+\frac{\psi_i(m)}{\varphi_j(m-\tau(m))}-\frac{\psi_i(m)}{\psi_j(m-\tau(m))}\\
\le&\left(\frac{\varphi_i(m)-\psi_i(m)}{D}\right)+\frac{\psi_i(m)\left[\psi_j(m-\tau(m))-\varphi_j(m-\tau(m))\right]}{\varphi_j(m-\tau(m))\psi_j(m-\tau(m))}\\
\le&\frac{\varphi_i(m)-\psi_i(m)}{D}+\frac{B\left[\left(\psi_j(m-\tau(m))-\varphi_j(m-\tau(m))\right)\right]}{D^2},
\end{aligned}$$

where $j=1,\dots,N$. Therefore,

$$\begin{aligned}
&\|\mathfrak{T}_i(\varphi(n))-\mathfrak{T}_i(\psi(n))\|\\
\le&\sup_{n\in\mathbb{Z}}\Bigg|\sum_{m=-\infty}^{n-1}\prod_{s=m+1}^{n-1}(1-r_i^-)\left[\sum_{j=1}^{N}a_{ij}^+\left(\frac{\varphi_i(m)-\psi_i(m)}{D}+\frac{B\left[\left(\psi_j(m)-\varphi_j(m)\right)\right]}{D^2}\right)\right.\\
&\left.+\sum_{j=1}^{N}b_{ij}^+\left(\frac{\varphi_i(m)-\psi_i(m)}{D}+\frac{B\left[\left(\psi_j(m-\tau(m))-\varphi_j(m-\tau(m))\right)\right]}{D^2}\right)\right]\Bigg|\\
\le&\sup_{n\in\mathbb{Z}}\sum_{m=-\infty}^{n-1}(1-r_i^-)^{n-m-1}\sum_{j=1}^{N}(a_{ij}^++b_{ij}^+)\left(\left|\frac{\varphi_i(m)-\psi_i(m)}{D}+\frac{B\left[\left(\psi_j(m)-\varphi_j(m)\right)\right]}{D^2}\right|\right)\\
\le&\frac{(D+B)\sum_{j=1}^{N}(a_{ij}^++b_{ij}^+)}{D^2r_i^-}\|\varphi-\psi\|.
\end{aligned}$$

Hence, by (H_2), the mapping $\mathfrak{T}$ is contractive on $\mathbf{B}$. It follows that the mapping $\mathfrak{T}$ possesses a unique fixed point $\varphi^*\in\mathbf{B}$, such that $\mathfrak{T}\varphi^*=\varphi^*$. thus φ^* is an almost periodic solution.

Proof of Theorem 2.2: Let $y(n)=u(n)-u^*(n)$, where $y_i(n)=\{y_1(n),\dots,y_N(n)\}$. Then, for $i=1,\dots,N$,

$$\begin{aligned}
\Delta y_i(n)=&-r_i(n)y_i(n)+\left[\sum_{j=1}^{N}a_{ij}(m)\left(\frac{u_i(n)}{u_j(n)}-\frac{u_i^*(n)}{u_j^*(n)}\right)\right.\\
&\left.+\sum_{j=1}^{N}b_{ij}(n)\left(\frac{u_i(n)}{u_j(n-\tau(n))}-\frac{u_i^*(n)}{u_j^*(n-\tau(n))}\right)\right].
\end{aligned}$$

Define a function $\Phi(\mu)$ by setting

$$\Phi(\mu)=e^\mu-r_i^-e^\mu+e^\mu\frac{(D+B)}{D^2}\sum_{j=1}^{N}a_{ij}^++(\frac{e^\mu}{D}+\frac{(D+B)e^{\mu(\tau^++1)}}{D^2})\sum_{j=1}^{N}b_{ij}^+,\quad \mu\in[0,1].$$

Obviously, Φ is continuous on $[0,1]$. Then, by (H_3), we have

$$\begin{aligned}
\Phi(0)=&1-r_i^-+\frac{(D+B)}{D^2}\sum_{j=1}^{N}a_{ij}^++(\frac{1}{D}+\frac{D+B}{D^2})\sum_{j=1}^{N}b_{ij}^+\\
=&1-r_i^-+\frac{(D+B)\sum_{j=1}^{N}(a_{ij}^++b_{ij}^+)}{D^2}<0,
\end{aligned}$$

which implies that there exist a constant $\lambda \in (0, 1]$, such that

$$\Phi(\lambda) = e^{\lambda} - r_i^- e^{\lambda} + e^{\lambda}\frac{(D+B)}{D^2}\sum_{j=1}^{N} a_{ij}^+ + (\frac{e^{\lambda}}{D} + \frac{(D+B)e^{\lambda(\tau^+ +1)}}{D^2})\sum_{j=1}^{N} b_{ij}^+ < 0.$$

