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Research article

Entire functions that share a small function with their linear difference polynomial

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Abstract: In this paper, we investigate the uniqueness of an entire function sharing a small function with its linear difference polynomial. Our results improve some results due to Li and Yi [11], Zhang, Chen and Huang [17], Zhang, Kang and Liao [18, 19] etc.

Keywords: entire functions; linear difference polynomial; small functions; unicity
Mathematics Subject Classification: 30D35

1. Introduction and main results

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory, see [6, 10, 14, 15]. In the following, a meromorphic function means meromorphic in the whole complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function $a(z)$ is said to be a small function of $f(z)$ if it satisfies $T(r, a) = S(r, f)$. If $a(z)$ is an entire function, then $a(z)$ is called an entire small function of $f(z)$. We say that two nonconstant meromorphic functions $f(z)$ and $g(z)$ share small function $a(z)$ CM(IM), if $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros counting multiplicities (ignoring multiplicities).

Let $f(z)$ be a nonconstant meromorphic function. Define

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r, f)}{\log r},$$

$$\rho_2(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ T(r, f)}{\log r},$$

$$\mu(g) = \underline{\lim_{r\to\infty}} \frac{\log^+ T(r, f)}{\log r},$$

by the order, the hyper-order and the lower order of $f(z)$, respectively.

If a meromorphic function $f(z)$ satisfies

$$\rho(f) = \mu(f) = \lim_{r\to\infty} \frac{\log^+ T(r, f)}{\log r},$$

then we say that $f(z)$ is of regular growth.

The exponents of convergence of zeros and poles of $f(z)$ are defined by

$$\lambda(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{f}\right)}{\log r},$$

and

$$\lambda\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log^+ N(r, f)}{\log r}.$$

Let $a(z)$ be an entire small function of $f(z)$. The exponent of convergence of zeros of $f(z) - a(z)$ is defined by

$$\lambda(f - a) = \overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{f-a}\right)}{\log r}.$$

If

$$\overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{f-a}\right)}{\log r} < \rho(f),$$

for $\rho(f) > 0$; and $N\left(r, \frac{1}{f-a}\right) = O(\log r)$, for $\rho(f) = 0$, then $a(z)$ is called a Borel exceptional value of $f(z)$. If $a = \infty$, then $N\left(r, \frac{1}{f-a}\right)$ is replaced by $N(r, f)$.

In addition, we also use the following notations [15]. We denote by $N_{k)}(r, f)$ the counting function for poles of $f(z)$ with multiplicity $\leq k$, and by $\overline{N}_{k)}(r, f)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, f)$ be the counting function for poles of $f(z)$ with multiplicity $\geq k$ and $\overline{N}_{(k}(r, f)$ be the corresponding one for which multiplicity is not counted. Set

$$N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f) + \cdots + \overline{N}_{(k}(r, f).$$

Similarly, we have the notations:

$$N_{k)}\left(r, \frac{1}{f}\right), \overline{N}_{k)}\left(r, \frac{1}{f}\right), N_{(k}\left(r, \frac{1}{f}\right), \overline{N}_{(k}\left(r, \frac{1}{f}\right), N_k\left(r, \frac{1}{f}\right), \cdots.$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $f(z)$ and $g(z)$ share 1 IM. We denote by $\overline{N}_L\left(r, \frac{1}{f-1}\right)$ the counting function for 1-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with multiplicity being not counted [16]. Similarly, we have the notation $\overline{N}_L\left(r, \frac{1}{g-1}\right)$. Especially, if $f(z)$ and $g(z)$ share 1 CM, then

$$\overline{N}_L\left(r, \frac{1}{f-1}\right) = \overline{N}_L\left(r, \frac{1}{g-1}\right) = 0.$$

If

$$N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) - 2N(r,a) \leq S(r,f) + S(r,g)$$

or

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) - 2\overline{N}(r,a) \leq S(r,f) + S(r,g),$$

then we call that $f(z)$ and $g(z)$ share a CM or IM almost.

For a nonzero complex constant η, we define the difference operators of $f(z)$ as $\triangle_\eta f(z) = f(z+\eta) - f(z)$ and $\triangle_\eta^n f(z) = \triangle_\eta(\triangle_\eta^{n-1} f(z))$, where $n(\geq 2)$ is an integer.

Let

$$L(z,f) = b_1(z)f(z+c_1) + b_2(z)f(z+c_2) + \cdots + b_n(z)f(z+c_n), \tag{1.1}$$

where $b_i(z)(\not\equiv 0)$ $(i = 1, 2, \cdots, n)$ are small functions of $f(z)$, and c_i $(i = 1, 2, \cdots, n)$ are distinct finite complex numbers.

In 1996, Brück [1] investigated the uniqueness question of entire functions sharing one value with its first derivatives and posed the following conjecture.

Conjecture 1. *Let $f(z)$ be a nonconstant entire function satisfying $\rho_2(f) < +\infty$, which is not a positive integer, and let a be a finite value. If $f(z)$ and $f'(z)$ share a CM, then*

$$f'(z) - a = c(f(z) - a),$$

where $c \neq 0$.

Brück proved that the conjecture is true provided $a \neq 0$ and $N(r, \frac{1}{f'}) = S(r,f)$ or $a = 0$. However, this conjecture is still an open question. Recently, many authors considered the uniqueness of meromorphic functions sharing a small function with their difference operators. For example, Liu and Yang [12] proved the following result.

Theorem A. *Let $f(z)$ be a transcendental entire function such that $\rho(f) < 1$, let n be a positive integer, let a be a finite value, and let η be a nonzero complex number. If $f(z)$ and $\Delta_\eta^n f(z)$ share a CM, then*

$$\Delta_\eta^n f - a = c(f - a),$$

where c is a nonzero finite complex number.

But Zhang et al. [18] found that such probability in the conclusion of Theorem A does not exist. They obtained the following result.

Theorem B. *Let $f(z)$ be a transcendental entire function such that $\rho(f) < 1$, let n be a positive integer, and let a and η be two finite values. Then $f(z)$ and $\Delta_\eta^n f(z)$ can not share a CM.*

In the proof of Theorem A, it is easy to know that the hypothesis $\rho(f) < 1$ plays an important role. Zhang, Kang and Liao [18] posed the following question: If $\rho(f) < 1$ is replaced by $\rho(f) \geq 1$ in Theorem A, Theorem A is valid or not?

In this direction, Zhang et al. [18] proved the following result.

Theorem C. *Let $f(z)$ be a transcendental entire function such that $\rho(f) < 2$, let n be a positive integer, let a and η be two nonzero constants, and let $\lambda(f-a) < \rho(f)$. If $f(z)$ and $\Delta_\eta^n f(z)$ share a CM, then*

$$f(z) = a + be^{cz},$$

where b, c are two nonzero constants such that $e^c = 1$.

But Zhang et al. [17] found out such probability in Theorem C does not exist. They proved the following result.

Theorem D. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$, let $L(z, f)$ be a linear difference polynomial of the form (1.1) with $b_i(z)(\not\equiv 0)$ $(i = 1, 2, \cdots, n)$ are entire small functions of $f(z)$, and let $a(z)$ be an entire small function of $f(z)$ satisfying $a(z) \not\equiv L(z, a)$ and $L(z, f) \not\equiv L(z, a)$. If $\delta(a, f) = 1$, then $f(z)$ and $L(z, f)$ can not share either $a(z)$ or $L(z, a)$ CM.*

By Theorem D, it is naturally to pose the following question.

Question 1: If $a(z), b_i(z)(\not\equiv 0)$ $(i = 1, 2, \cdots, n)$ are entire small functions of $f(z)$ is replaced by $a(z), b_i(z)(\not\equiv 0)$ $(i = 1, 2, \cdots, n)$ are small functions of $f(z)$, and $f(z)$ and $L(z, f)$ can not share either $a(z)$ or $L(z, a)$ CM is replaced by $f(z)$ and $L(z, f)$ can not share either $a(z)$ or $L(z, a)$ IM in Theorem D, Theorem D is valid or not?

In this paper, we give a positive answer to Question 1, and prove the following result.