We consider the discrete Lyapunov functional $V(n) = |y(n)|e^{\lambda n}$. We calculate the difference of $V_i(n)$, for all $n \geq n_0$, $i = 1, \ldots, N$, it yields

$$\begin{aligned}
\Delta V_i(n) =& \Delta(|y_i(n)|e^{\lambda n}) = \Delta|y_i(n)|e^{\lambda(n+1)} + |y_i(n)|\Delta e^{\lambda n} \\
\leq& - r_i(n)|y_i(n)|e^{\lambda(n+1)} + \left[\sum_{j=1}^{N} a_{ij}(n)\left|\frac{u_i(n)}{u_j(n)} - \frac{u_i^*(n)}{u_j^*(n)}\right|\right. \\
&\left. + \sum_{j=1}^{N} b_{ij}(n)\left|\frac{u_i(n)}{u_j(n-\tau(n))} - \frac{u_i^*(n)}{u_j^*(n-\tau(n))}\right|\right] e^{\lambda(n+1)} + |y_i(n)|(e^{\lambda(n+1)} - e^{\lambda n}) \\
\leq& |y_i(n)|e^{\lambda(n+1)} - r_i(n)|y_i(n)|e^{\lambda(n+1)} + \left[\sum_{j=1}^{N} a_{ij}(n)\left|\frac{u_i(n)}{u_j(n)} - \frac{u_i^*(n)}{u_j^*(n)}\right|\right. \\
&\left. + \sum_{j=1}^{N} b_{ij}(n)\left|\frac{u_i(n)}{u_j(n-\tau(n))} - \frac{u_i^*(n)}{u_j^*(n-\tau(n))}\right|\right] e^{\lambda(n+1)}.
\end{aligned}$$

Let $M_i = e^{\lambda n_0}(\max_{n\in\mathbb{Z}[n_0,\infty)} |x_i(n) - x_i^*(n)|)$, for all $n \geq n_0$. We claim that

$$V(n) = |y(n)|e^{\lambda n} < M, \tag{4.7}$$

where $M = \max M_i$. We prove our claim by contradiction. Let be a $n_* > n_0$, such that $V_i(n_* + 1) \geq M_i$, and $V_i(n) < M$, $\forall n \in \mathbb{Z}[n_0, n_*)$, which implies that $V_i(n_*) - M_i \geq 0$ and $V_i(n) - M_i < 0$, $\forall n \in \mathbb{Z}[n_0, n_*)$. However,

$$\begin{aligned}
0 \leq& \Delta(V_i(n_*) - M) = \Delta V_i(n_*) \\
\leq& |y_i(n_*)|e^{\lambda(n_*+1)} - r_i(n_*)|y_i(n_*)|e^{\lambda(n_*+1)} + \left[\sum_{j=1}^{N} a_{ij}(n)\left|\frac{u_i(n)}{u_j(n)} - \frac{u_i^*(n)}{u_j^*(n)}\right|\right. \\
&\left. + \sum_{j=1}^{N} b_{ij}(n)\left|\frac{u_i(n)}{u_j(n-\tau(n))} - \frac{u_i^*(n)}{u_j^*(n-\tau(n))}\right|\right] e^{\lambda(n_*+1)} \\
\leq& (1 - r_i^-)M_i e^{\lambda} + \left[\sum_{j=1}^{N} a_{ij}^+\left|\frac{y_i(n)e^{\lambda(n_*+1)}}{D} + \frac{By_j(n)e^{\lambda(n_*+1)}}{D^2}\right|\right. \\
&\left. + \sum_{j=1}^{N} b_{ij}^+\left|\frac{y_i(n)e^{\lambda(n_*+1)}}{D} + \frac{By_j(n-\tau(n))e^{\lambda(n_*+1)}}{D^2}\right|\right] \\
\leq& \left[e^{\lambda} - r_i^- e^{\lambda} + e^{\lambda}\frac{(D+B)}{D^2}\sum_{j=1}^{N} a_{ij}^+ + (\frac{e^{\lambda}}{D} + \frac{Be^{\lambda(\tau^+ +1)}}{D^2})\sum_{j=1}^{N} b_{ij}^+\right] M_i.
\end{aligned}$$

Thus,

$$e^{\lambda} - r_i^- e^{\lambda} + e^{\lambda}\frac{(D+B)}{D^2}\sum_{j=1}^{N} a_{ij}^+ + (\frac{e^{\lambda}}{D} + \frac{(D+B)e^{\lambda(\tau^+ +1)}}{D^2})\sum_{j=1}^{N} b_{ij}^+ \geq 0,$$

which is contradictory. Hence (4.7) holds, then $|y(n)| < Me^{-\lambda n}$ for all $n \geq n_0$. This complete the proof.