Theorem 1. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$, let $L(z, f)$ be a linear difference polynomial of the form (1.1) with $b_i(z)(\not\equiv 0)$ $(i = 1, 2, \cdots, n)$ are small functions of $f(z)$, and let $a(z)$ be a small function of $f(z)$ satisfying $a(z) \not\equiv L(z, a)$ and $L(z, f) \not\equiv L(z, a)$. If $\delta(a, f) = 1$, then $f(z)$ and $L(z, f)$ can not share either $a(z)$ or $L(z, a)$ IM.*

Corollary 2. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$, let $L(z, f)$ be a linear difference polynomial of the form (1.1) with $b_i(z)(\not\equiv 0)$ $(i = 1, 2, \cdots, n)$ are small functions of $f(z)$, and let $a(z)$ be a small function of $f(z)$ satisfying $a(z) \not\equiv L(z, a)$ and $L(z, f) \not\equiv L(z, a)$. If $a(z)$ is a Borel exceptional small entire function of $f(z)$, then $f(z)$ and $L(z, f)$ can not share either $a(z)$ or $L(z, a)$ IM.*

Li and Yi [11] proved the following result.

Theorem E. *Let $f(z)$ be a transcendental entire function with $\lambda(f) < \rho(f) < +\infty$, let $a(z)(\not\equiv 0)$ be an entire small function of $f(z)$ satisfying $\rho(a) < \rho(f)$, let η be a nonzero complex number, and let n be a positive integer. If $f(z)$ and $\Delta_\eta^n f(z)$ share $a(z)$ CM, then $\rho(f) = 1$, and $\Delta_\eta^n f(z) \equiv f(z)$.*

Zhang et al. [19] improved Theorem E as follows.

Theorem F. *Let $f(z)$ be a transcendental entire function with $\lambda(f) < \rho(f) < +\infty$, let $a(z)(\not\equiv 0)$ be an entire small function of $f(z)$ satisfying $\rho(a) < \rho(f)$, let η be a nonzero complex number, and let n be a positive integer. If $f(z)$ and $\Delta_\eta^n f(z)$ share $a(z)$ CM, then $f(z) = ce^{c_1 z}$, where c and c_1 are two nonzero constants.*

Recently, Zhang et al. [17] proved

Theorem G. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$, let $a_1(z), a_2(z)$ be two entire small functions of $f(z)$ satisfying $a_1(z) \not\equiv a_2(z)$ and $\rho(a_j) < 1 (j = 1, 2)$, and let $L(z, f)$ be a linear* *difference polynomial of the form (1.1) with $b_i(z)(\not\equiv 0)$ are entire small functions of $f(z)$, $\rho(b_i) < 1$ $(i = 1, 2, \cdots, n)$, and $a_1(z) \not\equiv L(z, a_2(z))$. If $\delta(a_2, f) + \delta(a_2, L(z, f)) > 1$, and $f(z)$ and $L(z, f)$ share $a_1(z)$ CM, then $f(z) \equiv L(z, f)$.*

In this paper, we improve Theorems E–G as follows.

Theorem 3. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$, let $a_1(z), a_2(z)$ be two small functions of $f(z)$ satisfying $a_1(z) \not\equiv a_2(z)$, and let $L(z, f)$ be a linear difference polynomial of the form (1.1) with $b_i(z)(\not\equiv 0)$ $(i = 1, 2, \cdots, n)$ are small functions of $f(z)$, and $a_1(z) \not\equiv L(z, a_2(z))$. If $\delta(a_2, f) + \delta(a_2, L(z, f)) > 1$, and $f(z)$ and $L(z, f)$ share $a_1(z)$ CM, then $L(z, f) \equiv f(z)$.*

Corollary 4. *Let $f(z)$ be a transcendental entire function with $\lambda(f) < \rho(f) < +\infty$, let $a(z)$ be an entire small function of $f(z)$ such that $a(z) \not\equiv 0$, let n be a positive integer, and let η be a nonzero finite value. If $f(z)$ and $\Delta_\eta^n f(z)$ share $a(z)$ IM, then $f(z) = ce^{c_1 z}$, where c and c_1 are two nonzero constants.*

By Corollary 4, we remove the condition $\rho(a) < \rho(f)$ in Theorem F.

2. Preliminary lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1. *[2, 3, 7–9] Let $f(z)$ be a meromorphic function with $\rho(f) < +\infty$, and let c be a nonzero finite constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f).$$

Lemma 2.2. *[13, 14] Let $f(z)$ be a nonconstant meromorhic function, and let $a_0, a_1, \cdots, a_n$ be small functions of $f(z)$ such that $a_n \not\equiv 0$. Then*

$$T(r, a_n f^n + \cdots + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.3. *[14] Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $\rho(f)$ and $\rho(g)$ be the order of $f(z)$ and $g(z)$, respectively. Then we have*

$$\rho(f \cdot g) \leq \max\{\rho(f), \rho(g)\}.$$

Lemma 2.4. *[14] Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, let $\rho(f)$ be the order of $f(z)$, and let $\mu(g)$ be the lower order of $g(z)$. If $\rho(f) < \mu(g)$, then we have*

$$T(r, f) = o(T(r, g)).$$

Lemma 2.5. *[5] Let $f(z)$ and $g(z)$ be two meromorphic functions. If $f(z)$ and $g(z)$ share 1 IM. Then one of the following cases must occur:*

(1)

$$\begin{aligned} T(r, f) + T(r, g) \leq & 2\left[N_2(r, f) + N_2(r, g) + N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right)\right] \\ & + 3\overline{N}_L\left(r, \frac{1}{f-1}\right) + 3\overline{N}_L\left(r, \frac{1}{g-1}\right) + S(r, f) + S(r, g); \end{aligned}$$

(2) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where* $a(\neq 0)$ *and* b *are two constants.*

Lemma 2.6. *[4] Let* n *be a positive integer, let* $f(z)$ *be a transcendental meromorphic function of finite order with two Borel exceptional values* 0 *and* ∞*, and let* η *be a nonzero constant such that* $\Delta_\eta^n f(z) \not\equiv 0$*. If* $f(z)$ *and* $\Delta_\eta^n f(z)$ *share* $0, \infty$ *CM, then* $f(z) = e^{cz+c_1}$*, where* $c(\neq 0)$, c_1 *are two constants.*

Lemma 2.7. *Let* $f(z)$ *be a nonconstant entire function with* $\rho(f) < +\infty$*, and let* $a(z)$ *be a small entire function of* $f(z)$*. If* $\lambda(f-a) < \rho(f)$*, then* $\delta(a, f) = 1$*.*

Proof. We consider two cases.

Case 1. $\rho(f) > 0$. Since $\lambda(f-a) < \rho(f)$, then

$$f - a = He^Q,$$

where H is the canonical product of zeros of $f-a$, and Q is a nonzero polynomial.

Since $a(z)$ is a small entire function of $f(z)$ and $\lambda(f-a) < \rho(f)$, by Hadamard's factorization theorem, we have $\rho(H) = \lambda(H) = \lambda(f-a) < \rho(f)$.

Obviously, $T(r, f-a) = T(r, f) + S(r, f)$. So we get $\rho(f-a) = \rho(f)$.

From Lemma 2.3 and e^Q is of regular growth, we obtain

$$\rho(H) < \rho(f) = \rho(f-a) \leq max\{\rho(H), \rho(e^Q)\} = \rho(e^Q),$$

and

$$\rho(e^Q) \leq max\{\rho(H), \rho(f)\} = \rho(f).$$

It follows that $\rho(f) = \rho(e^Q) = \mu(e^Q)$. From Lemma 2.4, we get $T(r, H) = o(T(r, e^Q))$. Then we have

$$\begin{aligned} T(r, f) =& T(r, f-a) + S(r, f) \\ =& T(r, He^Q) + S(r, f) \\ \leq& T(r, H) + T(r, e^Q) + S(r, f) \\ \leq& T(r, e^Q) + S(r, f) + S(r, e^Q). \end{aligned}$$

We also get $T(r, e^Q) \leq T(r, f) + S(r, f) + S(r, e^Q)$. Then we obtain $T(r, f) = T(r, e^Q) + S(r, f)$. So we have

$$\delta(a, f) = 1 - \varlimsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)} \leq 1 - \varlimsup_{r\to\infty} \frac{N\left(r, \frac{1}{H}\right)}{T(r, e^Q)} = 1.$$

Case 2. $\rho(f) = 0$. As defined in the introduction, when $\rho(f) = 0$, the number of zeros of $f-a$ is finite. We obtain $N\left(r, \frac{1}{f-a}\right) = o(T(r, f))$. So we have $\delta(a, f) = 1$.