5. Examples

We consider two specific examples.

Example 1: Consider the following system, when $N = 1$,

$$x'(t) = x(t)\left[\frac{1+\sin(t)}{20} - \frac{1}{40+\sin t}x(t)\right].$$

Discretization of the differential equations, we get

$$\Delta x(n) = x(n)\left[\frac{1+\sin(n)}{20} - \frac{1}{40+\sin t}x(n)\right]. \tag{5.1}$$

It is easy to verity that system (5.1) satisfies all the assumptions. Thus, system (5.1) admits an almost periodic solution that is quasi-uniformly asymptotically stable. Figure 1 shows the dynamics of system (5.1).

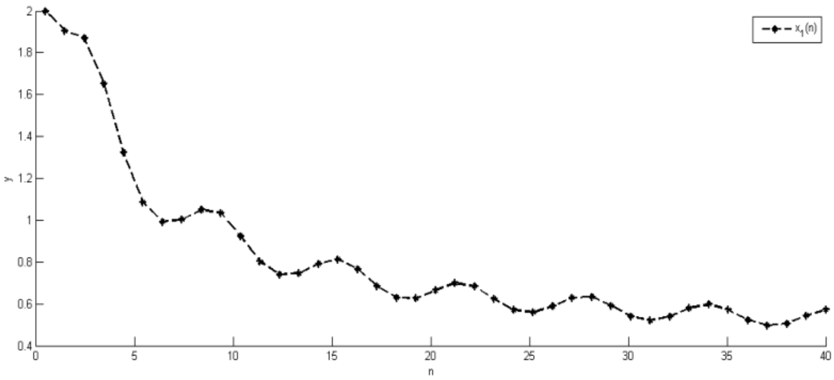

Figure 1. System(5.1) with x(1)=1 and $n \in \mathbb{Z}[0, 40)$.

Therefore, the system satisfies all the assumptions. It follows that this system has an almost periodic solution which is exponentially stable.

Example 2: Consider the following two dimensional predator-prey system, that is $N = 2$,

$$\begin{cases} x_1'(t) = x_1(t)\left[\frac{3+\sin t}{30} - x_1(t) - \frac{1+\sin t}{40}x_2(t)\right] \\ x_2'(t) = x_2(t)\left[1+\sin t - \frac{1}{40+\sin t}x_1(t) - \frac{2-\sin t}{20}x_2(t)\right]. \end{cases}$$

Discretization of the above differential equations, we obtain

$$\begin{cases} \Delta x_1(n) = x_1(n)\left[\frac{3+\sin n}{30} - x_1(n) - \frac{1+\sin n}{40}x_2(n)\right] \\ \\ \Delta x_2(n) = x_2(n)\left[1+\sin n - \frac{1}{40+\sin n}x_1(n) - \frac{2-\sin n}{20}x_2(n)\right]. \end{cases} \tag{5.2}$$

It is easy to verity that system (5.2) satisfies all the assumptions. Thus, system (5.2) admits an almost periodic solution that is exponential convergence. Figure 2 shows the dynamics of system (5.2).

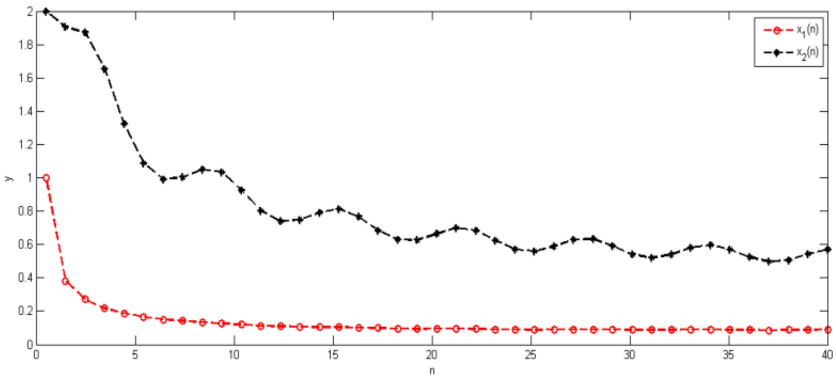

Figure 2. System(5.2) with $(x_1(1), x_2(1)) = (1, 2)$ and $n \in \mathbb{Z}[0, 40)$.

Therefore, the system satisfies all the assumptions. It follows that this system has an almost periodic solution which is exponentially stable.

6. Conclusions

In this paper, by applying exponential dichotomy, we obtain the existence and exponential convergence of positive non-autonomous discrete Lotka-Volterra systems. The method is different from the previous works [21–25]. I believe that this result has potential applications in population dynamics.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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