This completes the proof of Lemma 2.7. □

Lemma 2.8. *[6, 13–15] Let* $f(z)$ *be a transcendental meromorphic function, then*

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty.$$

3. Proof of Theorem 1

Considering that $f(z)$ is a transcendental entire function with $\rho_2(f) < 1$ and by Lemma 2.1, we have

$$m\left(r, \frac{1}{f-a}\right) = m\left(r, \frac{L(z,f)-L(z,a)}{f-a} \cdot \frac{1}{L(z,f)-L(z,a)}\right) \leq m\left(r, \frac{1}{L(z,f)-L(z,a)}\right) + S(r,f), \tag{3.1}$$

Then we obtain

$$\frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} \leq \frac{m\left(r, \frac{1}{L(z,f)-L(z,a)}\right)}{T(r,L(z,f))} \cdot \frac{T(r,L(z,f))}{T(r,f)} + \frac{S(r,f)}{T(r,f)}.$$

Thus we get

$$\varliminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} \leq \varliminf_{r\to\infty} \frac{m\left(r, \frac{1}{L(z,f)-L(z,a)}\right)}{T(r,L(z,f))} \cdot \varlimsup_{r\to\infty} \frac{T(r,L(z,f))}{T(r,f)} + \varlimsup_{r\to\infty} \frac{S(r,f)}{T(r,f)}.$$

Then from $\delta(a, f) = 1$, we obtain

$$1 = \delta(a, f) \leq \delta(L(z, a), L(z, f)) \leq 1. \tag{3.2}$$

So we have $\delta(a, f) = \delta(\infty, f) = 1$, $\delta(L(z, a), L(z, f)) = \delta(\infty, L(z, f)) = 1$. Suppose f and $L(z, f)$ share $a(z)$ IM. So we have

$$\overline{N}\left(r, \frac{1}{L(z,f)-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{f-a}\right) = S(r,f). \tag{3.3}$$

From the Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}
&T(r,L(z,f))\\
\leq&\overline{N}\left(r, \frac{1}{L(z,f)-a}\right) + \overline{N}\left(r, \frac{1}{L(z,f)-L(z,a)}\right) + \overline{N}(r,L(z,f)) + S(r,L(z,f))\\
\leq&\overline{N}\left(r, \frac{1}{L(z,f)-a}\right) + S(r,L(z,f))\\
=&\overline{N}_{1)}\left(r, \frac{1}{L(z,f)-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{L(z,f)-a}\right) + S(r,L(z,f))\\
\leq&N_{1)}\left(r, \frac{1}{L(z,f)-a}\right) + \frac{1}{2}N_{(2}\left(r, \frac{1}{L(z,f)-a}\right) + S(r,L(z,f))\\
\leq&N_{1)}\left(r, \frac{1}{L(z,f)-a}\right) + \frac{1}{2}\left[N\left(r, \frac{1}{L(z,f)-a}\right) - N_{1)}\left(r, \frac{1}{L(z,f)-a}\right)\right]\\
+&S(r,L(z,f))\\
\leq&\frac{1}{2}N\left(r, \frac{1}{L(z,f)-a}\right) + \frac{1}{2}N_{1)}\left(r, \frac{1}{L(z,f)-a}\right) + S(r,L(z,f))\\
\leq&\frac{1}{2}T(r,L(z,f)) + \frac{1}{2}N_{1)}\left(r, \frac{1}{L(z,f)-a}\right) + S(r,L(z,f)).
\end{aligned}$$

It follows that

$$\overline{N}\left(r,\frac{1}{L(z,f)-a}\right)=T(r,L(z,f))+S(r,L(z,f)). \tag{3.4}$$

By (3.3) and (3.4), we get

$$T(r,L(z,f))=S(r,f). \tag{3.5}$$

From (3.1), we obtain

$$\begin{aligned}T(r,f)=&m\left(r,\frac{1}{f-a}\right)+S(r,f)\\ \leq&m\left(r,\frac{1}{L(z,f)-L(z,a)}\right)+S(r,f)\\ \leq&T(r,L(z,f))+S(r,f).\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we get $T(r,L(z,f))=S(r,L(z,f))$, a contradiction.

Suppose $f(z)$ and $L(z,f)$ share $L(z,a)$ IM. Similarly, we get $T(r,f)=S(r,f)$, a contradiction.

This completes the proof of Theorem 1.

4. Proof of Corollary 2

Since a is a Borel exceptional small entire function of $f(z)$, by Lemma 2.7, it follows that $\delta(a,f)=1$. Hence by Theorem 1 we know that $f(z)$ and $L(z,f(z))$ can not share either $a(z)$ or $L(z,a(z))$ IM. This completes the proof of Corollary 2.

5. Proof of Theorem 3

According to the definition of $\delta(a_2,f)$ and $\delta(a_2,L(z,f))$, we have

$$\overline{\lim_{r\to\infty}}\frac{N\left(r,\frac{1}{f-a_2}\right)}{T(r,f)}=1-\delta(a_2,f)=\alpha,$$

$$\overline{\lim_{r\to\infty}}\frac{N\left(r,\frac{1}{L(z,f)-a_2}\right)}{T(r,L(z,f))}=1-\delta(a_2,L(z,f))=\beta.$$

Then we get

$$N\left(r,\frac{1}{f-a_2}\right)\leq(\alpha+\varepsilon)T(r,f), \tag{5.1}$$

$$N\left(r,\frac{1}{L(z,f)-a_2}\right)\leq(\beta+\varepsilon)T(r,L(z,f)), \tag{5.2}$$

where $\varepsilon=\frac{1}{6}[\delta(a_2,f)+\delta(a_2,L(z,f))-1]$.

From (5.1) and (5.2), we obtain

$$N\left(r,\frac{1}{f-a_2}\right)+N\left(r,\frac{1}{L(z,f)-a_2}\right)\leq(\alpha+\varepsilon)T(r,f)+(\beta+\varepsilon)T(r,L(z,f)). \tag{5.3}$$

According to the Nevanlinna's second fundamental theorem, Lemma 2.5 and (5.3), we get

$$\begin{aligned}&T(r,f)+T(r,L(z,f))\\ \leq&T(r,f-a_2)+T(r,L(z,f)-a_2)+S(r,f)\\ \leq&2\left[N_2\left(r,\frac{1}{f-a_2}\right)+N_2\left(r,\frac{1}{L(z,f)-a_2}\right)\right]+S(r,f)+S(r,L(z,f))\\ \leq&2(\alpha+\varepsilon)T(r,f)+2(\beta+\varepsilon)T(r,L(z,f))+S(r,f)+S(r,L(z,f))\\ \leq&2[1-\delta(a_2,f)+\varepsilon]T(r,f)+2[1-\delta(a_2,L(z,f))+\varepsilon]T(r,L(z,f))\\ +&S(r,f)+S(r,L(z,f)).\end{aligned}$$

Then we have

$$[2\delta(a_2,f)-2\varepsilon-1]T(r,f)+[2\delta(a_2,L(z,f))-2\varepsilon-1]T(r,L(z,f))\leq S(r,f)+S(r,L(z,f)). \tag{5.4}$$

Since $\delta(a_2,f)+\delta(a_2,L(z,f))>1$, we get $\alpha,\beta<1$. Hence, we obtain

$$2\delta(a_2,f)+2\delta(a_2,L(z,f))-4\varepsilon-2=\frac{4}{3}[\delta(a_2,f)+\delta(a_2,L(z,f))-1]>0.$$

If $T(r,f)\leq T(r,L(z,f))$, where $r\in I$ and I is a set of infinite logarithmic measure, then we have

$$\begin{aligned}&[2\delta(a_2,f)-2\varepsilon-1]T(r,f)+[2\delta(a_2,L(z,f))-2\varepsilon-1]T(r,L(z,f))\\ \geq&[2\delta(a_2,f)+2\delta(a_2,L(z,f))-4\varepsilon-2]T(r,f)\\ =&\frac{4}{3}[\delta(a_2,f)+\delta(a_2,L(z,f))-1]T(r,f).\end{aligned} \tag{5.5}$$

From (5.4) and (5.5), we have

$$\frac{4}{3}[\delta(a_2,f)+\delta(a_2,L(z,f))-1]T(r,f)\leq S(r,f)+S(r,L(z,f))\leq S(r,f).$$

It follows that $T(r,f)=S(r,f)$, a contradiction. If $T(r,L(z,f))\leq T(r,f)$, where $r\in I$ and I is a set of infinite logarithmic measure. Similarly, we get

$$T(r,L(z,f))=S(r,f). \tag{5.6}$$

Since $f(z)$ and $L(z,f)$ share $a_1(z)$ CM, we obtain

$$N\left(r,\frac{1}{f-a_1}\right)=N\left(r,\frac{1}{L(z,f)-a_1}\right). \tag{5.7}$$

From (5.1), we have

$$N\left(r,\frac{1}{f-a_2}\right)<\frac{1+\alpha}{2}T(r,f). \tag{5.8}$$

By the Nevanlinna's second fundamental theorem, (5.7) and (5.8), we get

$$T(r,f)\leq N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right)+S(r,f)$$

$$
\begin{aligned}
&< N\left(r, \frac{1}{L(z,f)-a_1}\right) + \frac{1+\alpha}{2} T(r,f) + S(r,f)\\
&\leq T(r, L(z,f)) + \frac{1+\alpha}{2} T(r,f) + S(r,f).
\end{aligned}
$$

It follows that

$$
\left(1-\frac{1+\alpha}{2}\right) T(r,f) \leq T(r, L(z,f)) + S(r,f). \tag{5.9}
$$

By (5.6) and (5.9), we have $T(r, L(z,f)) = S(r,f) = S(r, L(z,f))$, a contradiction.

Next we prove $L(z,a_2) \equiv a_2$. Suppose $L(z,a_2) \not\equiv a_2$. Similarly, from (3.2), it is easy to know that $\delta(a_2, f) \leq \delta(L(z,a_2), L(z,f))$. It follows that

$$
\begin{aligned}
&\delta(L(z,a_2), L(z,f)) + \delta(a_2, L(z,f)) + \delta(\infty, L(z,f))\\
\geq &\delta(a_2, f) + \delta(a_2, L(z,f)) + \delta(\infty, L(z,f)) > 2,
\end{aligned}
$$

a contradiction.

Set

$$
F = \frac{f(z)-a_2(z)}{a_1(z)-a_2(z)}, G = \frac{L(z,f)-a_2(z)}{a_1(z)-a_2(z)}. \tag{5.10}
$$

Since $\delta(a_2, f) + \delta(a_2, L(z,f)) > 1$, we have

$$
\delta(0, F) + \delta(0, G) > 1. \tag{5.11}
$$

From Lemma 2.5, we have

$$
F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \tag{5.12}
$$

where $a(\neq 0)$ and b are two constants.

Clearly, we have

$$
T(r, F) = T(r, G) + O(1). \tag{5.13}
$$

Next we consider three cases:

Case 1. $b \neq 0, -1$. In the following, we consider two subcases.

Case 1.1. $a - b - 1 \neq 0$.

From (5.12), we have

$$
\overline{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) = \overline{N}\left(r, \frac{1}{F}\right). \tag{5.14}
$$

Furthermore, by the Nevanlinna's second fundamental theorem, we have

$$
\begin{aligned}
T(r,G) &\leq \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) + S(r,G)\\
&\leq \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,G).
\end{aligned} \tag{5.15}
$$

According to the definition of $\delta(0, F)$ and $\delta(0, G)$, we have

$$
\overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{F}\right)}{T(r,F)} = 1 - \delta(0, F) = \alpha_1, \tag{5.16}
$$

$$\overline{\lim_{r\to\infty}}\frac{N\left(r,\frac{1}{G}\right)}{T(r,G}=1-\delta(0,G)=\beta_1. \tag{5.17}$$

Then we get

$$N\left(r,\frac{1}{F}\right)\leq(\alpha_1+\varepsilon_1)T(r,F), \tag{5.18}$$

$$N\left(r,\frac{1}{G}\right)\leq(\beta_1+\varepsilon_1)T(r,G), \tag{5.19}$$

where $\varepsilon_1=\frac{1}{4}[\delta(0,F)+\delta(0,G)-1]$.

From (5.13), (5.18), (5.19) and Lemma 2.8, we obtain

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)\\ \leq&N\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{F}\right)\\ \leq&(\beta_1+\varepsilon_1)T(r,G)+(\alpha_1+\varepsilon_1)T(r,F)\\ \leq&(\alpha_1+\beta_1+2\varepsilon_1)T(r,G)+S(r,G).\end{aligned}$$

It follows that

$$\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)\leq(\alpha_1+\beta_1+2\varepsilon_1)T(r,G)+S(r,G). \tag{5.20}$$

By (5.15)–(5.20) and $\overline{N}(r,G)\leq N(r,G)=S(r,G)$, we have

$$[1-(\alpha_1+\beta_1+2\varepsilon_1)]T(r,G)=\frac{1}{2}[\delta(0,F)+\delta(0,G)-1]T(r,G)=S(r,G).$$

It follows that $T(r,G)=S(r,G)$, a contradiction.

Case 1.2. $a-b-1=0$. By (5.12), we know

$$\overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right)=\overline{N}(r,F).$$

Similarly, we deduce a contradiction.

Case 2. $b=-1$. Then (5.12) becomes

$$F=\frac{a}{(a+1)-G}. \tag{5.21}$$

Next we consider two subcases:

Case 2.1. $a+1\neq 0$. By (5.21), we have

$$\overline{N}\left(r,\frac{1}{G-(a+1)}\right)=\overline{N}(r,F).$$

Similarly, we deduce a contradiction as in Case 1.1.

Case 2.2. $a+1=0$. From (5.21), we obtain $F\cdot G\equiv 1$. Then

$$(f-a_2)\cdot(L(z,f)-a_2)\equiv(a_1-a_2)^2.$$

Since f is an entire function, we know

$$\begin{aligned}N(r,\frac{1}{f-a_2})\leq& N(r,L(z,f)-a_2)+2N\left(r,\frac{1}{a_1-a_2}\right)\\ \leq& 2T(r,a_1-a_2)+S(r,f)=S(r,f).\end{aligned}$$

By Lemmas 2.1, 2.2 and 2.8, we have

$$\begin{aligned}2T(r,f)=&2T(r,f-a_2)=2T\left(r,\frac{1}{f-a_2}\right)+O(1)\\ =&2m\left(r,\frac{1}{f-a_2}\right)+2N\left(r,\frac{1}{f-a_2}\right)+O(1)\\ =&2m\left(r,\frac{1}{f-a_2}\right)+S(r,f)=m\left(r,\frac{1}{(f-a_2)^2}\right)+S(r,f)\\ \leq& m\left(r,\frac{(a_1-a_2)^2}{(f-a_2)^2}\right)+S(r,f)=m\left(r,\frac{L(z,f)-a_2}{f-a_2}\right)+S(r,f)\\ \leq& S(r,f).\end{aligned}$$

It gives $T(r,f)=S(r,f)$, a contradiction.

Case 3. $b=0$. From (5.12) we have

$$F=\frac{G+(a-1)}{a}. \tag{5.22}$$

Next we consider two subcases:

Case 3.1. $a+1\neq 0$. By (5.22), we have

$$\overline{N}\left(r,\frac{1}{G-(a+1)}\right)=\overline{N}(r,F).$$

Similarly, we deduce a contradiction as in Case 1.1.

Case 3.2. $a+1=0$. From (5.22), we obtain $F\equiv G$. So we have $f\equiv L(z,f)$.

This completes the proof of Theorem 3.

6. Proof of Corollary 4

Set

$$F_1=\frac{f}{a},G_1=\frac{\Delta_\eta^n f}{a}.$$

Since $a(z)$ is an entire small function of $f(z)$, by Lemma 2.7 it follows $\delta(0,f)=1$. Considering f and $\Delta_\eta^n f$ share a IM, we deduce that F_1 and G_1 share 1 CM almost. Thus by Theorem 3, we know $F_1\equiv G_1$. So we have $f\equiv\Delta_\eta^n f$. Since f and $\Delta_\eta^n f$ share $0,\infty$ CM, by Lemma 2.6, we deduce that $f(z)=ce^{c_1z}$, where c and c_1 are two nonzero constants. This completes the proof of Corollary 4.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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